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AN INDEFINITE CONCAVE-CONVEX EQUATION UNDER A NEUMANN BOUNDARY CONDITION II

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Abstract. We proceed with the investigation of the problem

$$(\mathrm{P}_\lambda) \quad -\Delta u = \lambda b(x)|u|^{q-2}u + a(x)|u|^{p-2}u \quad \text{in } \Omega, \qquad \frac{\partial u}{\partial \mathbf{n}} = 0 \quad \text{on } \partial\Omega,$$

where Ω is a bounded smooth domain in $\mathbb{R}^N$ $(N \geq 2)$, $1 < q < 2 < p$, $\lambda \in \mathbb{R}$, and $a, b \in C^\alpha(\overline{\Omega})$ with $0 < \alpha < 1$. Dealing now with the case $b \geq 0$, $b \not\equiv 0$, we show the existence (and several properties) of an unbounded subcontinuum of nontrivial nonnegative solutions of (P_λ). Our approach is based on *a priori* bounds, a regularisation procedure, and Whyburn's topological method.

1. Introduction and statements of main results

Let Ω be a bounded domain of $\mathbb{R}^N$ $(N \geq 2)$ with smooth boundary $\partial\Omega$. This paper is devoted to the study of nontrivial nonnegative solutions for the problem

$$(\mathrm{P}_\lambda) \qquad \begin{cases} -\Delta u = \lambda b(x) u^{q-1} + a(x) u^{p-1} & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega, \end{cases}$$

where

- $\Delta = \sum_{j=1}^{N} \partial^2/\partial x_j^2$ is the usual Laplacian in $\mathbb{R}^N$;

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- $\lambda \in \mathbb{R}$;
- $1 < q < 2 < p < \infty$;
- $a, b \in C^\alpha(\overline{\Omega})$ for some $\alpha \in (0,1)$, $a, b \not\equiv 0$, and $b \geq 0$;
- $\mathbf{n}$ is the unit outer normal to the boundary $\partial\Omega$.

By a *nonnegative* (*classical*) *solution* of (P_λ) we mean a nonnegative function $u \in C^{2+\theta}(\overline{\Omega})$ for some $\theta \in (0,1)$ which satisfies (P_λ) in the classical sense. When $\lambda \geq 0$, the strong maximum principle and the boundary point lemma apply to (P_λ), and as a consequence a nontrivial nonnegative solution of (P_λ) is positive on $\overline{\Omega}$. In the sequel we call it a *positive solution* of (P_λ).

In this article, we proceed with the investigation of (P_λ) made in [13]. We are now concerned with the case where $b \geq 0$ and we investigate the existence of an unbounded subcontinuum $\mathcal{C}_0 = \{(\lambda, u)\}$ of nontrivial nonnegative solutions of (P_λ), bifurcating from the trivial line $\{(\lambda, 0)\}$. Note that since $q < 2$ the nonlinearity in (P_λ) is not differentiable at $u = 0$, so that we cannot apply the standard local bifurcation theory from [5] directly. When $a \equiv 0$, $\Gamma_0 = \{(0, c) : c$ is a positive constant$\}$ is a continuum of positive solutions of (P_λ) bifurcating at $(0,0)$, and there is no positive solution for any $\lambda \neq 0$. Throughout this paper we shall then assume $a \not\equiv 0$, and we shall observe that the existence and behavior of $\mathcal{C}_0$ depend on the sign of a.

To state our main results we introduce the following sets:

$$\Omega^a_\pm = \{x \in \Omega : a(x) \gtrless 0\}, \qquad \Omega^b_+ = \{x \in \Omega : b(x) > 0\}.$$

We remark that $\Omega^a_\pm, \Omega^b_+$ are all open subsets of Ω. We shall use the following conditions on these sets:

(H_1) $\Omega^a_\pm$ are both smooth subdomains of Ω, with either

$$\overline{\Omega^a_+} \subset \Omega \quad \text{and} \quad \Omega = \overline{\Omega^a_+} \cup \Omega^a_-, \quad \text{or} \tag{1.1}$$

$$\overline{\Omega^a_-} \subset \Omega \quad \text{and} \quad \Omega = \overline{\Omega^a_-} \cup \Omega^a_+. \tag{1.2}$$

(H_2) Under (H_1) there exist a function α^+ which is continuous, positive, and bounded away from zero in a tubular neighbourhood of $\partial\Omega^a_+$ in Ω^a_+ and $\gamma > 0$ such that

$$a^+(x) = \alpha^+(x)\,\mathrm{dist}(x, \partial\Omega^a_+)^\gamma,$$

where $\mathrm{dist}(x, A)$ denotes the distance function to a set A, and moreover,

$$2 < p < \min\left\{\frac{2N}{N-2}, \frac{2N+\gamma}{N-1}\right\} \quad \text{if } N > 2.$$

Assumptions (H_1) and (H_2) are used to obtain *a priori* bounds on positive solutions of $(\mathrm{Q}_{\lambda,\varepsilon})$ below, cf. Amann and López-Gómez [2].

Remark 1.1. In (H_1) we may allow $\Omega^a_+ = \emptyset$ (resp. $\Omega^a_- = \emptyset$). In this case it is understood that $\Omega = \Omega^a_-$ (resp. $\Omega = \Omega^a_+$).

Let us recall that a positive solution u of (P_λ) is said to be *asymptotically stable* (resp. *unstable*) if $\gamma_1(\lambda, u) > 0$ (resp. < 0), where $\gamma_1(\lambda, u)$ is the smallest eigenvalue of the linearized eigenvalue problem at u, namely,

$$\begin{cases} -\Delta\phi = \lambda(q-1)b(x)u^{q-2}\phi + (p-1)a(x)u^{p-2}\phi + \gamma\phi & \text{in } \Omega, \\ \dfrac{\partial\phi}{\partial\mathbf{n}} = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

In addition, u is said to be *weakly stable* if $\gamma_1(\lambda, u) \geq 0$.

First we state a result on the existence of an unbounded subcontinuum of nontrivial nonnegative solutions of (P_λ), and its behavior and stability in the case $\int_\Omega a \geq 0$.

THEOREM 1.2. *Assume $\int_\Omega a \geq 0$ and $p \leq 2N/(N-2)$ if $N > 2$. Then (P_λ) possesses an unbounded subcontinuum of nonnegative solutions $\mathcal{C}_0 = \{(\lambda, u)\} \subset \mathbb{R} \times C(\overline{\Omega})$ bifurcating at $(0,0)$. Moreover, the following assertions hold:*

(a) *There is no positive solution of (P_λ) for any $\lambda \geq 0$. Consequently, if $(\lambda, u) \in \mathcal{C}_0 \setminus \{(0,0)\}$ then $\lambda < 0$.*
(b) *Any positive solution of (P_λ) is unstable.*
(c) *$\mathcal{C}_0 \cap \{(\lambda, 0) : \lambda \neq 0\} = \emptyset$. More precisely, for any $\Lambda > 0$ there exists $\delta_0 > 0$ such that $\max_{\overline{\Omega}} u > \delta_0$ for all nontrivial nonnegative solutions of (P_λ) with $\lambda \leq -\Lambda$.*
(d) *If (H_1) and (H_2) hold then for any $\Lambda > 0$ there exists $C_\Lambda > 0$ such that $\max_{\overline{\Omega}} u \leq C_\Lambda$ for all $(\lambda, u) \in \mathcal{C}_0$ with $\lambda \in [-\Lambda, 0)$. Consequently, $\{\lambda \in \mathbb{R} : (\lambda, u) \in \mathcal{C}_0 \setminus \{(0,0)\}\} = (-\infty, 0)$. In this case, (P_λ) has at least one nontrivial nonnegative solution for every $\lambda < 0$, see Figure 1.*

REMARK 1.3. The non-existence result in assertion (a) of Theorem 1.2 does not require the condition $p \leq 2N/(N-2)$ if $N > 2$.

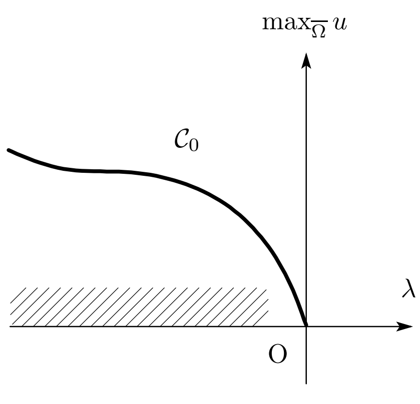

FIGURE 1. An unbounded subcontinuum of nontrivial nonnegative solutions in the case $\int_\Omega a \geq 0$.

To state our result corresponding to Theorem 1.2 in the case $\int_\Omega a < 0$ we consider the following eigenvalue problem:

$$\begin{cases} -\Delta\phi = \lambda b(x)\phi + \sigma\phi & \text{in } \Omega, \\ \dfrac{\partial\phi}{\partial\mathbf{n}} = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.4}$$

For $\lambda > 0$ we denote by σ_λ the smallest eigenvalue of (1.4), which is simple and principal, and by ϕ_λ a positive eigenfunction associated with σ_λ. Note that $\sigma_\lambda < 0$.

We shall deal with the following cases as well:

(H_{01}) $\Omega_+^a \cap \Omega_+^b \neq \emptyset$.
(H_{02}) $\Omega_+^a = \emptyset$.

Theorem 1.4. *Assume $\int_\Omega a < 0$ and $p < 2N/(N-2)$ if $N > 2$. Then* (P_λ) *possesses an unbounded subcontinuum of nontrivial nonnegative solutions $\mathcal{C}_0 = \{(\lambda, u)\} \subset \mathbb{R} \times C(\overline{\Omega})$ bifurcating at $(0,0)$ and such that $(\mathcal{C}_0 \setminus \{(0,0)\}) \cap ([0,\infty) \times C(\overline{\Omega}))$ consists of positive solutions of (P_λ) in $\lambda \geq 0$. Moreover, the following assertions hold*:

(a) *There exists $\delta_0 > 0$ such that $\max_{\overline{\Omega}} u > \delta_0$ for all nontrivial nonnegative solutions of (P_λ) with $\lambda \leq 0$. Consequently, $\mathcal{C}_0$ bifurcates to the region $\lambda > 0$ at $(0,0)$ and does not meet $\{(\lambda, 0) : \lambda < 0\}$.*

(b) *Let $\Lambda > 0$. Then there exists $c_\Lambda > 0$ such that $u \geq c_\Lambda \phi_\Lambda$ on $\overline{\Omega}$ for all positive solutions u of (P_λ) with $\lambda \geq \Lambda$. Consequently, $\mathcal{C}_0$ does not meet $\{(\lambda, 0) : \lambda > 0\}$.*

(c) *For some $\Lambda_0 \in (0, \infty]$, $\mathcal{C}_0$ contains $\{(\lambda, \underline{u}_\lambda) : 0 < \lambda < \Lambda_0\}$, where $\underline{u}_\lambda$ is the minimal positive solution of (P_λ) for $\lambda \in (0, \Lambda_0)$, i.e. $\underline{u}_\lambda \leq u$ on $\overline{\Omega}$ for all positive solutions u of (P_λ). In addition, we have*:
   - (c1) *$\lambda \mapsto \underline{u}_\lambda$ is increasing*;
   - (c2) *$\lambda \mapsto \underline{u}_\lambda$ is C^∞ from $(0, \Lambda_0)$ to $C^{2+\alpha}(\overline{\Omega})$*;
   - (c3) *$\underline{u}_\lambda \to 0$ and $\lambda^{-1/(p-q)}\underline{u}_\lambda \to c^*$ in $C^{2+\alpha}(\overline{\Omega})$ as $\lambda \to 0^+$, where $c^* = \left(\int_\Omega b / (-\int_\Omega a)\right)^{1/(p-q)}$*;
   - (c4) *$\underline{u}_\lambda$ is asymptotically stable for $\lambda \in (0, \Lambda_0)$.*

   Finally, there exists $\delta > 0$ such that if $|\lambda| \leq \delta$ and u is a positive solution of (P_λ) such that $\max_{\overline{\Omega}} u \leq \delta$ then $(\lambda, u) \in \mathcal{C}_0$.

(d) *If (H_{01}) holds then*

$$\Lambda_0 < \infty. \tag{1.5}$$

   Moreover, the following assertions hold:
   - (d1) *(P_λ) has a minimal positive solution $\underline{u}_{\Lambda_0}$ for $\lambda = \Lambda_0$, and $\lambda \mapsto \underline{u}_\lambda$ is continuous from $(0, \Lambda_0]$ to $C^{2+\alpha}(\overline{\Omega})$.*
   - (d2) *$\mathcal{C}_0$ consists of a smooth curve around $(\Lambda_0, \underline{u}_{\Lambda_0})$. More precisely, it is given by $(\lambda(s), u(s))$, $|s| < s_1$ (for some $s_1 > 0$) such that*

$\lambda(0) = \Lambda_0$, $\lambda'(0) = 0 > \lambda''(0)$, $u(0) = \underline{u}_{\Lambda_0}$. Moreover, $u(s) = \underline{u}_{\lambda(s)}$ for $s \in (-s_1, 0]$;

(d3) *There is no positive solution of* (P_λ) *for any* $\lambda > \Lambda_0$.

(d4) *The minimal positive solution* $\underline{u}_{\Lambda_0}$ *is weakly stable. More precisely,* $\gamma_1(\Lambda_0, \underline{u}_{\Lambda_0}) = 0$.

(d5) *Any positive solution* u *of* (P_λ) *except* $\underline{u}_\lambda$ *for* $0 < \lambda \le \Lambda_0$, *is unstable. In particular, any positive solution* u *of* (P_λ) *with* $(\lambda, u) \in \mathcal{C}_0 \setminus \{(\lambda, \underline{u}_\lambda) : 0 < \lambda \le \Lambda_0\}$ *is unstable.*

(e) *If* (H_{02}) *holds then* $\Lambda_0 = \infty$. *Moreover, the minimal positive solution* $\underline{u}_\lambda$ *is unique among the positive solutions of* (P_λ) *for* $\lambda > 0$.

(f) *If* (H_1) *and* (H_2) *hold, then for any* $\Lambda > 0$ *there exists* $C_\Lambda > 0$ *such that* $\max_{\overline{\Omega}} u \le C_\Lambda$ *for all* $(\lambda, u) \in \mathcal{C}_0$ *with* $\lambda \in [-\Lambda, \Lambda]$.

REMARK 1.5. (a) Assertion (b), assertions (c1)–(c4) and the uniqueness result in assertion (e) of Theorem 1.4 do not require the condition $p < 2N/(N-2)$ if $N > 2$.

(b) In the case $\int_\Omega a < 0$, it holds under (H_{01}), (H_1) and (H_2) that

$$\{\lambda \in \mathbb{R} : (\lambda, u) \in \mathcal{C}_0\} = (-\infty, \Lambda_0].$$

Consequently, (P_λ) has at least one nontrivial nonnegative solution for every $\lambda < 0$ and at least one positive solution for $\lambda = 0$, Λ_0, and at least two positive solutions for every $\lambda \in (0, \Lambda_0)$, see Figure 2.

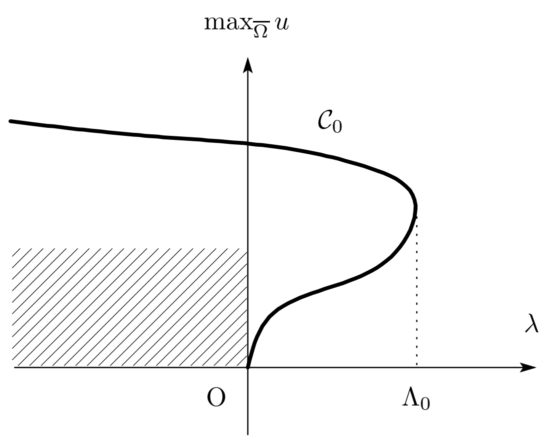

FIGURE 2. An unbounded subcontinuum of nontrivial nonnegative solutions in the case $\int_\Omega a < 0$.

1.1. Notation. Throughout this article we use the following notations and conventions:

- The infimum of an empty set is assumed to be ∞.

- Unless otherwise stated, for any $f \in L^1(\Omega)$ the integral $\int_\Omega f$ is considered with respect to the Lebesgue measure, whereas for any $g \in L^1(\partial\Omega)$ the integral $\int_{\partial\Omega} g$ is considered with respect to the surface measure.
- For $r \geq 1$ the Lebesgue norm in $L^r(\Omega)$ will be denoted by $\|\cdot\|_r$ and the usual norm of $H^1(\Omega)$ by $\|\cdot\|$.
- The strong and weak convergence are denoted by $\to$ and $\rightharpoonup$, respectively.
- The positive and negative parts of a function u are defined by $u^\pm := \max\{\pm u, 0\}$.
- If $U \subset \mathbb{R}^N$ then we denote the closure of U by $\overline{U}$ and the interior of U by $\operatorname{int} U$.
- The support of a measurable function f is denoted by $\operatorname{supp} f$.

The rest of this article is organized as follows. In Section 2 we prove some non-existence results. In Section 3, to bypass the difficulty that (P_λ) is not differentiable at $u = 0$, we consider a regularized problem with a new parameter $\varepsilon > 0$ and prove the existence of an unbounded subcontinuum of positive solutions for this problem. By the Whyburn topological technique, we shall deduce the existence of an unbounded subcontinuum of nontrivial nonnegative solutions for (P_λ), passing to the limit as $\varepsilon \to 0^+$. Section 4 is devoted to the proofs of Theorems 1.2 and 1.4.

2. Some non-existence results

First we prove the following non-existence result for the case $\int_\Omega a \geq 0$.

Proposition 2.1. *Assume $\int_\Omega a \geq 0$. Then the following two assertions hold*:

(a) *There is no positive solution of (P_λ) for any $\lambda \geq 0$.*

(b) *Assume $p \leq 2N/(N-2)$ if $N > 2$. Then, for any $\Lambda > 0$ there exists $\delta_0 > 0$ such that $\max_{\overline{\Omega}} u > \delta_0$ for all nontrivial nonnegative solutions of (P_λ) with $\lambda \leq -\Lambda$.*

Proof. (a) Let u be a positive solution of (P_λ) for some $\lambda \in \mathbb{R}$. We consider two cases:

Case 1. We assume that $a(x) \not\equiv cb(x)$ for any $c \in \mathbb{R}$. Then u is not a constant. The divergence theorem provides

$$\int_\Omega \frac{-\Delta u}{u^{p-1}} = \int_\Omega \nabla u \nabla\left(\frac{1}{u^{p-1}}\right) = -\int_\Omega (p-1)|\nabla u|^2 u^{-p} < 0.$$

It follows that

$$\int_\Omega \frac{-\Delta u}{u^{p-1}} = \int_\Omega a + \lambda \int_\Omega b u^{q-p} < 0.$$

Since $\int_\Omega b u^{q-p} > 0$, it should hold that $\lambda < 0$.

Case 2. We assume now that $a(x) \equiv cb(x)$ for some $c \in \mathbb{R}$. Since $\int_\Omega a \geq 0$ and $b \geq 0$, we have $c > 0$. If u is a constant then it is clear that $\lambda < 0$. Otherwise we argue as in Case 1.

(b) Let $\Lambda > 0$. Assume by contradiction that there exists a sequence (u_n) of nontrivial nonnegative solutions of (P_λ) with $\lambda = \lambda_n$ such that $\lambda_n \leq -\Lambda$ and $\max_{\overline{\Omega}} u_n \to 0$ ($\lambda_n \to -\infty$ may occur). It follows that

$$\int_\Omega |\nabla u_n|^2 = \int_\Omega a u_n^p + \lambda_n \int_\Omega b u_n^q \leq \int_\Omega a u_n^p \to 0, \tag{2.1}$$

and consequently $u_n \to 0$ in $H^1(\Omega)$. We set $v_n = u_n/\|u_n\|$, and we assume that $v_n \rightharpoonup v_0$ for some $v_0 \in H^1(\Omega)$. From

$$\int_\Omega \nabla u_n \nabla \phi = \int_\Omega a u_n^{p-1}\phi + \lambda_n \int_\Omega b u_n^{q-1}\phi \quad \text{for all } \phi \in H^1(\Omega),$$

we get $\lambda_n \int_\Omega b v_n^{q-1}\phi \to 0$ for every $\phi \in H^1(\Omega)$. It follows that $\int_\Omega b v_0^{q-1}\phi = 0$ for every $\phi \in H^1(\Omega)$, so that $b v_0^{q-1} \equiv 0$.

On the other hand, from (2.1) we get $\lim \int_\Omega |\nabla v_n|^2 = 0$, which implies $v_n \to v_0$ in $H^1(\Omega)$, and v_0 is a constant. Since $\|v_n\| = 1$, we have $v_0 > 0$. Hence, from $b v_0^{q-1} \equiv 0$ we obtain $b \equiv 0$, which is a contradiction. $\square$

Proposition 2.2. *Assume* $\int_\Omega a < 0$ *and* $p < 2N/(N-2)$ *if* $N > 2$. *Then there exists* $c_0 > 0$ *such that* $\max_{\overline{\Omega}} u \geq c_0$ *for all nontrivial nonnegative solutions* u *of* (P_λ) *with* $\lambda \leq 0$.

Proof. Similarly as in the proof of Proposition 2.1 (b), we argue by contradiction. Assume that there exists a sequence $\{(\lambda_n, u_n)\}$ of nontrivial nonnegative solutions u_n of (P_λ) with $\lambda = \lambda_n$ such that $\lambda_n \leq 0$ and $\max_{\overline{\Omega}} u_n \to 0$ ($\lambda_n \to -\infty$ may occur). It follows that $\|u_n\| \to 0$ using (2.1) again. Set $v_n = u_n/\|u_n\|$. We may assume that $v_n \rightharpoonup v_0$ for some $v_0 \in H^1(\Omega)$, and $v_n \to v_0$ in $L^p(\Omega)$. From (2.1) it follows that $\lim \int_\Omega |\nabla v_n|^2 = 0$. We deduce that v_0 is a positive constant, and $v_n \to v_0$ in $H^1(\Omega)$. On the other hand, from (2.1) we infer $\int_\Omega a u_n^p \geq 0$, so that $\int_\Omega a v_n^p \geq 0$. Since $v_n \to v_0$ in $L^p(\Omega)$, we have $0 \leq \int_\Omega a v_0^p = v_0^p \int_\Omega a$, which contradicts our assumption. $\square$

3. Positive solutions of a regularized problem

We consider now the existence of a subcontinuum of nontrivial nonnegative solutions for (P_λ) emanating from the trivial line. Since the mapping $t \mapsto t^{q-1}$ is not differentiable at $t = 0$ we cannot use the local and global bifurcation theory from simple eigenvalues [4] and [5]. To overcome this difficulty we investigate the existence of a subcontinuum of positive solutions emanating from the trivial line for a regularized version of (P_λ), which is formulated as

$$(\mathrm{Q}_{\lambda,\varepsilon}) \qquad \begin{cases} -\Delta u = \lambda b(x)(u+\varepsilon)^{q-2}u + a(x)u^{p-1} & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega, \end{cases}$$

where $0 < \varepsilon \leq 1$. Indeed, the mapping $t \mapsto (t+\varepsilon)^{q-2}t$ is analytic at $t = 0$. We remark that $(\mathrm{Q}_{\lambda,0}) = (\mathrm{P}_\lambda)$, so that (P_λ) is the limiting case of $(\mathrm{Q}_{\lambda,\varepsilon})$ as $\varepsilon \to 0^+$. To study the existence of bifurcation points on the trivial line $\{(\lambda,0)\}$ for $(\mathrm{Q}_{\lambda,\varepsilon})$, we consider the linearized eigenvalue problem at a nonnegative solution u of $(\mathrm{Q}_{\lambda,\varepsilon})$

$$(3.1) \qquad \begin{cases} -\Delta\phi = a(x)(p-1)u^{p-2}\phi \\ \qquad\qquad +\lambda b(x)\{(q-2)(u+\varepsilon)^{q-3}u + (u+\varepsilon)^{q-2}\}\phi + \sigma\phi & \text{in } \Omega, \\ \dfrac{\partial \phi}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega. \end{cases}$$

Plugging $u = 0$ into (3.1), we obtain the linearized eigenvalue problem

$$(3.2) \qquad \begin{cases} -\Delta\phi = \lambda\varepsilon^{q-2}b(x)\phi + \sigma\phi & \text{in } \Omega, \\ \dfrac{\partial \phi}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega. \end{cases}$$

This problem has a unique principal eigenvalue $\sigma_\varepsilon(\lambda)$, which is simple. Moreover, we see that $\sigma_\varepsilon(\lambda) > 0$ for $\lambda < 0$, $\sigma_\varepsilon(\lambda) = 0$ for $\lambda = 0$, and $\sigma_\varepsilon(\lambda) < 0$ for $\lambda > 0$. Note that (3.2) has a positive eigenfunction associated with $\sigma_\varepsilon(\lambda)$, which is a positive constant if $\lambda = 0$.

Proposition 3.1. *Let $0 < \varepsilon \leq 1$. Then the following two assertions hold*:

(a) *If u_n is a positive solution of $(\mathrm{Q}_{\lambda,\varepsilon})$ for $\lambda = \lambda_n$ such that $\max_{\overline{\Omega}} u_n \to 0$ and $\lambda_n \to \lambda^*$ for some $\lambda^* \in \mathbb{R}$ then $\lambda^* = 0$.*

(b) *$(\mathrm{Q}_{\lambda,\varepsilon})$ possesses an unbounded subcontinuum $\mathcal{C}_\varepsilon = \{(\lambda,u)\}$ in $\mathbb{R} \times C(\overline{\Omega})$ of positive solutions, which bifurcates at $(0,0)$ and does not meet $(\lambda,0)$ for any $\lambda \neq 0$.*

Proof. Assertion (a) is straightforward from the fact that $\sigma_\varepsilon(\lambda) > 0$ for $\lambda < 0$, and $\sigma_\varepsilon(\lambda) < 0$ for $\lambda > 0$. By using assertion (a), assertion (b) is a direct consequence of the global bifurcation theory from [9]. □

4. Proofs of Theorems 1.2 and 1.4

4.1. A priori upper bounds. The following *a priori* upper bound of λ for positive solutions of $(\mathrm{Q}_{\lambda,\varepsilon})$ follows from [13, Proposition 6.1].

Proposition 4.1. *If (H_{01}) holds then there exists $\overline{\lambda} > 0$ such that $(\mathrm{Q}_{\lambda,\varepsilon})$ has no positive solutions for $\lambda \geq \overline{\lambda}$ and $\varepsilon \in [0,1]$.*

The following *a priori* upper bound on the uniform norm of nonnegative solutions of $(\mathrm{Q}_{\lambda,\varepsilon})$ is obtained using a blow up technique from Gidas and Spruck [6] and follows from Amann and López-Gómez [2] and López-Gómez, Molina-Meyer and Tellini [7]:

PROPOSITION 4.2. *Assume* (H_1) *and* (H_2). *Then for any* $\Lambda > 0$ *there exists* $C_\Lambda > 0$ *such that* $\max_{\overline{\Omega}} u \le C_\Lambda$ *for all nonnegative solutions of* $(\mathrm{Q}_{\lambda,\varepsilon})$ *with* $\lambda \in [-\Lambda, \Lambda]$ *and* $\varepsilon \in [0,1]$. *In particular, the conclusion holds for* (P_λ).

PROOF. The case where (1.1) holds follows by means of Proposition A.1 as in the proof of [13, Proposition 6.5], whereas the case where (1.2) holds follows from the following lemma:

LEMMA 4.3. *Assume* (H_1) *with* (1.2). *Assume in addition that for any* $\Lambda > 0$ *there exists a constant* $C_1 > 0$ *such that* $\max_{\overline{\Omega^a_+}} u \le C_1$ *for all nonnegative solutions* u *of* $(\mathrm{Q}_{\lambda,\varepsilon})$ *with* $\lambda \in [-\Lambda, \Lambda]$ *and* $\varepsilon \in (0,1]$. *Then, for any* $\Lambda > 0$ *there exists a constant* C_2 *such that* $\max_{\overline{\Omega}} u \le C_2$ *for all nonnegative solutions* u *of* $(\mathrm{Q}_{\lambda,\varepsilon})$ *with* $\lambda \in [-\Lambda, \Lambda]$ *and* $\varepsilon \in (0,1]$.

PROOF. We use a comparison principle. For $\Lambda > 0$ we first consider the case $\lambda \in [0, \Lambda]$. Let u be a nonnegative solution of $(\mathrm{Q}_{\lambda,\varepsilon})$. Then, since $u \le C_1$ on $\partial\Omega^a_-$ by assumption, u is a subsolution of the problem

$$\begin{cases} -\Delta u = \lambda b(x)(u+\varepsilon)^{q-2}u - a^-(x)u^{p-1} & \text{in } \Omega^a_-, \\ u = C_1 & \text{on } \partial\Omega^a_-. \end{cases} \tag{4.1}$$

Let w_0 be the unique positive solution of the Dirichlet problem

$$\begin{cases} -\Delta w = 1 & \text{in } \Omega^a_-, \\ w = 0 & \text{on } \partial\Omega^a_-. \end{cases} \tag{4.2}$$

Set $w_1 = C(1+w_0)$ with $C > 0$. Then w_1 is a supersolution of (4.1) if we choose C such that

$$C^{2-q} = \max\left\{C_1^{2-q}, \Lambda\left(\max_{\overline{\Omega^a_-}} b\right)\left(1 + \max_{\overline{\Omega^a_-}} w_0\right)^{q-1}\right\}.$$

Indeed, we observe that

$$\begin{aligned} -\Delta w_1 + a^- w_1^{p-1} &- \lambda b(w_1+\varepsilon)^{q-2}w_1 \\ &\ge C - \Lambda\left(\max_{\overline{\Omega^a_-}} b\right)(C(1+w_0)+\varepsilon)^{q-2}C(w_0+1) \\ &\ge C - \Lambda\left(\max_{\overline{\Omega^a_-}} b\right)C^{q-1}(1+w_0)^{q-1} \\ &\ge C^{q-1}\left(C^{2-q} - \Lambda\left(\max_{\overline{\Omega^a_-}} b\right)\left(1 + \max_{\overline{\Omega^a_-}} w_0\right)^{q-1}\right) \ge 0. \end{aligned}$$

So, the comparison principle (Proposition A.1 in the Appendix) for (4.1) provides us with the assertion

$$u \leq C(1 + w_0) \leq C\left(1 + \max_{\overline{\Omega^a_-}} w_0\right) \quad \text{on } \overline{\Omega^a_-}.$$

Next we consider the case $\lambda \in [-\Lambda, 0]$. Let u be a nonnegative solution of $(\mathrm{Q}_{\lambda,\varepsilon})$. It is straightforward that u is a subsolution of the problem

$$\begin{cases} -\Delta u = -a^-(x)u^{p-1} & \text{in } \Omega^a_-, \\ u = C_1 & \text{on } \partial\Omega^a_-. \end{cases} \tag{4.3}$$

Using the unique positive solution w_0 of (4.2), we see that $C_1(1 + w_0)$ is a supersolution of (4.3), and thus, from the comparison principle, we deduce again

$$u \leq C_1(1 + w_0) \leq C_1\left(1 + \max_{\overline{\Omega^a_-}} w_0\right) \quad \text{on } \overline{\Omega^a_-}.$$

Summing up, $C_2 = C(1 + \max_{\overline{\Omega^a_-}} w_0)$ yields the desired conclusion. □

The following *a priori* upper bound of the uniform norm on $\overline{\Omega^a_+}$ for nonnegative solutions of $(\mathrm{Q}_{\lambda,\varepsilon})$ can be established in a similar manner as [7, Theorem 6.3].

Lemma 4.4. *Assume* (H_2) *in addition to* (H_1) *with* (1.2). *Then, for any* $\Lambda > 0$ *there exists a constant* $C_1 > 0$ *such that* $\max_{\overline{\Omega^a_+}} u \leq C_1$ *for all nonnegative solutions* u *of* $(\mathrm{Q}_{\lambda,\varepsilon})$ *with* $\lambda \in [-\Lambda, \Lambda]$ *and* $\varepsilon \in (0, 1]$.

Lemma 4.4 completes the proof of Proposition 4.2 in view of Lemma 4.3. □

4.2. Proof of Theorem 1.2. Assertions (a) and (c) follow from Proposition 2.1. By use of the Nehari manifold technique, assertion (b) can be verified in a similar way just as in [13, Remark 2.2], relying on the assumption that $\lambda < 0$, $b \geq 0$ and $b \not\equiv 0$.

We use now a topological method proposed by Whyburn [14] to prove the existence of an unbounded subcontinuum of nontrivial nonnegative solutions of (P_λ). Let $0 < \varepsilon \leq 1$ and $\Lambda > 0$ be fixed. By Proposition 3.1, there exists a subcontinuum $\mathcal{C}'_\varepsilon$ of positive solutions of $(Q_{\lambda,\varepsilon})$ such that

$$\mathcal{C}'_\varepsilon \subset \mathcal{C}_\varepsilon \cap \{(\lambda, u) \in \mathbb{R} \times C(\overline{\Omega}) : |\lambda| \leq \Lambda,\ \|u\|_{C(\overline{\Omega})} \leq C_\Lambda\},$$

where C_Λ is a positive constant given by Proposition 4.2. Then, we have we have $(0, 0) \in \mathcal{C}'_\varepsilon$ and there exists $(\lambda_\varepsilon, u_\varepsilon) \in \mathcal{C}'_\varepsilon$ such that $|\lambda_\varepsilon| = \Lambda$. Moreover, since we can prove that $(\mathrm{Q}_{\lambda,\varepsilon})$ with $\lambda \geq 0$ and $\varepsilon \in (0, 1]$ has no positive solution in the same way just as in the proof of Proposition 2.1 (a), we have that $\lambda < 0$ if $(\lambda, u) \in \mathcal{C}'_\varepsilon \setminus \{(0, 0)\}$. Consequently, $\lambda_\varepsilon = -\Lambda$, see Figure 3.

Arguing as in Section 3 of [11], we have the following facts:

- $\bigcup_{0<\varepsilon\leq 1} \mathcal{C}'_\varepsilon$ is precompact in $C(\overline{\Omega})$;

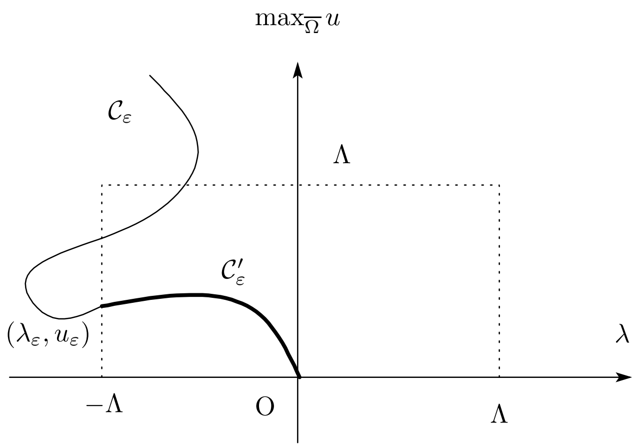

FIGURE 3. The subcontinuum $\mathcal{C}'_\varepsilon$.

- $(0,0) \in \liminf_{\varepsilon \to 0^+} \mathcal{C}'_\varepsilon$, i.e. it is nonempty;
- up to a subsequence, there holds $(\lambda_\varepsilon, u_\varepsilon) \to (-\Lambda, u_0)$ in $\mathbb{R} \times C(\overline{\Omega})$, and u_0 is a nonnegative solution of (P_λ) for $\lambda = -\Lambda$.

Hence we use (9.12) Theorem on page 11 of [14], to deduce that $\mathcal{C}_0 := \limsup_{\varepsilon \to 0^+} \mathcal{C}'_\varepsilon$ is nonempty, closed and connected, i.e. it is a subcontinuum. Furthermore, we can check that $\mathcal{C}_0$ is contained in the set of nonnegative weak solutions of (P_λ) (and therefore in the set of nonnegative solutions of (P_λ) by elliptic regularity).

Finally, we shall show that $\mathcal{C}_0 \setminus \{(0,0)\}$ consists of nontrivial nonnegative solutions of (P_λ). To this end, we prove the following lemma, see Proposition 2.1 (b).

LEMMA 4.5. *Assume* $p \leq 2N/(N-2)$ *if* $N > 2$. *Then, for any* $\Lambda > 0$, *there exists* $\delta_0 > 0$ *such that* $\max_{\overline{\Omega}} u > \delta_0$ *for all positive solutions of* $(\mathrm{Q}_{\lambda,\varepsilon})$ *with* $\lambda \leq -\Lambda$ *and* $\varepsilon \to 0^+$.

PROOF. The proof is carried out with a minor modification of that of Proposition 2.1 (b). Assume that u_n is a positive solution of $(\mathrm{Q}_{\lambda_n,\varepsilon_n})$ such that $\max_{\overline{\Omega}} u_n \to 0$, $\varepsilon_n \to 0^+$, and $\lambda_n \leq -\Lambda$. As in the proof of Proposition 2.1 (b), we deduce $u_n \to 0$ in $H^1(\Omega)$, and then, putting $v_n = u_n/\|u_n\|$, it follows that, up to a subsequence, $v_n \to v_0$ in $H^1(\Omega)$ for some positive constant v_0.

Now, from the assumption of u_n, we derive

$$\int_\Omega a u_n^{p-1} + \lambda_n \int_\Omega b(u_n + \varepsilon_n)^{q-2} u_n = 0.$$

By multiplying the left hand side by $\|u_n\|^{-1}$, we deduce

$$\int_\Omega a v_n^{p-1} \|u_n\|^{p-2} + \lambda_n \int_\Omega b(u_n + \varepsilon_n)^{q-2} v_n = 0,$$

so that

$$0 \le \frac{1}{(\max_{\overline{\Omega}} u_n + \varepsilon_n)^{2-q}} \int_\Omega b v_n \le \int_\Omega b(u_n + \varepsilon_n)^{q-2} v_n \to 0.$$

It follows that

$$\int_\Omega b v_n \to \int_\Omega b v_0 = 0.$$

Since v_0 is a positive constant, we have $\int_\Omega b = 0$, a contradiction. $\square$

Now, we end the proof of Theorem 1.2. By definition, $(-\Lambda, u_0) \in \mathcal{C}_0$. From Lemma 4.5, it follows that $u_0 \not\equiv 0$, so that u_0 is a nontrivial nonnegative solution of (P_λ) for $\lambda = -\Lambda$. Combining this assertion, Proposition 2.1, and the connectivity of $\mathcal{C}_0$, we deduce that $\mathcal{C}_0 \setminus \{(0,0)\}$ is contained in the set of nontrivial nonnegative solutions of (P_λ). Since Λ is arbitrary, assertion (d) of this theorem follows, and now, $\mathcal{C}_0$ is the desired subcontinuum. We have finished the proof of Theorem 1.2. $\square$

4.3. Proof of Theorem 1.4. The argument is similar. Assertion (a) follows from Proposition 2.2, whereas assertion (1.5) follows from Proposition 4.1. Assertions (b) through (d) except (1.5) and assertion (d5) can be proved similarly as [12, Theorem 1.1]. Assertion (d5) is verified carrying out the argument in [12, Proposition 5.2 (4)] for $\lambda > 0$, and the one in assertion (b) of Theorem 1.2 for $\lambda < 0$. Assertion (f) follows from Proposition 4.2.

Now it remains to verify assertion (e). To prove the uniqueness of a positive solution of (P_λ) for $\lambda > 0$, we first reduce (P_λ) to an equation with a nonlinear, compact and increasing mapping, as follows. If u is a positive solution of (P_λ) then, for a constant $\omega > 0$, we have

$$u = K(\omega u + a(x)u^{p-1} + \lambda b(x) u^{q-1}) =: K F_\omega(u) \quad \text{in } C(\overline{\Omega}),$$

where $K\colon C(\overline{\Omega}) \to C^1(\overline{\Omega})$ is the compact mapping defined as the resolvent of the linear Neumann problem

$$\begin{cases} (-\Delta + \omega) u = \psi & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega. \end{cases}$$

More precisely, for any $\psi \in C^\theta(\overline{\Omega})$, $\theta \in (0,1)$, $K\psi \in C^{2+\theta}(\overline{\Omega})$ is the unique solution of the linear problem above. Moreover, K is known to be *strongly positive*, i.e. for $u \ge 0$ satisfying $u \not\equiv 0$ we have $Ku > 0$ on $\overline{\Omega}$ (we denote it by $Ku \gg 0$).

Next we shall observe that

(4.4) for $C > 0$, $F_\omega(u)$ is nondecreasing in $0 \le u \le C$ if ω is large enough,

(4.5) $F_\omega(\tau u) \ge \tau F_\omega(u)$ (and $\not\equiv \tau F_\omega(u)$) for $\tau \in (0,1)$ and $u \gg 0$.

We derive (4.4) from the slope condition of F_ω. Indeed, we see that if $0 \le u \le v \le C$ then

$$\omega u + a(x)u^{p-1} - \{\omega v + a(x)v^{p-1}\} = (u-v)\left\{\omega + a(x)\,\frac{u^{p-1}-v^{p-1}}{u-v}\right\} \le 0,$$

provided that ω is large. We derive (4.5) by the direct computation

$$F_\omega(\tau u) - \tau F_\omega(u) = -a(x)\tau u^{p-1}(1-\tau^{p-2}) + \lambda b(x)\tau^{q-1}u^{q-1}(1-\tau^{2-q}) \ge 0$$

(and $\not\equiv 0$).

Now we use a uniqueness argument from the proof of [1, Theorem 24.2]. Let $\lambda > 0$, u_1 be the minimal positive solution of (P_λ), and u_2 another positive solution of (P_λ). Then we have $u_1 \le u_2$. Assume by contradiction that $u_1 \not\equiv u_2$. Then, since $u_1 \gg 0$, there exists $\tau_0 \in (0,1)$ such that $u_1 - \tau_0 u_2 \ge 0$ but $u_1 - \tau_0 u_2 \in \partial P$, where $P = \{u \in C(\overline{\Omega}) : u \ge 0\}$ denotes the positive cone of $C(\overline{\Omega})$ and ∂P the boundary of P. Note that if $u \gg 0$ then u is an interior point of P. Take a constant $C > 0$ such that $u_1, u_2 \le C$. Using (4.4), (4.5) and the fact that K is strongly positive, we deduce that

$$u_1 = KF_\omega(u_1) \ge KF_\omega(\tau_0 u_2) \gg \tau_0 KF_\omega(u_2) = \tau_0 u_2,$$

where $u \gg v$ means $u - v \gg 0$. Hence $u_1 - \tau_0 u_2$ is an interior point of P, which contradicts $u_1 - \tau_0 u_2 \in \partial P$. Consequently, $u_1 \equiv u_2$, and the uniqueness holds.

Moreover, under (H_{01}), the implicit function theorem is applicable at any positive solution of (P_λ) with $\lambda > 0$. Therefore, based on assertion (a), we deduce that $\mathcal{C}_0 \setminus \{(0,0)\} = \{(\lambda, \underline{u}_\lambda) : 0 < \lambda < \Lambda_0\}$.

To prove $\Lambda_0 = \infty$, we establish an *a priori* bound for positive solutions of (P_λ) in a similar way as Proposition 2.1 (b). For the sake of a contradiction we may assume $|\lambda_n| \le \Lambda$, $\|u_n\| \to \infty$, and u_n is a positive solution for $\lambda = \lambda_n$. Since

$$\int_\Omega |\nabla u_n|^2 = \int_\Omega a u_n^p + \lambda_n \int_\Omega b u_n^q \le \lambda_n \int_\Omega b u_n^q,$$

we deduce that $\limsup_n \int_\Omega |\nabla v_n|^2 \to 0$, where $v_n = u_n/\|u_n\|$. Hence we may assume that $v_n \to v_0$ in $H^1(\Omega)$ and v_0 is a positive constant. Also we have $v_n \to v_0$ in $L^{p-1}(\Omega)$. On the other hand, we see that

$$\int_\Omega \nabla u_n \nabla \phi = \int_\Omega a u_n^{p-1}\phi + \lambda_n \int_\Omega b u_n^{q-1}\phi \quad \text{for all } \phi \in H^1(\Omega).$$

It follows that $\int_\Omega a v_n^{p-1}\phi \to 0$, so that $\int_\Omega a v_0^{p-1}\phi = 0$ for every $\phi \in H^1(\Omega)$. Hence we have $a v_0^{p-1} \equiv 0$. Since v_0 is a positive constant, this contradicts the assumption $a \not\equiv 0$. Therefore we have proved that for any $\Lambda > 0$ there exists $C_\Lambda > 0$ such that if u is a positive solution of (P_λ) with $\lambda \in [-\Lambda, \Lambda]$ then $\|u\| \le C_\Lambda$, and thus, $\|u\|_{C(\overline{\Omega})} \le C$ for some $C > 0$ by elliptic regularity, as desired. By combining the *a priori* bound and the use of the implicit function theorem, we verify assertion (e). The proof of Theorem 1.4 is now complete. $\square$

We conclude with the following remark on Theorems 1.2 and 1.4:

Remark 4.6. Consider (P_λ) with $q = 1, 2$. These cases do not correspond to a concave-convex nonlinearity but it is worthwhile discussing the nontrivial nonnegative solutions set of (P_λ). We may check that (P_λ) still has a subcontinuum $\mathcal{C}_0$ of solutions such that $\mathcal{C}_0 \setminus \{(0,0)\}$ consists of nontrivial nonnegative solutions (with the same nature as in the case $q \in (1,2)$).

(a) Case $q = 1$. In this case, $\lambda b(x) u^{q-1} = \lambda b(x)$ does not depend on u, so that (P_λ) no longer possesses the trivial line of solutions $\{(\lambda, 0)\}$. However, when $\int_\Omega a < 0$, we can prove the existence of a subcontinuum $\mathcal{C}_1 = \{(\lambda, u)\}$ of nonnegative solutions bifurcating at $(0,0)$ to $\lambda > 0$ and such that $\mathcal{C}_1 \setminus \{(0,0)\}$ consists of positive solutions of (P_λ), when $\lambda \geq 0$. To this end we carry out again the Whyburn topological argument developed in Subsection 4.2. Let $\mathcal{C}_q = \{(\lambda, u)\}$, $q \in (1,2)$, be the unbounded subcontinuum of positive solutions of (P_λ) bifurcating at $(0,0)$, as provided by Theorem 1.4. Then, the topological argument in Subsection 4.2 holds with ε replaced by q for $\lambda \geq 0$. Note that $\overline{\lambda}$ given by Proposition 4.1 and C_Λ given by Proposition 4.2 are determined uniformly as $q \to 1^+$. Moreover, we can check in the same way that assertions (a) through (f) in Theorem 1.4 hold true for $q = 1$. Consequently, $\mathcal{C}_1 = \limsup_{q \to 1^+} \mathcal{C}_q|_{\lambda \geq 0}$ is our desired subcontinuum.

(b) Case $q = 2$. In this case, $\lambda b(x) u^{q-1} = \lambda b(x) u$ is linear. There is a large literature on this case, with many results on the positive solutions set. Indeed, the general global bifurcation theory due to Rabinowitz provides the existence of an unbounded subcontinuum $\mathcal{C}_2 = \{(\lambda, u)\}$ of solutions of (P_λ) bifurcating at $(0,0)$ and such that $\mathcal{C}_2 \setminus \{(0,0)\}$ consists of positive solutions. Furthermore, assertions (a) through (d) in Theorem 1.2 and assertions (a) through (f) in Theorem 1.4 are verified in the same way, except the assertion $\Lambda_0 = \infty$ in Theorem 1.4 (e). Actually, this assertion is not true in general for $q = 2$. Indeed, when (H_{02}) is satisfied, we know the following two results:

- If $a < 0$ on $\overline{\Omega}$ then $\Lambda_0 = \infty$ (see Amann [1, Theorem 25.4]).
- Assume that $\{x \in \Omega : a(x) = 0\} \neq \emptyset$ and $b \equiv 1$. Assume additionally that $D_0 := \Omega \setminus \overline{\Omega_-^a}$ is a smooth subdomain of Ω bounded away from $\partial\Omega$. Consider the smallest eigenvalue $\lambda_1(D_0) > 0$ of the Dirichlet eigenvalue problem

$$\begin{cases} -\Delta \phi = \lambda \phi & \text{in } D_0, \\ \phi = 0 & \text{on } \partial D_0. \end{cases}$$

  Then $\Lambda_0 = \lambda_1(D_0)$ and the minimal positive solution $\underline{u}_\lambda$ grows up to infinity in $C(\overline{\Omega})$ as $\lambda \to \lambda_1(D_0)^-$. Moreover, there is no positive solution of (P_λ) for any $\lambda \geq \lambda_1(D_0)$ (see Ouyang [8, Theorem 3]).

On the other hand, it would be difficult to consider the limiting case $p = 2$ by the same approach as in the cases $q = 1, 2$, since our argument essentially uses the condition $p > 2$. Indeed, we do not know whether Proposition 2.1 (b) and Proposition 2.2 remain true for the case $p = 2$. Thus, in the case $p = 2$, one should follow another approach to study bifurcation from zero.

Appendix A. A slight variant of the comparison principle for concave problems

In this appendix we provide a variant of the comparison principle proved by Ambrosetti, Brezis and Cerami [3, Lemma 3.3] to mixed Dirichlet and Neumann nonlinear boundary conditions. We consider the general boundary value problem

$$\begin{cases} -\Delta u = f(x,u) & \text{in } D, \\ \dfrac{\partial u}{\partial \mathbf{n}} = g(x,u) & \text{on } \Gamma_1, \\ u = C_1 & \text{on } \Gamma_0, \end{cases} \tag{A.1}$$

where

- D is a bounded domain of $\mathbb{R}^N$ with smooth boundary ∂D,
- $\Gamma_0, \Gamma_1 \subset \partial D$ are disjoint, open, and smooth $(N-1)$ dimensional surfaces of ∂D,
- $\overline{\Gamma_0}, \overline{\Gamma_1}$ are compact manifolds with $(N-2)$ dimensional closed boundary $\gamma = \overline{\Gamma_0} \cap \overline{\Gamma_1}$ such that $\partial D = \Gamma_0 \cup \gamma \cup \Gamma_1$,
- $f \colon \overline{\Omega} \times [0,\infty) \to \mathbb{R}$ and $g \colon \overline{\Gamma_1} \times [0,\infty) \to \mathbb{R}$ are continuous,
- C_1 is a nonnegative constant.

The result [10, Proposition A.1] can be slightly relaxed as follows:

Proposition A.1. *Under the above conditions, assume that for every $x \in D$, $t \mapsto f(x,t)/t$ is decreasing in $(0,\infty)$, and for every $x \in \Gamma_1$, $t \mapsto g(x,t)/t$ is nonincreasing in $(0,\infty)$. Let $u, v \in H^1(D) \cap C(\overline{D})$ be nonnegative functions satisfying $u \le C_1 \le v$ on Γ_0, and*

$$\int_D \nabla u \nabla \varphi - \int_D f(x,u)\varphi - \int_{\Gamma_1} g(x,u)\varphi \le 0 \quad \text{for all } \varphi \in H^1_{\Gamma_0}(D) \text{ such that } \varphi \ge 0, \tag{A.2}$$

$$\int_D \nabla v \nabla \varphi - \int_D f(x,v)\varphi - \int_{\Gamma_1} g(x,v)\varphi \ge 0 \quad \text{for all } \varphi \in H^1_{\Gamma_0}(D) \text{ such that } \varphi \ge 0. \tag{A.3}$$

If $v > 0$ in D, then $u \le v$ on $\overline{D}$.

Remark A.2. (a) In [10, Proposition A.1] the case $C_1 = 0$ has been considered.

(b) Assume additionally that f, g are smooth enough. If a nonnegative function $u \in C^2(\overline{\Omega})$ satisfies

$$\begin{cases} -\Delta u \leq f(x,u) & \text{in } D, \\ u \leq C_1 & \text{on } \Gamma_0, \\ \dfrac{\partial u}{\partial \mathbf{n}} \leq g(x,u) & \text{on } \Gamma_1, \end{cases}$$

then u satisfies (A.2). Similarly if the opposite inequalities hold then u satisfies (A.3).

(c) $\Gamma_0 = \emptyset$ (or alternatively $\Gamma_1 = \emptyset$) is allowed.

Proof. Let $\theta \colon \mathbb{R} \to \mathbb{R}$, be a nonnegative nondecreasing smooth function such that $\theta(t) = 0$ for $t \leq 0$ and $\theta(t) = 1$ for $t \geq 1$. For $\varepsilon > 0$ we set $\theta_\varepsilon(t) = \theta(t/\varepsilon)$. Since $u - v \leq 0$ on Γ_0, we have $v\theta_\varepsilon(u-v) \in H^1_{\Gamma_0}(D)$, so that

$$\text{(A.4)} \quad \int_D \nabla u \nabla(v\theta_\varepsilon(u-v)) - \int_D f(x,u) v\theta_\varepsilon(u-v) - \int_{\Gamma_1} g(x,u) v\theta_\varepsilon(u-v) \leq 0.$$

Likewise, since $u\theta_\varepsilon(u-v) \in H^1_{\Gamma_0}(D)$, we have

$$\text{(A.5)} \quad \int_D \nabla v \nabla(u\theta_\varepsilon(u-v)) - \int_D f(x,v) u\theta_\varepsilon(u-v) - \int_{\Gamma_1} g(x,v) u\theta_\varepsilon(u-v) \geq 0.$$

Let $\Gamma_1^+ = \{x \in \Gamma_1 : u, v > 0\}$, and $D^+ = \{x \in D : u > 0\}$. Since $t \mapsto g(x,t)/t$ is nonincreasing in $(0,\infty)$, we have $g(x,0) \geq 0$, which combined with (A.4) and (A.5) yields

$$\begin{aligned} &\int_D u\theta_\varepsilon'(u-v)\nabla v(\nabla u - \nabla v) - \int_D v\theta_\varepsilon'(u-v)\nabla u(\nabla u - \nabla v) \\ &\geq \int_{D^+} uv\bigg(\frac{f(x,v)}{v} - \frac{f(x,u)}{u}\bigg)\theta_\varepsilon(u-v) + \int_{\Gamma_1^+} uv\bigg(\frac{g(x,v)}{v} - \frac{g(x,u)}{u}\bigg)\theta_\varepsilon(u-v) \\ &\geq \int_{D^+} uv\bigg(\frac{f(x,v)}{v} - \frac{f(x,u)}{u}\bigg)\theta_\varepsilon(u-v). \end{aligned}$$

From $-\int_{D^+} u\theta_\varepsilon'(u-v)|\nabla(u-v)|^2 \leq 0$, it follows that

$$\text{(A.6)} \quad \int_D (u-v)\theta_\varepsilon'(u-v)\nabla u \nabla(u-v) \geq \int_{D^+} uv\bigg(\frac{f(x,v)}{v} - \frac{f(x,u)}{u}\bigg)\theta_\varepsilon(u-v).$$

Now, we introduce $\gamma_\varepsilon(t) = \int_0^t s\theta_\varepsilon'(s)\,ds$ for $t \in \mathbb{R}$. We have then $0 \leq \gamma_\varepsilon(t) \leq \varepsilon$, $t \in \mathbb{R}$. Note that $\nabla(\gamma_\varepsilon(u-v)) = (u-v)\theta_\varepsilon'(u-v)\nabla(u-v)$. Hence, from (A.6) we deduce that

$$\int_D \nabla u \nabla(\gamma_\varepsilon(u-v)) \geq \int_{D^+} uv\bigg(\frac{f(x,v)}{v} - \frac{f(x,u)}{u}\bigg)\theta_\varepsilon(u-v).$$

Now, since $\gamma_\varepsilon(u-v) \in H^1_{\Gamma_0}(D)$ and $\gamma_\varepsilon(u-v) \geq 0$, we note that

$$\int_D \nabla u \nabla(\gamma_\varepsilon(u-v)) - \int_D f(x,u)\gamma_\varepsilon(u-v) - \int_{\Gamma_1} g(x,u)\gamma_\varepsilon(u-v) \leq 0,$$

and combining the two latter assertions, we get

$$\int_D f(x,u)\gamma_\varepsilon(u-v) + \int_{\Gamma_1} g(x,u)\gamma_\varepsilon(u-v) \geq \int_{D^+} uv\bigg(\frac{f(x,v)}{v} - \frac{f(x,u)}{u}\bigg)\theta_\varepsilon(u-v).$$

Since $\gamma_\varepsilon(t) \leq \varepsilon$, there exists a constant $C > 0$ such that

$$C\varepsilon \geq \int_{D^+} uv\bigg(\frac{f(x,v)}{v} - \frac{f(x,u)}{u}\bigg)\theta_\varepsilon(u-v). \tag{A.7}$$

Since $t \mapsto f(x,t)/t$ is decreasing in $(0,\infty)$, we use Fatou's lemma to deduce from (A.7) that

$$\int_{D^+} \liminf_{\varepsilon\to 0^+} uv\bigg(\frac{f(x,v)}{v} - \frac{f(x,u)}{u}\bigg)\theta_\varepsilon(u-v) \leq 0.$$

Note that

$$\lim_{\varepsilon\to 0^+} \theta_\varepsilon(u-v) = \begin{cases} 1 & u > v, \\ 0 & u \leq v, \end{cases}$$

so that

$$\int_{D^+\cap\{u>v\}} uv\bigg(\frac{f(x,v)}{v} - \frac{f(x,u)}{u}\bigg) \leq 0.$$

Using again that $t \mapsto f(x,t)/t$ is decreasing in $(0,\infty)$, we conclude that $|D^+ \cap \{u > v\}| = 0$, and since $u \equiv 0 < v$ in $D \setminus D^+$, we have $u \leq v$ almost everywhere in D. By continuity, the desired conclusion follows. □

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