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ON SOME PROPERTIES OF THE SOLUTION SET MAP TO VOLTERRA INTEGRAL INCLUSION

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Abstract. For the multivalued Volterra integral equation defined in a Banach space, the set of solutions is proved to be R_δ, without auxiliary conditions imposed in Theorem 6 [J. Math. Anal. Appl. 403 (2013), 643–666]. It is shown that the solution set map, corresponding to this Volterra integral equation, possesses a continuous singlevalued selection; and the image of a convex set under the solution set map is acyclic. The solution set to the Volterra integral inclusion in a separable Banach space and the preimage of this set through the Volterra integral operator are shown to be absolute retracts.

1. Introduction

In [13], the author conducted the study of geometric properties of the solution set to the following Volterra integral inclusion:

$$x(t) \in h(t) + \int_0^t k(t,s)F(s,x(s))\,ds, \quad t \in I = [0,T], \tag{1.1}$$

in a Banach space E, with $h \in C(I,E)$, $k(t,s) \in \mathscr{L}(E)$ and $F\colon I \times E \multimap E$ a convex valued perturbation. It was proven that the solution set $S_F^p(h)$ of integral inclusion (1.1) is acyclic in the space $C(I,E)$ or is even R_δ, provided some additional conditions on the Banach space E or the kernel k and perturbation F are imposed. We will show that these auxiliary assumptions are redundant.

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Thus, the present work complements [13] by strengthening Theorems 6 and 7 of loc. cit. In Section 3, we also give some applications of investigated properties of the solution set to problem (1.1). One of them is the characterization of the solution set of an evolution inclusion in the so-called parabolic case. The second is the result on the existence of periodic trajectories of the integral inclusion under consideration.

Introducing the so-called *solution set map* $S_F^p\colon C(I,E) \multimap C(I,E)$, which associates with each inhomogeneity $h \in C(I,E)$ the set of all solutions to (1.1), we prove that under generic assumptions on F this multimap possesses a continuous singlevalued selection. With this aim we adapt a well-known construction taken from [8].

Already singlevalued examples show that in case of continuous f no more than connectedness of the image $f(M)$ of a connected set M can be expected. Also in the case of the solution set map it is clear only that the set $\bigcup_{h\in M} S_F^p(h)$ is connected, if $M \subset C(I,E)$ is connected. We exploit the admissibility of the solution set map and the result of Vietoris to demonstrate that the image of a compact convex $M \subset C(I,E)$ through S_F^p must be acyclic.

Since the solutions of inclusion (1.1) are understood in the sense of Aumann integral, it is natural to examine the issue of geometric structure of the set of these integrable selections of the perturbation F, which make up the solution set $S_F^p(h)$ being mapped by the Volterra integral operator. It has been shown that these selections form a retract of the space $L^p(I,E)$. In the context of stronger assumptions on the Volterra integral operator kernel, the solution set $S_F^p(h)$ turns out to be also an absolute retract.

2. Preliminaries

Denote by I the interval $[0,T]$ and by Σ the σ-algebra of Lebesgue measurable subsets of I. Let E be a real Banach space with the norm $|\cdot|$ and $\mathscr{B}(E)$ the family of Borel subsets of E. The space of bounded linear endomorphisms of E is denoted by $\mathscr{L}(E)$ and E^* stands for the normed dual of E. Given $S \in \mathscr{L}(E)$, $||S||_{\mathscr{L}}$ is the norm of S. The closure and the closed convex envelope of $A \subset E$ will be denoted by $\overline{A}$ and $\overline{\mathrm{co}}\,A$ and if $x \in E$ we set

$$d(x,A) = \inf\{|x-y| : y \in A\}.$$

Besides, for two nonempty closed bounded subsets A, B in E we denote by $d_H(A,B)$ the Hausdorff distance from A to B, i.e.

$$d_H(A,B) = \max\{\sup\{d(x,B) : x \in A\}, \sup\{d(y,A) : y \in B\}\}.$$

By $(C(I,E), ||\cdot||)$ we mean the Banach space of continuous maps $I \to E$ equipped with the maximum norm. Let $1 \le p < \infty$. Then $(L^p(I,E), ||\cdot||_p)$ is the

Banach space of all (Bochner) p-integrable maps $w\colon I \to E$, i.e. $w \in L^p(I,E)$ if and only if f is strongly measurable and

$$||w||_p = \bigg(\int_0^T |w(t)|^p \, dt \bigg)^{1/p} < \infty.$$

Notice that strong measurability is equivalent to the usual measurability in case E is separable.

Recall that a subset K of $L^1(I,E)$ is called decomposable if for every $u, v \in K$ and every $A \in \Sigma$, we have $u\,\mathbf{1}_A + v\,\mathbf{1}_{I\setminus A} \in K$, where $\mathbf{1}_A$ stands for the characteristic function of A.

A set-valued map $F\colon E \multimap E$ assigns to any $x \in E$ a nonempty subset $F(x) \subset E$. F is upper (lower) semicontinuous, if the small inverse image $F^{-1}(A) = \{x \in E : F(x) \subset A\}$ is open (closed) in E whenever A is open (closed) in E. A map $F\colon E \multimap E$ is upper hemicontinuous if for each $p \in E^*$, the function $\sigma(p, F(\,\cdot\,))\colon E \to \mathbb{R} \cup \{+\infty\}$ is upper semicontinuous as an extended real function, where $\sigma(p, F(x)) = \sup\limits_{y \in F(x)} \langle p, y\rangle$. Let X be a separable metric space and $\mathscr{A}$ be a σ-algebra of subsets of X. The map $F\colon X \multimap E$ is said to be $\mathscr{A}$-measurable if for every open $C \subset X$, we have $F^{-1}(C) \in \mathscr{A}$. A function $f\colon X \to E$ such that $f(x) \in F(x)$ for every $x \in X$ is called a selection of F.

By $H_*(\,\cdot\,)$ we denote the Čech homology functor with coefficients in the field of rational numbers $\mathbb{Q}$ from the category of compact pairs of metric spaces and continuous maps of such pairs to the category of graded vector spaces over $\mathbb{Q}$ and linear maps of degree zero. The space X having the property

$$H_q(X) = \begin{cases} 0 & \text{for } q \geq 1, \\ \mathbb{Q} & \text{for } q = 0, \end{cases}$$

is called acyclic. In other words its homology is exactly the same as the homology of a one point space. A compact (nonempty) space X is an R_δ-set if there is a decreasing sequence of contractible compacta $(X_n)_{n\geq 1}$ containing X as a closed subspace such that $X = \bigcap\limits_{n\geq 1} X_n$. From the continuity of the homology functor $H_*(\,\cdot\,)$ it follows that R_δ-sets are acyclic.

We say that a set-valued map $F\colon E \multimap M$, where M is a metric space, is a J-map if F is upper semicontinuous and the set $F(x)$ is R_δ. An upper semicontinuous map $F\colon E \multimap M$ is called acyclic if it has compact acyclic values. Let U be an open subset of E. A map $\Phi\colon U \multimap E$ is called decomposable, if there is a J-map $F\colon U \multimap M$, with M being a metric ANR, and a singlevalued continuous $f\colon M \to E$ such that $\Phi = f \circ F$. A set-valued map $F\colon E \multimap M$ is admissible (compare [9, Definition 40.1]) if there is a metric space X and two continuous functions $p\colon X \to E$, $q\colon X \to M$ from which p is a Vietoris map such that $F(x) = q(p^{-1}(x))$ for every $x \in E$. Clearly, every J-map, acyclic map or decomposable map is admissible.

Finally, a real function χ defined on the family of bounded subsets of E is called the Hausdorff measure of noncompactness if

$$\chi(\Omega) := \inf\big\{\varepsilon > 0 : \Omega \text{ admits a finite covering by balls of a radius } \varepsilon\big\}.$$

Assume that p is a real number from the interval $[1,\infty)$. Throughout this paper it is assumed that $q \in (1,\infty]$ satisfies $q^{-1} + p^{-1} = 1$. We will say that a multimap $F\colon I \times E \multimap E$ satisfies (F) if the following hypotheses are satisfied:

- (F_1) for every $(t,x) \in I \times E$ the set $F(t,x)$ is nonempty, closed and convex,
- (F_2) the map $F(\,\cdot\,,x)$ has a strongly measurable selection for every $x \in E$,
- (F_3) the map $F(t,\,\cdot\,)$ is upper hemicontinuous for almost all $t \in I$,
- (F_4) there is $c \in L^p(I,\mathbb{R})$ such that $||F(t,x)||^+ = \sup\{|y| : y \in F(t,x)\} \leq c(t)(1+|x|)$ for almost all $t \in I$ and for all $x \in E$,
- (F_5) there is a function $\eta \in L^p(I,\mathbb{R})$ such that for all bounded subsets $\Omega \subset E$ and for almost all $t \in I$ the following inequality holds

$$\chi(F(t,\Omega)) \leq \eta(t)\chi(\Omega).$$

We shall say that $F\colon I \times E \multimap E$ fulfills (H) if the following assumptions are satisfied:

- (H_1) the set $F(t,x)$ is nonempty, closed and bounded for every $(t,x) \in I \times E$,
- (H_2) the map $F(\,\cdot\,,x)$ is Σ-measurable for every $x \in E$,
- (H_3) there exists $\alpha \in L^p(I,\mathbb{R})$ such that $d_H(F(t,x),F(t,y)) \leq \alpha(t)|x-y|$, for all $x,y \in E$, almost everywhere in I,
- (H_4) there exists $\beta \in L^p(I,\mathbb{R})$ such that $d(0,F(t,0)) \leq \beta(t)$, almost everywhere in I.

Denote by $\triangle$ the set $\{(t,s) \in I \times I : 0 \leq s \leq t \leq T\}$. We shall also assume that the mapping $k\colon \triangle \to \mathscr{L}(E)$ possesses the following properties:

- (k_1) the function $k(\,\cdot\,,s)\colon [s,T] \to \mathscr{L}(E)$ is differentiable for every $s \in I$,
- (k_2) the function $k(t,\,\cdot\,)\colon [0,t] \to \mathscr{L}(E)$ is continuous for all $t \in I$,
- (k_3) the function $k(\,\cdot\,,\,\cdot\,)\colon \{(t,t) : t \in I\} \to \mathscr{L}(E)$ is continuous, whereas the operator $k(t,t)$ is invertible for all $t \in I$,
- (k_4) there exists $\mu \in L^q(I,\mathbb{R})$ such that for every $(t,s) \in \triangle$ we have

$$\left\|\frac{\partial}{\partial t}k(t,s)\right\|_{\mathscr{L}} \leq \mu(s),$$

- (k_5) for every $t \in I$, $k(t,\,\cdot\,) \in L^q([0,t],\mathscr{L}(E))$,
- (k_6) the function $I \ni t \mapsto k(t,\,\cdot\,) \in L^q([0,t],\mathscr{L}(E))$ is continuous in the norm $||\cdot||_q$ of the space $L^q(I,\mathscr{L}(E))$.

By a solution of the Volterra integral inclusion (1.1) we mean a function $x \in C(I, E)$, which satisfies the equation

$$x(t) = h(t) + \int_0^t k(t,s)w(s)\,ds, \quad t \in I,$$

for some $w \in L^p(I, E)$ such that $w(t) \in F(t, x(t))$ for almost all $t \in I$. Obviously, the set of all solutions to the integral inclusion under consideration coincides with the set of fixed points $\mathrm{Fix}(h + V \circ N_F^p(\,\cdot\,))$ of the multivalued operator

$$h + V \circ N_F^p \colon C(I, E) \multimap C(I, E),$$

where $N_F^p \colon C(I, E) \multimap L^p(I, E)$ is the Nemtyskiĭ operator corresponding to F, given by

$$N_F^p(x) = \{w \in L^p(I, E) : w(t) \in F(t, x(t)) \text{ for a.a. } t \in I\},$$

and $V \colon L^p(I, E) \to C(I, E)$ is the Volterra integral operator, defined by

$$V(w)(t) = \int_0^t k(t,s)w(s)\,ds, \quad t \in I.$$

The eponymous *solution set map* is the multivalued operator $S_F^p \colon C(I, E) \multimap C(I, E)$, given by the formula

$$S_F^p(h) := \{x \in C(I, E) : x \in h + V \circ N_F^p(x)\} = \mathrm{Fix}(h + V \circ N_F^p(\,\cdot\,)).$$

The core reasoning, which supports proofs of the results in [13], on both topological (nonemptiness, compactness) and geometric (acyclicity) properties of the solution set $S_F^p(h)$, constitutes the so-called convergence theorem for convex valued upper hemicontinuous multimaps. In order to improve these results, we will need the following "multivalued version" of this theorem.

Theorem 2.1. *Assume that $F \colon E \multimap E$ is upper hemicontinuous and $G \colon I \multimap E$ has compact values. If sequences $(G_n : I \multimap E)_{n \geq 1}$ and $(y_n : I \to E)_{n \geq 1}$ satisfy the following conditions*:

(a) $d_H(G_n(t), G(t)) \xrightarrow[n\to\infty]{} 0$ *for almost all* $t \in I$,
(b) $y_n \rightharpoonup y$ *in the space* $L^p(I, E)$, *where* $p \geq 1$,
(c) $y_n(t) \in \overline{\mathrm{co}}\, B(F(B(G_n(t), \varepsilon_n)), \varepsilon_n)$ *for almost all* $t \in I$, *where* $\varepsilon_n \to 0^+$ *as* $n \to \infty$,

then $y(t) \in \overline{\mathrm{co}}\, F(G(t))$ *for almost all* $t \in I$.

Proof. There is a subset I_1 of full measure in I such that (a) and (c) hold. Take $t \in I_1$. Let $\varepsilon > 0$ and $e^* \in E^* \setminus \{0\}$ be arbitrary. Using the upper hemicontinuity of F we see that for every $x \in G(t)$ there is $\delta_x > 0$ such that

$$\sigma(e^*, F(B(x, \delta_x))) < \sigma(e^*, F(x)) + \frac{\varepsilon}{2}.$$

Since G is compact, we have $x_1, \ldots, x_m \in G(t)$ such that

$$G(t) \subset \bigcup_{i=1}^{m} B(x_i, \delta_{x_i}/2).$$

Put $\delta := \min_{1 \le i \le m} \delta_{x_i}/2$. If $z \in B(G(t), \delta)$, then there is $i \in \{1, \ldots, m\}$ such that

$$\sigma(e^*, F(z)) < \sigma(e^*, F(x_i)) + \frac{\varepsilon}{2}.$$

Thus

$$\sigma(e^*, F(B(G(t), \delta))) < \sigma(e^*, F(G(t))) + \frac{\varepsilon}{2}.$$

In view of (a) there is an index N such that

$$B(F(B(G_n(t), \varepsilon_n)), \varepsilon_n) \subset B\bigg(F(B(G(t), \delta)), \frac{\varepsilon}{2||e^*||}\bigg)$$

for every $n \ge N$. Whence

$$\begin{aligned}
\sigma(e^*, &B(F(B(G_n(t), \varepsilon_n)), \varepsilon_n)) \le \sigma\bigg(e^*, B\bigg(F(B(G(t), \delta)), \frac{\varepsilon}{2||f^*||}\bigg)\bigg) \\
&= \sigma\bigg(e^*, F(B(G(t), \delta)) + B\bigg(0, \frac{\varepsilon}{2||e^*||}\bigg)\bigg) \\
&\le \sigma(e^*, F(B(G(t), \delta))) + \sigma\bigg(e^*, \frac{\varepsilon}{2||e^*||}\, \overline{B}(0,1)\bigg) \\
&\le \sigma(e^*, F(G(t))) + \frac{\varepsilon}{2} + \frac{\varepsilon}{2||e^*||}\, \sigma(e^*, \overline{B}(0,1)) = \sigma(e^*, F(G(t))) + \varepsilon
\end{aligned}$$

for $n \ge N$. Since $\varepsilon > 0$ was arbitrary we see that

$$\overline{\lim_{n\to\infty}}\, \sigma(e^*, B(F(B(G_n(t), \varepsilon_n)), \varepsilon_n)) \le \sigma(e^*, F(G(t))). \tag{2.1}$$

Since $y_n \rightharpoonup y$ in $L^p(I, E)$, there is a sequence $(z_n)_{n \ge 1}$ strongly convergent to y such that $z_n \in \operatorname{co}\{y_k\}_{k=n}^{\infty}$. Further, there is a subset I_2 of full measure in I and a subsequence (again denoted by) $(z_n)_{n \ge 1}$ pointwise convergent to y for every $t \in I_2$. Let us take $t \in I_1 \cap I_2$. Then

$$\begin{aligned}
z_n(t) \in \operatorname{co}\{y_k(t)\}_{k=n}^{\infty} &\subset \operatorname{co} \bigcup_{k=n}^{\infty} \overline{\operatorname{co}}\, B(F(B(G_k(t), \varepsilon_k)), \varepsilon_k) \\
&\subset \overline{\operatorname{co}} \bigcup_{k=n}^{\infty} B(F(B(G_k(t), \varepsilon_k)), \varepsilon_k)
\end{aligned}$$

and

$$\begin{aligned}
\langle e^*, z_n(t) \rangle &\le \sigma\bigg(e^*, \overline{\operatorname{co}} \bigcup_{k=n}^{\infty} B(F(B(G_k(t), \varepsilon_k)), \varepsilon_k)\bigg) \\
&= \sup_{k \ge n} \sigma(e^*, B(F(B(G_k(t), \varepsilon_k)), \varepsilon_k)).
\end{aligned}$$

As a result we get

$$\begin{aligned}\langle e^*, y(t)\rangle &= \lim_{n\to\infty}\langle e^*, z_n(t)\rangle \le \lim_{n\to\infty}\sup_{k\ge n}\sigma(e^*, B(F(B(G_k(t),\varepsilon_k)),\varepsilon_k)\\ &= \overline{\lim_{n\to\infty}}\,\sigma(e^*, B(F(G_n(t),\varepsilon_n)),\varepsilon_n)).\end{aligned}$$

Combining this estimation with (2.1) we see that

$$\langle e^*, y(t)\rangle \le \sigma(e^*, F(G(t))).$$

Bearing in mind that closed convex sets possess the following description

$$\overline{\mathrm{co}}\,F(G(t)) = \{v\in E : \langle e^*, v\rangle \le \sigma(e^*, \overline{\mathrm{co}}\,F(G(t))) \text{ for every } e^*\in E^*\},$$

we conclude finally that $y(t)\in\overline{\mathrm{co}}F(G(t))$ for almost all $t\in I$. □

Taking into account that the initial set of assumptions (F) ensures that the fixed point set $\mathrm{Fix}(h+V\circ N_F^p(\,\cdot\,))$ is acyclic, one can reverse the problem by asking the question about the structure of $\mathrm{Fix}(N_F^p(h+V(\,\cdot\,)))$. Thus, it is quite natural to define the *selection set map* $\mathscr{S}_F^p\colon C(I,E)\multimap L^p(I,E)$, corresponding to problem (1.1), by

$$\mathscr{S}_F^p(h) := \bigcup_{x\in S_F^p(h)}\{w\in N_F^p(x) : x = h+V(w)\} = \mathrm{Fix}\,\big(N_F^p(h+V(\,\cdot\,))\big).$$

Remark 2.2. Conditions (k_1)–(k_4) imply that $\{w\in N_F^p(x) : x=h+V(w)\}$ is a singleton for every $x\in S_F^p(h)$ (compare Lemma 2 in [13]).

Theorems concerning the geometrical structure of the set $\mathscr{S}_F^p(h)$ are direct conclusions of the following technical lemma.

Lemma 2.3. *Let E be a separable Banach space. Assume that $F\colon I\times E\multimap E$ satisfies* (H) *and $k\colon \triangle\to\mathscr{L}(E)$ satisfies* (k_5)–(k_6). *There is an equivalent norm on the space $L^p(I,E)$, in which the multivalued map $G_p\colon C(I,E)\times L^p(I,E)\multimap L^p(I,E)$ given by the formula*

$$\begin{aligned}G_p(h,u) := \bigg\{w\in L^p(I,E) : w(t)\in F\bigg(t, h(t)+\int_0^t k(t,s)u(s)\,ds\bigg) \text{ a.e. in } I\bigg\}\\ = N_F^p(h+V(u)),\end{aligned}$$

is continuous and contractive with respect to u.

Proof. Endow the space $L^p(I,E)$ of Bochner p-integrable functions with the following equivalent norm:

$$|||w|||_p := \bigg(\int_0^T e^{-2^{2p-1}Mr(t)}|w(t)|^p\,dt\bigg)^{1/p}, \tag{2.2}$$

where $M := \max\Big\{1, \sup_{t\in I}||k(t,\,\cdot\,)||_q^p\Big\}$ and $r(t) := \int_0^t \alpha(s)^p\,ds$.

It follows from (H_3) that $F(t,\cdot)$ is Hausdorff continuous. By virtue of [11, Theorem 3.3] the map F is $\Sigma\otimes\mathscr{B}(E)$-measurable. In particular, $F(\cdot,h(\cdot)+V(u)(\cdot))$ is measurable for $(h,u)\in C(I,E)\times L^p(I,E)$, since F is superpositionally measurable ([16, Theorem 1]). Thanks to (H_3) and (H_4) we have $\inf\{|z| : z\in F(t,h(t)+V(u)(t))\}\in L^p(I,\mathbb{R})$, which means that $G_p(h,u)\neq\emptyset$.

Take $(h_1,u_1),(h_2,u_2)\in C(I,E)\times L^p(I,E)$, $w_1\in G_p(h_1,u_1)$ and $\varepsilon>0$. It is easy to see that $t\mapsto d(w_1(t),F(t,h_2(t)+V(u_2)(t)))$ is measurable. By (H_3) we have

$$d(w_1(t),F(t,h_2(t)+V(u_2)(t)))\le\alpha(t)||h_1-h_2+V(u_1-u_2)||$$

almost everywhere in I, and so $d(w_1(\cdot),F(\cdot,h_2(\cdot)+V(u_2)(\cdot)))\in L^p(I,\mathbb{R})$.

Define $\psi=\operatorname{ess\,inf}\{|u(\cdot)| : u\in K\}$, where $K:=\{w_1\}-N_F^1(h_2+V(u_2))$. Considering that

$$\psi(t)\le d(w_1(t),F(t,h_2(t)+V(u_2)(t)))<d(w_1(t),F(t,h_2(t)+V(u_2)(t)))+\varepsilon$$

almost everywhere in I, it follows by [5, Proposition 2] that there is $w_2\in G_p(h_2,u_2)$ such that

$$|w_1(t)-w_2(t)|<d(w_1(t),F(t,h_2(t)+V(u_2)(t)))+\varepsilon$$

for almost all $t\in I$. Thus, we can estimate

$$\begin{aligned}
|||w_1-w_2|||_p^p &= \int_0^T e^{-2^{2p-1}Mr(t)}|w_1(t)-w_2(t)|^p\,dt\\
&\le\int_0^T e^{-2^{2p-1}Mr(t)}\big(d(w_1(t),F(t,h_2(t)+V(u_2)(t)))+\varepsilon\big)^p\,dt\\
&\le\int_0^T e^{-2^{2p-1}Mr(t)}\big(d_H(F(t,h_1(t)+V(u_1)(t)),F(t,h_2+V(u_2)(t)))+\varepsilon\big)^p\,dt\\
&\le\int_0^T e^{-2^{2p-1}Mr(t)}\left(\alpha(t)\left|h_1(t)-h_2(t)+\int_0^t k(t,s)(u_1(s)-u_2(s))\,ds\right|+\varepsilon\right)^p dt\\
&\le\int_0^T e^{-2^{2p-1}Mr(t)}2^{p-1}(\alpha(t)|h_1(t)-h_2(t)|+\varepsilon)^p\,dt\\
&\quad+\int_0^T e^{-2^{2p-1}Mr(t)}2^{p-1}\alpha(t)^p\left(\int_0^t||k(t,s)||_{\mathscr{L}}|u_1(s)-u_2(s)|\,ds\right)^p dt\\
&\le||h_1-h_2||^p\int_0^T 2^{2p-2}e^{-2^{2p-1}Mr(t)}\alpha(t)^p\,dt+2^{2p-2}\varepsilon^p\int_0^T e^{-2^{2p-1}Mr(t)}\,dt\\
&\quad+\int_0^T e^{-2^{2p-1}Mr(t)}\alpha(t)^p2^{p-1}||k(t,\cdot)||_q^p\int_0^t|u_1(s)-u_2(s)|^p\,ds\,dt\\
&\le\frac{1}{2M}\big(1-e^{-2^{2p-1}Mr(T)}\big)||h_1-h_2||^p+T2^{2p-2}\varepsilon^p\\
&\quad+M\int_0^T\int_s^T e^{-2^{2p-1}Mr(t)}\alpha(t)^p2^{p-1}|u_1(s)-u_2(s)|^p\,dt\,ds
\end{aligned}$$

$$
\begin{aligned}
&\le \frac{1}{2M}\,||h_1-h_2||^p + T2^{2p-2}\varepsilon^p \\
&\quad + \frac{1}{2^p}\int_0^T \big(e^{-2^{2p-1}Mr(s)} - e^{-2^{2p-1}Mr(T)}\big)|u_1(s)-u_2(s)|^p\,ds \\
&\le \frac{1}{2}\,||h_1-h_2||^p + \frac{1}{2^p}\int_0^T e^{-2^{2p-1}Mr(s)}|u_1(s)-u_2(s)|^p\,ds \\
&\quad - \frac{1}{2^p}\,e^{-2^{2p-1}Mr(T)}||u_1-u_2||_p^p + T2^{2p-2}\varepsilon^p \\
&\le \frac{1}{2}\big(||h_1-h_2||^p + |||u_1-u_2|||_p^p\big) + T2^{2p-2}\varepsilon^p.
\end{aligned}
$$

Since ε was arbitrarily small and w_1 was an arbitrary element of $G_p(h_1,u_1)$, it follows that

$$
\Big(\sup_{w\in G_p(h_1,u_1)} d(w,G_p(h_2,u_2))\Big)^p \le \frac{1}{2}\,(||h_1-h_2||^p + |||u_1-u_2|||_p^p)
$$

and consequently

$$
d_H(G_p(h_1,u_1),G_p(h_2,u_2)) \le \bigg(\frac{1}{2}\,(||h_1-h_2||^p + |||u_1-u_2|||_p^p)\bigg)^{1/p}.
$$

In particular,

$$
d_H(G_p(h,u_1),G_p(h,u_2)) \le \frac{1}{2^{p^{-1}}}\,|||u_1-u_2|||_p.
$$

Therefore G_p is Hausdorff continuous and contractive with respect to the second variable. □

3. Main results

The following result is the announced enhancement of Theorem 6 in [13], dedicated to the description of geometric structure of the solution set $S_F^p(h)$.

Theorem 3.1. *Let $p\in[1,\infty)$, while the space E is reflexive for $p\in(1,\infty)$. Assume that $F\colon I\times E\multimap E$ satisfies* (F) *and $k\colon\triangle\to\mathscr{L}(E)$ satisfies* (k_1)–(k_4). *Then the solution set $S_F^p(h)$ of integral inclusion* (1.1) *is an R_δ-set in the space $C(I,E)$.*

Proof. We adopt the notation and arguments used to justify [13, Theorem 6] in the context of assumption (E_2). Let us recall that there exists a nonempty convex compact set $X\subset C(I,E)$ possessing the following property:

$$
h(t) + \int_0^t k(t,s)\,\overline{\mathrm{co}}\,F(s,X(s))\,ds \subset X(t) \quad \text{on } I.
$$

Let $\mathrm{Pr}\colon I\times E\multimap E$ be the time-dependent metric projection, given by:

$$
\mathrm{Pr}(t,x) = \{y\in X(t) : |x-y| = \inf\{|x-z| : z\in X(t)\}\}.
$$

Observe that Pr is an upper semicontinuous multimap with compact values. Relying on the compactness of the set X, define the multimap $\widetilde{F}\colon I \times E \multimap E$ by the formula:

$$\widetilde{F}(t,x) = \overline{\mathrm{co}}\, F(t, \mathrm{Pr}(t,x)).$$

Note that $\widetilde{F}$ satisfies assumptions (F_1)–(F_5). Replacement of the map F by the map $\widetilde{F}$ does not change the set of solutions to inclusion (1.1), i.e. $S^p_F = S^p_{\widetilde{F}}$. Using a strictly analogous approach to that applied in the proof of case (E_1) we can represent the solution set $S^p_{\widetilde{F}}$ in the form of a countable intersection of a decreasing sequence of compact solution sets S^p_n to integral inclusions corresponding to multivalued approximations F_n such that $F_n(t,x) \subset \overline{\mathrm{co}}\, \widetilde{F}(t, B(x, 3^{-n+1})) \subset \overline{\mathrm{co}}\, F(t, X(t))$ for $(t,x) \in I \times E$. Resting on the uniqueness of solutions to integral equations of the form

$$x(t) = g(t) + \int_\tau^t k(t,s) f_n(s, x(s))\, ds,$$

one can show that sets S^p_n are contractible. As a result, we conclude that $S^p_{\widetilde{F}}$ is an R_δ-set. The choice of measurable-locally Lipschitzian mapping $f_n\colon I \times E \to E$ is based on the existence of strongly measurable selection of the map $\widetilde{F}(\,\cdot\,, x)$. Therefore the proof will be completed if we can show that the multimap $\overline{\mathrm{co}}\, F(\,\cdot\,, \mathrm{Pr}(\,\cdot\,, x))$ possesses a strongly measurable selection.

Since $\mathrm{Pr}(\,\cdot\,, x)\colon I \multimap E$ is a compact valued upper semicontinuous multimap, there exists a sequence $(\mathrm{Pr}_n\colon I \multimap E)_{n\geq 1}$ of compact valued upper semicontinuous step multifunctions such that for every $t \in I$ we have

$$\mathrm{Pr}(t,x) \subset \mathrm{Pr}_{n+1}(t) \subset \mathrm{Pr}_n(t,x) \quad \text{and} \quad d_H(\mathrm{Pr}_n(t), \mathrm{Pr}(t,x)) \xrightarrow{n\to\infty} 0.$$

Indeed, one can define the map Pr_n in the following way:

$$\mathrm{Pr}_n(t) := \sum_{i=0}^{2^n-1} \mathbf{1}_{[t^n_i, t^n_{i+1}]}(t)\, \mathrm{Pr}([t^n_i, t^n_{i+1}], x),$$

where $t^n_i = iT/2^n$. In view of assumptions (F_2) and (F_4) there are functions $w^n_i \in L^p(I,E)$ such that $w^n_i(t) \in F(t, a^n_i)$ almost everywhere on I for some $a^n_i \in \mathrm{Pr}([t^n_i, t^n_{i+1}], x)$. Let $w_n = \sum\limits_{i=0}^{2^n-1} w^n_i \mathbf{1}_{[t^n_i, t^n_{i+1}]}$. The function w_n is also Bochner integrable and more importantly $w_n(t) \in F(t, \mathrm{Pr}_n(t))$ for almost all $t \in I$. If $p = 1$, we see that for almost all $t \in I$ the set $\{w_n(t)\}_{n\geq 1}$ is contained in the weakly compact set $F\Big(t, \bigcup\limits_{t\in I} X(t)\Big)$. Thus the sequence $(w_n)_{n\geq 1}$ is relatively weakly compact in $L^1(I,E)$, thanks to [15, Proposition 11]. The Eberlein–Šmulian theorem implies the relative weak compactness of the sequence $(w_n)_{n\geq 1}$ for $p \in (1,\infty)$. Denote by w the weak limit of some subsequence of $(w_n)_{n\geq 1}$. Applying Theorem 2.1, we infer that $w(t) \in \overline{\mathrm{co}}\, F(t, \mathrm{Pr}(t,x))$ almost everywhere on I. Thus w is the sought strongly measurable selection of $\widetilde{F}(\,\cdot\,,x)$, completing the proof. $\square$

COROLLARY 3.2. *Let E be a real Banach space and let $p=1$. Assume that the multimap $F\colon I\times E\multimap E$ satisfies* (F) *and the linear operator $A\colon E\to E$ is the infinitesimal generator of a uniformly continuous semigroup of bounded linear operators. Then the set of all mild solutions of the Cauchy problem for the following semilinear differential inclusion:*

$$\begin{cases} \dot{x}(t)+Ax(t)\in F(t,x(t)) & \textit{on } I,\\ x(0)=x_0, \end{cases} \tag{3.1}$$

is an R_δ-subset of the space $C(I,E)$.

PROOF. Denote by $\{U(t)\}_{t\geq 0}$ the semigroup generated by the operator A. Recall that a continuous function $x\colon I\to E$ is a mild solution to (3.1) if there is an integrable $w\in N_F^1(x)$ such that

$$x(t)=U(t)x_0+\int_0^t U(t-s)w(s)\,ds \quad \text{for } t\in I.$$

Uniform continuity of the semigroup $\{U(t)\}_{t\geq 0}$ means that the Volterra kernel $k(t,s):=U(t-s)$ satisfies conditions (k_1)–(k_4), completing the proof. $\square$

REMARK 3.3. It is worth noting at this point that in the particular case of the semilinear differential inclusion (3.1) there are well-established results regarding the R_δ-structure of the solution set ([3, Theorem 5], [10, Corollary 5.3.1]) and upper semicontinuous dependence on initial conditions ([10, Corollary 5.2.2]), which are actually more general than Corollary 3.2, since they are based on the assumption that the operator A generates merely a C_0-semigroup.

In the subsequent propositions we will discuss possible further implications of the structure result proven above. Following the established terminology of [12], let us recall that we are dealing with the parabolic initial value problem for the evolution inclusion

$$\begin{cases} \dot{x}(t)+A(t)x(t)\in F(t,x(t)) & \text{on } I,\\ x(0)=x_0, \end{cases} \tag{3.2}$$

if the mentioned below assumptions are met:

(P_1) the domain $D(A(t))=D$ of $A(t)$, $t\in I$, is dense in E and independent of t,

(P_2) for $t\in I$, the resolvent $R(\lambda:A(t))$ of $A(t)$ exists for all λ with $\operatorname{Re}\lambda\leq 0$ and there is a constant M such that

$$||R(\lambda:A(t))||_{\mathscr{L}}\leq \frac{M}{|\lambda|+1} \quad \text{for } \operatorname{Re}\lambda\leq 0,\ t\in I,$$

(P_3) there exist constants L and $0 < \alpha \leq 1$ such that

$$||(A(t) - A(s))A(\tau)^{-1}||_{\mathscr{L}} \leq L|t - s|^{\alpha} \quad \text{for } s, t, \tau \in I.$$

Theorem 3.4. *Let E be a real Banach space and let* (P_1)–(P_3) *be satisfied. Moreover, assume that*

(a) *the domain $D(A(t)) = E$ for all $t \in I$ and independent of t,*
(b) *the transformation $I \ni t \mapsto A(t)^{-1}x \in E$ is continuous for every $x \in E$,*
(c) *the unique evolution system $U(t, s)$ generated by the family $\{A(t)\}_{t\in I}$ is continuous with respect to the second variable in the uniform operator topology, i.e. $U(t, \cdot)\colon [0, t] \to \mathscr{L}(E)$ is continuous for every $t \in I$.*

If the multivalued perturbation $F\colon I \times E \multimap E$ satisfies (F) *with $p = 1$, then the set $S_F(x_0) \subset C(I, E)$ of all mild solutions of the parabolic initial value problem* (3.2) *is an R_δ-set.*

Proof. Recall that $x \in S_F(x_0)$ if and only if

$$x(t) \in U(t, 0)x_0 + \int_0^t U(t, s)F(s, x(s))\, ds \quad \text{on } I.$$

Observe that all the hypotheses of Theorem 3.1 are fulfilled, except condition (k_4). In particular, the evolution system $U\colon \triangle \to \mathscr{L}(E)$ satisfies properties (k_1)–(k_3). The property (k_1) results from [12, Theorem 6.1] (item $(\mathrm{E}_2)^+$). Condition (k_2) is equivalent to (c) and (k_3) follows from the very definition of the evolution system.

Lemma 2 in [13] is crucial for the proof of Theorem 6 in [13] and, consequently, for an enhanced version given in Theorem 3.1. What is important, condition (k_4) is used exclusively to justify this particular lemma. From this it follows that the present proof will be completed if we show that the Volterra integral operator $V\colon L^1(I, E) \to C(I, E)$, given by $V(w)(t) := \int_0^t U(t, s)w(s)\, ds$, is a monomorphism.

Take $w \in L^1(I, E)$. By virtue of (b) the mapping $[0, t] \ni s \mapsto A(s)^{-1}w(s) \in E$ is strongly measurable. Note that

$$|U(t, s)A(s)^{-1}w(s)| \leq ||U(t, s)||_{\mathscr{L}}||A(s)^{-1}||_{\mathscr{L}}|w(s)| \leq CM|w(s)|,$$

in view of property (P_2) and $(\mathrm{E}_1)'$ in [12, Theorem 6.1]. Therefore, the following modified integral operator $\widetilde{V}\colon L^1(I, E) \to C(I, E)$:

$$\widetilde{V}(w)(t) := \int_0^t U(t, s)A(s)^{-1}w(s)\, ds,$$

is properly defined.

Let $\widetilde{V}(w_1) = \widetilde{V}(w_2)$ and $w = w_1 - w_2$. Basically, we will reproduce the approach used in the proof of Lemma 2 in [13]. Since $d\widetilde{V}(w)(t)/dt = 0$, we have

$$\lim_{n\to\infty}\bigg(\int_0^{t-1/n}\frac{U(t-1/n,s)A(s)^{-1}-U(t,s)A(s)^{-1}}{-1/n}\,w(s)\,ds + n\int_{t-1/n}^{t}U(t,s)A(s)^{-1}w(s)\,ds\bigg)=0 \tag{3.3}$$

for every $t\in I$. If

$$f_n(s)=\frac{U(t-1/n,s)A(s)^{-1}-U(t,s)A(s)^{-1}}{-1/n}\,w(s)\mathbf{1}_{[0,\,t-1/n]}(s),$$

then $f_n \in L^1([0,t],E)$ and $f_n(s) \xrightarrow{n\to\infty} \partial U(t,s)A(s)^{-1}w(s)/\partial t$ for $s \in [0,t)$. Using the findings of [12, Theorem 6.1], we can estimate

$$\begin{aligned}|f_n(s)| &\le \bigg\|\frac{U(t-1/n,s)A(s)^{-1}-U(t,s)A(s)^{-1}}{-1/n}\bigg\|_{\mathscr{L}}|w(s)|\\ &\le \sup_{\xi\in[t-1/n,t]}\bigg\|\frac{\partial}{\partial t}U(\xi,s)A(s)^{-1}\bigg\|_{\mathscr{L}}|w(s)|\\ &= \sup_{\xi\in[t-1/n,t]}\|A(\xi)U(\xi,s)A(s)^{-1}\|_{\mathscr{L}}|w(s)| \le C|w(s)|\end{aligned}$$

for all $n\ge 1$ and $s\in[0,t-1/n)$. Consequently, there is convergence

$$\int_0^{t-1/n}\frac{U(t-1/n,s)A(s)^{-1}-U(t,s)A(s)^{-1}}{-1/n}\,w(s)\,ds \xrightarrow{n\to\infty} \int_0^t\frac{\partial}{\partial t}U(t,s)A(s)^{-1}w(s)\,ds$$

for every $t\in I$. On the other side, we can evaluate the second term of (3.3) in the following way:

$$\begin{aligned}&n\int_{t-1/n}^{t}|U(t,s)A(s)^{-1}w(s)-U(t,t)A(t)^{-1}w(t)|\,ds\\ &\qquad\le ||U(t,\xi(n))-U(t,t)||_{\mathscr{L}}\,n\int_{t-1/n}^{t}|A(s)^{-1}w(s)|\,ds\\ &\qquad\quad+||U(t,t)||_{\mathscr{L}}\,n\int_{t-1/n}^{t}|A(s)^{-1}w(s)-A(t)^{-1}w(t)|\,ds\end{aligned}$$

for some $\xi(n)\in[t-1/n,t]$. Using the characteristic of Lebesgue points and the continuity of $U(t,\,\cdot\,)$, we obtain

$$\lim_{n\to\infty} n\int_{t-1/n}^{t}U(t,s)A(s)^{-1}w(s)\,ds = U(t,t)A(t)^{-1}w(t)$$

for almost all $t\in I$. Applying (3.3) one gets

$$A(t)^{-1}w(t) = -\int_0^t\frac{\partial}{\partial t}U(t,s)A(s)^{-1}w(s)\,ds \tag{3.4}$$

for almost all $t \in I$.

Observe that $\{A(t)\}_{t\in I} \subset \mathscr{L}(E)$, in view of assumptions (P_2) and (a). We claim that the family $\{A(t)\}_{t\in I}$ is strongly continuous, i.e. the map $I \ni t \mapsto A(t)x \in E$ is continuous for every fixed $x \in E$. Considering (P_3), we have

$$||\mathrm{id} - (\mathrm{id} - A(t)(A(t)^{-1} - A(t+h)^{-1}))||_{\mathscr{L}} = ||\mathrm{id} - A(t)A(t+h)^{-1}||_{\mathscr{L}} \leq L|h|^{\alpha}$$

and consequently,

$$\begin{aligned} ||(\mathrm{id} - A(t)(A(t)^{-1} - A(t+h)^{-1}))^{-1}||_{\mathscr{L}} \\ \leq \frac{1}{1 - ||\mathrm{id} - A(t)A(t+h)^{-1}||_{\mathscr{L}}} \leq \frac{1}{1 - L|h|^{\alpha}}. \end{aligned}$$

Thus

$$\begin{aligned} ||A(t+h)||_{\mathscr{L}} = ||(\mathrm{id} - A(t)(A(t)^{-1} - A(t+h)^{-1}))^{-1}A(t)||_{\mathscr{L}} \\ \leq \frac{||A(t)||_{\mathscr{L}}}{1 - L|h|^{\alpha}} \leq 2||A(t)||_{\mathscr{L}} \end{aligned}$$

for sufficiently small h. Fix $x \in E$. Thanks to the above estimation we get:

$$\begin{aligned} |A(t+h)x - A(t)x| &= |A(t+h)(A(t)^{-1} - A(t+h)^{-1})A(t)x| \\ &\leq ||A(t+h)||_{\mathscr{L}}|(A(t)^{-1} - A(t+h)^{-1})A(t)x| \\ &\leq 2||A(t)||_{\mathscr{L}}|x - A(t+h)^{-1}(A(t)x)|. \end{aligned}$$

for sufficiently small h. From (b) it follows that the last quantity tends to zero as $h \to 0$. The uniform boundedness principle and the strong continuity of the family $\{A(t)\}_{t\in I}$ imply $R := \sup_{t\in I} ||A(t)||_{\mathscr{L}} < \infty$.

As a consequence of (3.4) and [12, Theorem 6.1] we obtain the following estimation:

$$|w(t)| \leq ||A(t)||_{\mathscr{L}} \int_0^t \left\| \frac{\partial}{\partial t} U(t,s)A(s)^{-1} \right\|_{\mathscr{L}} |w(s)|\, ds \leq R \int_0^t C|w(s)|\, ds$$

for almost all $t \in I$. In view of the Gronwall inequality, it is clear that $|w(t)| = 0$ for almost all $t \in I$. Thus $w_1 = w_2$ in $L^1(I,E)$ and the operator $\widetilde{V}$ is injective.

Let $V(w_1) = V(w_2)$. Taking into account previously demonstrated properties of $\{A(t)\}_{t\in I}$ we see that $A(\,\cdot\,)w_1, A(\,\cdot\,)w_2 \in L^1(I,E)$. Now it suffices to note that $\widetilde{V}(A(\,\cdot\,)w_1) = \widetilde{V}(A(\,\cdot\,)w_2)$. Hence $A(t)w_1(t) = A(t)w_2(t)$ for almost all $t \in I$. Eventually, $w_1(t) = A(t)^{-1}A(t)w_1(t) = A(t)^{-1}A(t)w_2(t) = w_2(t)$ almost everywhere on I, completing the proof. □

Let us recall that a set-valued map $F\colon U \subset E \multimap E$ is strongly upper semicontinuous if, for every sequence $(x_n)_{n\geq 1}$ in U such that $x_n \rightharpoonup x_0$ and any sequence $(y_n)_{n\geq 1}$ satisfying $y_n \in F(x_n)$ for all $n \geq 1$, there is a subsequence $(y_{k_n})_{n\geq 1}$ such that $y_{k_n} \to y_0 \in F(x_0)$. Providing another example of the application of Theorem 3.1, we will refer to the following fixed point result:

Lemma 3.5. *Let E be a reflexive Banach space and let $\varphi\colon \overline{B}(0,R) \subset E \multimap \overline{B}(0,R/2)$ be a decomposable strongly upper semicontinuous map. Assume there is $S \in \mathscr{L}(E)$ with $||S|| \leq 1/2$. Then the multimap $\Phi := S + \varphi\colon \overline{B}(0,R) \multimap E$ has a fixed point.*

Proof. Observe that S is a nonexpansive J-map and Φ satisfies Yamamuro's condition, i.e.

$$\exists\, z \in B(0,R)\ \forall\, x \in \partial B(0,R) \quad \lambda(x-z) \in \Phi(x) - \{z\} \Rightarrow \lambda \leq 1.$$

Indeed, set $z := 0$ and note that for $x \in \partial B(0,R)$ we have

$$|\lambda| = \frac{|\lambda x|}{|x|} \leq \frac{|\lambda x - Sx| + |Sx|}{|x|} \leq \left(\frac{R}{2} + \frac{R}{2}\right)\frac{1}{R} = 1.$$

Proceed along the proof of [2, Corollary 11] to convince oneself that $\operatorname{Fix}(\Phi) \neq \emptyset$.□

Remark 3.6. Of course, the condition of Opial used to justify [2, Corollary 11] is superfluous in our case, when S is a singlevalued bounded endomorphism.

Theorem 3.7. *Let E be a reflexive Banach space and $p \in [1,\infty)$. Assume that conditions* (F) *and* (k_1)–(k_4) *are satisfied. If $F(t,\,\cdot\,)$ is strongly upper semicontinuous for almost all $t \in I$, then there exists a continuous function $h\colon I \to E$, for which integral inclusion* (1.1) *possesses a T-periodic solution.*

Proof. At the beginning, note that we can assume, without loss of generality, that the map F is integrably bounded, i.e. there exists $\mu \in L^p(I,\mathbb{R})$ such that $||F(t,x)||^+ \leq \mu(t)$ for every $x \in E$ and for almost all $t \in I$ (see commentary at the beginning of Section 3 in [13]). Let $\{U(t)\}_{t\in I} \subset \mathscr{L}(E)$ be a family of operators possessing the following properties:

(i) $U(0) = \mathrm{id}$,
(ii) $\{U(t)\}_{t\in I}$ is strongly continuous, i.e. the map $I \ni t \mapsto U(t)x \in E$ is continuous for every $x \in E$,
(iii) $||U(T)||_{\mathscr{L}} \leq 1/2$.

Let us define the multimap $\varphi\colon E \multimap E$ as $\varphi(x) := V(\mathscr{S}_F^p(U(\,\cdot\,)x))(T)$. Clearly,

$$|V(w)(T)| \leq R := ||k(T,\,\cdot\,)||_q ||\mu||_p$$

for every $w \in \mathscr{S}_F^p(U(\,\cdot\,)x)$ and for all $x \in E$. Therefore $\varphi(\overline{B}(0,2R)) \subset \overline{B}(0,R)$. We claim that the Poincaré-like operator $P\colon \overline{B}(0,2R) \multimap E$, given by $P(x) := U(T)x + \varphi(x)$, possesses a fixed point $x_0 \in P(x_0)$. This fixed point is associated with a T-periodic solution to problem (1.1), with the inhomogeneity $h \in C(I,E)$ given by $h(t) = U(t)x_0$. In view of Lemma 3.5 the present proof will be completed if we can only show that φ is strongly usc decomposable map.

Since the geometry of R_δ-type sets is invariant under translation, the multivalued map $E \ni x \mapsto S_F^p(U(\,\cdot\,)x) - U(\,\cdot\,)x \subset C(I,E)$ possesses values of R_δ-type (by virtue of Theorem 3.1). Note that φ is a composition of this operator and the evaluation at time T.

Take $(x_n)_{n\geq 1}$ in $\overline{B}(0,2R)$ such that $x_n \rightharpoonup x_0$. Let $\widetilde{y}_n \in \varphi(x_n)$ for $n \geq 1$. Clearly, $\widetilde{y}_n = y_n(T)$, where $y_n = V(w_n)$ for some $w_n \in N_F^p(z_n)$ such that $z_n = U(\,\cdot\,)x_n + V(w_n)$. The image $F(t,\{z_n(t)\}_{n\geq 1})$ is relatively compact for almost all $t \in I$ due to the following bound:

$$\sup_{n\geq 1}|z_n(t)| \leq \sup_{t\in I}||U(t)||_{\mathscr{L}}\sup_{n\geq 1}|x_n| + \sup_{t\in I}||k(t,\cdot)||_q||\mu||_p$$

and in view of the strong upper semicontinuity of $F(t,\,\cdot\,)$. From [16, Proposition 11] it follows that $(w_n)_{n\geq 1}$ is relatively weakly compact in $L^1(I,E)$. Observe that the sequence $(k(t,\,\cdot\,)w_n(\,\cdot\,))_{n\geq 1} \subset L^1(I,E)$ is integrably bounded and the following inequality holds:

$$\begin{aligned}\chi(\{k(t,s)w_n(s)\}_{n\geq 1}) &\leq \|k(t,s)\|_{\mathscr{L}}\chi(\{w_n(s)\}_{n\geq 1})\\ &\leq \|k(t,s)\|_{\mathscr{L}}\chi(F(s,\{z_n(s)\}_{n\geq 1})) = 0\end{aligned}$$

almost everywhere on I. Applying the latter context of [11, Theorem 2.2.3] one obtains, for every $t \in I$,

$$\chi(\{y_n(t)\}_{n\geq 1}) \leq \chi\bigg(\bigg\{\int_0^t k(t,s)w_n(s)\,ds\bigg\}_{n\geq 1}\bigg) \leq 0.$$

Consequently, the set $\{y_n(t)\}_{n\geq 1}$ is relatively compact in E for every $t \in I$. It is easy to notice that the family $\{y_n\}_{n\geq 1}$ is equicontinuous. It follows from

$$\sup_{n\geq 1}|y_n(t)-y_n(\tau)| \leq ||k(t,\,\cdot\,)-k(\tau,\,\cdot\,)||_q||\mu||_p + \sup_{t\in I}||k(t,\,\cdot\,)||_q\bigg(\int_\tau^t \mu(s)^p\,ds\bigg)^{1/p}.$$

Therefore, passing to a subsequence if necessary we get the convergence $y_n \rightrightarrows y_0$.

Denote by $w_0 \in L^p(I,E)$ the weak limit of some subsequence of $(w_n)_{n\geq 1}$. Put $z_0(t) := U(t)x_0 + V(w_0)(t)$ for $t \in I$. It suffices to show that $w_0 \in N_F^p(z_0)$ to complete the proof. Indeed, for then $y_n = V(w_n) \rightrightarrows V(w_0) \in V(\mathscr{S}_F^p(U(\,\cdot\,)x_0))$. This means that the operator $x \mapsto V(\mathscr{S}_F^p(U(\,\cdot\,)x))$ is a J-mapping and eventually φ is decomposable. Moreover, setting $\widetilde{y}_0 := V(w_0)(T)$ we obtain in the result that $\widetilde{y}_n \to \widetilde{y}_0 \in \varphi(x_0)$.

In order to demonstrate that $w_0(t) \in F(t,z_0(t))$ almost everywhere on I, it is enough to duplicate the routine argumentation involving Mazur's lemma. We omit the details of the proof and instead of this evoke some version of the convergence theorem given in Lemma 1 in [7]. It is obvious that for every sequence $(x_n)_{n\geq 1}$ in E converging weakly to some x_0 there holds the inequality: $\limsup_{n\to\infty}\sigma(e^*,F(t,x_n)) \leq \sigma(e^*,F(t,x_0))$ for every $e^* \in E^*$ and for almost all $t \in I$. This indicates that the multimap $F(t,\,\cdot\,)$ is weakly sequentially upper hemicontinuous (within the meaning of Definition 1 in [7]) almost everywhere on I. Now, observe that $z_n(t) = U(t)x_n + V(w_n)(t) \rightharpoonup z_0(t)$ for $t \in I$ (in fact, $z_n \rightharpoonup z_0$

in $C(I,E)$, but this knowledge is redundant). Therefore, all the hypotheses of [7, Lemma 1] are met (confront Remark 1 in [13]) and $w_0(t) \in F(t, z_0(t))$ almost everywhere on I. $\square$

Below we formulate the thesis about the existence of a continuous selection of the solution set map S_F^p, which is part of a broader research trend initiated in [6] and [8].

Theorem 3.8. *Let $p \in [1,\infty)$ and E be a separable Banach space. Assume that $F\colon I \times E \multimap E$ satisfies* (H) *and $k\colon \triangle \to \mathscr{L}(E)$ satisfies* (k_5)–(k_6)*. Then the solution set map S_F^p possesses a continuous selection.*

Proof. Fix $\varepsilon > 0$, set $\varepsilon_n := (n+1)\varepsilon/(n+2)$, $M := \sup_{t\in I} ||k(t,\,\cdot\,)||_q$ and $m(t) := \int_0^t \alpha(s)^p\,ds$. Let $\gamma\colon C(I,E) \to L^1(I,\mathbb{R})$ be a continuous function, given by

$$\gamma(h)(t) := 2^p \max\{\beta(t)^p, \alpha(t)^p |ev(t,h)|\},$$

where $ev(t,h) = h(t)$ is the evaluation. Define $\beta_n\colon C(I,E) \to L^1(I,\mathbb{R})$ in the following way:

$$\beta_n(h)(t) = M^{np}\bigg(\int_0^t \gamma(h)(s)\,\frac{(m(t)-m(s))^{n-1}}{(n-1)!}\,ds + T\varepsilon_n\,\frac{m(t)^{n-1}}{(n-1)!}\bigg).$$

It follows from (H_3) that $F(t,\,\cdot\,)$ is Hausdorff continuous. Therefore, F is $\Sigma \otimes \mathscr{B}(E)$-measurable in view of [11, Theorem 3.3]. Theorem 1 in [16] implies that the set-valued map $F(\,\cdot\,, ev(\,\cdot\,,h))$ is Σ-measurable. Thanks to (H_3) the multimap $F(t, ev(t,\,\cdot\,))$ is Hausdorff continuous. Hence, applying again Theorem 3.3. in [11] we see that $F(\,\cdot\,, ev(\,\cdot\,,\,\cdot\,))$ is $\Sigma \otimes \mathscr{B}(C(I,E))$-measurable. Observe that the multivalued map $F_0\colon C(I,E) \multimap L^1(I,E)$ such that

$$F_0(h) := \{w \in L^1(I,E) : w(t) \in F(t, ev(t,h)) \text{ a.e. in } I\}$$

is lower semicontinuous and possesses nonempty closed decomposable values. This is a consequence of Proposition 2.1. in [8], since

$$d(0, F(t, ev(t,h))) \le \beta(t) + \alpha(t)|ev(t,h)|^{1/p} \quad \text{for a.a. } t \in I.$$

In fact, $F_0(h) \subset L^p(I,E)$, because $\inf\{|z| : z \in F(t, ev(t,h))\} \in L^p(I,\mathbb{R})$.

Let $H_0\colon C(I,E) \multimap L^1(I,E)$ be a multimap described by the formula

$$H_0(h) := \big\{w \in F_0(h) : |w(t)| < (\gamma(h)(t) + \varepsilon_0)^{1/p} \text{ for a.a. } t \in I\big\}.$$

We claim that it has nonempty values. Indeed, define the nonempty closed decomposable set $K := N_F^1(ev(\,\cdot\,,h))$. Let $\psi\colon I \to \mathbb{R}_+$ be defined by

$$\psi = \operatorname{ess\,inf}\{|w(\,\cdot\,)| : w \in K\}.$$

Using the Castaing representation we may write $F(t, ev(t,h)) = \overline{\{g_k(t)\}}_{k\ge 1}$. Thus, for every $k \ge 1$, $\psi(t) \le |g_k(t)|$ almost everywhere in I. Obviously, there is a subset $J \subset I$ of full measure such that for every $t \in J$, $\psi(t) \leq \inf_{k\geq 1} |g_k(t)|$. Consequently,

$$\psi(t) \leq d(0, \{g_k(t)\}_{k\geq 1}) = d(0, \overline{\{g_k(t)\}}_{k\geq 1}) = d(0, F(t, ev(t,h)))$$
$$\leq (2^p \max\{\beta(t)^p, \alpha(t)^p |ev(t,h)|\})^{1/p} < (\gamma(h)(t) + \varepsilon_0)^{1/p}$$

almost everywhere in I. Applying Proposition 2 in [5] we get $w \in L^p(I,E)$ such that $w(t) \in F(t, ev(t,h))$ and $|w(t)| < (\gamma(h)(t) + \varepsilon_0)^{1/p}$ almost everywhere in I.

Proposition 4 and Theorem 3 in [5] imply the existence of a continuous selection $g_0\colon C(I,E) \to L^p(I,E)$ of H_0. Define $f_0(t,h) := g_0(h)(t)$ and

$$x_1(t,h) := h(t) + \int_0^t k(t,s) f_0(s,h)\, ds, \quad t \in I.$$

Observe that

$$|x_1(t,h) - ev(t,h)|^p \leq \bigg(\int_0^t ||k(t,s)||_{\mathscr{L}} |f_0(s,h)|\, ds \bigg)^p$$
$$\leq ||k(t,\,\cdot\,)||_q^p \int_0^t (\gamma(h)(s) + \varepsilon_0)\, ds < M^p \int_0^t (\gamma(h)(s) + \varepsilon_1)\, ds = \beta_1(h)(t)$$

for each $h \in C(I,E)$ and $t \in (0,T]$.

We maintain that there exist Cauchy sequences $(f_n(\,\cdot\,,h))_{n\geq 1} \subset L^p(I,E)$ and $(x_n(\,\cdot\,,h))_{n\geq 1} \subset C(I,E)$, such that for all $n \geq 1$ the following properties hold:

(i) $C(I,E) \ni h \mapsto f_n(\,\cdot\,,h) \in L^p(I,E)$ is continuous,

(ii) $f_n(t,h) \in F(t, x_n(t,h))$ for every $h \in C(I,E)$ and almost every $t \in I$,

(iii) $|f_n(t,h) - f_{n-1}(t,h)| \leq \alpha(t)(\beta_n(h)(t))^{1/p}$ for almost all $t \in I$,

(iv) $x_{n+1}(t,h) = h(t) + \int_0^t k(t,s) f_n(s,h)\, ds$ for $t \in I$.

Suppose that $(f_k(\,\cdot\,,h))_{k=1}^n$ and $(x_k(\,\cdot\,,h))_{k=1}^n$ are constructed. Define

$$x_{n+1}(\,\cdot\,,h)\colon I \to E$$

in accordance with (iv). By virtue of (i) it follows that

$$x_n(\,\cdot\,,h_0) = h_0 + V(f_{n-1}(\,\cdot\,,h_0)) = \lim_{h\rightrightarrows h_0} h + V(f_{n-1}(\,\cdot\,,h)) = \lim_{h\rightrightarrows h_0} x_n(\,\cdot\,,h),$$

i.e. the mappings $C(I,E) \ni h \mapsto x_n(\,\cdot\,,h) \in C(I,E)$ are also continuous. Following calculations contained in [1, formula (14), p. 122] we see that

$$\text{(3.5)} \qquad M^p \int_0^t \alpha(s)^p \beta_n(h)(s)\, ds$$
$$= M^{(n+1)p} \bigg(\int_0^t \int_0^s \alpha(s)^p \gamma(h)(\tau) \frac{(m(s) - m(\tau))^{n-1}}{(n-1)!}\, d\tau\, ds + \int_0^t T\varepsilon_n \alpha(s)^p \frac{m(s)^{n-1}}{(n-1)!}\, ds \bigg)$$

$$
\begin{aligned}
&= M^{(n+1)p}\bigg(\int_0^t \gamma(h)(\tau)\int_\tau^t \frac{1}{n!}\,\frac{d}{ds}\,(m(s)-m(\tau))^n\,ds\,d\tau \\
&\qquad + T\varepsilon_n\int_0^t \frac{1}{n!}\,\frac{d}{ds}\,m(s)^n\,ds\bigg) \\
&< M^{(n+1)p}\bigg(\int_0^t \gamma(h)(s)\,\frac{(m(t)-m(s))^n}{n!}\,ds + T\varepsilon_{n+1}\,\frac{m(t)^n}{n!}\bigg) \\
&= \beta_{n+1}(h)(t).
\end{aligned}
$$

Hence

$$
\begin{aligned}
|x_{n+1}(t,h)-x_n(t,h)|^p &\le M^p\int_0^t |f_n(s,h)-f_{n-1}(s,h)|^p\,ds \\
&\le M^p\int_0^t \alpha(s)^p\beta_n(h)(s)\,ds < \beta_{n+1}(h)(t)
\end{aligned}
$$

for $t\in(0,T]$. Let $F_{n+1}\colon C(I,E)\multimap L^1(I,E)$ be such that

$$F_{n+1}(h) := \big\{w\in L^1(I,E) : w(t)\in F(t,x_{n+1}(t,h)) \text{ a.e. in } I\big\}.$$

Then F_{n+1} is lower semicontinuous with nonempty closed decomposable values, by exactly the same arguments as the multimap F_0. Define $H_{n+1}\colon C(I,E)\multimap L^1(I,E)$ in the following way:

$$H_{n+1}(h) := \big\{w\in F_{n+1}(h) : |w(t)-f_n(t,h)| < \alpha(t)(\beta_{n+1}(h)(t))^{1/p} \text{ a.e. in } I\big\}.$$

Notice that, almost everywhere in I,

$$
\begin{aligned}
d(f_n(t,h),&F(t,x_{n+1}(t,h))) \\
&\le d(f_n(t,h),F(t,x_n(t,h))) + d_H(F(t,x_n(t,h)),F(t,x_{n+1}(t,h))) \\
&\le \alpha(t)|x_n(t,h)-x_{n+1}(t,h)| < \alpha(t)(\beta_{n+1}(h)(t))^{1/p}.
\end{aligned}
$$

Imitating previous arguments one can show that $H_{n+1}(h)\neq\emptyset$ (applying these to the nonempty closed decomposable set $K := \{f_n(\,\cdot\,,h)\} - N_F^1(x_{n+1}(\,\cdot\,,h))$). Moreover, $H_{n+1}(h)\subset L^p(I,E)$. Let $g_{n+1}\colon C(I,E)\to L^p(I,E)$ be a continuous selection of the multimap H_{n+1}. Now, if we put $f_{n+1}(t,h) := g_{n+1}(h)(t)$ for $(t,h)\in I\times C(I,E)$, then it is obvious that the function f_{n+1} possesses properties (i)–(iii).

Making use of inequality (3.5) we can easily estimate

$$
\begin{aligned}
&||x_{n+1}(\,\cdot\,,h)-x_n(\,\cdot\,,h)|| \\
&\quad\le \sup_{t\in I}||k(t,\,\cdot\,)||_q||f_n(\,\cdot\,,h)-f_{n-1}(\,\cdot\,,h)||_p \le M\left(\int_0^T \alpha(t)^p\beta_n(h)(t)\,dt\right)^{1/p} \\
&\quad\le \left(M^{(n+1)p}\bigg(\int_0^T \gamma(h)(t)\,\frac{(m(T)-m(t))^n}{n!}\,dt + T\varepsilon_{n+1}\,\frac{m(T)^n}{n!}\bigg)\right)^{1/p} \\
&\quad\le \frac{M^{n+1}||\alpha||_p^n}{(n!)^{1/p}}\,(||\gamma(h)||_1+T\varepsilon)^{1/p}.
\end{aligned}
$$

Since the series $\sum_{n=1}^{\infty} M^{n+1}||\alpha||_p^n/(n!)^{1/p}$ is convergent, the sequences $(x_n(\,\cdot\,,h))_{n\geq 1}$ and $(f_n(\,\cdot\,,h))_{n\geq 1}$ are fundamental and ultimately converge to some $x(\,\cdot\,,h) \in C(I,E)$ and $f(\,\cdot\,,h) \in L^p(I,E)$, respectively.

Moreover, sequences $(h \mapsto x_n(\,\cdot\,,h))_{n\geq 1}$ and $(h \mapsto f_n(\,\cdot\,,h))_{n\geq 1}$ tend locally uniformly to the limit functions $h \mapsto x(\,\cdot\,,h)$ and $h \mapsto f(\,\cdot\,,h)$, because $||\gamma(\,\cdot\,)||_1$ is locally bounded. Therefore mappings $h \mapsto x(\,\cdot\,,h)$ and $h \mapsto f(\,\cdot\,,h)$ are also continuous.

Taking into account the estimation

$$\begin{aligned} d(f(t,h),&F(t,x(t,h))) \\ &\leq d(f_n(t,h),F(t,x_n(t,h))) + d_H(F(t,x(t,h)),F(t,x_n(t,h))) \\ &\quad + |f_n(t,h)-f(t,h)| \leq \alpha(t)|x_n(t,h)-x(t,h)| + |f_n(t,h)-f(t,h)| \end{aligned}$$

almost everywhere in I and almost everywhere convergence of $(f_n(\cdot,h))_{n\geq 1}$ along some subsequence, we infer that $f(t,h) \in F(t,x(t,h))$ for each $h \in C(I,E)$ and for almost all $t \in I$. Passing to the limit in (iv), uniformly with respect to t, we obtain

$$\begin{aligned} x(\,\cdot\,,h) &= \lim_{n\to\infty} x_n(\,\cdot\,,h) = \lim_{n\to\infty} h + V(f_n(\,\cdot\,,h)) \\ &= h + V(f(\,\cdot\,,h)) \in h + V \circ N_F^p(x(\,\cdot\,,h)), \end{aligned}$$

i.e. $x(\,\cdot\,,h) \in S_F^p(h)$, completing the proof. □

It is obvious that under the conditions of Theorem 3.1. the sum $\bigcup_{h\in M} S_F^p(h)$ is connected, provided M is connected. We can say more about the image $S_F^p(M)$ if $M \subset C(I,E)$ is convex, namely:

Theorem 3.9. *Let $p \in [1,\infty)$, while the space E is reflexive for $p \in (1,\infty)$. Assume that conditions* (F) *and* (k_1)–(k_4) *are fulfilled. Then the image $S_F^p(M)$ of a compact convex set M by the solution set map S_F^p is compact acyclic.*

Proof. The image $S_F^p(M)$ is obviously compact since the solution set map S_F^p is compact valued upper semicontinuous multifunction ([13, Proposition 4.]). In view of Theorem 3.1 it is an acyclic multimap. Therefore the projection $\pi_1^{S_F^p}\colon \Gamma(S_F^p) \to M$ of the graph $\Gamma(S_F^p) = \{(h,x) \in M \times C(I,E) : x \in S_F^p(h)\}$ onto the set M is a Vietoris map. Applying the Vietoris–Begle theorem ([9, Theorem 8.7]) we see that the induced map $(\pi_1^{S_F^p})_*\colon H_*(\Gamma(S_F^p)) \xrightarrow{\approx} H_*(M)$ is an isomorphism.

Observe that the solution set map S_F^p has convex fibers. Indeed, take $h_1,h_2 \in (S_F^p)^{-1}(\{x\})$ and $\lambda \in [0,1]$. Then there are $w_1,w_2 \in N_F^p(x)$ such that $h_1 + V(w_1) = x = h_2 + V(w_2)$. Thus $x = \lambda x + (1-\lambda)x = \lambda h_1 + (1-\lambda)h_2 + V(\lambda w_1 + (1-\lambda)w_2) \in \lambda h_1 + (1-\lambda)h_2 + V \circ N_F^p(x)$, since N_F^p is convex-valued. Therefore $\lambda h_1 + (1-\lambda)h_2 \in (S_F^p)^{-1}(\{x\})$.

Define a multimap $G\colon S_F^p(M) \multimap M$ by $G(x) := \{h \in M : x \in S_F^p(h)\}$. Clearly G is a compact valued upper semicontinuous map. Moreover, it has convex values, because $G(x) = M \cap (S_F^p)^{-1}(\{x\})$. Since the projection $\pi_1^G\colon \Gamma(G) \to S_F^p(M)$ of the graph of G onto the domain $S_F^p(M)$ is a Vietoris map, the induced map $(\pi_1^G)_*\colon H_*(\Gamma(G)) \xrightarrow{\approx} H_*(S_F^p(M))$ gives an isomorphism.

It is easy to indicate the homeomorphism between the graphs $\Gamma(S_F^p)$ and $\Gamma(G)$. Indeed, let $f\colon \Gamma(S_F^p) \to \Gamma(G)$ be the product mapping $f = \pi_2^{S_F^p} \times \pi_1^{S_F^p}$, i.e. $f(h,x) = (\pi_2^{S_F^p}(h,x), \pi_1^{S_F^p}(h,x)) = (x,h)$. Then $\pi_2^G \times \pi_1^G\colon \Gamma(G) \to \Gamma(S_F^p)$ provides the continuous inverse of f. Consequently, $f_*\colon H_*(\Gamma(S_F^p)) \xrightarrow{\approx} H_*(\Gamma(G))$ gives an isomorphism of Čech homology spaces. Combining above findings, we see that the homologies $H_*(S_F^p(M))$ and $H_*(M)$ are isomorphic. Therefore the set $S_F^p(M)$ must be acyclic. $\square$

Let us pass on to description of the properties of the selection set map $\mathscr{S}_F^p\colon C(I,E) \multimap L^p(I,E)$. We start with a fairly obvious:

Proposition 3.10. *Let $p \in [1,\infty)$, while the space E is reflexive for $p \in (1,\infty)$. Assume that conditions* (F) *and* (k$_5$)–(k$_6$) *are fulfilled. Then the multivalued map $\mathscr{S}_F^p$ is weakly upper semicontinuous and has weakly compact values.*

Proof. The claim is a straightforward consequence of upper semicontinuity of the solution set map S_F^p and weak upper semicontinuity of the Nemytskiĭ operator N_F^p (Propositions 4 and 1 in [13], respectively). $\square$

Theorem 3.11. *Let E be a separable Banach space. Assume that conditions* (H) *and* (k$_5$)–(k$_6$) *are fulfilled. Then the set $\mathscr{S}_F^1(h)$ is an absolute retract for every $h \in C(I,E)$. In other words, the selection set map $\mathscr{S}_F^1$ is an AR-valued multimap. Moreover, the selection set map $\mathscr{S}_F^1$ admits a continuous singlevalued selection.*

Proof. Observe that the Banach spaces $(C(I,E), \|\cdot\|)$, $(L^1(I,E), |||\cdot|||_1)$, with $|||\cdot|||_1$ given by (2.2), are separable. The set-valued map $(C(I,E), \|\cdot\|) \times (L^1(I,E), |||\cdot|||_1) \ni (h,u) \mapsto N_F^1(h + V(u)) \subset (L^1(I,E), |||\cdot|||_1)$ has nonempty closed bounded and decomposable values. In view of Lemma 2.3 it is continuous and contractive with respect to the second variable. It follows by [4, Theorem 1] that there exists a continuous function $g\colon C(I,E) \times L^1(I,E) \to L^1(I,E)$ such that $g(h,u) \in \operatorname{Fix}(N_F^1(h + V(\cdot)))$ for all $u \in L^1(I,E)$ and $g(h,u) = u$ for all $u \in \operatorname{Fix}(N_F^1(h + V(\cdot)))$. Therefore, the set $\mathscr{S}_F^1(h)$ is an AR-space as a retract of the normed space $L^1(I,E)$. With the second argument fixed, the function g becomes a continuous selection of the selection set map $\mathscr{S}_F^1$. $\square$

Theorem 3.12. *Let E be a separable Banach space and $p \in (1,\infty)$. Assume that F has convex values and satisfies* (H). *Let k satisfy conditions* (k$_5$)–(k$_6$). *Then the selection set map $\mathscr{S}_F^p$ is an AR-valued multimap.*

PROOF. In the context of the assumptions and content of Lemma 2.3 the set-valued operator $G_p(h,\,\cdot\,)\colon (L^p(I,E),|||\cdot|||_p) \multimap (L^p(I,E),|||\cdot|||_p)$, given by $G_p(h,\,\cdot\,)(u) = N_F^p(h+V(u))$, is contractive, with convex closed values. Now Theorem 1 in [14] applied to the space $L^p(I,E)$ with the norm $|||\cdot|||_p$ gives the desired result. $\square$

REMARK 3.13. There is a strong presumption that the thesis of Theorem 3.11 remains true also for exponents $p > 1$. This obviously depends on whether the results of papers [4], [5] can be expressed in the setting of p-integrable maps.

COROLLARY 3.14. *Let E be a separable Banach space and $p \in [1,\infty)$. Assume that F satisfies* (H) *and F has convex values for $p \in (1,\infty)$. Then*:

(a) *the solution set $S_F^p(h)$ is arcwise connected, provided that k satisfies conditions* (k_5)–(k_6),

(b) *the solution set $S_F^p(h)$ is an absolute retract, provided that k satisfies conditions* (k_1)–(k_4).

PROOF. (a) Observe that the solution set $S_F^p(h)$ and the continuous (affine) image $h+V(\mathscr{S}_F^p(h))$, of the arcwise connected set $\mathscr{S}_F^p(h)$, coincide.

(b) Consider $V\colon L^p(I,E) \to \operatorname{Im} V \subset C(I,E)$. Conditions (k_1)–(k_4) imply the thesis of [13, Lemma 2]. Let $||\cdot||_{\mathrm{Im}}\colon \operatorname{Im} V \to \mathbb{R}_+$ be such that

$$||V(w)||_{\mathrm{Im}} := \max\{||V(w)||, ||w||_p\}.$$

Since V is a monomorphism, the mapping $||\cdot||_{\mathrm{Im}}$ determines the norm on the subspace $\operatorname{Im} V$. Of course, the normed space $(\operatorname{Im} V, ||\cdot||_{\mathrm{Im}})$ is complete. In view of the bounded inverse theorem the inverse map

$$V^{-1}\colon (\operatorname{Im} V, ||\cdot||_{\mathrm{Im}}) \to (L^p(I,E), ||\cdot||_p)$$

is continuous.

Consider metric spaces $(\mathscr{S}_F^p(h), d)$ and $(S_F^p(h),\rho)$ with the metrics d, ρ given by the formulae: $d(w,u) := ||w-u||_p$, $\rho(x,y) := ||x-y||_{\mathrm{Im}}$. Then it is obvious that the mapping $V_h\colon (\mathscr{S}_F^p(h), d) \to (S_F^p(h),\rho)$, such that $V_h(w) := h+V(w)$, is continuous. Define the inverse function $V_h^{-1}\colon (S_F^p(h),\rho) \to (\mathscr{S}_F^p(h), d)$ in the following way $V_h^{-1}(x) := V^{-1}(x-h)$. This map is also continuous, due to the continuity of V^{-1}. In conclusion, the solution set $S_F^p(h)$ of the integral inclusion (1.1) must be an absolute retract as an homeomorphic image of another absolute retract $\mathscr{S}_F^p(h)$. $\square$

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