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FIRST ORDER LINEAR DIFFERENTIAL EQUATIONS WITH INVOLUTIVE DELAY AND HYPERGEOMETRIC FUNCTIONS

Stefan Czekalski — Henryk Żołądek

Dedicated to the memory of Marek Burnat

Abstract. We present an alternative approach to functions satisfying second order linear ordinary differential equations. It turns out that many of them satisfy a first order ordinary differential equation with an involution. The involution acts on the argument as well as on parameters. Basic examples involve the hypergeometric functions and their descendants.

1. Introduction

Let $t \in \mathbb{C}$ be complex time and $\lambda \in \mathbb{C}^k$ denote parameter(s). Assume that we have an *involution* of $\mathbb{C} \times \mathbb{C}^k$ of the form

$$I\colon (t,\lambda) \mapsto (s,\mu) = (T(t), \Sigma(\lambda)), \tag{1.1}$$

thus $T \circ T = \mathrm{Id}$ and $\Sigma \circ \Sigma = \mathrm{Id}$. By a *linear first order differential equation with an involution* we mean the following equation:

$$\dot{x}_\lambda(t) = a_\lambda(t) x_\mu(s), \tag{1.2}$$

i.e. $\partial x(t;\lambda)/\partial t = a(t;\lambda) \cdot x(T(t);\Sigma(\lambda))$. The function $a(t;\lambda) = a_\lambda(t)$ is called the *directing coefficient*.

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Equation (1.2) is an analogue of an ODE with delay, i.e. $\dot{x}(t) = f(x(t), x \circ T(t))$, but there $T(t) = t - u$ (with fixed $u > 0$) and the dynamics of the delay map T is of different character.

Applying the involution (1.1) to the both sides of the latter equation we obtain

$$\dot{x}_\mu(s) = a_\mu(s) x_\lambda(t) \tag{1.3}$$

(which can be called the dual first order equation with an involution). Next, after differentiating the both sides of equation (1.2) with respect to t and using equation (1.3), we arrive at the standard *second order linear differential equation*

$$\ddot{x}_\lambda = b_\lambda(t)\dot{x}_\lambda + c_\lambda(t) x_\lambda, \tag{1.4}$$

$$b_\lambda(t) = \dot{a}_\lambda(t)/a_\lambda(t), \tag{1.5}$$

$$c_\lambda(t) = a_\lambda(t) a_\mu(s) \cdot ds/dt. \tag{1.6}$$

In some sense equation (1.2) is a 'primitive version' of equation (1.4). It turns out that many classical second order equations, like the Gauss hypergeometric equation, are obtained in this way.

If we start with equation (1.3) then we arrive at the following *dual second order equation* (i.e. dual to (1.4)):

$$\ddot{x}_\mu(s) = d_\mu(s)\dot{x}_\mu(s) + e_\mu(s) x_\mu(s), \tag{1.7}$$

$$d_\mu = \dot{a}_\mu / a_\mu, \tag{1.8}$$

$$e_\mu = a_\lambda(t) a_\mu(s) \cdot dt/ds. \tag{1.9}$$

This sort of duality takes place in the family of hypergeometric equations.

The hypergeometric equations appear in many areas of mathematics and mathematical physics (see [1], [2], [6]–[8]) and were thoroughly studied. In particular, a lot of symmetries and recurrent formulas were found and organized into some special groups (see [3]–[5]). It is possible to obtain our dualities for the Gauss hypergeometric equation using compositions of those symmetries and recurrences. More precisely, our involutions in the parameters of the hypergeometric equation are expressed as compositions of corresponding actions on the parameters of the relations from [**?**]–[**?**] (see Remark 4.1 in Section 4.1). But the underlying involutions in the very differential equations are different sort (they are realized at the level of the first order equations) than the above mentioned symmetries and recurrence relations. Probably this phenomenon awaits for an application.

The plan of our paper is following. In Section 2 we present an example of equation (1.2) related with the Chebyshev equation. In Section 3 we prove some general results about behavior of solutions to (1.2) near singular points and about conditions for the directing coefficient a_λ and the involution I to get 'good' equation (1.4), e.g. with rational coefficients. Section 4 is devoted to the hypergeometric equation and its descendants (the Legendre equation, the Chebyshev equation and the confluent hypergeometric equation). In Section 5 we consider the situation when the map T is not an involution, but is periodic with period 4, and we show its connection with the Hermite equation; this section does not belong directly to the above framework, but is related with it.

2. An equation with an involution related with the Chebyshev equation

Let

$$s = T(t) = \sqrt{1-t^2}, \quad \mu = \Sigma(\lambda) = 1/\lambda, \tag{2.1}$$

be the involutions in the time and the parameter ([1]), and take the following directing coefficient:

$$a_\lambda(t) = n\lambda/\sqrt{1-t^2} = n\lambda/s \tag{2.2}$$

(where n is a fixed constant). The corresponding equation (1.2) is

$$\dot{x}_\lambda(t) = (\lambda n/s) \cdot x_\mu(s) \tag{2.3}$$

and the corresponding second order equation is the *Chebyshev equation*

$$\ddot{x} = \frac{t}{1-t^2}\,\dot{x} - \frac{n^2}{1-t^2}\,x \tag{2.4}$$

(see [3], [7]).

The general solution to equation (2.4) is of the form

$$x(t) = C_1 \mathrm{e}^{\mathrm{i}n \arccos t} + C_2 \mathrm{e}^{-\mathrm{i}n \arccos t}. \tag{2.5}$$

Here $\mathrm{i} = \sqrt{-1} = \mathrm{e}^{\mathrm{i}\pi/2}$ and we used $d \arccos t/dt = -1/\sqrt{1-t^2} = -1/s$. Equation (2.5) indicates that the Chebyshev equation is equivalent with the harmonic oscillator equation via the change $t = \cos\varphi$ (see also [3]).

Inserting the right-hand side of the latter equation into equation (2.3), with use of the relation $\arccos\sqrt{1-x^2} = \pi/2 - \arccos x$, we obtain the condition

$$C_2 + (1/\lambda)\mathrm{i}^{n+1}C_1 = 0;$$

hence the general solution to equation (2.3) is of the form

$$x_\lambda(t) = C\psi_\lambda(t) = C\big\{\mathrm{e}^{\mathrm{i}n \arccos t} - (\mathrm{i}^{n+1}/\lambda)\,\mathrm{e}^{-\mathrm{i}n \arccos t}\big\}, \tag{2.6}$$

where C is a constant and $\mathrm{i}^{n+1} = \mathrm{e}^{\mathrm{i}(n+1)\pi/2}$. If $\lambda' \neq \lambda$ then the functions ψ_λ and $\psi_{\lambda'}$ form a basis of the space of solutions to the Chebyshev equation.

([1]) The mapping $t \mapsto T(t)$ is multivalued; we choose its branch such that $T(1/\sqrt{2}+\tau) = 1/\sqrt{2} - \tau + O(\tau^2)$ near the fixed point $t = 1/\sqrt{2}$.

Note finally that the solution (2.6) is proportional to the *Chebyshev polynomial*

$$T_n(x) = \cos(n \arccos x), \tag{2.7}$$

$n = 0, 1, 2, \ldots$, only when n is integer and $\lambda = \mathrm{i}^{n-1}$. Then the solution to the initial value problem $x_\lambda(0) = 1$ for equation (2.3) is $x_\lambda(t) = T_n(t)$.

3. General results

We are interested in first order equations with an involution which lead to second order equations with rational coefficients. One of the aims of this section is to find conditions for the involution and the directing coefficient which are sufficient for this.

3.1. Introduction of a multiplicative parameter. We begin with the following observation. Suppose that we have an equation (1.2), with an involution

$$I\colon (t, \lambda) \mapsto (T(t), \Sigma(\lambda)), \quad \lambda = (\lambda_1, \ldots, \lambda_k),$$

defined by a directing coefficient $a_\lambda(t)$ and leading to a second order equation (1.4). One can extend the parameter space to $\mathbb{C}^* \times \mathbb{C}^k = \{\widetilde{\lambda} = (\lambda_0, \lambda)\}$ and the involution to $\widetilde{I} = (T, \widetilde{\Sigma})$, $\widetilde{\Sigma}(\lambda_0, \lambda) = (1/\lambda_0, \Sigma(\lambda))$, and define the new directing coefficient

$$\widetilde{a}_{\widetilde{\lambda}}(t) = \lambda_0 a_\lambda(t).$$

Then equation (1.4) does not change.

Indeed, from equation (1.5) it follows that b_λ does not depend on the multiplicative constant before a_λ. Next, by the special form of our extension, we have $\widetilde{a}_{\widetilde{\lambda}}(t)\widetilde{a}_{\widetilde{\mu}}(s) = a_\lambda(t)a_\mu(s)$ and from equation (1.6) we find that c_λ is unchanged. Therefore, in the sequel we shall assume the directing coefficient in the form

$$a_\lambda(t) = \lambda_0 A_\lambda(t), \quad \lambda_0 \in \mathbb{C}^* = \mathbb{C} \setminus 0, \tag{3.1}$$

where A_λ depends on $\lambda_1, \ldots, \lambda_{k-1}$ and

$$\Sigma\colon \lambda = (\lambda_0, \ldots, \lambda_{k-1}) \mapsto \mu = (\mu_0, \ldots, \mu_{k-1})$$

is such that $\mu_0 = 1/\lambda_0$.

The principal reason for introducing λ_0 is that in our main examples the directing coefficient a_λ is a multivalued function, the action of the corresponding monodromy maps results in its multiplication by a nonzero constant factor (see Section 3.3 below).

3.2. Principal involutions and their pull-backs. The involution in time should be algebraic. A general algebraic involution $s = T(t)$ would be of the form

$$P(s,t) = 0, \tag{3.2}$$

where P is a symmetric polynomial, $P(t,s) = P(s,t)$. We do not know how to deal with the case of general polynomials P (e.g. with $P = s^2 + st + t^2 - 3$) and which of them are useful in obtaining a second order differential equation with rational coefficients (for suitable directing coefficients). Therefore we distinguish some of them.

There are two *principal involutions*:

$$T_1(t) = -t, \tag{3.3}$$

$$T_2(t) = 1/t, \tag{3.4}$$

i.e. defined by elementary symmetric polynomials $P_1(s,t) = s+t$, $P_2(s,t) = st-1$.

These two involutions serve as generators for a larger class of algebraic involutions, by taking the so-called pull-backs of them via rational maps defined as follows. Namely, for a rational function R we consider the equations

$$R(s) + R(t) = 0, \tag{3.5}$$

$$R(s)R(t) = 1. \tag{3.6}$$

The corresponding *pull-back involutions* (*via* R) are denoted by R^*T_1 and R^*T_2; they are multivalued functions (interpreted like in the footnote in Section 2). For a general involution T defined by equation (3.2) the pull-back involution R^*T is defined by the equation $P(R(s), R(t)) = 0$.

Let us look how the pull-back operation acts on the first order equations with involution and on the corresponding second order equations. We put $\tau = R(t)$ and $\sigma = R(s)$ with $\sigma = T(\tau)$ and assume that we have the corresponding first and second order equations

$$(dy_\lambda/d\tau)(\tau) = \alpha_\lambda(\tau) y_\mu(\sigma), \qquad d^2 y_\lambda/d\tau^2 = \beta_\lambda dy_\lambda/d\tau + \gamma_\lambda y_\lambda \tag{3.7}$$

(with $\mu = \Sigma(\lambda)$). Denoting $x_\lambda(t) = y_\lambda \circ R(t)$ and $x_\mu(s) = y_\mu \circ R(s)$ and differentiating we obtain the *pull-back equations*

$$\dot{x}_\lambda(t) = a_\lambda(t) x_\mu(s), \qquad \ddot{x}_\lambda = b_\lambda \dot{x}_\lambda + c_\lambda x_\lambda$$

with

$$\begin{aligned} a_\lambda(t) &= R'(t) \cdot \alpha_\lambda \circ R(t), \\ b_\lambda(t) &= R'(t) \cdot \alpha'_\lambda \circ R(t)/\alpha_\lambda \circ R(t) + R''(t)/R(t), \\ c_\lambda(t) &= (R'(t))^2 \cdot \alpha_\lambda \circ R(t) \cdot \alpha_\mu \circ R(s) \cdot d\sigma/d\tau. \end{aligned} \tag{3.8}$$

We can summarize this in the following

Proposition 3.1. *If we have an equation with an involution and a corresponding second order equation* (3.7), *with an involution* $I = (T, \Sigma)$ *and defined by the directing coefficient* α_λ, *then the pull-backs by means of a rational map* R *of these equations have the following coefficients:*

$$a_\lambda = R' \cdot \alpha_\lambda \circ R, \qquad b_\lambda = R' \cdot \beta_\lambda \circ R + R''/R', \qquad c_\lambda = (R')^2 \cdot \gamma_\lambda \circ R.$$

Example 3.2. The involution (2.1) is a pull-back of the first principal involution, $T = R^* T_1$ for $R(t) = 2t^2 - 1 =: q$, i.e. $r := T_1(q) = -q$. Then, with $t^2 = (1+q)/2$, $s^2 = 1 - t^2 = (1-q)/2 = (1+r)/2$, $z_\lambda(q) = x_\lambda(t)$, we find that equations (2.3)–(2.4) are the pull-backs of the following equations:

$$\begin{aligned} \frac{dz_\lambda}{dq}(q) &= \frac{\lambda n/2}{\sqrt{1-q^2}}\, z_\mu(r), \\ \frac{d^2 z}{dq^2} &= \frac{q}{1-q^2}\,\frac{dz}{dq} - \frac{n^2/4}{1-q^2}\, z. \end{aligned}$$

The latter equation is also Chebyshev, but the parameter n is replaced with $n/2$.

3.3. Darbouxian directing coefficients. Let the involution in time be one of the principal involutions, $T = T_{1,2}$, or $T = T_0 := \mathrm{Id}$.

We assume that the directing coefficient is in the so-called *Darboux form*

$$a_\lambda(t) = \lambda_0 K t^{\theta_0} (t - t_1)^{\theta_1} \ldots (t - t_m)^{\theta_m}; \tag{3.9}$$

also the function a_μ should be of similar form,

$$a_\mu(s) = \lambda_0^{-1} K s^{\vartheta_0} (s - s_1)^{\vartheta_1} \ldots (s - s_m)^{\vartheta_m}. \tag{3.10}$$

Here $s_j = T(t_j)$ for $j > 0$ (and $t_0 = s_0 = 0$), K is a fixed constant and the exponents θ_j, ϑ_j depend on λ_i, $i > 0$; in fact, we will treat θ_j and ϑ_j as genuine parameters, but we stick to the notations in Eqs. (3.9)–(3.10) (formally, $\lambda = (\lambda_0, \theta_0, \ldots, \theta_m)$ and $\mu = (\mu_0, \vartheta_1, \ldots, \vartheta_m)$).

More precisely, if $T = T_0 = \mathrm{Id}$ then the formulas (3.9)–(3.10) are completely correct. If $T = T_1$ then the system $\mathrm{Sing} = \{t_0, \ldots, t_m\} = \{s_0, \ldots, s_m\}$ is invariant under T_1 and we assume $m = 2l$ and

$$t_{l+1} = s_1 = -t_1, \quad \ldots, \quad t_{2l} = s_l = -t_l. \tag{3.11}$$

If $T = T_2$ then the system Sing is invariant under T_2, i.e., $m = 2l + 2$ and

$$\begin{gathered} t_{l+1} = s_1 = 1/t_1, \quad \ldots, \quad t_{2l} = s_l = 1/t_l, \\ t_{2l+1} = s_{2l+1} = 1, \quad t_{2l+2} = s_{2l+2} = -1 \end{gathered} \tag{3.12}$$

(if $t = 1$ or $t = -1$ is not a singular point than the corresponding exponent vanishes).

The function (3.9) is multivalued with ramification points t_j. Its monodromy, as the argument t runs along a loop surrounding a single singular point t_j, results in the following change of the parameter λ_0 :

$$\mathcal{M}_j \colon \lambda_0 \mapsto \mathrm{e}^{2\pi \mathrm{i}\theta_j} \lambda_0. \tag{3.13}$$

LEMMA 3.3. *If* a_λ *is like in equation* (3.3) *then the coefficient* b_λ *in equation* (1.4) *equals*

$$b_\lambda(t) = \sum_{j=1}^{m} \frac{\theta_j}{t - t_j}. \tag{3.14}$$

PROOF. By equation (1.5) the function b_λ is the logarithmic derivative of the function a_λ. □

Now we want to describe the behavior of coefficients of differential equations and of their solutions near a given singular point. The analysis depends on the type of the time involution.

Assume firstly that $T = T_0$. Then

$$a_\mu(s) = a_\mu(t) = \lambda_0^{-1} K t^{\vartheta_0} (t - t_1)^{\vartheta_1} \dots (t - t_m)^{\vartheta_m}.$$

Near a point $t = t_j$ we have

$$a_\lambda(t) = \lambda_0 E (t - t_j)^{\theta_j} (1 + \dots), \qquad a_\mu(t) = \lambda_0^{-1} F (t - t_j)^{\vartheta_j} (1 + \dots),$$

where E, F are constants and $1 + \dots$ denote germs of holomorphic functions. From equation (1.6) we find that

$$c_\lambda(t) = EF (t - t_j)^{\theta_j + \vartheta_j} (1 + \dots). \tag{3.15}$$

Near $t = \infty$ we get

$$c_\lambda(t) \sim \mathrm{const} \cdot t^{\sum(\theta_j + \vartheta_j)}. \tag{3.16}$$

Let $T = T_1$. Then near $t = 0$ we have

$$a_\lambda(t) = \lambda_0 E t^{\theta_0} (1 + \dots), \qquad a_\mu(s) = \lambda_0^{-1} F t^{\vartheta_0} (1 + \dots)$$

and $ds/dt = -1$, and hence

$$c_\lambda(t) = -EF t^{\theta_0 + \vartheta_0} (1 + \dots). \tag{3.17}$$

Near t_j, $j = 1, \dots, l$, we have

$$a_\lambda(t) = \lambda_0 E (t - t_j)^{\theta_j} (1 + \dots), \qquad a_\mu(s) = \lambda_0^{-1} F (t - t_j)^{\vartheta_{l+j}} (1 + \dots)$$

and hence

$$c_\lambda = -EF (t - t_j)^{\theta_j + \vartheta_{l+j}} (1 + \dots). \tag{3.18}$$

Analogously, we find

$$c_\lambda = -EF (t - t_{l+j})^{\theta_{l+j} + \vartheta_j} (1 + \dots) \tag{3.19}$$

near t_{l+j}, $j=1,\dots,l$. Near $t=\infty$ we have the behavior (3.16).

Let $T=T_2$. Then near $t=0$ we have

$$a_\lambda(t)=\lambda_0 E t^{\theta_0}(1+\dots),\qquad a_\mu(s)=\lambda_0^{-1}Ft^{-\sum\vartheta_j}(1+\dots)$$

and $ds/dt=-t^{-2}$, and hence

$$c_\lambda=-EFt^{\theta_0-\sum\vartheta_j-2}(1+\dots). \tag{3.20}$$

Near $t=1$ (respectively, $t=-1$) we have

$$c_\lambda\sim(t-1)^{\theta_{2l+1}+\vartheta_{2l+1}}\quad(\text{resp. } c_\lambda\sim(t+1)^{\theta_{2l+2}+\vartheta_{2l+2}}). \tag{3.21}$$

Near t_j (or t_{l+j}), $j=1,\dots,l$, we get equations (3.18) (or (3.19)). Finally, near $t=\infty$ we have

$$c_\lambda\sim t^{\sum\theta_j-\vartheta_0-2}. \tag{3.22}$$

The above implies the following result.

Theorem 3.4. *Assume the directing coefficient in the Darboux form* (3.9) *and the time involution* $T_{0,1,2}$. *Then, in order to get a second order equation with rational coefficients, the involution* $\Sigma\colon\theta\mapsto\vartheta$ *in parameters should act on the exponents as follows*:

(a) *for* $T=T_0$

$$\vartheta_0=p_0-\theta_0,\quad\dots,\quad\vartheta_m=p_m-\theta_m,$$

with $p_j\in\mathbb{Z}$;

(b) *for* $T=T_1$

$$\begin{aligned}\vartheta_0&=p_0-\theta_0, & \vartheta_1&=p_1-\theta_{l+1}, & &\dots,\\ \vartheta_l&=p_l-\theta_{2l}, & \vartheta_{l+1}&=p_{l+1}-\theta_1, & &\dots, & \vartheta_{2l}&=p_{2l}-\theta_l,\end{aligned}$$

with $p_j\in\mathbb{Z}$;

(c) *for* $T=T_2$

$$\begin{aligned}\vartheta_0&=p_0+\sum\theta_j, & \vartheta_1&=p_1-\theta_{l+1}, & &\dots,\\ \vartheta_l&=p_l-\theta_{2l}, & \vartheta_{l+1}&=p_{l+1}-\theta_1, & &\dots,\\ \vartheta_{2l}&=p_{2l}-\theta_l, & \vartheta_{2l+1}&=p_{2l+1}-\theta_{2l+1}, & \vartheta_{2l+2}&=p_{2l+2}-\theta_{2l+2},\end{aligned}$$

where $p_j\in\mathbb{Z}$ *and satisfy*

$$p_0+\sum_{j\ge0}p_j=0 \tag{3.23}$$

(*the latter condition guarantees the involutivity*).

Remark 3.5. One can extend the class of Darboux functions (serving as directing coefficients) to so-called generalized Darboux functions

$$a_\lambda(t) = \lambda_0 K \mathrm{e}^{h_\lambda(t)} \prod (t - t_j)^{\theta_j}, \tag{3.24}$$

where $h_\lambda(t)$ is a rational function of t. These functions are limits of the usual Darboux functions.

Then the coefficient $b_\lambda(t) = \sum \theta_j/(t-t_j) + \dot{h}_\lambda(t)$ is rational in t. In order that the coefficient $c_\lambda(t) = a_\lambda a_\mu \circ T \cdot ds/dt$ be rational it is necessary that $a_\mu(s)$ contains the factor $\mathrm{e}^{-h_\lambda(t)}$. This can be realized in two ways:

- $h_\lambda(t) = tH(t^2)$ (with a rational function H) and $T = T_1$;
- $h_\lambda(t) = \lambda_1 t$, $T = T_0$ and Σ: $\lambda_1 \mapsto \mu_1 = -\lambda_1$.

The statements (a) and (b) of Theorem 3.4 hold true in these cases.

Remark 3.6. In our analysis we do not exclude the situation when the involution is trivial in the sense that the only change is $\lambda_0 \mapsto 1/\lambda_0$. Then, with $a_\lambda(t) = \lambda_0 A(t)$ and $a_\mu(s) = \lambda_0^{-1} A(t)$, we find from equation (1.6) that $c_\lambda(t) = c(t) = A^2(t)$. Hence equation (1.2) takes the form

$$\dot{x}_\lambda = \lambda \sqrt{c(t)} \cdot x_{1/\lambda}$$

for a rational function c and $\lambda = \lambda_0$.

3.4. Regularity. Recall that a singular point t_j of an analytic linear differential equation is *regular* if and only if all its solutions $\varphi(t)$, for t near t_j and with bounded $\arg(t - t_j)$, obey the following bound:

$$|\varphi(t)| \le C|t - t_j|^{-N}$$

for some constants $C, N > 0$. If all singular points, including $t = \infty$, are regular then the equation is *Fuchsian*.

For a linear second order equation $\ddot{x} + b(t)\dot{x} + c(t)x = 0$ with meromorphic singular point t_j the regularity condition is equivalent to the following property (see [1], [2]):

the order of pole t_j of $b(t)$ (respectively, of $c(t)$) is ≤ 1 (respectively, is ≤ 2).

It turns out that this criterion holds also for solutions of the linear equations with an involution. In fact, such an equation is the same as the following linear system

$$\dot{x} = a(t)y, \qquad \dot{y} = d(t)x \tag{3.25}$$

(where $x = x_\lambda$, $a = a_\lambda$, $y = x_\mu \circ T$ and $d = a_\mu \circ T \cdot ds/dt$) which, in order, is equivalent to equation (1.4).

We can state the following result.

Theorem 3.7. *Assume the directing coefficient as in Theorem* 3.4. *Then*:

(a) *the singular point* $t = t_j \neq 0,\ \infty$ *is regular if and only if* $p_j \geq -2$;
(b) *the singular point* $t = t_0 = 0$ *is regular if and only if*
- $p_0 \geq -2$ *if* $T = T_{0,1}$,
- $p_0 \geq 0$ *if* $T = T_2$;

(c) *the singular point* $t = \infty$ *is regular if and only if*
- $\sum p_j \leq -2$ *if* $T = T_{0,1}$,
- $p_0 \geq 0$ *if* $T = T_2$.

Proof. The coefficient b_λ in equation (1.4) has only first order poles.

Thus the point $t = t_j \neq 0,\ \infty$ is regular if and only if: $\theta_j + \vartheta_k \geq -2$ if $T(t_j) = t_k$ (see (3.15), (3.18), (3.19) and (3.21)).

The point $t = 0$ is regular if and only if: $\theta_0 + \vartheta_0 \geq -2$ if $T = T_{0,1}$ and $\theta_0 - \sum_{j\geq 0} \vartheta_j - 2 \geq -2$ (see (3.17) and (3.20)).

Finally the point $t = \infty$ is regular if and only if: $\sum_{j\geq 0} \theta_j + \vartheta_0 - 2 \leq -2$ if $T = T_{0,1}$ and $\sum_{j\geq 0} \theta_j - \vartheta_0 - 2 \leq -2$ (see (3.16) and (3.22)).

By Theorem 3.4 we have $\theta_j + \vartheta_j = p_j$ if $T = T_{0,1}$ or $T = T_2$ and $j > 0$. Next, $\theta_0 - \sum \vartheta_j - 2 = -\sum_{j\geq 0} p_j - 2 = p_0 - 2$ (see (3.23)) and $\sum \theta_j - \vartheta_0 - 2 = -p_0 - 2$.□

Finally we note that, when the directing coefficient is a generalized Darboux function (like in Remark 3.5), then one of the singular points is irregular (compare Section 4.4. below). This follows from analysis of the situation near some pole of the rational function h_λ.

3.5. Basic solutions and monodromy. Assume that our equation with an involution has Darbouxian directing coefficient and has regular singular points (is Fuchsian).

Then for each singularity t_j we can write down the *defining equation* for exponents of its solutions. This means that, when we assume the solution $\psi(t) \sim (t - t_j)^\nu$, then (like in the case of a second order equation) we get the following equation for ν:

$$\nu(\nu - 1) = \theta_j \nu + G_j, \tag{3.26}$$

where $G_j = G_j(\lambda)$ is the coefficient before $(t - t_j)^{-2}$ in c_λ. For $t = \infty$ the corresponding defining equation is

$$\nu(\nu - 1) = -\theta_\infty \nu + G_\infty, \quad \theta_\infty = -\sum \theta_j. \tag{3.27}$$

Suppose that each defining equation (3.26)–(3.27) has two different solutions: $\nu = \nu_j$ and $\nu = \zeta_j \neq \nu_j$, $j = 0, \ldots, m, \infty$. Then we define the *basic solutions*:

$$\psi_j(t; \lambda_0) = (t - t_j)^{\nu_j}(1 + \ldots) + D_j(\lambda_0)(t - t_j)^{\zeta_j}(1 + \ldots) = \varphi_j + D_j \phi_j, \tag{3.28}$$

where $\varphi_j(t) = (t-t_j)^{\nu_j}(1+\dots)$ and $\phi_j(t) = (t-t_j)^{\zeta_j}(1+\dots)$ are two independent solutions to the second order equation (1.4), well defined near t_j and prolonged analytically to a suitable larger domain.

We have underlined the dependence of the constant $D_j(\lambda_0)$ (before ϕ_j) on the multiplicative constant λ_0 in the directing coefficient. Recall that the second order equation (1.4) does not depend on λ_0 (see Section 3.1), but the solutions to the first order equation essentially depend on it (compare the Chebyshev equation case). On the other hand, the parameter λ_0 is not uniquely defined (see equation (3.13)).

The theory of linear equations with meromorphic coefficients has many faces. In some sources, like [3], [4], the operator approach is principal. For us important are analytic properties of solutions. One of the tools to study it is the monodromy group defined below.

Let us fix $t_* \neq t_0, \dots, t_m, \infty$. We define loops $\gamma_j \in \pi_1(\mathbb{C} \setminus \{t_0, \dots, t_m\}, t_*)$ which begin and end at t_* and surround just one singular point t_j, $j \neq \infty$, in positive direction; the loop γ_∞ equals $(\gamma_0 \dots \gamma_m)^{-1}$. For a solution $\psi(t)$ to equation (1.2), analytic near t_*, we consider its analytic continuation along γ_j (together with the continuation of a_λ). After returning to a neighbourhood of t_* we obtain a new function

$$\mathcal{M}_j\psi(t) \tag{3.29}$$

which is is a solution to equation (1.2) but with changed directing coefficient, $\mathcal{M}_j a_\lambda$ (see equation (3.13)). The map $\psi \mapsto \mathcal{M}_j\psi$ is the *monodromy transformation* associated with the loop γ_j. The maps $\mathcal{M}_j$ generate the *monodromy group* of the first order equation with an involution.

The action of the monodromy transformations on the basic solutions is described in the next statement.

THEOREM 3.8. *Under the above assumptions we have*

$$\mathcal{M}_j\psi_j(t;\lambda_0) = \mathrm{e}^{2\pi\mathrm{i}\nu_j}\psi_j(t;\mathrm{e}^{2\pi\mathrm{i}\theta_j}\lambda_0).$$

Important are also the *connection coefficients* $C_{ij}(\lambda_0)$ between the basic solutions at different singular points,

$$\psi_i(t;\lambda_0) = C_{ij}(\lambda_0)\psi_j(t;\lambda_0). \tag{3.30}$$

Each such coefficient C_{ij} depends on the path of continuation of the solution ψ_j from a neighbourhood of t_j to a neighbourhood of t_i. Therefore it is defined modulo action of the monodromy transformations. In examples we choose the continuation paths as the straight segments $[t_i, t_j]$.

4. The Gauss hypergeometric equation

This is

$$\ddot{x} = \left(\frac{c-a-b-1}{t-1} - \frac{c}{t} \right) \dot{x} - \frac{ab}{t(t-1)}\, x. \tag{4.1}$$

When we are looking for a corresponding equation with an involution, then we firstly notice that the directing coefficient is of the Darboux form,

$$a_\lambda(t) = \mathrm{const} \cdot \int^t b_\lambda(s)\, ds = \lambda_0 K t^{\theta_0} (1-t)^{\theta_1} \tag{4.2}$$

with

$$\theta_0 = -c, \qquad \theta_1 = c-a-b-1. \tag{4.3}$$

Note that equation (4.1) essentially depends on the following parameters: θ_0, θ_1 and $\theta_2 := ab$.

In the sequel the cases with different time involutions are considered separately.

4.1. The case $T = T_0$. Here $a_\mu(s) = a_\mu(t) = \lambda_0^{-1} K t^{\vartheta_0}(1-t)^{\vartheta_1}$, $c_\lambda(t) = K^2 t^{\theta_0+\vartheta_0}(1-t)^{\theta_1+\vartheta_1} = abt^{-1}(1-t)^{-1}$, and we get the following conditions:

$$K^2 = ab, \tag{4.4}$$

$$\vartheta_0 = -1-\theta_0, \qquad \vartheta_1 = -1-\theta_1. \tag{4.5}$$

Equations (4.5) describe the involution Σ in the parameters; it is the central symmetry in the (θ_0, θ_1)-plane with the center at $(-1/2, -1/2)$. If a', b', c' are the new parameters of equation (4.1) then they satisfy

$$c' = 1-c, \qquad a' = -a \qquad b' = -b. \tag{4.6}$$

The second and third of equations (4.6) mean that $\vartheta_2 = \theta_2$, i.e. that the parameters' involution act trivially on the third parameter $\theta_2 = ab$ ([2]). Equations (4.6) define parameters of the dual hypergeometric equation.

The hypergeometric equation has the following solutions near $t = 0$:

$$\begin{aligned} \varphi_0(t) &= F(a,b;c;t), \\ \phi_0(t) &= t^{1-c} F(a-c+1, b-c+1; 2-c; t), \end{aligned} \tag{4.7}$$

where

$$F(a,b;c;t) = \sum_{n\geq 0} \frac{(a)_n (b)_n}{(c)_n n!}\, t^n$$

([2]) Of course, it is possible to impose any involution in the θ_2-parameter line, which means that there exists a whole family of equations with an involution giving the same hypergeometric equation. But we stick to the simplest (trivial) involution of θ_2.

is the hypergeometric series (with the Pochhammer symbols $(a)_n = a(a+1)\dots(a+n-1)$). Accordingly to equation (3.28) we choose the basic solution to the corresponding equation with an involution (1.2) in the form

$$\psi_0(t;\lambda_0) = \varphi_0(t) + D_0(\lambda_0)\phi_0(t). \tag{4.8}$$

To determine the constant $D_0(\lambda_0)$ we insert the latter function into equation (1.2) and compare the terms with t^0 (provided $c \notin \mathbb{Z}$). We get $ab/c + \ldots = \lambda_0\sqrt{ab}t^{-c}\,\{(1+\ldots) + D_0(\lambda_0)t^{1-c'}(1+\ldots)\}$, thus

$$D_0(\lambda_0) = \sqrt{ab}/\lambda_0 c. \tag{4.9}$$

Near $t = 1$ the basic solution is (compare [2])

$$\begin{aligned}
\psi_1(t) &= \varphi_1(t) + D_1(\lambda_0)\phi_1(t),\\
\varphi_1 &= F(a,b;a+b-c+1;1-t),\\
\phi_1 &= (1-t)^{c-a-b}F(c-b,c-a;1-a-b+c;1-t),\\
D_1(\lambda_0) &= -\sqrt{ab}/\lambda_0(a+b-c+1).
\end{aligned}$$

Near $t = \infty$ the basic solution is

$$\begin{aligned}
\psi_\infty(t) &= \varphi_\infty(t) + D_\infty(\lambda_0)\phi_\infty(t),\\
\varphi_\infty &= t^{-a}F(a,1+a-c;1+a-b;1/t),\\
\phi_\infty &= t^{-b}F(b,1+b-c;1-a+b;1/t),\\
D(\lambda_0) &= \mathrm{e}^{\mathrm{i}\pi(a+b-c)}\sqrt{a/b}/\lambda_0.
\end{aligned}$$

We can also find the connection coefficients ([3])

$$\begin{aligned}
C_{0,1} &= \psi_{0,1}(1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} + \frac{\sqrt{ab}}{\lambda_0 c}\,\frac{\Gamma(2-c)\Gamma(c-a-b)}{\Gamma(1-a)\Gamma(1-b)},\\
C_{\infty,1} &= \frac{\Gamma(1+a-b)\Gamma(c-a-b)}{\Gamma(1-b)\Gamma(c-b)} + \mathrm{e}^{\mathrm{i}\pi(a+b-c)}\frac{\sqrt{a/b}}{\lambda_0}\,\frac{\Gamma(1+b-a)\Gamma(c-a-b)}{\Gamma(1-a)\Gamma(c-a)}.
\end{aligned}$$

Of course, these formulas hold for generic a, b, c.

Remark 4.1. A standard alternative choice of the parameters in the Gauss equation is the following:

$$\lambda = 1-c, \qquad \mu = b-a, \qquad \nu = c-a-b, \tag{4.10}$$

i.e. the differences of the exponents of basic solutions at the singular points (see [1], [2, Section 2.7.2] and [8]). (J. Dereziński [3] uses other notations: $\alpha = c-1$, $\beta = a+b-c$, $\mu = b-a$).

([3]) The formula $F(a,b;c;1) = \Gamma(c)\Gamma(c-\alpha-b)/\Gamma(c-\alpha)\Gamma(c-b)$ follows from the Euler formula $F(a,b;c;1) = ((\Gamma(c)/\Gamma(b)\Gamma(c-b))\int_{[0,1]} z^{b-1}(1-z)^{c-b-1}(1-zt)^{-\alpha}\,dz$, when $\mathrm{Re}(c) > \mathrm{Re}(b) > 0$ and $\mathrm{Re}(a) < 1$. For other generic values of the parameters we use the analytic continuation of the gamma function.

Our involution (4.6) takes the form

$$(\lambda,\mu,\nu)\mapsto(-1-\lambda,-\mu,-1-\nu). \tag{4.11}$$

Symmetries and so-called recurrence relations act on the latter parameters as follows:

$$(\lambda,\mu,\nu)\mapsto(\pm\lambda+l,\pm\mu+m,\pm\nu+n), \tag{4.12}$$

where l,m,n are integers such that $l+m+n$ is even. The monodromy properties of the Gauss equation do not change after such modification of parameters ([4]).

These relations follow from the following facts.

1. The symmetry $a\leftrightarrow b$ does not change the Gauss equation and gives the reflection

$$(\lambda,\mu,\nu)\mapsto(\lambda,-\mu,\nu). \tag{4.13}$$

2. Define the differential operator

$$\mathcal{F}(a,b,c)=\mathcal{F}=t(1-t)\mathcal{D}^2+(c-(a+b+1)t)\mathcal{D}-ab, \qquad \mathcal{D}=d/dt,$$

such that $\mathcal{F}x=0$ is the hypergeometric equation. If some parameter is shifted, e.g. $a\mapsto a+1$, then we write $\mathcal{F}(a+1)$, etc. Let also

$$\mathcal{A}(\kappa)=t^{\kappa}, \qquad \mathcal{B}(\kappa)=(t-1)^{\kappa}$$

be multiplication operators. We have

$$\ker\mathcal{F}(a+1,b+1,c+1)\mathcal{D}=\ker\mathcal{F}\oplus\mathbb{C}\cdot 1.$$

It follows from the action of the operator $\mathcal{D}$ on the basic solutions (4.7) to the Gauss equation:

$$\begin{aligned}\mathcal{D}\varphi_0 &= t(ab/c)F(a+1,b+1;c+1;t),\\ \mathcal{D}\phi_0 &= (1-c)t^{-c}F(a+1-c,b+1-c;1-c;t)=(1-c)\phi_0(a+1,b+1,c+1).\end{aligned}$$

This gives the symmetry

$$(\lambda,\mu,\nu)\mapsto(\lambda-1,\mu,\nu-1). \tag{4.14}$$

([4]) In the case of real parameters this allows to reduce the range of the parameters to $\{0\le\lambda,\mu,\nu<1,\ 0\le\lambda+\mu,\lambda+\nu,\mu+\nu\le 1\}$.

In this case one has a geometrical interpretation of the monodromy group (a subgroup of $\mathrm{PGL}(2,\mathbb{C})$ modulo conjugation by an element of $\mathrm{PGL}(2,\mathbb{C})$) of the ratio $w(t)=\varphi(t)/\phi(t)$ of two independent solutions to the hypergeometric equation. One takes a triangle Δ with angles $\lambda\pi,\mu\pi,\nu\pi$ in the hyperbolic plane, when $\lambda+\mu+\nu<1$, or in the euclidean plane, when $\lambda+\mu+\nu=1$, or in the elliptic plane (sphere), when $\lambda+\mu+\nu>1$. Next, one defines the group generated by inversions with respect to the sides of Δ. The monodromy group is the index 2 subgroup of the latter group consisting of compositions of even number of inversions.

3. We have $\mathcal{A}(\lambda)\mathcal{F}\mathcal{A}(-\lambda) = \mathcal{F}(a+1-c, b+1-c, 2-c)$ (see [3]), which implies the change

$$(\lambda, \mu, \nu) \mapsto (-\lambda, \mu, \nu). \tag{4.15}$$

4. We have $\mathcal{B}(-\nu)\mathcal{F}\mathcal{B}(\nu) = \mathcal{F}(c-a, c-b, c)$ (see [3]), which implies

$$(\lambda, \mu, \nu) \mapsto (\lambda, -\mu, -\nu). \tag{4.16}$$

5. We have the recurrence relation

$$\ker \mathcal{F}(a+1)\mathcal{A}(1-a)\mathcal{D}\mathcal{A}(a) = \ker \mathcal{F} \oplus \mathbb{C} \cdot t^{-a},$$

which follows from $at^{a-1}F(a+1, b; c; t) = \mathcal{D}(t^a F(a, b; c; t))$ (see [2, v. 1]). It implies

$$(\lambda, \mu, \nu) \mapsto (\lambda, \mu - 1, \nu - 1). \tag{4.17}$$

The changes (4.13)–(4.17) generate all the changes (4.12). Our involution is a composition of them, (4.11)=(4.14) $\circ$ (4.16) $\circ$ (4.15).

Other relations between the hypergeometric functions follow from changes in the Euler integral (see the footnote 3) and are known as the Kummer table (see [2], [4]). But we did not find their application in our subject.

4.2. The case $T = T_1$. We use not exactly the involution T_1, but modified in the way that its fixed point is $t = 1/2$. Thus $T(t) = 1 - t$, i.e. $T = R^* T_1$ with $R(t) = t - 1/2$. Then we have $a_\mu(s) = \lambda_0^{-1} K t^{\vartheta_1}(1-t)^{\vartheta_0}$. Therefore we get the conditions

$$K^2 = -ab, \tag{4.18}$$

$$\vartheta_0 = -1 - \theta_1, \qquad \vartheta_1 = -1 - \theta_0, \qquad \vartheta_2 = \theta_2. \tag{4.19}$$

The involution (4.14) is a reflection in the θ-space with respect to the fixed surface $\{\theta_0 + \theta_1 + 1 = 0\}$. We have

$$c' = c - a - b, \qquad a' = -a, \qquad b' = -b.$$

In terms of the parameters (4.11) the reflection (4.19) becomes

$$(\lambda, \mu, \nu) \mapsto (1 - \nu, -\mu, 1 - \lambda).$$

This change belongs to the collection (4.12).

Again, the basic solutions are chosen as in Section 4.1, but with different constants $D_j(\lambda_0)$. For example, for $t_j = t_0 = 0$ we have $\psi_0(t) = \varphi_0(t) + D_0(\lambda_0)\phi_0(t)$ with φ_0 and ϕ_0 defined in equation (4.7). However, the computation of the constant $D_0 = D_0(\lambda_0)$ is trickier.

We evaluate at $t = 1$ the both sides of the first order equation, i.e. of

$$\frac{d}{dt}\{F(a,b;c;t) + D_0 t^{1-c} F(a-c+1, b-c+1; 2-c; t)\}$$
$$= \lambda_0 \sqrt{-ab}\, t^{\theta_0}(1-t)^{\theta_2}\{F(a',b';c';1-t)$$
$$+ D_0(1-t)^{1-c'} F(a'-c'+1, b'-c'+1; 2-c'; 1-t)\}.$$

We get

$$D_0(\lambda_0) = -\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \bigg/ \left\{\frac{\lambda_0}{\sqrt{-ab}} + \frac{\Gamma(2-c)\Gamma(c-a-b)}{\Gamma(1-a)\Gamma(1-b)}\right\}. \tag{4.20}$$

Also it is possible to compute other constants $D_j(\lambda_0)$ and the connection coefficients $C_{ij}(\lambda_0)$. We skip it.

In this case sometimes it is natural to change the time variable to τ via the formulas

$$t = (1+\tau)/2, \qquad 1 - t = (1-\tau)/2. \tag{4.21}$$

Then equation (4.1) becomes

$$\frac{d^2x}{d\tau^2} = \left(\frac{\theta_0}{1+\tau} - \frac{\theta_1}{1-\tau}\right)\frac{dx}{d\tau} + \frac{\theta_2}{1-\tau^2}\, x. \tag{4.22}$$

Consider two examples.

Example 4.2 (The Chebyshev equation revisited). This is the case when $\theta_0 = \theta_1 = -1/2$ and $\theta_3 = -n^2$, i.e. at the fixed points surface $\{\theta_0 + \theta_1 + 1 = 0\}$. The corresponding first order equation with an involution is

$$\dot{x}_\lambda(t) = \frac{\lambda_0 n}{\sqrt{1-\tau^2}} \cdot x_\mu(s).$$

Therefore the Chebyshev equation can be obtained from two sorts of first equations with two different involutions.

Example 4.3 (The Legendre equation). This equation, i.e.

$$(1-\tau^2)\ddot{x} - 2\tau\dot{x} + n(n+1)x = 0 \tag{4.23}$$

(where the dot denotes derivation with respect to τ) is a special case of equation (4.22) with $\theta_0 = \theta_1 = -1$ and $\theta_2 = -n(n+1)$. Therefore $T(\tau) = -\tau$, $\vartheta_0 = \vartheta_1 = 0$ and $K = \sqrt{n(n+1)}$. The defining equation for the both exponents (at $\tau = \pm 1$) is $\nu^2 = 0$; hence the general solution contains a logarithm.

But there is another approach to equation (4.23). Consider the equation

$$\dot{y}_\lambda(\tau) = \lambda_0(-1)^n \frac{n}{\tau - 1} \cdot y_\mu(-\tau), \tag{4.24}$$

which gives equation (4.22) with $\theta_0 = 0$, $\theta_1 = -1$ and $\theta_2 = -n^2$, i.e., with $a = -b = n$ and $c = 0$. The corresponding hypergeometric equation has the polynomial solution $\chi_1(\tau) = F(n, -n; 0; (1+\tau)/2)$ and a solution $\chi_2(t)$ with a logarithm. For positive integer n the function χ_1 is a polynomial which solves equation (4.24) for $\lambda_0 = 1$.

But one can check, that the function

$$z(\tau) = \{y(\tau) + (-1)^n y(-\tau)\}|_{\lambda_0=1} \tag{4.25}$$

satisfies equation (4.23). Therefore for $n \in \mathbb{Z}$ and positive the function (4.25) is proportional to the Legendre polynomial $P_n(\tau)$.

4.3. The case $T = T_2$. The involution T_2 can be applied either to the variable t in equation (4.1) or to the variable τ in equation (4.21).

Assume firstly $T(t) = 1/t$. Then we get the following relations between exponents in equation (3.9)–(3.10):

$$(\vartheta_0, \vartheta_1) = \Sigma(\theta_0, \theta_1) = (\theta_0 + \theta_1, -1 - \theta_1). \tag{4.26}$$

But the second iteration of this map differs from identity, $\Sigma \circ \Sigma\colon (\theta_0, \theta_1) \mapsto (\theta_0 - 1, \theta_1)$; this map is even not periodic. The obstacle to the involutivity of Σ is directly related with condition (3.23) in Theorem 3.4.

Assume then that $\sigma = T(\tau) = 1/\tau$. Here we assume

$$a_\lambda(\tau) = \lambda_0 K \tau^0 (1-\tau)^{\theta_1}(1+\tau)^{\theta_2}, \qquad a_\mu(\sigma) = \lambda_0^{-1} K \sigma^{\vartheta_0}(1-\sigma)^{\vartheta_1}(1+\sigma)^{\vartheta_2},$$

i.e. $\theta_0 = 0$. The conditions for the exponents, i.e.

$$\vartheta_0 = \theta_1 + \theta_2, \qquad \vartheta_1 = -1 - \theta_1, \qquad \vartheta_2 = -1 - \theta_2, \tag{4.27}$$

imply that the corresponding transformation Σ cannot be an involution. The conclusion is that: *the transformation T_2 is not useful in application to the hypergeometric equation.*

4.4. The confluent hypergeometric equation. Now we assume

$$a_\lambda(t) = \lambda_0 K t^{-\gamma} \mathrm{e}^t$$

(compare Remark 3.5) and the involutions: $T\colon t \mapsto -t$ and $\Sigma\colon \gamma \mapsto 1-\gamma$. Then $a_\mu(-t) = \lambda_0^{-1}(-1)^{1+\gamma} K t^{\gamma-1} \mathrm{e}^{-t}$ and hence $b_\lambda(t) = 1 - \gamma/t$ and $c_\lambda(t) = (-1)^\gamma K^2/t$. The corresponding second order equation

$$t\ddot{x} + (\gamma - t)\dot{x} - ax = 0, \tag{4.28}$$

$a = \mathrm{e}^{\mathrm{i}\pi\gamma} K^2$, is known as the *confluent hypergeometric equation* (or degenerate hypergeometric equation).

5. Equation with the delay map of order 4 and the Hermite equation

Consider the following first order equation:

$$\dot{x}(t) = \lambda_0 K \mathrm{e}^{t^2} x(\mathrm{i}t), \quad \mathrm{i} = \sqrt{-1}. \tag{5.1}$$

Therefore the time delay map $T\colon t \mapsto \mathrm{i}t$ is periodic of order 4. So, this equation falls out of the main scope of our work, but below we find a corresponding connection.

It is not difficult to show that the elimination of the delay leads to the following standard equation of order 4:

$$\ddot{\ddot{x}} = 4t\dddot{x} - 4t\dot{x} - K^4 x.$$

On the other hand, the functions $x(t)$ and $\widetilde{x}(t) = x(-t)$ satisfy the following second order equation with the involution T_1:

$$\ddot{x}(t) = 2t\dot{x}(t) + \mathrm{i}K^2 x(-t).$$

It follows that the function $y(t) = x(t) + x(-t)$ satisfies the *Hermite equation*

$$\ddot{y} = 2t\dot{y} + \mathrm{i}K^2 y.$$

For $K = \sqrt{2\mathrm{i}n}$, with integer and positive n, it has solution $H_n(t)\mathrm{e}^{t^2}$, where $H_n(t)$ are Hermite polynomials.

But one can look at this subject from still another point of view. The map $T\colon t \mapsto s = \mathrm{i}t$, defined by the equation $t^2 + s^2 = 0$, is the pull-back R^*T_1 of the involution T_1 by means of $R(t) = t^2$. With $\tau = R(t)$ from the formulas in Proposition 3.1 we have

$$a_\lambda(t) = R'(t) \cdot \alpha_\lambda(t^2), \qquad \alpha_\lambda(\tau) = \lambda_0 K e^\tau/\sqrt{\tau}.$$

Therefore $\alpha_\mu(-\tau) = -\mathrm{i}\lambda_0 K\mathrm{e}^{-\tau}/\sqrt{\tau}$ and $\beta_\lambda = 1 - 1/2\tau$, $\gamma_\lambda = \mathrm{i}K^2/\tau$. The function $z(\tau) = x_\lambda(\sqrt{\tau})$ satisfies the confluent hypergeometric equation equation

$$\ddot{z} = (1 - 1/2\tau)\dot{z} + (\mathrm{i}K^2/\tau)z.$$

The Hermite equation is a pull-back of this equation via the map $t \mapsto \tau = t^2$.

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STEFAN CZEKALSKI
ul. Marszałkowska 1 m. 80
00-624 Warsaw, POLAND

HENRYK ŻOŁĄDEK
Institute of Mathematics
University of Warsaw
ul. Banacha 2
02-097 Warsaw, POLAND

E-mail address: zoladek@mimuw.edu.pl