Topological Methods in Nonlinear Analysis
Volume 49, No. 2, 2017, 647–664
DOI: 10.12775/TMNA.2017.001

INFINITELY MANY POSITIVE SOLUTIONS OF FRACTIONAL BOUNDARY VALUE PROBLEMS

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Abstract. We are concerned with the qualitative analysis of solutions of a class of fractional boundary value problems with Dirichlet boundary conditions. By combining a direct variational approach with the theory of the fractional derivative spaces, we establish the existence of infinitely many distinct positive solutions whose E^α-norms and L^∞-norms tend to zero (to infinity, respectively) whenever the nonlinearity oscillates at zero (at infinity, respectively).

1. Introduction and statement of main result

In this paper, we consider the fractional boundary value problem of the following form:

$$\text{(P)}\quad \begin{cases} \dfrac{d}{dt}\bigg(\dfrac{1}{2}\,{}_0D_t^{-\beta}(u'(t)) + \dfrac{1}{2}\,{}_0D_T^{-\beta}(u'(t))\bigg) + \nabla F(t,u(t)) = 0 \quad \text{a.a. } t \in [0,T], \\ u(0) = u(T) = 0, \end{cases}$$

2010 *Mathematics Subject Classification.* Primary: 35J20, 35J70; Secondary: 35R20.

Key words and phrases. Fractional differential equation; oscillatory nonlinearities; infinitely many solutions; variational methods.

Supported by the National Natural Science Foundation of China (Nos. 11201095, 11201098), the Youth Scholar Backbone Supporting Plan Project of Harbin Engineering University, the Fundamental Research Funds for the Central Universities, Postdoctoral research startup foundation of Heilongjiang (No. LBH-Q14044) and the Science Research Funds for Overseas Returned Chinese Scholars of Heilongjiang Province (No. LC201502).

V. Rădulescu was supported by Partnership Program in Priority Areas PN II, MEN UEFISCDI, project number PN-II-PT-PCCA-2013-4-0614.

where ${}_0D_t^{-\beta}$ and ${}_0D_T^{-\beta}$ are the left and right Riemann–Liouville fractional integrals of order $0 \leq \beta < 1$, respectively, $F\colon [0,T] \times \mathbb{R}^N \to \mathbb{R}$ is a given function and $\nabla F(t,x)$ is the gradient of F at x.

Fractional differential equations are very efficient tools for the mathematical description of numerous phenomena in various fields of science and engineering, such as, viscoelasticity, electrochemistry, electromagnetism, economics, optimal control, porous media, etc., see [2], [4], [7], [14], [16]. In consequence, the subject of fractional differential equations is gaining much importance and attention. For details and examples, we refer to [8], [23], [13], [21], [10] and the references therein.

During the past years, there are many papers dealing with the existence of multiple solutions of fractional boundary value problems, for example [1], [3], [5], [6], [9], [15], [18]–[20], [22]. Chen and Tang [1] studied the existence and multiplicity of solutions for system (P) when the nonlinearlity F is superquadradic, asymptotically quadratic, and subquadratic. Jiao and Zhou [11] obtained the existence of solutions for (P) by the mountain pass theorem under the Ambrosetti–Rabinowitz condition. Nyamoradi obtained in [18] the existence of infinitely many non-negative solutions of problem (P). In [3], by using the variational method, the existence of at least two nontrivial solutions for (P) is established.

The aim of the present paper is to prove the existence of infinitely many distinct positive solutions for problem (P) under suitable oscillatory assumptions on the potential F at zero or at infinity. Indeed, our main results (see Theorems 1.3 and 1.6 below) give sufficient conditions on the oscillatory terms such that problem (P) has infinitely many positive solutions. As a byproduct, these solutions can be constructed in such a way that their norms in a suitable space E^α tend to zero (to infinity, respectively) whenever the nonlinearity oscillates at zero (at infinity, respectively). These results correspond to the existence of infinitely many low-energy (respectively, high-energy) solutions, according to the oscillation properties of the nonlinear term.

In the present paper, in order to establish the existence of infinitely many solutions of system (P) we consider two distinct cases, according to the growth of the nonlinear term near the origin, respectively, in a neighborhood of infinity.

1.1. Oscillation near the origin. Now we are in the position to state our first main result which deals with the case when the nonlinearity F exhibits an oscillation at the origin. For this case, we make the following assumptions.

H(F)$_1$ $F\colon [0,T] \times \mathbb{R}^N \to \mathbb{R}$ is a function such that $F(t,0) = 0$ for almost all $t \in [0,T]$ and it satisfies the following founded facts:

(1) For all $x \in \mathbb{R}^N$, $t \mapsto F(t,x)$ is measurable.

(2) For almost all $t \in [0,T]$, $x \mapsto F(t,x)$ is continuously differentiable.

(3) There exist $c \in C([0,T],\mathbb{R})$ and $0 < \alpha_0 < +\infty$ such that

$$|\nabla F(t,x)|,\ |F(t,x)| \le c(t)(1+|x|^{\alpha_0})$$

for almost all $t \in [0,T]$ and all $x \in \mathbb{R}^N$.

(4) $-\infty < \liminf_{|x|\to 0^+} \dfrac{F(t,x)}{|x|^2} \le \limsup_{|x|\to 0^+} \dfrac{F(t,x)}{|x|^2} = +\infty$ uniformly for almost every $t \in [0,T]$.

(5) For every $k \in \mathbb{N}$, there exists $e_k \in \mathbb{R}^N$ with $|e_k| = 1$ and there are two sequences $\{a_k\}$ and $\{b_k\}$ in $(0,+\infty)$ with $a_k < b_k$, $\lim_{k\to+\infty} b_k = 0$ such that $\nabla F(t,\xi)\cdot e_k \le 0$ for all $\xi \in [a_k,b_k]e_k$.

REMARK 1.1. Hypotheses $\mathrm{H(F)}_1$ (4) and $\mathrm{H(F)}_1$ (5) imply an oscillatory behaviour of F near the origin.

REMARK 1.2. A simple example of a potential function satisfying hypothesis $\mathrm{H(F)}_1$ is

$$F(t,x) = \begin{cases} 0 & \text{if } x = 0, \\ |x|^{\alpha(t)}[1+\sin(-5\ln|x|)] & \text{if } |x| > 0, \end{cases}$$

where $1 < \alpha(t) < 2$ for any $t \in [0,T]$.

PROOF. It is easy to verify conditions (1) and (2) in $\mathrm{H(F)}_1$.

Obviously, $x \mapsto F(t,x)$ is continuously differentiable. By straightforward computation, we have

$$\nabla F(t,x) = x\alpha(t)|x|^{\alpha(t)-2}[1+\sin(-5\ln|x|)] - 5x|x|^{\alpha(t)-2}\cos(-5\ln|x|).$$

Set $\alpha_0 = \max_{t\in[0,T]} \alpha(t)$, then

$$\begin{aligned} |F(t,x)| &\le \big||x|^{\alpha(t)}[1+\sin(-5\ln|x|)]\big| \le 2|x|^{\alpha(t)} \le 2(1+|x|^{\alpha_0}), \\ |\nabla F(t,x) &\le 2\alpha_0|x|^{\alpha(t)-1} + 5|x|^{\alpha(t)-1} \le (2\alpha_0+5)(1+|x|^{\alpha_0}). \end{aligned}$$

So condition $\mathrm{H(F)}_1$ (3) holds. Then, for any $1 \le k \in \mathbb{N}$ we can choose

$$a_k := \exp\bigg(-\frac{2k\pi}{5} - \frac{\pi}{20}\bigg), \qquad b_k := \exp\bigg(-\frac{2k\pi}{5}\bigg),$$

which means $a_k < b_k$, $\lim_{k\to+\infty} b_k = 0$. Then, for any $\xi \in [a_k,b_k]e_k$, $e_k \in \mathbb{R}^N$ and $|e_k| = 1$, there exists $t_0 \in [0,1]$ such that $\xi = [t_0a_k + (1-t_0)b_k]e_k$. Thus,

$$\begin{aligned} &\nabla F(t,\xi)\cdot e \\ &\quad = \big(\xi\alpha(t)|\xi|^{\alpha(t)-2}[1+\sin(-5\ln|\xi|)] - 5\xi|\xi|^{\alpha(t)-2}\cos(-5\ln|\xi|)\big)\cdot e_k \\ &\quad = \big(\alpha(t)|\xi|^{\alpha(t)-2}[1+\sin(-5\ln|\xi|)] - 5|\xi|^{\alpha(t)-2}\cos(-5\ln|\xi|)\big)\xi\cdot e_k \\ &\quad = \big(\alpha(t)[1+\sin(-5\ln|\xi|)] - 5\cos(-5\ln|\xi|)\big)[t_0a_k+(1-t_0)b_k]^{\alpha(t)-2} \le 0. \end{aligned}$$

So condition $\mathrm{H(F)}_1$ (5) is satisfied.

To verify H(F)$_1$ (4), we can choose

$$x_k = \exp\left(-\frac{2k\pi}{5} - \frac{3\pi}{10}\right)e_k \quad \text{and} \quad y_k = \exp\left(-\frac{2k\pi}{5} - \frac{\pi}{10}\right)e_k, \quad k \geq 1,$$

such that

$$\begin{aligned}\liminf_{|x|\to 0^+} \frac{F(t,x)}{|x|^2} &= \lim_{k\to+\infty} \frac{F(t,x_k)}{|x_k|^2} \\ &= \lim_{k\to+\infty} \left[\exp\left(-\frac{2k\pi}{5} - \frac{3\pi}{10}\right)\right]^{\alpha(t)-2} \left(1 + \sin\left(2k\pi + \frac{3}{2}\pi\right)\right) = 0 > -\infty, \\ \limsup_{|x|\to 0^+} \frac{F(t,x)}{|x|^2} &= \lim_{k\to+\infty} \frac{F(t,y_k)}{|y_k|^2} \\ &= \lim_{k\to+\infty} \left[\exp\left(-\frac{2k\pi}{5} - \frac{\pi}{10}\right)\right]^{\alpha(t)-2} \left(1 + \sin\left(2k\pi + \frac{1}{2}\pi\right)\right) = +\infty\end{aligned}$$

uniformly for almost every $t \in [0,T]$. So condition H(F)$_1$ (4) holds. $\square$

Theorem 1.3. *Suppose that* H(F)$_1$ *holds. Then there exists a sequence* $\{u_n\} \subset E^\alpha$ *of distinct positive solutions of problem* (P) *such that*

$$\lim_{n\to+\infty} \|u_n\|_\alpha = \lim_{n\to+\infty} |u_n|_\infty = 0.$$

1.2. Oscillation at infinity. Next, we state the counterpart of Theorem 1.3 when the nonlinearity oscillates at infinity. The hypotheses on the potential F are the following:

H(F)$_2$ $F\colon [0,T] \times \mathbb{R}^N \to \mathbb{R}$ is a function such that $F(t,0) = 0$ for almost all $t \in [0,T]$ and it satisfies the following founded facts:

(1) For all $x \in \mathbb{R}^N$, $t \mapsto F(t,x)$ is measurable.

(2) For almost all $t \in [0,T]$, $x \mapsto F(t,x)$ is continuously differentiable.

(3) There exist $c \in C([0,T],\mathbb{R})$ and $0 < \alpha_0 < +\infty$ such that

$$|\nabla F(t,x)|,\ |F(t,x)| \leq c(t)(1 + |x|^{\alpha_0})$$

for almost all $t \in [0,T]$ and all $x \in \mathbb{R}^N$.

(4) $-\infty < \liminf_{|x|\to+\infty} \frac{F(t,x)}{|x|^2} \leq \limsup_{|x|\to+\infty} \frac{F(t,x)}{|x|^2} = +\infty$ uniformly for almost every $x \in \mathbb{R}^N$.

(5) For every $k \in \mathbb{N}$, there exist $e_k \in \mathbb{R}^N$ with $|e_k| = 1$ and there are two sequences $\{a_k\}$ and $\{b_k\}$ in $(0,+\infty)$ with $a_k < b_k$, $\lim_{k\to+\infty} a_k = +\infty$ such that $\nabla F(t,\xi) \cdot e_k \leq 0$ for all $\xi \in [a_k, b_k]e_k$.

Remark 1.4. Hypotheses H(F)$_2$ (4) and H(F)$_2$ (5) imply an oscillatory behaviour of F near the infinity.

REMARK 1.5. A simple example of a potential function satisfying hypothesis $\mathrm{H(F)}_2$ is

$$F(t,x)=\begin{cases}0 & \text{if } x=0,\\ |x|^{\alpha(t)}(1+\sin|x|) & \text{if } |x|>0,\end{cases}$$

where $2<\alpha(t)<\pi$ for any $t\in[0,T]$.

PROOF. One can easily check that hypotheses (1) and (2) of $\mathrm{H(F)}_2$ are satisfied.

Obviously, $x\mapsto F(t,x)$ is continuously differentiable. Then

$$\nabla F(t,x)=\begin{cases}0 & \text{if } x=0,\\ \alpha(t)x|x|^{\alpha(t)-2}(1+\sin|x|)+x|x|^{\alpha(t)-1}\cos|x| & \text{if } |x|>0.\end{cases}$$

Let $\alpha_0=\max\limits_{t\in[0,T]}\alpha(t)$. Then

$$|F(t,x)|\le\big||x|^{\alpha(t)}(1+\sin(|x|)\big|\le 2|x|^{\alpha(t)}\le 2(1+|x|^{\alpha_0})$$

for all $x\in\mathbb{R}^N$ and all $t\in[0,T]$, and

$$\begin{aligned}|\nabla F(t,x)|&=\big|\alpha(t)x|x|^{\alpha(t)-2}(1+\sin|x|)+x|x|^{\alpha(t)-1}\cos|x|\big|\\ &\le 2|\alpha(t)||x|^{\alpha(t)-1}+|x|^{\alpha(t)}\le(2\alpha_0+1)(1+|x|^{\alpha_0}).\end{aligned}$$

So condition $\mathrm{H(F)}_2$ (3) holds. Then, for any $1\le k\in\mathbb{N}$ we can choose

$$a_k:=(2k+1)\pi,\qquad b_k:=\bigg(2k+\frac{3}{2}\bigg)\pi,$$

which means $a_k<b_k$, $\lim\limits_{k\to+\infty}a_k=+\infty$ and, for any $\xi\in[a_k,b_k]e_k$ there exists $\lambda_0\in[0,1]$ such that $\xi=[\lambda_0a_k+(1-\lambda_0)b_k]e_k$, so

$$\begin{aligned}&\nabla F(t,\xi)\cdot e_k\\ &\quad=\big(\alpha(t)\xi|\xi|^{\alpha(t)-2}(1+\sin|\xi|)+\xi|\xi|^{\alpha(t)-1}\cos|\xi|\big)\cdot e_k\\ &\quad=\big(\alpha(t)|\xi|^{\alpha(t)-2}(1+\sin|\xi|)+|\xi|^{\alpha(t)-1}\cos|\xi|\big)\xi\cdot e_k\\ &\quad=\big(\alpha(t)|\xi|^{\alpha(t)-2}(1+\sin|\xi|)+|\xi|^{\alpha(t)-1}\cos|\xi|\big)(\lambda_0a_k+(1-\lambda_0)b_k)\le 0.\end{aligned}$$

So condition $\mathrm{H(F)}_2$ (5) is satisfied. To verify $\mathrm{H(F)}_2$ (4), we can choose

$$x_k:=\bigg(2k+\frac{1}{2}\bigg)\pi e_k,\qquad y_k:=\bigg(2k+\frac{3}{2}\bigg)\pi e_k,$$

which implies

$$\begin{aligned}
\liminf_{k\to+\infty}\frac{F(t,x)}{|x|^2} &= \lim_{k\to+\infty}\frac{F(t,y_k)}{|y_k|^2}\\
&= \lim_{k\to+\infty}\frac{|(2k+3/2)\pi e_k|^{\alpha(t)}(1+\sin|(2k+3/2)\pi e_k|)}{|(2k+3/2)\pi e_k|^2}\\
&= \lim_{k\to+\infty}\bigg[\bigg(2k+\frac{3}{2}\bigg)\pi\bigg]^{\alpha(t)-2}\bigg(1+\sin\bigg[\bigg(2k+\frac{3}{2}\bigg)\pi\bigg]\bigg) = 0 > -\infty,\\
\limsup_{k\to+\infty}\frac{F(t,x)}{|x|^2} &= \lim_{k\to+\infty}\frac{F(t,y_k)}{|y_k|^2}\\
&= \lim_{k\to+\infty}\frac{|(2k+1/2)\pi e_k|^{\alpha(t)}(1+\sin|(2k+3/2)\pi e_k|)}{|(2k+1/2)\pi e_k|^2}\\
&= \lim_{k\to+\infty}\bigg[\bigg(2k+\frac{1}{2}\bigg)\pi\bigg]^{\alpha(t)-2}\bigg(1+\sin\bigg[(2k+\frac{1}{2}\bigg)\pi\bigg]\bigg) = +\infty
\end{aligned}$$

uniformly for almost every $t\in[0,T]$. So condition $\mathrm{H(F)}_2$ (4) holds. □

THEOREM 1.6. *Suppose that* $\mathrm{H(F)}_2$ *holds. Then there exists a sequence* $\{u_n\}\subset E^\alpha$ *of distinct positive solutions of problem* (P) *such that*

$$\lim_{n\to+\infty}\|u_n\|_\alpha = \lim_{n\to+\infty}|u_n|_\infty = +\infty.$$

2. Preliminaries

In this part, we recall some definitions and display the variational setting which has been established for our problem.

DEFINITION 2.1 ([12]). Let f be a function defined on $[a,b]$ and $\tau>0$. The left and right Riemann–Liouville fractional integrals of order τ for the function f denoted by ${}_aD_t^{-\tau}f$ and ${}_tD_b^{-\tau}f$, respectively, are defined by

$$\begin{aligned}
{}_aD_t^{-\tau}f(t) &= \frac{1}{\Gamma(\tau)}\int_a^t (t-s)^{\tau-1}f(s)\,ds, \quad t\in[a,b],\\
{}_tD_b^{-\tau}f(t) &= \frac{1}{\Gamma(\tau)}\int_t^b (t-s)^{\tau-1}f(s)\,ds, \quad t\in[a,b],
\end{aligned}$$

provided the right-hand sides are pointwise defined on $[a,b]$, where Γ is the gamma function.

DEFINITION 2.2 ([12]). Let f be a function defined on $[a,b]$. The left and right Riemann–Liouville fractional derivatives of order τ for the function f denoted by ${}_aD_t^{\tau}f$ and ${}_tD_b^{\tau}f$, respectively, are defined by

$${}_aD_t^{\tau}f(t) = \frac{d^n}{dt^n}\,{}_aD_t^{\tau-n}f(t) = \frac{1}{\Gamma(n-\tau)}\,\frac{d^n}{dt^n}\bigg(\int_a^t (t-s)^{n-\tau-1}f(s)\,ds\bigg),$$

$$
\begin{aligned}
{}_tD_b^{\tau} f(t) &= (-1)^n \frac{d^n}{dt^n}\, {}_tD_b^{\tau-n} f(t) \\
&= \frac{1}{\Gamma(n-\tau)} \frac{d^n}{dt^n} \bigg(\int_t^b (t-s)^{n-\tau-1} f(s)\, ds \bigg),
\end{aligned}
$$

where $t \in [a,b]$, $n-1 \le \tau < n$ and $n \in \mathbb{N}$.

The left and right Caputo fractional derivatives are defined via the above Riemann–Liouville fractional derivatives. In particular, they are defined for functions belonging to the space of absolutely continuous functions, which we denote by $AC([a,b],\mathbb{R}^N)$. $AC^k([a,b],\mathbb{R}^N)$ $(k = 1,2,\ldots)$ is the space of functions f such that $f \in C^k([a,b],\mathbb{R}^N)$. In particular, $AC([a,b],\mathbb{R}^N) = AC^1([a,b],\mathbb{R}^N)$.

Definition 2.3 ([12]). Let $\tau \ge 0$ and $n \in \mathbb{N}$. If $\tau \in [n-1,n)$ and $f \in AC^n([a,b],\mathbb{R}^N)$, then the left and right Caputo fractional derivatives of order τ for the function f denoted by ${}_a^cD_t^{\tau} f$ and ${}_t^cD_b^{\tau} f$, respectively, exist almost everywhere on $[a,b]$. ${}_a^cD_t^{\tau} f(t)$ and ${}_t^cD_b^{\tau} f(t)$ are represented by

$$
\begin{aligned}
{}_a^cD_t^{\tau} f(t) &= {}_aD_t^{\tau-n} f^{(n)}(t) = \frac{1}{\Gamma(n-\tau)} \bigg(\int_a^t (t-s)^{n-\tau-1} f^{(n)}(s)\, ds \bigg), \\
{}_t^cD_b^{\tau} f(t) &= (-1)^n\, {}_t^nD_b^{\tau-n} f^{(n)}(t) = \frac{1}{\Gamma(n-\tau)} \bigg(\int_t^b (t-s)^{n-\tau-1} f^{(n)}(s)\, ds \bigg),
\end{aligned}
$$

respectively, where $t \in [a,b]$.

Definition 2.4 ([11]). Let $0 < \alpha \le 1$ and $1 < p < \infty$. The fractional derivative space $E_0^{\alpha,p}$ is defined by the closure of $C_0^{\infty}([0,T],\mathbb{R}^N)$ with respect to the norm

$$
\|u\|_{\alpha,p} = \bigg(\int_0^T |u(t)|^p\, dt + \int_0^T |{}_0^cD_t^{\alpha} u(t)|^p\, dt \bigg)^{1/p} \quad \text{for all } u \in E_0^{\alpha,p},
$$

where $C_0^{\infty}([0,T],\mathbb{R}^N)$ denotes the set of all functions $u \in C^{\infty}([0,T],\mathbb{R}^N)$ with $u(0) = u(T) = 0$. It is obvious that the fractional derivative space $E_0^{\alpha,p}$ is the space of functions $u \in L^p([0,T],\mathbb{R}^N)$ having an α-order Caputo fractional derivative ${}_0^cD_t^{\alpha} u \in L^p([0,T],\mathbb{R}^N)$ and $u(0) = u(T) = 0$.

Proposition 2.5 ([11]). *Let $0 < \alpha \le 1$ and $1 < p < \infty$. The fractional derivative space $E_0^{\alpha,p}$ is a reflexive and separable space.*

Proposition 2.6 ([11]). *Let $0 < \alpha \le 1$ and $1 < p < \infty$. For all $u \in E_0^{\alpha,p}$ we have*

$$
\|u\|_{L^p} \le \frac{T^{\alpha}}{\Gamma(\alpha+1)} \|{}_0^cD_t^{\alpha} u\|_{L^p}.
$$

Moreover, if $\alpha > 1/p$ and $1/p + 1/q = 1$, then

$$
\|u\|_{\infty} \le \frac{T^{(\alpha-1)/p}}{\Gamma(\alpha)((\alpha-1)q+1)^{1/q}} \|{}_0^cD_t^{\alpha} u\|_{L^p}. \tag{2.1}
$$

According to (2.1), *we can consider* $E_0^{\alpha,p}$ *with respect to the norm*

$$\|u\|_{\alpha,p} = \|{}_0^cD_t^\alpha u\|_{L^p} = \bigg(\int_0^T |{}_0^cD_t^\alpha u|^p\,dt\bigg)^{1/p}.$$

Proposition 2.7 ([11]). *Let* $0 < \alpha \leq 1$ *and* $1 < p < \infty$. *Assume that* $\alpha > 1/p$ *and the sequence* u_k *converges weakly to* $u \in E_0^{\alpha,p}$, *i.e.* $u_k \rightharpoonup u$. *Then* $u_k \to u$ *in* $C([0,T],\mathbb{R}^N)$, *i.e.* $\|u_k - u\|_\infty \to 0$, *as* $k \to \infty$.

Making use of Definition 2.3, for any $u \in AC([0,T],\mathbb{R}^N)$ problem (P) is equivalent to the following problem:

$$(\mathrm{P}_1)\quad \begin{cases} \dfrac{d}{dt}\bigg(\dfrac{1}{2}\,{}_0D_t^{\alpha-1}({}_0^cD_t^\alpha u(t)) - \dfrac{1}{2}\,{}_tD_T^{\alpha-1}({}_t^cD_T^\alpha u(t))\bigg) + \nabla F(t,u(t)) = 0 \\ \hfill \text{a.e. } t \in [0,T], \\ u(0) = u(T) = 0, \end{cases}$$

where $\alpha = 1 - \beta \in (1/2, 1]$. In the following, we will treat problem (P_1) in the Hilbert space $E^\alpha = E_0^{\alpha,2}$ with the corresponding norm $\|u\|_\alpha = \|u\|_{\alpha,2}$. It follows from Theorem 4.1 in [11] that the functional φ given by

$$\varphi(u) = \int_0^T \bigg[-\frac{1}{2}\,({}_0^cD_t^\alpha u(t),\, {}_t^cD_T^\alpha u(t))\bigg]\,dt - \int_0^T F(t,u(t))\,dt$$

is continuously differentiable on E^α. Moreover, for $u, v \in E^\alpha$ we have

$$\begin{aligned}\langle \varphi'(u), v\rangle = &-\int_0^T \frac{1}{2}\Big[({}_0^cD_t^\alpha u(t),\, {}_t^cD_T^\alpha v(t)) + ({}_t^cD_T^\alpha u(t),\, {}_0^cD_t^\alpha v(t))\Big]dt \\ &-\int_0^T \nabla F(t,u(t))\cdot v(t)\,dt.\end{aligned}$$

Definition 2.8 ([11]). A function $u \in AC([0,T],\mathbb{R}^N)$ is called a solution of (P) if

(a) $D^\alpha(u(t))$ is derivative for almost every $t \in [0,T]$, and
(b) u satisfies (P),

where

$$D^\alpha(u(t)) := \frac{1}{2}\,{}_0D_t^{\alpha-1}({}_0^cD_t^\alpha u(t)) - \frac{1}{2}\,{}_tD_T^{\alpha-1}({}_t^cD_T^\alpha u(t)).$$

Proposition 2.9 ([11]). *If* $1/2 < \alpha \leq 1$, *then for any* $u \in E^\alpha$ *we have*

$$|\cos(\pi\alpha)|\|u\|_\alpha^2 \leq -\int_0^T \big({}_0^cD_t^\alpha u(t),\, {}_t^cD_T^\alpha u(t)\big)\,dt \leq \frac{1}{|\cos(\pi\alpha)|}\,\|u\|_\alpha^2.$$

Now, we are going to prove our main results.

3. Proof of Theorem 1.3

For every fixed $k \in \mathbb{N}$, consider the set

$$S_k = \big\{u \in E^\alpha : u(t) \neq 0 \text{ and } u(t) \in [0, b_k]e_k \text{ a.e. } t \in [0,T]\big\},$$

where b_k is from $\mathrm{H(F)}_1$ (5). The proof is divided into four steps as follows.

STEP 1. We claim that φ is bounded from below on S_k and its infimum m_k on S_k is attained at $u_k \in S_k$.

On account of $\mathrm{H(F)}_1$ (3) and Proposition 2.9, for every $u \in S_k$ we have

$$\begin{aligned}
\varphi(u) &= \int_0^T \bigg[-\frac{1}{2}\,({}_0^cD_t^\alpha u(t),\, {}_t^cD_T^\alpha u(t))\bigg]\,dt - \int_0^T F(t,u(t))\,dt \\
&\geq \frac{|\cos(\pi\alpha)|}{2}\,\|u\|_\alpha^2 - \int_0^T c(t)(1+|u(t)|^{\alpha_0})\,dt \\
&\geq \frac{|\cos(\pi\alpha)|}{2}\,\|u\|_\alpha^2 - \int_0^T c_0(1+|u(t)|^{\alpha_0})\,dt \\
&\geq \frac{|\cos(\pi\alpha)|}{2}\,\|u\|_\alpha^2 - c_0T - c_0\int_0^T |u(t)|^{\alpha_0}\,dt \\
&\geq \frac{|\cos(\pi\alpha)|}{2}\|u\|_\alpha^2 - c_0T - c_0T|b_k|^{\alpha_0} \geq -c_0T - c_0T|b_k|^{\alpha_0},
\end{aligned}$$

where $c_0 = \max\limits_{t\in[0,T]} c(t)$. It is clear that S_k is convex and closed, thus weakly closed in E^α. Let $m_k = \inf\limits_{S_k}\varphi$, and $\{u_k^n\}_{n=1}^\infty$ be a sequence in S_k such that $m_k \leq \varphi(u_k^n) \leq m_k + 1/n$ for all $n \in \mathbb{N}$. Then

$$m_k + \frac{1}{n} \geq \varphi(u_k^n) = \int_o^T \bigg[-\frac{1}{2}\,({}_0^cD_t^\alpha u_k^n(t),\, {}_t^cD_T^\alpha u_k^n(t))\bigg]dt - \int_0^T F(t,u_k^n(t))\,dt,$$

which implies that

$$\begin{aligned}
\frac{|\cos(\pi\alpha)|}{2}\,\|u_k^n\|_\alpha^2 &\leq \int_0^T \bigg[-\frac{1}{2}\,({}_0^cD_t^\alpha u_k^n(t), {}_t^c D_T^\alpha u_k^n(t))\bigg]\,dt \\
&\leq m_k + \frac{1}{n} + \int_0^T F(t,u_k^n(t))\,dt \\
&\leq m_k + \frac{1}{n} + \int_0^T c(t)(1+|u_k^n(t)|^{\alpha_0})\,dt \\
&\leq m_k + \frac{1}{n} + c_0T + c_0T|b_k|^{\alpha_0}
\end{aligned}$$

for all $n \in \mathbb{N}$, thus $\{u_k^n(t)\}_{n=1}^\infty$ is bounded in E^α.

By Proposition 2.5, one can easily see that there exists $\{u_k^n\}_{n=1}^\infty \in E^\alpha$ such that $u_k^n \rightharpoonup u_k$ in E^α. As φ is weak lower semicontinuous (see [17, Theorem 3.1, Step 1]), $\varphi(u_k) \leq \underline{\lim}_{n\to+\infty} \varphi(u_k^n)$. Then,

$$m_k \leq \varphi(u_k) \leq \underline{\lim}_{n\to+\infty} \varphi(u_k^n) \leq m_k + \frac{1}{n},$$

which implies that $\varphi(u_k) = m_k$. Hence, u_k is a minimum point of φ over S_k.

Step 2. We show that $u_k(t) \in [0, a_k]e_k$ for almost every $t \in [0, T]$.

Let $A = \{t \in [0,T] : u_k(t) \notin [0, a_k]e_k\} = \{t \in [0,T] : u_k(t) \in [a_k, b_k]e_k\}$ and we can suppose that $\operatorname{meas}(A) > 0$. Define the function $h\colon [0,+\infty)e_k \to [0,+\infty)e_k$ by

$$h(s) = \begin{cases} a_k e_k & \text{if } s \in [a_k, +\infty]e_k, \\ s & \text{if } s \in [0, a_k]e_k. \end{cases}$$

Now, we set $v_k = h \circ u_k$. Since h is a Lipschitz function and $h(0) = 0$, the theorem of Marcus–Mizel [17] shows that $v_k \in E^\alpha$. Moreover, $v_k(t) \in [0, a_k]e_k$ for almost every $t \in [0, T]$. Consequently, $v_k \in S_k$ and

$$v_k(x) = \begin{cases} u_k(t) & \text{if } t \in [0,T] \setminus A, \\ a_k e_k & \text{if } t \in A. \end{cases}$$

By straightforward computation, we obtain

$$\begin{aligned}
\varphi(v_k) - \varphi(u_k) &= \int_{[0,T]} \bigg[-\frac{1}{2} \big({}_0^c D_t^\alpha v_k(t),\ {}_t^c D_T^\alpha v_k(t)\big) \bigg] \, dt - \int_{[0,T]} F(t, v_k(t)) \, dt \\
&\quad - \int_{[0,T]} \bigg[-\frac{1}{2} \big({}_0^c D_t^\alpha u_k(t),\ {}_t^c D_T^\alpha u_k(t)\big) \bigg] \, dt + \int_{[0,T]} F(t, u_k(t)) \, dt \\
&= \int_{[0,T]\setminus A} \bigg[-\frac{1}{2} \big({}_0^c D_t^\alpha u_k(t),\ {}_t^c D_T^\alpha u_k(t)\big) \bigg] \, dt \\
&\quad + \int_A \bigg[-\frac{1}{2} \big({}_0^c D_t^\alpha a_k e_k,\ {}_t^c D_T^\alpha a_k e_k\big) \bigg] \, dt \\
&\quad - \int_{[0,T]\setminus A} F(t, u_k(t)) \, dt - \int_A F(t, a_k e_k) \, dt \\
&\quad - \int_{[0,T]\setminus A} \bigg[-\frac{1}{2} \big({}_0^c D_t^\alpha u_k(t),\ {}_t^c D_T^\alpha u_k(t)\big) \bigg] \, dt \\
&\quad - \int_A \bigg[-\frac{1}{2} \big({}_0^c D_t^\alpha u_k(t),\ {}_t^c D_T^\alpha u_k(t)\big) \bigg] \, dt \\
&\quad + \int_{[0,T]\setminus A} F(t, u_k(t)) \, dt + \int_A F(t, u_k(t)) \, dt \\
&= - \int_A \bigg[-\frac{1}{2} \big({}_0^c D_t^\alpha u_k(t),\ {}_t^c D_T^\alpha u_k(t)\big) \bigg] \, dt \\
&\quad - \int_A [F(t, a_k e_k) - F(t, u_k(t))] \, dt.
\end{aligned}$$

For every $t \in A$, $u_k(t) \in [a_k, b_k]e_k$, there exists a map $\lambda\colon A \to [0,1]$ such that $u_k(t) = a_k e_k + \lambda(t)(b_k - a_k)e_k$. By the Mean Value Theorem, it holds

$$\begin{aligned}\int_A [F(t, a_k e_k) - F(t, u_k(t))]\, dt &= \int_A \nabla F(t, \xi_k(t)) \cdot (a_k e_k - u_k(t))\, dt \\ &= \int_A \nabla F(t, \xi_k(t)) \cdot [a_k e_k - a_k e_k - \lambda(t)(b_k - a_k)e_k]\, dt \\ &= \int_A \nabla F(t, \xi_k(t)) \cdot \lambda(t)(a_k - b_k)e_k\, dt.\end{aligned}$$

By $\mathrm{H(F)}_1$ (5), we have $\xi_k(t) \in [a_k, b_k]e_k$ for almost every $t \in A$. Consequently,

$$\int_A [F(t, a_k e_k) - F(t, u_k(t))]\, dt \geq 0.$$

In conclusion, every term of the expression $\varphi(v_k) - \varphi(u_k) \leq 0$. On the other hand, since $v_k \in S_k$, then $\varphi(v_k) \geq \varphi(u_k) = \inf_{S_k} \varphi$. So, $\varphi(v_k) - \varphi(u_k) = 0$. Namely,

$$-\int_A \bigg[-\frac{1}{2}\,({}_0^cD_t^\alpha u_k(t), {}_t^cD_T^\alpha u_k(t))\bigg]\, dt - \int_A [F(t, a_k e_k) - F(t, u_k(t))]\, dt = 0,$$

which implies that $\mathrm{meas}(A) = 0$.

STEP 3. We show that u_k is a local minimum point in E^α.

Let $A' = \{t \in [0,T] : u(t) \notin [0, a_k]e_k\} = \{t \in [0,T] : u(t) \in (a_k, b_k]e_k\}$. Set $v = h \circ u$, then we have

$$\begin{aligned}(3.1)\quad \varphi(u) &- \varphi(v) \\ &= \int_{[0,T]} \bigg[-\frac{1}{2}\,({}_0^cD_t^\alpha u(t),\ {}_t^cD_T^\alpha u(t))\bigg]\, dt - \int_{[0,T]} F(t, u(t))\, dt \\ &\quad - \int_{[0,T]} \bigg[-\frac{1}{2}\,({}_0^cD_t^\alpha v(t),\ {}_t^cD_T^\alpha v(t))\bigg]\, dt + \int_{[0,T]} F(t, v(t))\, dt \\ &= \int_{[0,T]\setminus A'} \bigg[-\frac{1}{2}\,({}_0^cD_t^\alpha u(t),\ {}_t^cD_T^\alpha u(t))\bigg]\, dt \\ &\quad + \int_{A'} \bigg[-\frac{1}{2}\,({}_0^cD_t^\alpha u(t),\ {}_t^cD_T^\alpha u(t))\bigg]\, dt - \int_{[0,T]\setminus A'} F(t, u(t))\, dt \\ &\quad - \int_{A'} F(t, u(t))\, dt - \int_{[0,T]\setminus A'} \bigg[-\frac{1}{2}\,({}_0^cD_t^\alpha u(t),\ {}_t^cD_T^\alpha u(t))\bigg]\, dt \\ &\quad - \int_{A'} \bigg[-\frac{1}{2}\,({}_0^cD_t^\alpha a_k e_k,\ {}_t^cD_T^\alpha a_k e_k)\bigg]\, dt \\ &\quad + \int_{[0,T]\setminus A'} F(t, u(t))\, dt + \int_{A'} F(t, a_k e_k)\, dt \\ &= \int_{A'} \bigg[-\frac{1}{2}\,({}_0^cD_t^\alpha u(t),\ {}_t^cD_T^\alpha u(t))\bigg]\, dt + \int_{A'} [F(t, a_k e_k) - F(t, u(t))]\, dt.\end{aligned}$$

From assumption $\mathrm{H(F)}_1$ (5), we have

$$\int_{A'} [F(t, a_k e_k) - F(t, u(t))]\, dt = \int_{A'} \nabla F(t, \xi(t)) \cdot (a_k e_k - u(t))\, dt \geq 0$$

for almost every $t \in A'$, where $\xi(t) \in [a_k e_k, u(t)] \subseteq [a_k, b_k] e_k$ for almost every $t \in A'$. Consequently, $\varphi(u) - \varphi(v) \geq 0$. On the other hand, by $v \in S_k$, we have $\varphi(v) \geq \varphi(u_k)$. In view of (3.1), we derive

$$\varphi(u) - \varphi(v) \geq \int_{A'} \bigg[-\frac{1}{2} \left({}_0^c D_t^\alpha u(t),\ {}_t^c D_T^\alpha u(t)\right) \bigg]\, dt.$$

Moreover, we have

$$\begin{aligned}
\varphi(u) &\geq \varphi(v) + \int_{A'} \bigg[-\frac{1}{2} \left({}_0^c D_t^\alpha u(t),\ {}_t^c D_T^\alpha u(t)\right) \bigg]\, dt \\
&\geq \varphi(u_k) + \int_{A'} \bigg[-\frac{1}{2} \left({}_0^c D_t^\alpha u(t),\ {}_t^c D_T^\alpha u(t)\right) \bigg]\, dt \\
&\geq \varphi(u_k) + \int_{[0,T]} \bigg[-\frac{1}{2} \left({}_0^c D_t^\alpha u(t),\ {}_t^c D_T^\alpha u(t)\right) \bigg]\, dt \\
&\quad - \int_{[0,T]\setminus A'} \bigg[-\frac{1}{2} \left({}_0^c D_t^\alpha u(t),\ {}_t^c D_T^\alpha u(t)\right) \bigg]\, dt \\
&\geq \varphi(u_k) + \int_{[0,T]} \bigg[-\frac{1}{2} \left({}_0^c D_t^\alpha (u(t) - v(t)),\ {}_t^c D_T^\alpha (u(t) - v(t))\right) \bigg]\, dt \\
&\geq \varphi(u_k) + \frac{|\cos(\pi\alpha)|}{2} \|u - v\|_\alpha^2.
\end{aligned}$$

Since h is continuous, there exists $\delta > 0$ such that for every $u \in E^\alpha$, $\|u - v\|_\alpha < \delta$, which implies that u_k is a local minimum of φ.

Step 4. In this step the fact that $m_k = \inf_{S_k} \varphi < 0$ and $\lim_{k\to+\infty} m_k = 0$ is proved.

Let $B_{r_0}(t_0) \subset [0, T]$ be the ball with radius $r_0 \in (0, 1)$ and center $t_0 \in [0, T]$. For $\xi \in \mathbb{R}^N$, define

$$\eta_\xi(t) = \begin{cases} 0 & \text{if } t \in [0, T] \setminus B_{r_0}(t_0), \\ \xi & \text{if } t \in B_{r_0/2}(t_0), \\ \dfrac{2\xi}{r_0}(r_0 - |t - t_0|) & \text{if } t \in B_{r_0}(t_0) \setminus B_{r_0/2}(t_0). \end{cases} \tag{3.2}$$

It is clear that $\eta_\xi \in E^\alpha$ and $|\eta_\xi(t)| \leq 2|\xi|/r_0$;

$$\begin{aligned}
|{}_0^c D_t^\alpha \eta_\xi(t)| &= \bigg| \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \eta_\xi'\, ds \bigg| \\
&\leq \frac{1}{\Gamma(1-\alpha)} \bigg(\int_0^t (t-s)^{-\alpha} |\eta_\xi'|\, ds \bigg) \leq \frac{1}{\Gamma(1-\alpha)} \frac{2|\xi|}{r_0} \frac{t^{1-\alpha}}{1-\alpha};
\end{aligned}$$

$$\begin{aligned}
(3.3)\qquad \|\eta_\xi\|_\alpha^2 &= \int_0^T |{}_0^cD_t^\alpha \eta_\xi(t)|^2\,dt \le \int_0^T \frac{1}{\Gamma^2(1-\alpha)}\,\frac{4|\xi|^2}{r_0^2}\,\frac{t^{2-2\alpha}}{(1-\alpha)^2}\,dt\\
&\le \frac{1}{\Gamma^2(1-\alpha)}\,\frac{4\xi^2}{r_0^2}\,\frac{1}{(1-\alpha)^2}\int_0^T t^{2-2\alpha}\,dt\\
&\le \frac{4|\xi|^2}{\Gamma^2(1-\alpha)r_0^2(1-\alpha)^2(3-2\alpha)}\,T^{3-2\alpha}.
\end{aligned}$$

From the left part of $\mathrm{H(F)}_1$ (4) we deduce the existence of some $l_0 > 0$ and $\lambda_0 \in [0, a_k]e_k$ such that

$$(3.4)\qquad \operatorname*{essinf}_{t\in[0,T]} F(t,x) \ge -l_0|x|^2 \quad \text{for all } x \in [0,\lambda_0]e_k.$$

There exists $L_0 > 0$ large enough to enable

$$(3.5)\qquad C(r_0,\alpha,T) + l_0T < L_0r_0,$$

where

$$C(r_0,\alpha,T) = \frac{1}{2|\cos(\pi\alpha)|}\,\frac{4T^{3-2\alpha}}{\Gamma^2(1-\alpha)r_0^2(3-2\alpha)}.$$

Taking into account the right part of $\mathrm{H(F)}_1$ (4), there is a sequence $\{\xi_k\} \in [0,\lambda_0]$ such that $\{\xi_k\} \in [0, a_k]e_k$ and

$$(3.6)\qquad \operatorname*{esssup}_{t\in[0,T]} F(t,\xi_k) > L_0|\xi_k|^2 \quad \text{for all } k \in N.$$

Clearly, $|t-t_0| \in (r_0/2, r_0)$, $r_0 - |t-t_0| \in (0, r_0/2)$ for all $t \in B_{r_0}(t_0)\setminus B_{r_0/2}(t_0)$. Therefore,

$$\frac{2\xi_k}{r_0}\,(r_0 - |t-t_0|) \in [0,\xi_k] \subset [0,\lambda_0]e_k, \quad \text{for all } t \in B_{r_0}(t_0)\setminus B_{r_0/2}(t_0).$$

In view of Proposition 2.9 and (3.3), we deduce

$$\begin{aligned}
(3.7)\qquad \int_0^T \Big[-\frac{1}{2}\,({}_0^cD_t^\alpha \eta_{\xi_k}(t),\ {}_t^cD_T^\alpha \eta_{\xi_k}(t))\Big]\,dt &\le \frac{1}{2|\cos(\pi\alpha)|}\,\|\eta_{\xi_k}(t)\|_\alpha^2\\
&\le \frac{1}{2|\cos(\pi\alpha)|}\,\frac{4T^{3-2\alpha}}{\Gamma^2(1-\alpha)r_0^2(3-2\alpha)}\,|\xi_k|^2 = C(r_0,\alpha,T)|\xi_k|^2.
\end{aligned}$$

Combining (3.4) with (3.6), we obtain

$$\begin{aligned}
(3.8)\qquad &\int_0^T F(t,\eta_\xi(t))\,dt\\
&= \int_{B_{r_0/2}(t_0)} F(t,\eta_{\xi_k}(t))\,dt + \int_{B_{r_0}(t_0)\setminus B_{r_0/2}(t_0)} F(t,\eta_{\xi_k}(t))\,dt\\
&\ge \int_{B_{r_0/2}(t_0)} F(t,\xi_k(t))\,dt + \int_{B_{r_0}(t_0)\setminus B_{r_0/2}(t_0)} F\Big(t, \frac{2\xi_k}{r_0}\,(r_0-|t-t_0|)\Big)\,dt\\
&\ge L_0r_0|\xi_k|^2 - l_0T|\xi_k|^2.
\end{aligned}$$

Let $k \in \mathbb{N}$ be a fixed number and let $\eta_{\xi_k} \in E^\alpha$ be the function from (3.2) corresponding to the value $|\xi_k| > 0$. Then $\eta_{\xi_k} \in S_k$, and on account of (3.5), (3.7) and (3.8), one has

$$\begin{aligned}
\varphi(\eta_{\xi_k}) &= \int_0^T \bigg[-\frac{1}{2}\, ({}_0^c D_t^\alpha \eta_{\xi_k}(t),\ {}_t^c D_T^\alpha \eta_{\xi_k}(t)) \bigg]\, dt - \int_0^T F(t, \eta_\xi(t))\, dt \\
&\leq C(r_0, \alpha, T)|\xi_k|^2 - L_0 r_0 |\xi_k|^2 + l_0 T |\xi_k|^2 \\
&\leq (C(r_0, \alpha, T) + l_0 T - L_0 r)|\xi_k|^2 < 0.
\end{aligned} \tag{3.9}$$

Due to Step 3 and (3.9), we deduce that $m_k = \varphi(u_k) = \inf_{S_k} \varphi \leq \varphi(\eta_{\xi_k}) < 0$.

Now we prove that $\lim_{k \to +\infty} m_k = 0$. Observe that by assumption $\mathrm{H(F)}_1$ (3), one can find a constant $c_0 = \max_{t \in [0,T]} > 0$ such that

$$|\nabla F(t, x)| \leq c_0 (1 + |x|^{\alpha_0}) \quad \text{for all } t \in [0, T],\ x \in \mathbb{R}^N.$$

Applying the Mean Value Theorem and the above inequality for every $x \in [0, a_k] e_k$ and all $t \in [0, T]$, one can find a constant $c_0 > 0$ such that

$$\begin{aligned}
|F(t, x)| &= |F(t, x) - F(t, 0)| = |\nabla F(t, \xi) \cdot x| = |\nabla F(t, \xi)||x| \\
&\leq c_0 (1 + |\xi|^{\alpha_0})|x| \leq c_0 |x| + c_0 \lambda^{\alpha_0} |x|^{\alpha_0 + 1},
\end{aligned}$$

where $\lambda \in [0, 1]$ is such that $\xi = \lambda x$. Therefore

$$\begin{aligned}
m_k = \varphi(u_k) &= \int_0^T \bigg[-\frac{1}{2}\, ({}_0^c D_t^\alpha u_k(t), {}_t^c D_T^\alpha u_k(t)) \bigg]\, dt - \int_0^T F(t, u_k(t))\, dt \\
&\geq \frac{|\cos(\pi\alpha)|}{2} \|u_k\|_\alpha^2 - \int_0^T F(t, u_k(t))\, dt \geq - \int_0^T F(t, u_k(t))\, dt \\
&\geq - \int_0^T \big[c_0 |u_k(t)| + c_0 \lambda^{\alpha_0} |u_k(t)|^{\alpha_0 + 1} \big]\, dt \geq -c_0 T (|b_k| + \lambda^{\alpha_0} |b_k|^{\alpha_0 + 1}).
\end{aligned}$$

Since $\lim_{k \to +\infty} b_k = 0$, we have $\lim_{k \to +\infty} m_k \geq 0$. Note that $m_k < 0$, hence $\lim_{k \to +\infty} m_k = 0$.

Finally, since u_k are local minima of φ, they are critical points of φ, thus weak solutions of (P). Due to Step 2, there are infinitely many distinct u_k with $\lim_{k \to +\infty} |u_k|_\infty = 0$. Moreover, we have

$$\begin{aligned}
\frac{|\cos(\pi\alpha)|}{2} \|u_k\|_\alpha^2 &\leq \int_0^T \bigg[-\frac{1}{2}\, ({}_0^c D_t^\alpha u_k(t),\ {}_t^c D_T^\alpha u_k(t)) \bigg]\, dt \\
&= m_k + \int_0^T F(t, u_k(t))\, dt \leq m_k + cT (|b_k| + \lambda^{\alpha_0} |b_k|^{\alpha_0 + 1}),
\end{aligned}$$

which means that $\lim_{k \to +\infty} \|u_k\|_\alpha = 0$. $\square$

Next, we prove Theorem 1.6 when the nonlinearity oscillates at infinity.

Proof of Theorem 1.6. For every fixed $k \in \mathbb{N}$, consider the set

$$T_k = \big\{ u \in E^\alpha : u(x) \neq 0 \text{ and } u(x) \in [0, b_k] e_k \text{ for a.e. } x \in \mathbb{R}^N \big\},$$

where b_k is from $\mathrm{H(F)}_2\,(5)$. The first part of the proof is similar to that of Theorem 1.3. Indeed, we can prove that the functional φ is bounded from below on T_k and its infimum on T_k is attained (see Step 1 of Theorem 1.3). Moreover, if $u_k \in T_k$ is chosen such that $\varphi(u_k) = \inf_{T_k} \varphi$, then $u_k(t) \in [0, a_k]e_k$ for almost every $t \in [0,T]$ (see Step 2 of Theorem 1.3), and u_k is a local minimum point of φ in E^α (see Step 3 of Theorem 1.3). Instead of Step 4, we prove

STEP 4′. Let $\vartheta_k = \inf_{T_k} \varphi = \varphi(u_k)$, then $\lim_{k\to+\infty} \vartheta_k = -\infty$.

From $\mathrm{H(F)}_2\,(4)$, we deduce that there exist $l_\infty > 0$ and $\lambda_\infty > 0$ such that

$$\operatorname*{essinf}_{t\in[0,T]} F(t,x) \geq -l_\infty |x|^2 \quad \text{for all } |x| > l_\infty. \tag{3.10}$$

There exists $L_\infty > 0$ large enough to enable

$$C(r_0,\alpha,T) + l_\infty T < L_\infty r_0. \tag{3.11}$$

From the right hand side of $\mathrm{H(F)}_2\,(4)$, there is a sequence $\{\xi_k\} \subset \mathbb{R}^N$ such that $\lim_{k\to+\infty} \xi_k = +\infty$ and

$$\operatorname*{esssup}_{t\in[0,T]} F(t,\xi_k) > L_\infty |\xi_k|^2 \quad \text{for all } k \in \mathbb{N}. \tag{3.12}$$

It is easy to see that

$$|\eta_{\xi_k}(t)| \leq |\xi_k| \quad \text{for all } t \in B_{r_0}(t_0) \setminus B_{r_0/2}(t_0),$$

since

$$\eta_{\xi_k}(t) = \frac{2\xi_k}{r_0}\,(r_0 - |t-t_0|) \quad \text{for all } t \in B_{r_0}(t_0) \setminus B_{r_0/2}(t_0).$$

Let $k \in \mathbb{N}$ be fixed and let $\eta_{\xi_k} \in E^\alpha$ be the function from (3.2) corresponding to the value $\xi_k \in \mathbb{R}^N$. Then $\eta_{\xi_k} \in T_{b_k}$, and on account of (3.10) and (3.12), we have

$$\begin{aligned}
\varphi(\eta_{\xi_k}) &= \int_0^T \Big[-\frac{1}{2}\,({}_0^cD_t^\alpha \eta_{\xi_k}(t),\ {}_t^cD_T^\alpha \eta_{\xi_k}(t)) \Big]\,dt - \int_0^T F(t,\eta_\xi(t))\,dt \\
&\leq \frac{1}{2|\cos(\pi\alpha)|}\,\|\eta_{\xi_k}\|_\alpha^2 - \int_{B_{r_0/2}(t_0)} F(t,\eta_{\xi_k}(t))\,dt \\
&\quad - \int_{(B_{r_0}(t_0)\setminus B_{r_0/2}(t_0))\cap\{t:|\eta_{\xi_k}(t)|>\lambda_\infty\}} F(t,\eta_{\xi_k}(t))\,dt \\
&\quad - \int_{(B_{r_0}(t_0)\setminus B_{r_0/2}(t_0))\cap\{t:|\eta_{\xi_k}(t)|\leq\lambda_\infty\}} F(t,\eta_{\xi_k}(t))\,dt \\
&\leq \frac{1}{2|\cos(\pi\alpha)|}\,\frac{4T^(3-2\alpha)}{\Gamma^2(1-\alpha)r_0^2(3-2\alpha)}\,|\xi_k|^2 \\
&\quad - L_\infty r_0|\xi_k|^2 + l_\infty T|\xi_k|^2 - cT(1+\lambda_\infty^{\alpha_0}) \\
&= (C(r_0,\alpha,T) - L_\infty + l_\infty T)|\xi_k|^2 + cT\lambda_\infty^{\alpha_0}.
\end{aligned} \tag{3.13}$$

From (3.11), (3.13) and $\lim_{k\to+\infty}\xi_k=+\infty$, we conclude that

$$\lim_{k\to+\infty}\varphi(\eta_{\xi_k})=-\infty. \tag{3.14}$$

On the other hand, from $\varphi(u_{m_k})=\min_{T_{b_{m_k}}}\varphi$, we have $\varphi(u_{m_k})\le\varphi(\eta_{\xi_k}(t))$. Therefore, on account of (3.14), we have

$$\lim_{k\to+\infty}\varphi(u_{m_k})=-\infty. \tag{3.15}$$

Since the sequence $\{\varphi(u_k)\}$ is non-increasing, so $\lim_{k\to+\infty}\vartheta_k=\lim_{k\to+\infty}\varphi(u_k)=-\infty$.

Step 5. In this step the fact $\lim_{k\to+\infty}|u_k|_\infty=\lim_{k\to+\infty}\|u_k\|_\alpha=+\infty$ is proved.

Arguing by contradiction, assume that there exists a subsequence $\{u_{n_k}\}$ of $\{u_k\}$ such that $|u_{n_k}|_\infty\le M$ for some $M>0$. In particular, $\{u_{n_k}\}\subset T_{b_l}$ for some $l\in\mathbb{N}$. Thus, for every $n_k>l$ we have

$$\vartheta_l\ge\vartheta_{n_k}=\inf_{T_{n_k}}\varphi=\varphi(u_{n_k})\ge\inf_{T_l}\varphi=\vartheta_l.$$

Consequently, $\vartheta_{n_k}=\vartheta_l$ for every $n_k>l$. This fact contradicts with (3.15), which completes the first part of the proof.

Next, we prove $\lim_{k\to+\infty}\|u_k\|_\alpha=+\infty$.

Note that $1<\alpha_0<+\infty$, then by Proposition 2.7, we have $E^\alpha\hookrightarrow C([0,T],\mathbb{R}^N)$ (compact embedding). Furthermore, there exists $c_2>0$ such that $|u_k|_\infty\le c_2\|u_k\|_\alpha$. Hence, there exists a constant $c_3>0$ such that

$$\begin{aligned}\int_0^T F(t,u_k(t))\,dt&\le\int_0^T c_0(1+|u_k(t)|^{\alpha_0})\,dt\le c_0T+c_0|u_k(t)|_\infty^{\alpha_0}T\\&\le c_0T+c_0c_2^{\alpha_0}\|u_k\|_\alpha^{\alpha_0}T\le c_0T+c_3\|u_k\|_\alpha^{\alpha_0}T.\end{aligned}$$

Let us assume that there exists a subsequence $\{u_{n_k}\}$ of $\{u_k\}$ such that for some $M>0$, we have $\|u_{n_k}\|_\alpha\le M$. In particular, due to the above inequality,

$$\begin{aligned}|\varphi(u_{n_k})|&=\bigg|\int_0^T\Big[-\frac12\,({}_0^cD_t^\alpha u_{n_k}(t),\ {}_t^cD_T^\alpha u_{n_k}(t))\Big]\,dt-\int_0^T F(t,u_{n_k}(t))\,dt\bigg|\\&\le\bigg|\int_0^T\Big[-\frac12\,({}_0^cD_t^\alpha u_{n_k}(t),\ {}_t^cD_T^\alpha u_{n_k}(t))\Big]\,dt\bigg|+\bigg|\int_0^T F(t,u_{n_k}(t))\,dt\bigg|\\&\le\frac{1}{2|\cos(\pi\alpha)|}\,\|u_{n_k}\|_\alpha^2+c_0T+c_3\|u_k\|_\alpha T\end{aligned}$$

is bounded. Hence $\vartheta_{n_k}=\varphi(u_{n_k})$ is also bounded. This fact contradicts with $\lim_{k\to+\infty}\vartheta_k=-\infty$. $\square$

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Manuscript received March 26, 2016

accepted June 21, 2016

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