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RELATIVE INDEX THEORIES AND APPLICATIONS

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Abstract. We develop some relative index theories for abstract operator equations. As applications, we prove a new Galerkin approximation formula and a new saddle point reduction formula for the P-index. We apply these new formulas to the minimal periodic problem for P-symmetric periodic solutions of nonlinear Hamiltonian systems.

1. Introduction

Many problems can be displayed as a self-adjoint operator equation

$$Au = F'(u), \quad u \in D(A) \subset H, \tag{O.E.}$$

where H is an infinite-dimensional separable Hilbert space, A is a self-adjoint operator on H with domain $D(A)$, F is a nonlinear functional on H. For example, the Dirichlet problem for Laplace's equation on bounded domain, periodic problem for periodic solutions of Hamiltonian systems, Schrödinger equations, periodic problem for periodic solutions of wave equations and so on. By the variational method, we know that the solutions of (O.E.) correspond to the critical points of a functional on a Hilbert space. For any critical point of a functional, one can define its so-called Morse index (may be infinite). In many cases with

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the help of Morse theory the relationship between the global and local behavior of the functional can be established. However in the case of the so-called strongly indefinite functionals, such as the functionals related to periodic solutions of first order Hamiltonian systems, Schrödinger equations, wave equations, etc., their Morse indices are infinite. Hence one needs to define some relative Morse indices which could replace the classical Morse index.

For example, basing on the analytic approach, by using a Galerkin approximation sequence, one can define a kind of relative Morse index, which can be used in place of the Morse index when dealing with variational problems, see e.g. [8], [18], [24], [35], [37], [44], etc. Similarly, by using the so-called saddle point reduction method (a kind of the Lyapunov–Schmidt procedure, see e.g. [1], [2] and [7]), one can define a kind of relative Morse index, which in many cases coincides with the relative Morse index defined via the Galerkin approximation method (cf. [35] for the case of symplectic paths related to the periodic solutions of Hamiltonian systems). In the case of convex Hamiltonian systems, due to the dual variational method and convex analysis theory (see e.g. [4], [14], [17]) one can define a Morse index for any critical point of the corresponding dual functionals (cf. [13]–[16]). In [42], Wang and the author developed an index theory for linear self-adjoint operator equation where the operator A in (O.E.) may contain a nonempty essential spectrum.

Basing on the algebraic approach, for a linear Hamiltonian system its fundamental solution is a path in a symplectic group starting from the identity. Here the symplectic group is defined as $\mathrm{Sp}(2n) = \{M \in \mathcal{L}(2n) : M^T J M = J\}$, $J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$, I_n is the $n \times n$ identity matrix. The set of symplectic paths starting from the identity is denoted by $\mathcal{P}_\tau(2n) = \{\gamma : \gamma \in C([0,\tau], \mathrm{Sp}(2n)), \gamma(0) = I_{2n}\}$. We say that a symplectic path $\gamma \in \mathcal{P}_\tau(2n)$ is non-degenerate if $\dim \ker_{\mathbf{C}}(\gamma(\tau) - I) = 0$. In 1984, Conley and Zehnder in [9] developed an index theory for the non-degenerate symplectic paths with $n \geq 2$. In 1990, Long and Zehnder in [36] generalized it to the non-degenerate case with $n = 1$. Long in [31], [32] and Viterbo in [41] extended this Maslov-type index theory to the degenerate case, they assigned a pair of integers $(i_1(\gamma), \nu_1(\gamma)) \in \mathbf{Z} \times \{0, 1, \ldots, 2n\}$ to $\gamma \in \mathcal{P}_\tau(2n)$. In [33], the index pair $(i_1(\gamma), \nu_1(\gamma))$ was further extended to an index function $(i_\omega(\gamma), \nu_\omega(\gamma)) \in \mathbf{Z} \times \{0, 1, \ldots, 2n\}$ with $\omega \in \mathbf{U} = \{z \in \mathbf{C} : |z| = 1\}$. It was proved that this index pair in fact coincides with the relative index pair defined via the Galerkin approximation method [35] and the relative index pair defined via the saddle point reduction method [35]. So in this case, the relative Morse index introduced via the analytic approach is the same as the index introduced via the algebraic approach.

For any $P \in \mathrm{Sp}(2n)$, the author in [24] and Dong in [10] independently and with different methods defined the so-called P-index pair $(i^P(\gamma), \nu^P(\gamma)) \in$

$\mathbf{Z} \times \{0, 1, \dots, 2n\}$. Tang and the author in [29], [30] defined the so-called (P, ω)-index pair $(i_\omega^P(\gamma), \nu_\omega^P(\gamma)) \in \mathbf{Z} \times \{0, 1, \dots, 2n\}$ and studied the iteration theory for the P-index pair.

In this paper, we first develop some relative index theories for abstract operator equation (O.E.) with A being a self-adjoint Fredholm operator in two directions: the Galerkin approximation scheme and the saddle point reduction scheme. Then we show that the relative indices along these two directions are equal, in Theorem 3.5. We apply these index theories to a special case in which the P-index is well defined. In Theorem 4.6 we prove a new Galerkin approximation formula and a new saddle point reduction formula for the P-index. As further application, in Theorem 5.1, we consider the minimal periodic problem for P-symmetric solutions of nonlinear Hamiltonian systems and prove a result which is an improvement of the main result of [30] (in [30] there was a restricting condition on the P-symmetric period, but in Theorem 5.1 it is dropped).

In his pioneer work [38], Rabinowitz posed a problem whether a superquadratic Hamiltonian system possesses a periodic solution with a prescribed minimal period. This question has been thoroughly studied by many mathematicians, we refer to [12], [14], [18]–[20], [28], [35]. In this paper, we consider the minimal periodic problem for the P-symmetric solution of a superquadratic Hamiltonian system with P-invariant Hamiltonian functions.

2. Relative index via the Galerkin approximation sequence

Let E be a separable Hilbert space, $A\colon E \to E$ be a bounded self-adjoint Fredholm linear operator and $B\colon E \to E$ be a compact self-adjoint linear operator. We set $Q = A - B$ and denote by $\mathcal{L}_{\rm cs}(E)$ the set of all compact self-adjoint linear operators on E. Suppose that $N = \ker Q$ and $\dim N < +\infty$. $Q|_{N^\perp}$ is invertible. $P\colon E \to N$ is the orthogonal projection. Suppose $\Gamma = \{P_m : m = 1, 2, \dots\}$ is the Galerkin approximation sequence of A with $P_m\colon E \to E_m$ satisfying:

(1) $E_m := P_m E$ is finite dimensional for all $m \in \mathbb{N}$,
(2) $P_m \to I$ strongly as $m \to +\infty$,
(3) $P_m A = A P_m$.

In applications, we usually have $E_m \subset E_{m+1}$ for all $m \in \mathbb{N}$.

For a self-adjoint operator T, we denote by $M^*(T)$ the eigenspaces of T with eigenvalues belonging to $(0, +\infty)$, $\{0\}$ and $(-\infty, 0)$ with $* = +, 0$ and $* = -$, respectively. We denote $m^*(T) = \dim M^*(T)$. Similarly, we choose $0 < d \le \|(Q|_{N^\perp})^{-1}\|^{-1}/4$ and denote by $M_d^*(T)$ the d-eigenspaces of T with eigenvalues belonging to $(d, +\infty)$, $(-d, d)$ and $(-\infty, -d)$ with $* = +, 0$ and $* = -$, respectively. We denote $m_d^*(T) = \dim M_d^*(T)$. For any self-adjoint operator L, we denote $L^\sharp = (L|_{\operatorname{Im} L})^{-1}$. In the following lemma we recall that $P_m(Q+P)P_m\colon E_m \to E_m$.

Lemma 2.1. *There exists $m_0 \in \mathbb{N}$ such that, for all $m \ge m_0$, there hold*

$$m^-(P_m(Q+P)P_m) = m_d^-(P_m(Q+P)P_m), \tag{2.1}$$

$$m^-(P_m(Q+P)P_m) = m_d^-(P_mQP_m). \tag{2.2}$$

Proof. We note that $\dim\ker(Q+P) = 0$. Consider the operators $Q+sP$ and $Q-sP$ for small $s > 0$, for example $s < \min\{1, d/2\}$, then there exists $m_1 \in \mathbb{N}$ such that

$$m_d^-(P_mQP_m) \le m^-(P_m(Q+sP)P_m), \tag{2.3}$$

$$m_d^-(P_mQP_m) \ge m^-(P_m(Q-sP)P_m) - m_d^0(P_mQP_m), \tag{2.4}$$

for all $m \ge m_1$. Indeed, (2.3) follows from

$$P_m(Q+sP)P_m = P_mQP_m + sP_mPP_m$$

and

$$(P_m(Q+sP)P_mx, x) \le -d\|x\|^2 + s\|x\|^2 \le -\frac{d}{2}\,\|x\|^2,$$

for $x \in M_d^-(P_mQP_m)$. Inequality (2.4) follows from

$$(P_mQP_mx, x) \le s(P_mPP_mx, x) < d\|x\|^2,$$

for $x \in M^-(P_m(Q-sP)P_m)$. From the Floquet theory, for $m \ge m_1$, we have

$$m_d^0(P_mQP_m) = \dim N = \dim \operatorname{Im}(P_mPP_m),$$

and as $\operatorname{Im}(P_mPP_m) \subseteq M_d^0(P_mQP_m)$ we have

$$\operatorname{Im}(P_mPP_m) = M_d^0(P_mQP_m).$$

It is easy to see that

$$M_d^0(P_mQP_m) \subseteq M_d^+(P_m(Q+sP)P_m).$$

Since $P_m(Q-sP)P_m = P_m(Q+sP)P_m - 2sP_mPP_m$, we have

$$m^-(P_m(Q-sP)P_m) \ge m^-(P_m(Q+sP)P_m) + m_d^0(P_mQP_m), \tag{2.5}$$

for all $m \ge m_1$. Now (2.2) follows from (2.3)–(2.5). □

Since $M^-(Q+P) = M^-(Q)$ and the two operators $Q+P$ and Q have the same negative spectrum, moreover, $P_m(Q+P)P_m \to Q+P$ and $P_mQP_m \to Q$ strongly, one can prove (2.2) applying the spectrum decomposition theory.

Lemma 2.2. *Let $B \in \mathcal{L}_{\mathrm{cs}}(E)$. Then $m_d^0(P_m(A-B)P_m)$ eventually becomes a constant independent of m and, for m large enough, there holds*

$$m_d^0(P_m(A-B)P_m) = m^0(A-B). \tag{2.6}$$

Proof. It is easy to show that there is a constant $m_1 > 0$ such that $\dim P_m \ker(A-B) = \dim\ker(A-B)$, for $m \geq m_1$. Since B is compact, there is $m_2 \geq m_1$ such that $\|(I-P_m)B\| \leq 2d$ for $m \geq m_2$.

Take $m \geq m_2$, let $E_m = P_m \ker(A-B) \oplus Y_m$, then $Y_m \subseteq \operatorname{Im}(A-B)$. For $y \in Y_m$, we have

$$y = (A-B)^\sharp(A-B)y = (A-B)^\sharp(P_m(A-B)P_m y + (I-P_m)By).$$

It implies

$$\|P_m(A-B)P_m y\| \geq 2d\|y\|, \quad \text{for all } y \in Y_m.$$

Thus we have

$$m_d^0(P_m(A-B)P_m) \leq m^0(A-B). \tag{2.7}$$

On the other hand, for $x \in P_m \ker(A-B)$, there exists $y \in \ker(A-B)$ such that $x = P_m y$. Since $P_m \to I$ strongly, there exists $m_3 \geq m_2$ such that, for $m \geq m_3$, there hold

$$\|I-P_m\| < \frac{1}{2}, \qquad P_m(A-B)(I-P_m) \leq \frac{d}{2}.$$

So we have

$$\|P_m(A-B)P_m x\| = \|P_m(A-B)(I-P_m)y\| \leq \frac{d}{2}\,\|y\| < d\|x\|.$$

Thus

$$m_d^0(P_m(A-B)P_m) \geq m^0(A-B). \tag{2.8}$$

Now (2.6) due to (2.7) and (2.8). □

From the above proof, we see that for any two operators $B_1, B_2 \in \mathcal{L}_{\mathrm{cs}}(E)$, there exist $d, m^* > 0$ such that

$$m_d^0(P_m(A-B(s))P_m) = m^0(A-B(s)), \quad m > m^*,$$

where $B(s) = (1-s)B_1 + sB_2$, $s \in [0,1]$.

Theorem 2.3. *For any two operators $B_1, B_2 \in \mathcal{L}_{\mathrm{cs}}(E)$ with $B_1 < B_2$, there is $m^* > 0$ such that*

$$m_d^-(P_m(A-B_2)P_m) - m_d^-(P_m(A-B_1)P_m) = \sum_{s\in[0,1)} m^0(A-B(s)), \tag{2.9}$$

for $m > m^*$.

Proof. We can understand $P_m(A-B(s))P_m$ as a continuous symmetric matrix function defined on $s \in [0,1]$. So we can determine (at least locally) some continuous spectral lines for this continuous operator path. Denote

$$\begin{aligned} m_d^-(s) &= m_d^-(P_m(A-B(s))P_m), \\ m_d^0(s) &= m_d^0(P_m(A-B(s))P_m) = m^0(A-B(s)). \end{aligned}$$

If $m^0(A - B(s_0)) = 0$, then there is a neighbourhood $B(s_0,\delta)$ of s_0 such that for $s \in B(s_0,\delta)$, $m_d^0(P_m(A - B(s))P_m) = m^0(A - B(s)) = 0$. Thus $m_d^-(s)$ is constant in $B(s_0,\delta)$.

If $m^0(A-B(s_0)) \neq 0$, we claim that $m_d^-(s_0+0) - m_d^-(s_0) = \nu(s_0)$. Indeed, on one hand, by the continuity of the eigenvalue of a continuous operator function, we have $m^-(s_0+0) - m^-(s_0) \leq \nu(s_0)$. On the other hand, since $(A - B(s_0)) > (A - B(s)))$, for $s_0 < s$, we see that $m^0(A - B(s)) = 0$, for $s > s_0$, but $s - s_0$ is small. So $m_d^0(s) = m_d^0(P_m(A - B(s))P_m) = 0$. Since $P_m(A - B(s_0))P_m \geq P_m(A - B(s))P_m$, we have

$$m_d^-(s_0+0) + m_d^0(s_0+0) \geq m_d^-(s_0) + m_d^0(s_0).$$

Thus the claim is true. Therefore we have equality (2.9). □

LEMMA 2.4. *Let $B \in \mathcal{L}_{\mathrm{cs}}(E)$. Then the difference of the d-Morse indices*

$$m_d^-(P_m(A - B)P_m) - m_d^-(P_m A P_m) \tag{2.10}$$

eventually becomes a constant independent of m, where $d > 0$ is determined by the operators A and $A - B$.

A similar result was proved in [8].

PROOF. We can choose $B_0 < 0$ and $B_0 < B$, so that we have

$$\begin{aligned} m_d^-(P_m(A - B)P_m) &- m_d^-(P_m A P_m) \\ &= m_d^-(P_m(A - B)P_m) - m_d^-(P_m(A - B_0)P_m) \\ &\quad - (m_d^-(P_m A P_m) - m_d^-(P_m(A - B_0)P_m)). \end{aligned}$$ □

DEFINITION 2.5. For a bounded self-adjoint Fredholm operator A with a Galerkin approximation sequence Γ and a self-adjoint compact operator B on Hilbert space E, we define the *relative index* by

$$I(A, A - B) = m_d^-(P_m(A - B)P_m) - m_d^-(P_m A P_m), \quad m \geq m^*, \tag{2.11}$$

where $m^* > 0$ is a constant large enough such that the difference in (2.10) becomes a constant independent of $m \geq m^*$.

By Lemma 2.4 we have the following

REMARK 2.6. Let $\widetilde{E}$ be another separable Hilbert space, $\widetilde{A}$ be a linear self-adjoint Fredholm operator on $\widetilde{E}$ and B be a compact linear self-adjoint operator on $\widetilde{E}$. There holds

$$I(A \oplus \widetilde{A}, (A \oplus \widetilde{A}) - (B \oplus \widetilde{B})) = I(A, A - B) + I(\widetilde{A}, \widetilde{A} - \widetilde{B}),$$

where $(A \oplus \widetilde{A})(x \oplus y) = Ax \oplus \widetilde{A}y$ and $(B \oplus \widetilde{B})(x \oplus y) = Bx \oplus \widetilde{B}y$ for $x \oplus y \in E \oplus \widetilde{E}$.

The spectral flow for a parameter family of linear self-adjoint Fredholm operators was introduced by Atiyah, Patodi and Singer in [3]. The following result shows that the relative index in Definition 2.5 is a spectral flow. It is obvious that $A_s = A - sB$, $s \in [0,1]$, is admissible in the sense of Definition 2.3 of [44].

Lemma 2.7. *For the operators A and B in Definition 2.5, there holds*

$$I(A, A-B) = -\mathrm{sf}\{A - sB,\ 0 \le s \le 1\}, \tag{2.12}$$

where $\mathrm{sf}\{A - sB,\ 0 \le s \le 1\}$ *is the* spectral flow *of the operator family $A - sB$, $s \in [0,1]$ (cf. [44]).*

Proof. For simplicity, we set $I_{\mathrm{sf}}(A, A-B) = -\mathrm{sf}\{A-sB,\ 0 \le s \le 1\}$ which is exactly the relative Morse index defined in [44]. By the Galerkin approximation formula in Theorem 3.1 of [44],

$$I_{\mathrm{sf}}(A, A-B) = I_{\mathrm{sf}}(P_m A P_m,\ P_m(A-B)P_m) \tag{2.13}$$

if $\ker A = \ker(A-B) = 0$, where m is big enough.

By (2.17) from [44], we have

$$\begin{aligned} I_{\mathrm{sf}}(P_m A P_m,\ P_m(A-B)P_m) &= m^-(P_m(A-B)P_m) - m^-(P_m A P_m) \\ &= m_d^-(P_m(A-B)P_m) - m_d^-(P_m A P_m) = I(A, A-B), \end{aligned} \tag{2.14}$$

for $d > 0$ small enough. Hence (2.12) holds in the non-degenerate case. In general, if $\ker A \ne 0$ or $\ker(A-B) \ne 0$, we can choose $d > 0$ small enough such that $\ker(A + d\,\mathrm{Id}) = \ker(A - B + d\,\mathrm{Id}) = 0$, here $\mathrm{Id}\colon E \to E$ is the identity operator. By (2.14) from [44], we have

$$\begin{aligned} I_{\mathrm{sf}}(A, A-B) &= I_{\mathrm{sf}}(A, A + d\,\mathrm{Id}) \\ &\quad + I_{\mathrm{sf}}(A + d\,\mathrm{Id}, A - B + d\,\mathrm{Id}) + I_{\mathrm{sf}}(A - B + d\,\mathrm{Id}, A - B) \\ &= I_{\mathrm{sf}}(A + d\,\mathrm{Id}, A - B + d\,\mathrm{Id}) = I(A + d\cdot\mathrm{Id}, A - B + d\cdot\mathrm{Id}) \\ &= m^-(P_m(A - B + d\,\mathrm{Id})P_m) - m^-(P_m(A + d\,\mathrm{Id})P_m) \\ &= m_d^-(P_m(A-B)P_m) - m_d^-(P_m A P_m) = I(A, A-B). \end{aligned} \tag{2.15}$$

In the second equality of (2.15) used the fact that

$$I_{\mathrm{sf}}(A, A + d\,\mathrm{Id}) = I_{\mathrm{sf}}(A - B + d\,\mathrm{Id}, A - B) = 0 \quad \text{for } d > 0 \text{ small enough},$$

since the spectrum of A is discrete and B is a compact operator, in the third and the forth equalities of (2.15) we have applied (2.14). □

A similar approach to definition of the relative index of two operators appeared in [8]. A different one can be found in [18].

3. Relative index via the saddle point reduction

Let H be a Hilbert space with the inner product $(\cdot\,,\cdot)_H$ and norm $\|\cdot\|_H$. Let A be a self-adjoint linear operator with compact resolvent and dense domain $D(A)\subset H$ (for short, $A\in\mathcal{O}(H)$). Let B be a bounded self-adjoint linear operator on H (for short, $B\in\mathcal{L}_{\rm bs}(H)$) with its operator norm $\|B\|_H<c$, $\pm c\notin\sigma(A)$. Denote by $N=\ker A$ the kernel of A and by $P_0=H\to N$ the projection. We set $\widetilde{A}=A+P_0$. Denote by E_λ the spectral resolution of the self-adjoint operator $\widetilde{A}$ and define the following projections on H:

$$\mathcal{P}=\int_{-c}^{c}dE_\lambda,\qquad \mathcal{P}^+=\int_{c}^{+\infty}dE_\lambda,\qquad \mathcal{P}^-=\int_{-\infty}^{-c}dE_\lambda.$$

The Hilbert space H possesses an orthogonal decomposition

$$H=H^+\oplus H^-\oplus X,$$

where $H^\pm=\mathcal{P}^\pm H$ and $X=\mathcal{P}H$ is a finite dimensional space. Consider the quadratic functional

$$f(z)=\frac{1}{2}\,((A-B)z,z)_H,\quad z\in D(A)\subset H.$$

The following theorem is a kind of the saddle point reduction for this quadratic functional. Its proof is much simpler than the general cases (see [35] for the functionals related with nonlinear Hamiltonian systems). It transfers the infinitely dimensional problem to a finitely dimensional problem. In the finitely dimensional case, the Morse index is well defined.

THEOREM 3.1. *There exist a function $a\in C^2(X,\mathbb{R})$ and a linear map $u\colon X\to H$ satisfying the following conditions*:

(a) *the map u has the form $u(x)=w(x)+x$ with $\mathcal{P}w(x)=0$;*

(b) *the function a satisfies*

$$a(x)=f(u(x))=\frac{1}{2}\,((A-B)u(x),u(x))_H=\frac{1}{2}\,((A-B')x,x)_H,$$

where $B'\colon X\to X$ is defined in (3.4) *below*;

(c) *$x\in X$ is a critical point of a if and only if $z=u(x)$ is a critical point of f, i.e. $z=u(x)\in\ker(A-B)$.*

PROOF. We follow the ideas of [42]. Denote $E=D(|\widetilde{A}|^{1/2})$. Since $0\notin\sigma(\widetilde{A})$, E is a Hilbert space with the inner product $(\cdot\,,\cdot)_E$ and corresponding norm $\|\cdot\|_E$ defined by

$$(x,y)_E:=(|\widetilde{A}|^{1/2}x,|\widetilde{A}|^{1/2}y)_H,\quad \text{for all } x,y\in E,$$
$$\|x\|_E^2:=(x,x)_E,\qquad \text{for all } x\in E.$$

We also have the following decomposition:

$$E=E_0\oplus E_1, \tag{3.1}$$

with $E_0 = E \cap X$ and $E_1 = E \cap (H^+ \cup H^-)$. Consider the following bounded self-adjoint operators $\overline{A}$ and $\overline{B}$ on E:

$$(\overline{A}x, y)_E := (Ax, y)_H, \quad \text{for all } x, y \in E,$$
$$(\overline{B}x, y)_E := (Bx, y)_H, \quad \text{for all } x, y \in E.$$

It is easy to see that $\overline{A} = |\widetilde{A}|^{-1}A$ and $\overline{B} = |\widetilde{A}|^{-1}B$. Thus

$$\ker(\overline{A} - \overline{B}) = \ker(A - B).$$

Furthermore, we can write $\overline{A}$ and $\overline{B}$ in the following block form:

$$\overline{A} = \begin{pmatrix} \overline{A}_1 & 0 \\ 0 & \overline{A}_2 \end{pmatrix}, \qquad \overline{B} = \begin{pmatrix} \overline{B}_{11} & \overline{B}_{12} \\ \overline{B}_{21} & \overline{B}_{22} \end{pmatrix}, \tag{3.2}$$

with respect to decomposition (3.1). For any $u \in E$, $u = x + y$ with $x \in E_0$ and $y \in E_1$, the equation $\overline{A}u = \overline{B}u$ can be rewritten as

$$\begin{pmatrix} \overline{A}_1 & 0 \\ 0 & \overline{A}_2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \overline{B}_{11} & \overline{B}_{12} \\ \overline{B}_{21} & \overline{B}_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

That is

$$\begin{cases} \overline{A}_1 x = \overline{B}_{11}x + \overline{B}_{12}y, \\ \overline{A}_2 y = \overline{B}_{21}x + \overline{B}_{22}y. \end{cases}$$

From the definitions of $\mathcal{P}, \mathcal{P}^{\pm}, \overline{A}_2$ and $\overline{B}_{22}$, it is easy to see that $\overline{A}_2$ is invertible on E_1 and $\|\overline{A}_2^{-1}\overline{B}_{22}\|_E < 1$. Thus we have $y = w(x) = (\overline{A}_2 - \overline{B}_{22})^{-1}\overline{B}_{21}x$, and

$$\overline{A}u - \overline{B}u = 0 \quad \Leftrightarrow \quad \overline{A}_1 x - [\overline{B}_{11} + \overline{B}_{12}(\overline{A}_2 - \overline{B}_{22})^{-1}\overline{B}_{21}]x = 0. \tag{3.3}$$

So $u(x) = x + w(x) = x + (\overline{A}_2 - \overline{B}_{22})^{-1}\overline{B}_{21}x$ satisfies all the required properties with B' defined as

$$B' = \overline{B}_{11} + \overline{B}_{12}(\overline{A}_2 - \overline{B}_{22})^{-1}\overline{B}_{21}. \tag{3.4}$$

□

We note that in the nonlinear case, the same result is true. But in the proof one should use the contraction mapping principle and the implicit function theorem. We refer to the papers [1], [2], [6], [7] and [35] for general settings.

Definition 3.2. For any $B \in \mathcal{L}_{\mathrm{bs}}(H)$ with $\|B\|_H < c$, we define

$$\mu_A^c(B) = m^-(\overline{A}|_{E_0} - B'), \qquad \nu_A(B) = \dim\ker(A - B).$$

Theorem 3.3. *For any two operators $B_1, B_2 \in \mathcal{L}_{\mathrm{bs}}(H)$ with $\|B_i\|_H < c$, $i = 1, 2$, and $B_1 < B_2$, there holds*

$$\mu_A^c(B_2) - \mu_A^c(B_1) = \sum_{s \in [0,1)} \nu_A((1-s)B_1 + sB_2). \tag{3.5}$$

PROOF. We set $B_s = (1-s)B_1 + sB_2$, $i(s) = \mu_A^c((1-s)B_1 + sB_2)$, $\nu(s) = \nu_A^c((1-s)B_1 + sB_2)$ and

$$a_s(x) = \frac{1}{2}\,((\overline{A}_1 - B_s')x, x)_E,$$

where $B_s' = ((1-s)\overline{B}_1 + s\overline{B}_2)_{11} + ((1-s)\overline{B}_1 + s\overline{B}_2)_{12}(\overline{A}_2 - ((1-s)\overline{B}_1 + s\overline{B}_2)_{22})^{-1}((1-s)\overline{B}_1 + s\overline{B}_2)_{21}$. We denote $b(s) = \overline{A}_1 - B_s'$.

For any $s_0 \in [0,1]$, if $\nu(s_0) = 0$, that is to say $b(s_0)$ has a zero nullity subspace of E_0, due to continuous dependence of the quadratic function a_s on s, there exists a neighbourhood $U(s_0)$ of s_0 in $[0,1]$ such that

$$i(s) = i(s_0) \quad \text{and} \quad \nu(s) = \nu(s_0) = 0, \quad \text{for all } s \in U(s_0). \tag{3.6}$$

If $\nu(s_0) \neq 0$, we have the following decomposition: $E_0 = E_0^- \oplus E_0^0 \oplus E_0^+$ such that $b(s_0)$ is negative definite, zero and positive definite on E_0^-, E_0^0 and E_0^+, respectively. For any $x_0 \in \ker b(s_0)$ with $\|x_0\| = 1$, that is $b(s_0)x_0 = 0$, define a smooth function $a(s)\colon [0,1] \to \mathbb{R}$ by

$$a(s) := (b(s)x_0, x_0)_E.$$

We have $a(s_0) = 0$. From the definition of $b(s)$ and denoting $\xi(s) := (\overline{A}_2 - \overline{B}_{22}(s))^{-1}\overline{B}_{21}(s)$ for simplicity, we have

$$a(s) = \big((\overline{A} - \overline{B}(s))(x_0 + \xi(s)x_0), (x_0 + \xi(s)x_0)\big)_E,$$

and

$$(\overline{A} - \overline{B}(s_0))(x_0 + \xi(s_0)x_0) = 0.$$

So,

$$\begin{aligned} a'(s_0) &= -\big(\overline{B}'(s_0)(x_0 + \xi(s_0)x_0), (x_0 + \xi(s_0)x_0)\big)_E \\ &= -\big((B_2 - B_1)(x_0 + \xi(s_0)x_0), (x_0 + \xi(s_0)x_0)\big)_H. \end{aligned}$$

Since $B_1 < B_2$ and $x_0 \neq 0$, we have $a'(s_0) < 0$. Summing up, there exists $\delta > 0$ such that $a(s) < 0$ for any $s \in (s_0, s_0 + \delta)$. So from the continuity of $b(s)$, there exists $\overline{\delta} \leq \delta$ such that

$$\begin{aligned} (b(s)x, x)_E &< 0, \quad \text{for all } x \in E_0^- \oplus E_0^0,\ s \in (s_0, s_0 + \overline{\delta}), \\ (b(s)x, x)_E &> 0, \quad \text{for all } x \in E_0^0, \qquad s \in (s_0 - \overline{\delta}, s_0), \\ (b(s)x, x)_E &> 0, \quad \text{for all } x \in E_0^+, \qquad s \in (s_0, s_0 + \overline{\delta}). \end{aligned}$$

That is to say

$$i(s) = i(s_0) + \nu(s_0) \quad \text{and} \quad \nu(s) = 0, \quad \text{for all } s \in (s_0, s_0 + \overline{\delta}), \tag{3.7}$$

$$i(s) = \nu(s_0) \quad \text{and} \quad \nu(s) = 0, \quad \text{for all } s \in (s_0 - \overline{\delta}, s_0). \tag{3.8}$$

So from (3.6), (3.7) and (3.8), we have

$$\mu_A^c(B_2) - \mu_A^c(B_1) = \sum_{s\in[0,1)} \nu_A((1-s)B_1 + sB_2). \qquad \square$$

Remark that $B_s = (1-s)B_1 + sB_2 = B_1 + s(B_2 - B_1)$ satisfies $B_s' = B_2 - B_1 > 0$, i.e. B_s is monotonically dependent on $s \in [0,1]$. So if we replace B_s with another operator path $\mathcal{B}(s)$ satisfying $\mathcal{B}(0) = B_1$, $\mathcal{B}(1) = B_2$, and $\mathcal{B}'(s) > 0$, then the same result is still true. Namely we have

$$\mu_A^c(B_2) - \mu_A^c(B_1) = \sum_{s\in[0,1)} \nu_A(\mathcal{B}(s)). \tag{3.9}$$

From Theorem 3.3, we know that $\mu_A^c(B_2) - \mu_A^c(B_1)$ is independent of c for any $B_1, B_2 \in \mathcal{L}_{\mathrm{bs}}(H)$ with $c > \max\{\|B_1\|_H, \|B_2\|_H\}$. In fact, for any two such operators, we can choose an operator $B_0 \in \mathcal{L}_{\mathrm{bs}}(H)$ such that $B_0 < B_i$, $i = 1,2$, and $\|B_0\|_H < c$. Then we have

$$\mu_A^c(B_2) - \mu_A^c(B_1) = \mu_A^c(B_2) - \mu_A^c(B_0) - (\mu_A^c(B_1) - \mu_A^c(B_0)),$$

which is independent of c.

Definition 3.4. For any $B \in \mathcal{L}_{\mathrm{bs}}(H)$, we define

$$\mu_A(B) = \mu_A^c(B) - \mu_A^c(0), \quad c > \|B\|_H. \tag{3.10}$$

So the index pair $(\mu_A(B), \nu_A(B))$ is well defined.

Now formula (3.5) can be written as

$$\mu_A(B_2) - \mu_A(B_1) = \sum_{s\in[0,1)} \nu_A((1-s)B_1 + sB_2). \tag{3.11}$$

From Theorems 2.3 and 3.3, we have the following result.

Theorem 3.5. *Suppose that both the indices $I(A, A-B)$ and $\mu_A(B)$ are well defined for the operator pair (A,B). Then we have*

$$I(A, A-B) = \mu_A(B). \tag{3.12}$$

Proof. Firstly, we claim that for the positively definite operator $B > 0$, (3.12) is true. Indeed, since $I(A, A-0) = \mu_A(0) = 0$, there holds

$$I(A, A-B) = \sum_{s\in[0,1)} m^0(A - sB) = \sum_{s\in[0,1)} \nu_A(sB) = \mu_A(B).$$

In general, we choose a positively definite operator B_0 such that $B < B_0$, so we have

$$I(A, A-B_0) - I(A, A-B) = \sum_{s\in[0,1)} m^0(A - (1-s)B - sB_0) = \mu_A(B_0) - \mu_A(B).$$

Therefore from $I(A, A-B_0) = \mu_A(B_0)$, we have the desired equality (3.12). $\square$

4. The P-Maslov type index theory

Let $B \in C([0,\tau],\mathcal{L}_{\rm s}(2n))$ be a continuous symmetric $2n\times 2n$ matrix valued function, where we have denoted the set of symmetrical $2n\times 2n$ matrices by $\mathcal{L}_{\rm s}(2n)$. For $\omega \in \mathbf{U} = \{z\in\mathbb{C} : |z| = 1\}$, we denote by $(i_\omega(B),\nu_\omega(B)) = (i_\omega(\gamma_B),\nu_\omega(\gamma_B))$ the ω-index of γ_B which was defined by Long in [33] (see also [35]), where γ_B is the fundamental solution of the linear Hamiltonian system $\dot{x}(t) = JB(t)x(t)$, $J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$, I_n is the $n\times n$ identity matrix. Let $\gamma_B\colon [0,\tau]\to \mathrm{Sp}(2n)$ be a symplectic path satisfying $\gamma_B(0) = I_{2n}$, where $\mathrm{Sp}(2n)$ is the symplectic group defined as $\mathrm{Sp}(2n) = \{M\in\mathcal{L}_{\rm s}(2n) : M^TJM = J\}$.

Denote the set of all symplectic paths starting from I_{2n} by $\mathcal{P}_\tau(2n)$, i.e. $\mathcal{P}_\tau(2n) = \{\gamma : \gamma\in C([0,\tau],\mathrm{Sp}(2n)),\ \gamma(0) = I_{2n}\}$. The definition of the ω-nullity $\nu_\omega(\gamma_B)$ is very simple:

$$\nu_\omega(\gamma_B) = \dim_{\mathbb{C}} \ker_{\mathbb{C}}(\gamma_B(\tau) - \omega\cdot I_{2n}).$$

But the definition of the part $i_\omega(\gamma_B)$ is somewhat complicated. Roughly speaking it is the algebraic intersection number of the symplectic path $\gamma_B(t)$, $t\in[0,\tau)$, with the ω-singular set $\mathrm{Sp}^0(2n) := \{M\in\mathrm{Sp}(2n) : \mathrm{detSp(2n)}(M-\omega\cdot I_{2n}) = 0\}$ (see [33] or [35] for details). For the ω-index $i_\omega(B)$, we have the following result.

LEMMA 4.1. *Suppose $B_0, B_1\in C(\mathbb{R},\mathcal{L}_{\rm s}(2n))$ such that $B_0 < B_1$ and $B_i(t+\tau) = B_i(t)$, $i = 0,1$, then there holds*

$$i_\omega(B_1) - i_\omega(B_0) = \sum_{s\in[0,1)} \nu_\omega((1-s)B_0 + sB_1). \tag{4.1}$$

PROOF. By the saddle point reduction formula for the ω-index (see [35, p. 134, Theorem 6.1.1], there holds

$$m^-(B_i) = d_\omega + i_\omega(B_i),\quad i = 0,1, \tag{4.2}$$

where $2d_\omega = \dim_{\mathbb{C}} Z^\omega$ is the dimension of the truncation space and $m^-(B)$ is the Morse index of the reduction functional

$$a_{B,\omega} = f_\omega(u_\omega(z)),\qquad f_\omega(y) = \frac{1}{2}\,\langle (A-B)y,y\rangle,\qquad A = -J\,\frac{d}{dt}.$$

Therefore, due to the boundary condition from the definition of the operators A, B_i in Section 3 there holds

$$i_\omega(B_1) - i_\omega(B_0) = m^-(B_1) - m^-(B_0) = \mu_A^c(B_1) - \mu_A^c(B_0).$$

The remainder is the same as the proof of Theorem 3.3 with the nullity in (3.5) replaced by the ω-nullity. □

The proof here looks in some sense a bit clumsy and unclear, but formula (4.2) in fact is a result of the saddle point reduction and the relative Morse index defined in (3.10) with the ω-boundary condition in the function space (there maybe a constant difference). So essentially formula (4.1) is nothing but (3.11) with a special boundary condition on the function space.

For $\tau > 0$ and any two paths $f\colon [0,\tau] \to \mathrm{Sp}(2n)$ and $g\colon [0,\tau] \to \mathrm{Sp}(2n)$ with $f(\tau) = g(0)$, we define their joint path by

$$g * f(t) = \begin{cases} f(2t), & 0 \le t \le \tau/2, \\ g(2t-\tau), & \tau/2 \le t \le \tau. \end{cases}$$

Definition 4.2 (see [29]). For any $\tau > 0$, $\omega \in \mathbf{U}$, $P \in \mathrm{Sp}(2n)$ and $\gamma \in \mathcal{P}_\tau(2n)$, we define the *Maslov (P,ω)-index* as

$$i^P_\omega(\gamma) = i_\omega(P^{-1}\gamma * \xi) - i_\omega(\xi), \qquad \nu^P_\omega(\gamma) = \dim_{\mathbb{C}} \ker_{\mathbb{C}}(\gamma(\tau) - \omega P), \tag{4.3}$$

where $\xi \in \mathcal{P}_\tau(2n)$ is such that $\xi(\tau) = P^{-1}\gamma(0) = P^{-1}$.

Note that the index $i^P_\omega(\gamma)$ is well defined, it does not depend on the choice of $\xi \in \mathcal{P}_\tau(2n)$.

For any $B \in C(\mathbb{R}, \mathcal{L}_{\mathrm{s}}(2n))$ with $B(t+\tau) = (P^{-1})^T B(t) P^{-1}$, we define its (P,ω)-index as $(i^P_\omega(B), \nu^P_\omega(B)) = (i^P_\omega(\gamma_B), \nu^P_\omega(\gamma_B))$. Here $\gamma_B \in \mathcal{P}(2n)$ is the fundamental solution of the linear system $\dot z(t) = JB(t)z(t)$. We write $(i(\gamma), \nu(\gamma)) = (i_1(\gamma), \nu_1(\gamma))$ and $(i^P(\gamma), \nu^P(\gamma)) = (i^P_1(\gamma), \nu^P_1(\gamma))$ for $\omega = 1$. For the iteration paths γ^k defined in [33], we write $(i(\gamma,k), \nu(\gamma,k)) = (i(\gamma^k), \nu(\gamma^k))$. In [29], it was claimed that if $P = I$, there holds $(i^I_\omega(\gamma), \nu^I_\omega(\gamma)) = (i_\omega(\gamma), \nu_\omega(\gamma))$ for $\omega \ne 1$, and $(i^I(\gamma), \nu^I(\gamma)) = (i(\gamma) + n, \nu(\gamma))$ in the case of $\omega = 1$. So the (I,ω)-index is the classical Maslov ω-index defined by Long in [33] (there maybe a constant difference, see also [35]).

For $P \in \mathrm{Sp}(2n)$, we know that there exists a unique polar decomposition $P = AU$, where $A = \exp(M_1)$, M_1 satisfies

$$M_1^T J + J M_1 = 0 \quad \text{and} \quad M_1^T = M_1, \tag{4.4}$$

U is a symplectic orthogonal matrix. $\mathrm{Sp}(2n) \cap O(2n)$ is a connected compact Lie group and its Lie algebra is $\mathrm{Sp}(2n) \cap o(2n)$ constituted by the matrices M_2 satisfying

$$M_2^T J + J M_2 = 0 \quad \text{and} \quad M_2^T + M_2 = 0. \tag{4.5}$$

Then there exists a matrix $M_2 \in \mathrm{Sp}(2n) \cap o(2n)$ such that $U = \exp(M_2)$. So P takes the form $P = \exp(M_1)\exp(M_2)$. We set $\gamma^P(t) = \exp(tM_1/\tau)\exp(tM_2/\tau)$. It is clear that $\gamma^P(0) = I_{2n}$ and $\gamma^P(\tau) = P$.

Lemma 4.3. *Suppose $B \in C(\mathbb{R}, \mathcal{L}_s(2n))$ satisfies $B(t+\tau) = (P^{-1})^T B(t) P^{-1}$, then there holds*

$$\nu^P_\omega(B) = \nu_\omega(\widetilde{B}), \tag{4.6}$$

where $\widetilde{B}(t) = \gamma^P(t)^T J \dot\gamma^P(t) + \gamma^P(t)^T B(t) \gamma^P(t)$.

PROOF. It is easy to check that the fundamental solution of the linear Hamiltonian system $\dot{x}(t) = J\widetilde{B}(t)x(t)$ is the following symplectic path $\gamma_2(t) := \gamma^P(t)^{-1}\gamma_B(t)$ with γ_B the fundamental solution of $\dot{x}(t) = JB(t)x(t)$. But $\gamma_2(\tau) = \gamma^P(\tau)^{-1}\gamma_B(\tau) = P^{-1}\gamma_B(\tau)$. Thus by definition, there holds

$$\nu_\omega^P(B) = \dim\ker(\gamma_B(\tau) - \omega P) = \dim\ker(P^{-1}\gamma_B(\tau) - \omega I_{2n}) = \nu_\omega(\widetilde{B}). \qquad \square$$

The following result was proved in [29].

LEMMA 4.4. *Let $\gamma_B, \gamma^P, \gamma_2 \in \mathcal{P}_\tau(2n)$ be defined as above, then there holds*

$$i_\omega^P(\gamma_B) - i_\omega^P(\gamma^P) = \begin{cases} i_\omega(\gamma_2), & \omega \neq 1, \\ i_\omega(\gamma_2) + n, & \omega = 1. \end{cases} \tag{4.7}$$

Thus the number $i_\omega(\gamma_2) + i_\omega^P(\gamma^P)$ depends only on P but not on the choice of M_1 and M_2, where M_1 and M_2 are appearing in $P = \exp(M_1)\exp(M_2)$.

LEMMA 4.5. *Suppose $B_0, B_1 \in C(\mathbb{R}, \mathcal{L}_s(2n))$ satisfy*

$$B(t+\tau) = (P^{-1})^T B(t) P^{-1} \quad \textit{and} \quad B_0 < B_1,$$

then there holds

$$i_\omega^P(B_1) - i\omega^P(B_0) = \sum_{s\in[0,1)} \nu\omega^P((1-s)B_0 + sB_1). \tag{4.8}$$

PROOF. From Lemma 4.4, we have

$$i_\omega^P(B_1) - i_\omega^P(B_0) = i_\omega(\widetilde{B}_1) - i_\omega(\widetilde{B}_0) = \sum_{s\in[0,1)} \nu_\omega((1-s)\widetilde{B}_0 + s\widetilde{B}_1).$$

We see that $(1-s)\widetilde{B}_0(t) + s\widetilde{B}_1(t) = \gamma^P(t)^T J\dot{\gamma}^P(t) + \gamma^P(t)^T[(1-s)B_0(t) + sB_1(t)]\gamma^P(t) = \widetilde{B}_s(t)$ with $B_s(t) = (1-s)B_0(t) + sB_1(t)$. Now from Lemma 4.3, we get the result. $\square$

Let in the following theorem the operator A be defined in the corresponding Hilbert spaces by $-J\frac{d}{dt}$ and the operator B be defined by the matrix function $B(t)$ as in Sections 2 and 3. We show that the indices defined in the Sections 2–4 are essentially the same.

THEOREM 4.6. *Suppose $B \in C(\mathbb{R}, \mathcal{L}_s(2n))$ satisfies*

$$B(t+\tau) = (P^{-1})^T B(t) P^{-1},$$

then there holds

$$I(A, A-B) = \mu_A(B) = i^P(B). \tag{4.9}$$

PROOF. We only need to prove the case $P \neq I$. From the definition, we have $i^P(0) = 0$. So, by Lemma 4.5 and similar computations as in the proof of Theorem 3.5, we have (4.9). $\square$

5. The minimal periodic problem for P-symmetric solutions

In this section, we apply the P-index theory and its iteration theory to the P-boundary problem of the following autonomous Hamiltonian system:

$$\begin{cases} \dot{x} = JH'(x), \qquad x \in \mathbb{R}^{2n}, \\ x(\tau) = Px(0), \end{cases} \tag{5.1}$$

where $P \in \mathrm{Sp}(2n)$ satisfies $P^k = I$, here k is assumed to be the smallest positive integer such that $P^k = I$ (this condition for P is called the $(P)_k$ condition in the sequel); and $H(x) \in C^2(\mathbb{R}^{2n}, \mathbb{R})$ satisfies $H(Px) = H(x)$, H' denotes the gradient of H. We note that the matrix $P \in \mathrm{Sp}(2n)$ satisfying $P^k = I$ is not necessary orthogonally symplectic,

$$P = \begin{pmatrix} a & b \\ -\dfrac{a^2+a+1}{b} & -a-1 \end{pmatrix}$$

is an example with $k = 3$ and $n = 1$. A solution (τ, x) of problem (5.1) is called a *P-solution* of the Hamiltonian system. Since $P^k = I$, the P-solution (τ, x) can be extended as a $k\tau$-periodic solution $(k\tau, x^k)$. We say that a T-periodic solution (T, x) of a Hamiltonian system in (5.1) is *P-symmetric* if $x(T/k) = Px(0)$. T is the *P-symmetric period* of x. We say that T is the *minimal P-symmetric period* of x if $T = \min\{\tau > 0 : x(t + \tau/k) = Px(t), \text{ for all } t \in \mathbb{R}\}$.

Theorem 5.1. *Suppose $P \in \mathrm{Sp}(2n)$ satisfies the $(\mathrm{P})_k$ condition and the Hamiltonian function H satisfies the following conditions*:

- (H1) *$H \in C^2(\mathbb{R}^{2n}, \mathbb{R})$ and $H(Px) = H(x)$, for all $x \in \mathbb{R}^{2n}$;*
- (H2) *there exist constants $\mu > 2$ and $R_0 > 0$ such that*

$$0 < \mu H(x) \le H'(x) \cdot x, \quad \text{for all } |x| \ge R_0;$$

- (H3) *$H(x) = o(|x|^2)$ at $x = 0$;*
- (H4) *$H(x) \ge 0$, for all $x \in \mathbb{R}^{2n}$.*

Then for every $\tau > 0$, system (5.1) possesses a nonconstant P-solution (τ, x) satisfying

$$\dim \ker_{\mathbb{R}}(P - I) + 2 - \nu^P(x) \le i^P(x) \le \dim \ker_{\mathbb{R}}(P - I) + 1. \tag{5.2}$$

Moreover, if this solution x also satisfies the following condition:

- (HC) *$H''(x(t)) > 0$ for every $t \in \mathbb{R}$,*

then the minimal P-symmetric period of x is $k\tau$ or $k\tau/(k+1)$.

Suppose $\overline{\gamma}(t) \in \mathcal{P}_\tau(2n)$ is the fundamental solution of the Hamiltonian system $\dot{z}(t) = JB(t)z(t)$ with $B(t) = H''(x(t))$. If $\overline{\gamma} \notin {}_P\mathcal{P}^e_\tau(2n) = \{\overline{\gamma} \in \mathcal{P}_\tau(2n) : P^{-1}\overline{\gamma}(\tau) \in \mathrm{Sp}^e(2n)\}$, then the minimal P-symmetric period of x is $k\tau$, i.e. the

P-symmetric periodic solution $(k\tau, x^k)$ generated from x possesses the minimal P-symmetric period.

We recall that $\mathrm{Sp}^e(2n) = \{M \in \mathrm{Sp}(2n) : \sigma(M) \subset \mathbf{U}\}$, i.e. $M \in \mathrm{Sp}^e(2n)$ if and only if $e(M) = 2n$. Here $e(M)$ is the elliptic height of M which is defined as the total number of eigenvalues of M on the unit circle $\mathbf{U}$ in $\mathbb{C}$ (counted with multiplicity) (see [35]).

The main points of the proof of Theorem 5.1 are the following three aspects. Firstly we get the variational setting of problem (5.1) and transfer it to the existence of a suitable critical point. Then we apply a critical point theorem and the index theories developed in this paper to find a solution of problem (5.1) satisfying index estimate (5.2). Finally, using the iteration inequalities developed in [30], we estimate the minimal period of the solution.

In order to estimate the the Maslov-type P-index of a critical point of the functional, we need the following saddle point theorem which was proved in [21], [23], [40].

THEOREM 5.2. *Let E be a real Hilbert space with the orthogonal decomposition $E = X \oplus Y$, where $\dim X < +\infty$. Suppose $f \in C^2(E, \mathbb{R})$ satisfies the* (PS) *condition and the following conditions*:

(F1) *there exist $\rho, \alpha > 0$ such that*
$$f(w) \geq \alpha, \quad \text{for all } w \in \partial B_\rho(0) \cap Y;$$
(F2) *there exist $e \in \partial B_1(0) \cap Y$ and $R > \rho$ such that*
$$f(w) < \alpha, \quad \text{for all } w \in \partial Q,$$
where $Q = (\overline{B_R(0)} \cap X) \oplus \{re : 0 \leq r \leq R\}$.

Then

(a) *f possesses a critical value $c \geq \alpha$ which is given by*
$$c = \inf_{h \in \Lambda} \max_{w \in Q} f(h(w)),$$
where $\Lambda = \{h \in C(\overline{Q}, E) : h = \mathrm{id} \text{ on } \partial Q\}$.

(b) *If $f''(w)$ is Fredholm for $w \in \mathcal{K}_c(f) = \{w \in E : f'(w) = 0,\ f(w) = c\}$, then there exists an element $w_0 \in \mathcal{K}_c(f)$ such that the negative Morse index $m^-(w_0)$ and nullity $m^0(w_0)$ of f at w_0 satisfy*
$$m^-(w_0) \leq \dim X + 1 \leq m^-(w_0) + m^0(w_0). \tag{5.3}$$

(c) *Suppose that there is an S^1 action on E, f is S^1-invariant, and for w_0 defined in* (b) *the set $S^1 * w_0$ is not a single point. Then* (5.3) *can be further improved to*
$$m^-(w_0) \leq \dim X + 1 \leq m^-(w_0) + m^0(w_0) - 1.$$

For a continuous symplectic path $\gamma\colon [0,\tau] \to \mathrm{Sp}(2n)$, $m \in \mathbb{N}$, we define the m-times iteration path $\gamma^m\colon [0, m\tau] \to \mathrm{Sp}(2n)$ of γ as

$$(5.4)\qquad \gamma^m(t) = \begin{cases} \gamma(t), & t \in [0,\tau], \\ P\gamma(t-\tau)P^{-1}\gamma(\tau), & t \in [\tau, 2\tau], \\ P^2\gamma(t-2\tau)(P^{-1}\gamma(\tau))^2, & t \in [2\tau, 3\tau], \\ P^3\gamma(t-3\tau)(P^{-1}\gamma(\tau))^3, & t \in [3\tau, 4\tau], \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots & \\ P^{m-1}\gamma(t-(m-1)\tau)(P^{-1}\gamma(\tau))^{m-1}, & t \in [(m-1)\tau, m\tau]. \end{cases}$$

We set $((i^{P^m}(\gamma,m), \nu^{P^m}(\gamma,m))) = (i^{P^m}(\gamma^m), \nu^{P^m}(\gamma^m))$. The following estimate (proved in [30]) will be used in the proof of Theorem 5.1. For ξ from Definition 4.2, $\nu(\xi,m) = \dim_{\mathbb{C}} \ker_{\mathbb{C}}(P^{-m} - I_{2n}) = \dim_{\mathbb{C}} \ker_{\mathbb{C}}(P^m - I_{2n})$ for $m \in \mathbb{N}$.

LEMMA 5.3 ([30]). *For any path $\gamma \in \mathcal{P}_\tau(2n)$, $P \in \mathrm{Sp}(2n)$, set $M = \gamma(\tau)$. Then, for any $m \in \mathbb{N}$, we have*

$$\begin{aligned} &\nu^{P^m}(\gamma,m) - \nu(\xi,1) + \nu(\xi,m+1) - \frac{e(P^{-1}M)}{2} - \frac{e(P^{-1})}{2} \\ &\quad\le i^{P^{(m+1)}}(\gamma,m+1) - i^{P^m}(\gamma,m) - i^P(\gamma,1) \\ &\quad\le \nu^P(\gamma,1) - \nu^{P^{(m+1)}}(\gamma,m+1) - \nu(\xi,m) + \frac{e(P^{-1}M)}{2} + \frac{e(P^{-1})}{2}. \end{aligned}$$

Let $W_P = \gamma^P W^{1/2,2}(S_\tau, \mathbb{R}^{2n})$, it is the space of all $W^{1/2,2}$ functions z defined on $\mathbb{R}$ satisfying $z(t+\tau) = Pz(t)$. It is an inner product space with the inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. We will denote the L^s norm by $\|\cdot\|_s$ for $s \ge 1$. The space W_P can be continuously embedded into $L^s([0,\tau], \mathbb{R}^{2n})$, i.e. there is $\alpha_s > 0$ such that

$$(5.5)\qquad \|z\|_s \le \alpha_s \|z\|, \quad \text{for all } z \in W_P.$$

Let A and B be the self-adjoint operators defined on W_P by the following bilinear forms:

$$(5.6)\qquad \langle Ax, y\rangle = \int_0^\tau (-J\dot{x}(t), y(t))\,dt, \qquad \langle Bx, y\rangle = \int_0^\tau (B(t)x(t), y(t))\,dt.$$

Suppose that $\ldots \le \lambda_{-k} \le \ldots \le \lambda_{-1} < 0 < \lambda_1 \le \ldots \le \lambda_k \le \ldots$ are all nonzero eigenvalues of the operator A (counted with multiplicity), and, correspondingly, e_j is the eigenvector of λ_j satisfying $\langle e_j, e_l\rangle = \delta_{jl}$. We denote the kernel of the operator A by W_P^0 which is exactly the space $\ker_{\mathbb{R}}(P - I)$. We define the subspaces of W_P by

$$W_P^m = W_m^- \oplus W_P^0 \oplus W_m^+$$

with

$$W_m^- = \Big\{ z \in W_P : z(t) = \sum_{j=1}^m a_{-j} e_{-j}(t),\ a_{-j} \in \mathbb{R} \Big\},$$
$$W_m^+ = \Big\{ z \in W_P : z(t) = \sum_{j=1}^m a_j e_j(t),\ a_j \in \mathbb{R} \Big\}.$$

For $z \in W_P$, we define

$$f(z) = \frac{1}{2} \int_0^{k\tau} (-J\dot{z}(t), z(t))\, dt - \int_0^{k\tau} H(z)\, dt = k\bigg(\frac{1}{2} \langle Az, z\rangle - \int_0^{\tau} H(z)\, dt \bigg). \tag{5.7}$$

It is well known that $f \in C^2(W_P, \mathbb{R})$ whenever

$$H \in C^2(\mathbb{R}^{2n}, \mathbb{R}) \quad \text{and} \quad |H''(z)| \le a_1 |z|^s + a_2, \tag{5.8}$$

for some $s \in (1, +\infty)$ and all $z \in \mathbb{R}^{2n}$. Looking for solutions of (5.1) is equivalent to looking for critical points of f on W_P.

Proof of Theorem 5.1. We follow the ideas of [12] and [28] and carry out the proof in several steps.

Step 1. Truncating the Hamiltonian function H. Since growth condition (5.8) has not been assumed for H, we need to truncate the function H suitably to get a function H_K satisfying condition (5.8).

We follow the method in Rabinowitz's pioneering work [38] (cf. also [19], [39]). Let $K > R_0$ and select $\chi \in C^\infty(\mathbb{R}, \mathbb{R})$ such that $\chi(y) \equiv 1$ if $y \le K$, $\chi(y) \equiv 0$ if $y \ge K+1$, and $\chi'(y) < 0$ if $y \in (K, K+1)$, where K is free for now. Set

$$H_K(z) = \chi(|z|)H(z) + (1 - \chi(|z|))R_K |z|^4, \tag{5.9}$$

where the constant R_K satisfies

$$R_K \ge \max_{K \le |z| \le K+1} \frac{H(z)}{|z|^4}.$$

Then $H_K \in C^2(\mathbb{R}^{2n}, \mathbb{R})$ satisfies (H3), (H4) and (5.8) with $s = 2$. Moreover, a straightforward computation shows that (H2) holds with μ replaced by $\nu = \min\{\mu, 4\}$, i.e. there exists $R_0 > 0$ such that

$$0 < \nu H_K(z) \le H_K'(z) \cdot z, \quad \text{for all } |z| \ge R_0. \tag{5.10}$$

Since $H_K \in C^2(\mathbb{R}^{2n}, \mathbb{R})$, then $H_K(z)$ is bounded for $|z| \le R_0$. Thus for $K > R_0$ there exist positive constants K_1, K_2 independent of K such that

$$R_K \nu |z|^4 - K_1 \le \nu H_K(z) \le H_K'(z) \cdot z + K_2, \quad \text{for all } z \in \mathbb{R}^{2n}, \tag{5.11}$$

via (5.9) and (5.10). Integrating (5.10) then yields

$$H_K(z) \ge a_3 |z|^\nu - a_4, \tag{5.12}$$

for all $z \in \mathbb{R}^{2n}$, where $a_3, a_4 > 0$ are independent of K.

Define a functional f_K on W_P by

$$f_K(z) = \frac{k}{2}\langle Az, z\rangle - \int_0^{k\tau} H_K(z)\,dt, \quad \text{for all } z \in W_P, \tag{5.13}$$

then $f_K \in C^2(W_P, \mathbb{R})$. Here since we do not have $H_K(Pz) = H_K(z)$, the equality

$$f_K(z) = k\left(\frac{1}{2}\langle Az, z\rangle - \int_0^{\tau} H_K(z)\,dt\right)$$

does not hold generally (compare (5.7)).

Step 2. Proving the linking conditions in Theorem 5.2. For $m > 0$, let $f_{K,m} = f_K|_{W_P^m}$. We will show that $f_{K,m}$ satisfies the hypotheses of Theorem 5.2. Indeed, by (H3), for any $\varepsilon > 0$, there is $\delta > 0$ such that $H_K(z) \le \varepsilon|z|^2$ for $|z| \le \delta$. Since $H_K(z)|z|^{-4}$ is uniformly bounded as $|z| \to +\infty$, there is $M_1 = M_1(\varepsilon, K)$ such that $H_K(z) \le M_1|z|^4$ for $|z| \ge \delta$. Hence

$$H_K(z) \le \varepsilon|z|^2 + M_1|z|^4, \quad \text{for all } z \in \mathbb{R}^{2n}. \tag{5.14}$$

Therefore by (5.14) and the Sobolev embedding theorem,

$$\int_0^{k\tau} H_K(z)\,dt \le C_K(\varepsilon\|z\|_2^2 + M_1\|z\|_4^4) \le C_K(\varepsilon\alpha_2 + M_1\alpha_4\|z\|^2)\|z\|^2, \tag{5.15}$$

where C_K is a constant depending on K. Let

$$X_m = W_m^- \oplus W_P^0, \qquad Y_m = W_m^+. \tag{5.16}$$

Consequently, for $z \in Y_m$, we have

$$\begin{aligned} f_{K,m}(z) &= \frac{k}{2}\langle Az, z\rangle - \int_0^{k\tau} H_K(z)\,dt \\ &\ge \frac{k\lambda_1}{2}\|z\|^2 - C_K(\varepsilon\alpha_2 + M_1\alpha_4\|z\|^2)\|z\|^2. \end{aligned}$$

So there are constants $\rho = \rho(K) > 0$ and $\alpha = \alpha(K) > 0$, which are sufficiently small and independent of m, such that

$$f_{K,m}(z) \ge \alpha, \quad \text{for all } z \in \partial B_\rho(0) \cap Y_m. \tag{5.17}$$

Let $e = e_1 \in \partial B_1(0) \cap Y_m$ and set

$$Q_m = \{re : 0 \le r \le r_1\} \oplus (B_{r_1} \cap X_m),$$

where r_1 is free for the moment. Let $z = z^- + z^0 \in W_m^- \oplus W_P^0$, then

$$\begin{aligned} f_{K,m}(z + re) &= \frac{k}{2}\langle Az^-, z^-\rangle + \frac{k}{2}r^2\langle Ae, e\rangle - \int_0^{k\tau} H_K(z + re)\,dt \\ &\le \frac{k\lambda_{-1}}{2}\|z^-\|^2 + \frac{k\lambda_1}{2}r^2 - \int_0^{k\tau} H_K(z + re)\,dt. \end{aligned} \tag{5.18}$$

If $r = 0$, due to condition (H4), there holds

$$f_{K,m}(z+re) \le \frac{k\lambda_{-1}}{2}\,\|z^-\|^2 \le 0. \tag{5.19}$$

If $r = r_1$ or $\|z\| = r_1$, by (5.12), there holds

$$\begin{aligned}\int_0^{k\tau} H_K(z+re)\,dt &\ge \int_0^{\tau} H_K(z+re)\,dt \\ &\ge a_3\int_0^{\tau} |z+re|^{\nu}dt - \tau a_4 \ge a_5\left(\int_0^{\tau}|z+re|^2 dt\right)^{\nu/2} - a_6 \\ &= a_5\left(\int_0^{\tau}\left(|z^0|^2+|z^-|^2+r^2|e|^2\right)dt\right)^{\nu/2} - a_6 \ge a_7(|z^0|^{\nu}+r^{\nu}) - a_6.\end{aligned} \tag{5.20}$$

Combining (5.20) with (5.18), we get

$$f_{K,m}(z+re) \le \frac{k\lambda_1 r^2}{2} + \frac{k\lambda_{-1}}{2}\,\|z^-\|^2 - a_7(\|z^0\|^{\nu}+r^{\nu}) + a_6.$$

So we can choose r_1 large enough which is independent of K and m such that

$$f_{K,m}(z+re) \le 0, \quad \text{for all } z \in \partial Q_m.$$

Next we will show that $f_{K,m}$ satisfies the (PS) condition on W_P^m for $m > 0$, i.e. any sequence $\{z_j\} \subset W_P^m$ possesses a convergent subsequence in W_P^m, provided $f_{K,m}(z_j)$ is bounded and $f'_{K,m}(z_j) \to 0$ as $j \to \infty$. We suppose $\|f_{K,m}(z_j)\| \le C$, then for j large enough:

$$\begin{aligned}C + \|z_j\| &\ge f_{K,m}(z_j) - \frac{1}{2}\,f'_{K,m}(z_j)z_j \\ &= \int_0^{k\tau}\left[\frac{1}{2}\,H'_K(z_j)\cdot z_j - H_K(z_j)\right]dt \\ &\ge \nu(2^{-1}-\nu^{-1})\int_0^{k\tau} H_K(z_j)\,dt - C_1 \\ &\ge \nu(2^{-1}-\nu^{-1})\int_0^{\tau} H_K(z_j)\,dt - C_1 \ge C_2\|z_j\|_4^4 - C_3\end{aligned} \tag{5.21}$$

due to (5.11). In (5.21), C_1 is independent of K, but both C_2 and C_3 depend on K. So $\{z_j\}$ is bounded in W_P^m. Since W_P^m is finite dimensional, the sequence $\{z_j\}$ has a convergent subsequence.

We have verified all the conditions of Theorem 5.2, hence $f_{K,m}$ has a critical value $c_{K,m} \ge \alpha$ which is given by

$$c_{K,m} = \inf_{g\in\Lambda_m}\max_{w\in Q_m} f_{K,m}(g(w)), \tag{5.22}$$

where $\Lambda_m = \{g \in C(Q_m, W_P^m) : g = \mathrm{id} \text{ on } \partial Q_m\}$. Note that there is a natural S^1-invariant on W_P and W_P^m defined by

$$\theta * x(t) = x(t+\theta), \quad \text{for all } x \in W_P,\ \theta \in [0,k\tau]/\{0,k\tau\} = S^1. \tag{5.23}$$

Now, since W_P^m is finite dimensional, $f''_{K,m}(x)$ is Fredholm for any critical point x, and $f_{K,m}$ is S^1-invariant under the above S^1-action (5.23) on W_P^m. So there is a critical point $x_{K,m}$ of $f_{K,m}$ which satisfies

$$\begin{aligned}(5.24)\quad m^-(x_{K,m}) &\le \dim X_m + 1\\ &= m + \dim\ker_{\mathbb{R}}(P-I) + 1 \le m^-(x_{K,m}) + m^0(x_{K,m}) - 1.\end{aligned}$$

Step 3. Proving that the critical point $x_{K,m}$ converges to x_K which is a critical point of f_K. We prove that there exists a nonconstant P-solution (τ, x_K) of the following problem:

$$\begin{cases}\dot{x} = JH'_K(x),\\ x(\tau) = Px(0).\end{cases} \tag{5.25}$$

On the one hand, since id $\in \Lambda_m$, by (5.18) and (H4), we have

$$c_{K,m} \le \sup_{w\in Q_m} f_{K,m}(w) \le \frac{k\lambda_1}{2}\, r_1^2. \tag{5.26}$$

Then in the sense of a subsequence we have

$$c_{K,m} \to c_K, \quad \alpha \le c_K \le \frac{k\lambda_1}{2}\, r_1^2. \tag{5.27}$$

On the other hand, we need to prove that f_K satisfies the (PS)* condition on W_P, i.e. for any sequence $\{z_m\} \subset W_P$ satisfying $z_m \in W_P^m$ $f_{K,m}(z_m)$ is bounded and $f'_{K,m}(z_m) \to 0$ possesses a convergent subsequence in W_P. It is a well-known result in the case of general periodic solution. For the reader's convenience, we give the proof following the idea in the appendix of [5].

The convergence $f'_{K,m}(z_m) \to 0$ as $m \to +\infty$ implies

$$-J\dot{z}_m - P_m H'_K(z_m) = \varepsilon_m, \tag{5.28}$$

with $\|\varepsilon_m\|_{(W_P^m)'} \to 0$ as $m \to +\infty$. Here W' denotes the dual space of W. Denote $z_m = z_m^0 + z_m^+ + z_m^-$. Using the same arguments as for (5.21) and by some direct estimates, we see that $\{z_m\}$ is bounded in W_P. Thus by passing to a subsequence, we may assume that

$$\begin{aligned} z_m &\to z \quad \text{in } W_P \text{ weakly},\\ z_m &\to z \quad \text{in } L^p \text{ strongly for } 1 \le p < +\infty,\\ z_m^0 &\to z^0 \quad \text{in } \mathbb{R}^{2n}.\end{aligned}$$

By (5.9), there exists a constant M_2 such that

$$|H'_K(z)| \le M_2|z|^3 + M_2, \quad \text{for all } z \in \mathbb{R}^{2n}.$$

This implies that $H'_K(z_m) \to H'_K(z)$ strongly in L^2. Thus $P_m H'_K(z_m) \to H'_K(z)$ strongly in L^2, and thus in W'_P. Therefore (5.28) implies that $\dot{z}_m = \varsigma_m + \varepsilon_m$

holds in W_P', where $\varsigma_m \to \varsigma = JH_K'(z)$ in L^2. This implies

$$\dot{z} = \varsigma \tag{5.29}$$

in W_P'. Since $\varsigma \in L^2$, $z \in W^{1,2}$ and thus $z \in C^2$, i.e. (5.29) holds in the classical sense. As W_P^m is a subspace of W_P, $P_m \colon W_P \to W_P^m$ is the projection:

$$\|z_m - P_m z\|_{W_P^m}^2 = \|\dot{z}_m - P_m \dot{z}\|_{(W_P^m)'}^2 + |z_m^0 - z^0|^2.$$

Then

$$\|z_m - P_m z\|_{W_P^m}^2 \leq \big(\|\varsigma_m - P_m\varsigma\|_{(W_P^m)'} + \|\varepsilon_m\|_{(W_P^m)'}\big)^2 + |z_m^0 - z^0|^2.$$

From

$$\|\varsigma_m - P_m\varsigma\|_{(W_P^m)'} \leq M_3\|\varsigma_m - P_m\varsigma\|_{L^2} \to 0$$

for some $M_3 > 0$ independent of m, we obtain

$$\|z_m - P_m z\|^2 = \|z_m - P_m z\|_{W_P^m}^2 \to 0.$$

This proves that $z_m \to z$ in W strongly. We have thus proved that f_K satisfies the (PS)* condition. Hence in the sense of a subsequence we have

$$x_{K,m} \to x_K, \qquad f_K(x_K) = c_K, \qquad f_K'(x_K) = 0. \tag{5.30}$$

From the above we conclude that f_K possesses a critical value $c_K \geq \alpha = \alpha(K) > 0$ with the corresponding critical point x_K. By the standard arguments similar to (6.35)–(6.37) in [39], x_K is a classical nonconstant P-solution of (5.25).

Indeed, if $x_K(t)$ is a constant solution of (5.25), then it should belong to $\ker_{\mathbb{R}}(P - I)$ and

$$f_K(x_K) = \frac{k}{2}\,\langle Ax_K, x_K\rangle - \int_0^{k\tau} H_K(x_K)\,dt \leq 0.$$

This contradicts to $f_K(x_K) = c_K \geq \alpha > 0$.

Step 4. Proving that for large K, the critical point x_K is a P-solution of problem (5.1). We show that there is $K_0 > 0$ such that for all $K \geq K_0$, $\|x_K\|_{L^\infty} < K$. Then $H_K'(x_K) = H'(x_K)$ and $x = x_K$ is a nonconstant P-solution of (5.1). By (5.27), $c_K \leq k\lambda_1 r_1^2/2$ independently of K. By (5.11), we obtain

$$\begin{aligned}\frac{k\lambda_1}{2}\,r_1^2 &\geq f_K(x_K) - \frac{1}{2}\,f_K'(x_K)x_K \\ &\geq (2^{-1} - \nu^{-1})\int_0^{k\tau} H_K'(x_K)\cdot x_K\,dt - C\end{aligned} \tag{5.31}$$

with $C = \nu^{-1}K_2\tau$ independent of K. Therefore (5.31) provides a K independent upper bound for $\int_0^\tau H_K'(x_K)\cdot x_K\,dt$. By (5.11),

$$H_K(\zeta) \leq \nu^{-1}H_K'(\zeta)\cdot\zeta + C/\tau, \quad \text{for all } \zeta \in \mathbb{R}^{2n}. \tag{5.32}$$

Recalling that $H_K(x_K) \equiv$ const since x_K satisfies an autonomous Hamiltonian system, replacing ζ by x_K, integrating (5.32) over $[0, \tau]$, and (5.32) yield

$$k\tau H_K(x_K) \leq \nu^{-1} \int_0^{k\tau} H_K'(x_K) \cdot x_K \, dt + C. \tag{5.33}$$

The right-hand side of (5.33) is bounded from above independently of K. Then (5.12) and (5.33) yield a K independent L^∞ bound for x_K. So choose K large enough such that $\|x_K\|_{L^\infty} < K$ thus x_K is a P-solution of problem (5.1). We denote it simply by $x := x_K$.

Step 5. Index estimate. We prove that

$$\dim \ker_{\mathbb{R}}(P - I) + 2 - \nu^P(x) \leq i^P(x) \leq \dim \ker_{\mathbb{R}}(P - I) + 1.$$

Let $B(t) = H''(x(t))$ and B be the operator defined by (5.6) corresponding to $B(t)$. By direct computation, we get

$$\langle f_K''(z)w, w\rangle - k\langle (A - B)w, w\rangle = \int_0^{k\tau} \left[(H_K''(x_K(t))w, w) - (H_K''(z(t))w, w) \right] dt,$$

for all $w \in W_P$. Then by the continuity of H_K'',

$$\|f_K''(z) - k(A - B)\| \to 0 \quad \text{as } \|z - x_K\| \to 0. \tag{5.34}$$

Let $d = \|(A - B)^\sharp\|^{-1}/4$. By (5.34), there exists $r_0 > 0$ such that

$$\|f_K''(z) - k(A - B)\| < \frac{1}{2}\, d, \quad \text{for all } z \in V_{r_0} = \{z \in W_P : \|z - x_K\| \leq r_0\}.$$

Hence for m large enough, there holds

$$\|f_{K,m}''(z) - kP_m(A - B)P_m\| < \frac{1}{2}\, d, \quad \text{for all } z \in V_{r_0} \cap W_P^m. \tag{5.35}$$

For $x_{K,m} \in V_{r_0} \cap W_P^m$ and all $w \in M_d^-(P_m(A - B)P_m) \setminus \{0\}$, from (5.35) we have

$$\begin{aligned} \langle f_{K,m}''(x_{K,m})w, w\rangle &\leq k\langle P_m(A - B)P_m w, w\rangle \\ &\qquad + \|f_{K,m}''(x_{K,m}) - kP_m(A - B)P_m\| \cdot \|w\|^2 \\ &\leq -kd\|w\|^2 + \frac{1}{2}\, d\|w\|^2 = -\frac{1}{2}\, d\|w\|^2 < 0. \end{aligned}$$

Then

$$\dim M^-(f_{K,m}''(x_{K,m})) \geq \dim M_d^-(P_m(A - B)P_m). \tag{5.36}$$

By (5.24), (5.30), (5.36) and Theorem 4.6, for m large enough, we have

$$\begin{aligned} m + \dim \ker_{\mathbb{R}}(P - I) + 1 &= \dim X_m + 1 \geq m^-(x_{K,m}) \\ &\geq \dim M_d^-(P_m(A - B)P_m) = m + i^P(x_K). \end{aligned}$$

Then $i^P(x_K) \leq \dim \ker_{\mathbb{R}}(P - I) + 1$.

Similarly, for all $w \in M_d^+(P_m(A-B)P_m) \setminus \{0\}$, from (5.35) we have

$$\begin{aligned}\langle f''_{K,m}(x_{K,m})w, w\rangle &\geq \langle P_m(A-B)P_m w, w\rangle \\ &\qquad - \|f''_{K,m}(x_{K,m}) - P_m(A-B)P_m\| \cdot \|w\|^2 \\ &\geq kd\|w\|^2 - \frac{1}{2}\, d\|w\|^2 > 0.\end{aligned}$$

Then

$$\dim M^+(f''_{K,m}(x_{K,m})) \geq \dim M_d^+(P_m(A-B)P_m). \tag{5.37}$$

By (5.24), (5.30), (5.37) and Theorem 4.6, for m large enough, we have

$$\begin{aligned}m + \dim\ker_{\mathbb{R}}(P-I) + 1 &= \dim X_m + 1 \\ &\leq m^-(x_{K,m}) + m^0(x_{K,m}) - 1 = \dim W_P^m - m^+(x_{K,m}) - 1 \\ &\leq 2m + \dim\ker_{\mathbb{R}}(P-I) - \dim M_d^+(P_m(A-B)P_m) - 1 \\ &= m + i_P(x_K) + \nu^P(x_K) - 1.\end{aligned}$$

This implies $\dim\ker_{\mathbb{R}}(P-I) + 2 \leq i_P(x_K) + \nu^P(x_K)$.

Step 6. Estimate of the minimal P-symmetric period of x. If $k\tau$ is not the minimal P-symmetric period of x, i.e. $\tau > \min\{\lambda > 0 : x(t+\lambda) = Px(t),\ \text{for all } t \in \mathbb{R}\}$, then there exists some l such that

$$T \equiv \frac{\tau}{l} = \min\big\{\lambda > 0 : x(t+\lambda) = Px(t),\ \text{for all } t \in \mathbb{R}\big\}.$$

Thus $x(\tau - T) = x(0)$, both $(l-1)T$ and kT are the periods of x. Since kT is the minimal P-symmetric period, we obtain $kT \leq (l-1)T$ and then $k \leq l-1$.

Note that $x|_{[0,kT]}$ is the k-th iteration of $x|_{[0,T]}$. Suppose $\gamma \in \mathcal{P}_T(2n)$ is the fundamental solution of the following linear Hamiltonian system:

$$\dot{z}(t) = JB(t)z(t)$$

with $B(t) = H''(x|_{[0,T]}(t))$. Suppose ξ is any symplectic path in $\mathcal{P}_T(2n)$ such that $\xi(T) = P^{-1}$. Since $P^k = I$,

$$\nu(\xi, 1) = \nu(\xi, k+1) = \nu(\xi, l). \tag{5.38}$$

All eigenvalues of P and P^{-1} are on the unit circle, then the elliptic height

$$e(P^{-1}) = e(P) = 2n. \tag{5.39}$$

Since system (5.1) is autonomous, we have

$$\nu_1(x|_{[0,kT]}) \geq 1 \quad \text{and} \quad \nu^{P^{l-1}}(\gamma, l-1) = \nu_1(x|_{[0,(l-1)T]}) \geq 1. \tag{5.40}$$

By Lemma 5.3, $P^{l-1}=I$ and (5.38)–(5.39), we have

$$
\begin{aligned}
i^I(\gamma,l-1) &= i^{P^{l-1}}(\gamma,l-1)\\
&\le i^{P^l}(\gamma,l)-i^P(\gamma,1)+\nu(\xi,1)-\nu(\xi,l)\\
&\quad+\frac{e(P^{-1}\gamma(T))}{2}+\frac{e(P^{-1})}{2}-\nu^{P^{l-1}}(\gamma,l-1)\\
&\le i^{P^l}(\gamma,l)-i^P(\gamma,1)+\frac{e(P^{-1}\gamma(T))}{2}+n-1\\
&\le i^{P^l}(\gamma,l)-i^P(\gamma,1)+2n-1.
\end{aligned}
\tag{5.41}
$$

Note that $i^{P^l}(\gamma,l)=i^P_{[0,\tau]}(x_K)\le \dim\ker_{\mathbb{R}}(P-I)+1$, here we write $i^P_{[0,\tau]}(x_K)$ for $i^P(x_K)$ to remind that the solution x_K is defined in the interval $[0,\tau]$. By the definition of the Maslov P-index, $i^I(\gamma,l-1)=i_1(\gamma,l-1)+n$. So we get

$$
i_1(\gamma,l-1)\le \dim\ker_{\mathbb{R}}(P-I)-i^P(\gamma,1)+n. \tag{5.42}
$$

By condition (HC) and (4.8) in Lemma 4.5, we have

$$
i^P(\gamma,1)=i^P(B)=\sum_{s\in[0,1)}\nu^P(sB)=\sum_{s\in[0,1)}\dim\ker_{\mathbb{R}}(\gamma_B(sT)-P). \tag{5.43}
$$

Here we recall that $B(t)=H''(x|_{[0,T]}(t))$ and γ_B is the fundamental solution of the linear Hamiltonian system

$$
\dot z(t)=JB(t)z(t).
$$

Since $\gamma_B(0)=I$, $\dim\ker_{\mathbb{R}}(\gamma_B(s\tau)-P)=\dim\ker_{\mathbb{R}}(P-I)$ when $s=0$. Thus we have

$$
i^P(\gamma,1)\ge \dim\ker_{\mathbb{R}}(P-I). \tag{5.44}
$$

From (5.42), we have

$$
i_1(\gamma,l-1)\le n. \tag{5.45}
$$

By convex condition (HC), we also have

$$
i_1(x|_{[0,kT]})\ge n \quad \text{and} \quad i_1(x|_{[0,(l-1)T]})\ge n. \tag{5.46}
$$

Set $m=(l-1)/k$. Note that $x|_{[0,(l-1)T]}$ is the m-th iteration of $x|_{[0,kT]}$. By (5.40), (5.46), (5.45) and Lemma 4.1 in [28], we obtain $m=1$ and then $k=l-1$. From the process of the proof, we see that only if $e(P^{-1}\gamma(T))=2n$, we can obtain $k=l-1$. In this case, the minimal P-symmetric period of x is $k\tau/(k+1)$.

Note that $\overline{\gamma}(\tau)=P^{l-1}\gamma(T)(P^{-1}\gamma(T))^{l-1}=P(P^{-1}\gamma(T))^l$. So we have

$$
e(P^{-1}\gamma(T))=e((P^{-1}\gamma(T))^l)=e(P^{-1}\overline{\gamma}(\tau)).
$$

If $\overline{\gamma}\notin {}_P\mathcal{P}^e_\tau(2n)$, then $e(P^{-1}\gamma(T))\le 2n-2$. We get $i_1(\gamma,l-1)<n$ by repeating the same process as in (5.41)–(5.42). It contradicts to the second inequality of (5.46). Thus the minimal P-symmetric period of x is $k\tau$. □

Let us note that in the step 6 above methods of [12] and [28] are not applicable. The reason is that the iteration inequalities for the P-index are more complicated (cf. [30]) and the lower bound of $i^P(\gamma)+\nu^P(\gamma)$ for the convex Hamiltonian system is not big enough to estimate the iteration number.

We believe that an alternative result about the minimal P-symmetric period of the P-symmetric periodic solution in Theorem 5.1 can be improved to that *the minimal P-symmetric period of x is $k\tau$* (the cases of $P=\pm I$, are true cf. [28] for $P=I$ and [43] for $P=-I$).

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