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POINCARÉ RECURRENCE THEOREM IN IMPULSIVE SYSTEMS

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Abstract. In this article, we generalize the Poincaré recurrence theorem to impulsive dynamical systems in $\mathbb{R}^n$. For a measure preserving system, we present some sufficient conditions to establish an impulsive system that is also measure preserving. Then, two recurrence theorems are proved. Finally, we use two examples to illustrate our results.

1. Introduction

Consider the differential equation $\dot{x} = f(x)$ on $\mathbb{R}^n$, where $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is a C^1 vector field. Let the vector field define a dynamical system or flow $\varphi\colon \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$. For a subset $\mathcal{D} \subset \mathbb{R}^n$, the flow φ is volume-preserving on $\mathcal{D}$ if for every measurable set $A \subset \mathcal{D}$ and every $t \in \mathbb{R}$ the set $\varphi^t(A) = \varphi(A \times \{t\})$ is measurable and $\mu(\varphi^t(A)) = \mu(A)$, where μ is a measure. For example, a Hamiltonian flow is volume-preserving (Liouville's Theorem). One of the most significant consequences of volume preservation is the following result.

Theorem 1.1 (Poincaré Recurrence Theorem). *If φ is a volume-preserving flow on an invariant bounded subset $\mathcal{D}$ of $\mathbb{R}^n$, then each point in $\mathcal{D}$ is nonwandering.*

This celebrated theorem has many generalizations, for example, see [1] and [8]–[12]. Our goal in this article is to generalize the Poincaré Recurrence Theorem

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to the impulsive systems. Since an impulsive system admits abrupt perturbations, its dynamical behavior is much richer than that of the corresponding system. Now, the theory of impulsive systems is an important and flourishing area of investigation, see [2]–[7]. In the next section, we establish some sufficient conditions to guarantee that an impulsive system is also volume-preserving. Then, in Section 3, we generalize the Poincaré Recurrence Theorem to impulsive dynamical systems. Finally, two examples are presented in Section 4 to illustrate the recurrence theorems.

2. Impulsive system

We first recall the definition and basic properties of a system with impulse action, and the reader may consult [2]–[7] for instance. Let φ be a flow on $\mathbb{R}^n$ defined by the vector field f, i.e. $\varphi\colon \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is continuous such that $\varphi(x,0) = x$ for all $x \in \mathbb{R}^n$, and $\varphi(\varphi(x,t),s) = \varphi(x,t+s)$ for all $x \in \mathbb{R}^n$, $t,s \in \mathbb{R}$. Note that $(\partial/\partial t)\varphi(x,t) = f(\varphi(x,t))$. The mappings $\varphi_x\colon \mathbb{R} \to \mathbb{R}^n$ $(t \mapsto \varphi(x,t))$ for $x \in \mathbb{R}^n$ are called the *motions* of the flow, and the mappings $\varphi^t\colon \mathbb{R}^n \to \mathbb{R}^n$ $(x \mapsto \varphi(x,t))$ for $t \in \mathbb{R}$ are called the *transitions*. If $A \subset \mathbb{R}^n$ and $J \subset \mathbb{R}$, then we write $\varphi(A \times J) = A \cdot J$, in particular, $\varphi(x,t) = x \cdot t$. If $x \in \mathbb{R}^n$, the *orbit* of x is the set $\gamma(x) = x \cdot \mathbb{R}$, or $\varphi_x(\mathbb{R})$. The *positive and negative semi-orbits* of x are the sets $\gamma^+(x) = x \cdot \mathbb{R}^+$ and $\gamma^-(x) = x \cdot \mathbb{R}^-$, respectively.

A set $M \subset \mathbb{R}^n$ is called a *smooth submanifold* of codimension one or a surface with dimension $n-1$ if it can be written as $M = \{x \in U : g(x) = 0\}$, where $U \subset \mathbb{R}^n$ is open, $g\colon U \to \mathbb{R}$ is a smooth function, and $\partial g/\partial x \neq 0$ for all $x \in U$. The submanifold M is said to be *transversal* to the vector field f if the dot product $\partial g/\partial x \cdot f(x) \neq 0$ for all $x \in M$, it is also called a cross section (see [10]).

Let M be a smooth $(n-1)$-dimensional surface in $\mathbb{R}^n$, and denote $\Omega = \mathbb{R}^n \backslash M$. Let $I\colon M \to \Omega$ be a diffeomorphism, then $N = I(M)$ is also a smooth $(n-1)$-dimensional surface. If $x \in M$, we shall denote $I(x)$ by x^+ and say that x *jumps* to x^+. Meanwhile, M is said to be an *impulsive set* and I is called an *impulsive function*. For each $x \in \Omega$, by $M^+(x)$ we mean the set $\gamma^+(x) \cap M$. We can define a function $\psi\colon \Omega \to \mathbb{R}^+ \cup \{+\infty\}$ (the space of extended positive reals) by

$$\psi(x) = \begin{cases} s & \text{if } x \cdot s \in M \text{ and } x \cdot t \notin M \text{ for } t \in [0,s), \\ +\infty & \text{if } M^+(x) = \emptyset. \end{cases}$$

In general, $\psi\colon \Omega \to \mathbb{R}^+ \cup \{+\infty\}$ is not continuous. Ciesielski [2] has established some easy conditions to guarantee the continuity of ψ. In this paper, we always assume that ψ is continuous on Ω.

We define an impulsive system $(\Omega, \widetilde{\varphi})$ by portraying the trajectory of each point in Ω. Let $x \in \Omega$, the *impulsive trajectory* of x is an Ω-valued function $\widetilde{\varphi}_x$ defined on a subset of $\mathbb{R}^+$. If $M^+(x) = \emptyset$, then $\psi(x) = +\infty$, and we set $\widetilde{\varphi}_x(t) =$

$x \cdot t$ for all $t \in \mathbb{R}^+$. If $M^+(x) \neq \emptyset$, it is easy to see that there is a positive number t_0 such that $x \cdot t_0 = x_1 \in M$ and $x \cdot t \notin M$ for $0 \leq t < t_0$. Thus, we define $\widetilde{\varphi}_x$ on $[0, t_0]$ by

$$\widetilde{\varphi}_x(t) = \begin{cases} x \cdot t & \text{if } 0 \leq t < t_0, \\ x_1^+ & \text{if } t = t_0, \end{cases}$$

where $\psi(x) = t_0$ and $x_1^+ = I(x_1) \in \Omega$. Since $t_0 < +\infty$, we continue the process by starting with x_1^+. Similarly, if $M^+(x_1^+) = \emptyset$, i.e. $\psi(x_1^+) = +\infty$, we define $\widetilde{\varphi}_x(t) = x_1^+ \cdot (t - t_0)$ for $t_0 < t < +\infty$. Otherwise, let $\psi(x_1^+) = t_1$ and $x_1^+ \cdot t_1 = x_2 \in M$, then we define $\widetilde{\varphi}_x(t)$ on $[t_0, t_0 + t_1]$ by

$$\widetilde{\varphi}_x(t) = \begin{cases} x_1^+ \cdot (t - t_0) & \text{if } t_0 \leq t < t_0 + t_1, \\ x_2^+ & \text{if } t = t_0 + t_1, \end{cases}$$

where $x_2^+ = I(x_2)$. Thus, continuing inductively, the process above either ends after a finite number of steps, whenever $M^+(x_n^+) = \emptyset$ for some n, or it continues infinitely, if $M^+(x_n^+) \neq \emptyset$ for $n = 1, 2, \ldots$, and $\widetilde{\varphi}_x$ is defined on the interval $[0, T(x))$, where $T(x) = \sum_{i=0}^{+\infty} t_i$. After setting each trajectory $\widetilde{\varphi}_x$ for every point x in Ω, we let $\widetilde{\varphi}(x, t) = \widetilde{\varphi}_x(t)$ for $x \in \Omega$ and $t \in [0, T(x))$, and then we get a discontinuous system $(\Omega, \widetilde{\varphi})$ satisfying the following properties:

(i) $\widetilde{\varphi}(x, 0) = x$ for $x \in \Omega$,

(ii) $\widetilde{\varphi}(\widetilde{\varphi}(x, t), s) = \widetilde{\varphi}(x, t + s)$ for $x \in \Omega$ and t, $s \in [0, T(x))$, such that $t + s \in [0, T(x))$.

We call $(\Omega, \widetilde{\varphi})$, with $\widetilde{\varphi}$ as defined above, an *impulsive dynamical system* associated with φ. Also for simplicity of exposition, we denote $\widetilde{\varphi}(x, t)$ by $x * t$. Thus, (ii) reads $(x * t) * s = x * (t + s)$. Now, for $x \in \Omega$ the mapping $\widetilde{\varphi}_x \colon \mathbb{R}^+ \to \Omega$ defined by $t \mapsto x * t$ and for $t \in \mathbb{R}^+$ the mapping $\widetilde{\varphi}^t \colon \Omega \to \Omega$ defined by $x \mapsto x * t$ may not be continuous. Similarly, if $A \subset \Omega$ and $J \subset \mathbb{R}^+$, we denote $A * J = \{x * t : x \in A$ and $t \in J\}$. In particular, if $J = \{t\}$, we let $A * t = A * \{t\} = \widetilde{\varphi}^t(A)$.

From the point of view of impulsive systems, the trajectories that are of interest are those with an infinite number of discontinuities and with $[0, +\infty)$ as the interval of definition. Following Kaul [7], we call them infinite trajectories. For an impulsive dynamical system, Ciesielski [4] uses the time reparametrization to get an isomorphic system whose impulsive trajectories are global, i.e. the resulting dynamics is defined for all positive times. Hence, in this paper we always assume $T(x) = +\infty$ for each $x \in \Omega$.

Remark 2.1. Indeed, we can similarly define an impulsive semidynamical system $(\mathbb{R}^n, \widetilde{\varphi})$ which admits $(\Omega, \widetilde{\varphi})$ as a subsystem. However, we are just interested in the system $(\Omega, \widetilde{\varphi})$ for the following reasons. First, each point in M of the system $(\mathbb{R}^n, \widetilde{\varphi})$ is a start point from the point of view of an impulsive system.

There is not much interesting dynamical behavior at the points of M, since impulsive trajectories jump away from M. Second, it destroys the consistency with the classical theory, e.g. the closure of an invariant set may not be invariant and a limit set may not be invariant, etc. Furthermore, ψ is not continuous on M, see [2].

For $x \in \Omega$, let $\widetilde{\gamma}^+(x) = x * \mathbb{R}^+$ or $\widetilde{\varphi}_x(\mathbb{R}^+)$ be the positive orbit of x. A point x in Ω is a *rest point* if $x * t = x$ for every $t \in \mathbb{R}^+$. Clearly, a point $x \in \Omega$ is a rest point of $(\Omega, \widetilde{\varphi})$ if and only if it is a rest point of φ. If a point x is not a rest point, we call it a *regular point*. An orbit $\widetilde{\gamma}^+(x)$ is said to be *periodic* of period τ and order k if $\widetilde{\gamma}^+(x)$ has k components and τ is the least positive number such that $x * \tau = x$.

In order to get a volume-preserving system, we now recall the concept of a measure on $\mathbb{R}^n$ (compare [13, Chapter 3]). Let $J \subset \mathbb{R}$ be a bounded interval with endpoints a and b such that $a \leq b$. The length of J is $l(J) = b - a$. A set $U \subset \mathbb{R}^n$ is a rectangle if there exist bounded intervals $J_1, \ldots, J_n$ such that $U = J_1 \times \ldots \times J_n$, in which case the volume of U is $\mathrm{vol}(U) = l(J_1) \ldots l(J_n)$. Let A be a set in $\mathbb{R}^n$, A has measure zero if for each $\varepsilon > 0$ there exists a cover $\{U_1, U_2, \ldots\}$ of A by rectangles such that $\sum_{i=1}^{+\infty} \mathrm{vol}(U_i) < \varepsilon$. Now, a set $A \subset \mathbb{R}^n$ is said to be *Jordan-measurable* if A is bounded and the boundary of A has measure zero. For a set $C \subset \mathbb{R}^n$, the characteristic function χ_C of C is defined by

$$\chi_C(x) = \begin{cases} 0 & \text{if } x \notin C, \\ 1 & \text{if } x \in C. \end{cases}$$

Then, a set A is Jordan-measurable if and only if its characteristic function χ_A is Riemann-integrable on $\mathbb{R}^n$, see Spivak [13]. If A is a Jordan-measurable subset of $\mathbb{R}^n$, then the volume or measure of A is $\mathrm{vol}(A) = \int_{\mathbb{R}^n} \chi_A(x)\, dx$. For brevity, in the sequel a Jordan-measurable set will be just called a measurable set.

Definition 2.2. Let $\mathcal{D} \subset \mathbb{R}^n$ be a Jordan-measurable set. A flow φ on $\mathcal{D}$ is *volume-preserving* if for every measurable set $A \subset \mathcal{D}$ and every $t \in \mathbb{R}$ the set $\varphi^t(A) = A \cdot t$ is measurable and $\mathrm{vol}(A \cdot t) = \mathrm{vol}(A)$.

From now on, we suppose that φ is a volume-preserving flow on $\mathbb{R}^n$. To get a volume-preserving impulsive system, we need some hypotheses on the impulsive set M and impulsive function I, namely:

(1) M and N are orientable surfaces, and they are transversal to the flow φ.
(2) $\psi(x) < +\infty$ for each $x \in N = I(M)$.
(3) For $p \in M$, let $\vec{n}_p$ be the unit normal vector of M at p. The dot product

$$f(p) \cdot \vec{n}_p = f(I(p)) \cdot \vec{n}_{I(p)} |\det I'(p)|$$

holds for every $p \in M$, where $\overrightarrow{n}_{I(p)}$ is the unit normal at $I(p) \in N$ and $I'(p)$ is the Jacobian matrix of I at p.

Clearly, condition (1) implies that each trajectory of φ passes M in one direction as the time t increases, and also it means that ψ is continuous (see [2]). By condition (2), we see that each positive orbit $\gamma^+(x)$ of $x \in N$ meets M at finite positive time $\psi(x)$. If $\mathcal{D} = \{x \cdot t : x \in N \text{ and } 0 \leq t < \psi(x)\}$, then the impulsive subsystem $(\mathcal{D}, \widetilde{\varphi})$ is well defined. Further, since I is a diffeomorphism, it is easy to see that for every $x \in \mathcal{D}$, the negative orbit $\widetilde{\gamma}^-(x)$ is unique, i.e. for any $t < 0$ there exists a unique $y \in \mathcal{D}$ such that $y * (-t) = x$. Consequently, for the impulsive system $(\mathcal{D}, \widetilde{\varphi})$, each orbit $\widetilde{\gamma}(x)$ of $x \in \mathcal{D}$ is uniquely determined. Now, for $(\mathcal{D}, \widetilde{\varphi})$, the semigroup axiom (ii) can be strengthened to the group axiom: $(x * t) * s = x * (t + s)$, being fulfilled for any $x \in \mathcal{D}$ and t, $s \in \mathbb{R}$. Finally, let S be an arbitrary small region of M, the flux of f on S is $\int_S (f \cdot \overrightarrow{n})\, dS$. On the other hand, the flux of f on $I(S)$ is

$$\int_{I(S)} (f \cdot \overrightarrow{n})\, dS.$$

By the change of variables, we have

$$\int_{I(S)} (f \cdot \overrightarrow{n})\, dS = \int_S \big(f(I(p)) \cdot \overrightarrow{n}_{I(p)}\big) |\det I'(p)|\, dS,$$

where $I'(p)$ is the Jacobian matrix of $I(p)$ at p. Then, it follows from condition (3) that

$$\int_{I(S)} (f \cdot \overrightarrow{n})\, dS = \int_S (f \cdot \overrightarrow{n})\, dS$$

is true for any small region $S \subset M$. Consequently, for a measurable set $A \subset \mathcal{D}$ and $t \in \mathbb{R}$, the set $A * t$ is measurable and $\operatorname{vol}(A * t) = \operatorname{vol}(A)$.

Definition 2.3. The impulsive system $(\mathcal{D}, \widetilde{\varphi})$ is volume-preserving if for every measurable subset $A \subset \mathcal{D}$ and every $t \in \mathbb{R}$ the set $A * t$ is measurable and $\operatorname{vol}(A * t) = \operatorname{vol}(A)$.

We conclude the above result as follows.

Theorem 2.4. *If conditions* (1)–(3) *are true, then the impulsive system* $(\mathcal{D}, \widetilde{\varphi})$ *is volume-preserving.*

Proof. Let $A \subset \mathcal{D}$ be a measurable set and $t \in \mathbb{R}$. If $\tau = \sup\{\psi(x) : x \in A\} < t$, then $A * t = A \cdot t$. We have $\operatorname{vol}(A * t) = \operatorname{vol}(A)$, since φ is a volume-preserving flow. Next, if $\tau = \inf\{\psi(x) : x \in A\} \geq t$, then the whole A passes through the surface M under φ. On the other hand, the flux of going out from $\mathcal{D}$ through any part S on M is the same as that of going into $\mathcal{D}$ from $I(S)$ on N. Consequently, we obtain $\operatorname{vol}(A*t) = \operatorname{vol}(A)$. Finally, let $A = B \cup C$, where $B = \{x \in A : \psi(x) < t\}$ and $C = \{x \in A : \psi(x) \geq t\}$. Since ψ is continuous on $\mathcal{D}$,

B and C are measurable and disjoint. Then, it follows that $A*t = B\cdot t \cup C*t$, and $\mathrm{vol}(A*t) = \mathrm{vol}(B\cdot t) + \mathrm{vol}(C*t) = \mathrm{vol}(B) + \mathrm{vol}(C) = \mathrm{vol}(A)$. So, $(\mathcal{D}, \widetilde{\varphi})$ is volume-preserving. In the above argument, we omit discussion of boundaries of those sets, since boundaries of measurable sets have measure zero. □

3. Recurrence theorems

Let $(\mathcal{D}, \widetilde{\varphi})$ be a volume-preserving impulsive system defined in the previous section. A point x is *nonwandering* for $\widetilde{\varphi}$ if for each open neighbourhood V of x in $\mathcal{D}$ there exists a real number $\tau > 1$ such that $V \cap V*\tau \neq \emptyset$. Clearly, fixed points and periodic points are nonwandering.

Theorem 3.1. *Let $\mathcal{D}$ be a bounded measurable subset in $\mathbb{R}^n$. If $\widetilde{\varphi}$ is a volume-preserving impulsive dynamical system on $\mathcal{D}$, then each point in $\mathcal{D}$ is nonwandering.*

Proof. Let U be a nonempty open subset of $\mathcal{D}$. An open set is the union of nonempty open balls that are measurable, so U is measurable. Since $\widetilde{\varphi}$ is volume-preserving, the sets $\widetilde{\varphi}^1(U), \widetilde{\varphi}^2(U), \ldots$ are measurable and have equal volumes, where $\widetilde{\varphi}^n(U) = U*n$. Consequently, $\sum_{n=1}^{+\infty} \mathrm{vol}(\widetilde{\varphi}^n(U))$ is infinite. By means of contradiction, assume that the sets $\widetilde{\varphi}^1(U), \widetilde{\varphi}^2(U), \ldots$ are pairwise disjoint. In this case,

$$\sum_{n=1}^{+\infty} \mathrm{vol}(\widetilde{\varphi}^n(U)) = \mathrm{vol}\bigg(\bigcup_{n=1}^{+\infty} \widetilde{\varphi}^n(U)\bigg),$$

where we consider these quantities as extended real numbers. Since $\mathcal{D}$ is bounded and $\bigcup_{n=1}^{+\infty} \widetilde{\varphi}^n(U) \subset \mathcal{D}$, we obtain

$$\sum_{n=1}^{+\infty} \mathrm{vol}(\widetilde{\varphi}^n(U)) = \mathrm{vol}\bigg(\bigcup_{n=1}^{+\infty} \widetilde{\varphi}^n(U)\bigg) \leq \mathrm{vol}(\mathcal{D}) < +\infty.$$

This is a contradiction. It follows that the sets $\widetilde{\varphi}^1(U), \widetilde{\varphi}^2(U), \ldots$ are not pairwise disjoint. Hence, there exist integers $j \geq 1$ and $i > j$ such that $\widetilde{\varphi}^i(U) \cap \widetilde{\varphi}^j(U) \neq \emptyset$. Since $\widetilde{\varphi}$ satisfies the group axiom, let $k = i - j$, we have $k \geq 1$ and $\widetilde{\varphi}^k(U) \cap U \neq \emptyset$. Therefore, every point in $\mathcal{D}$ is nonwandering. □

In the following, we see that a stronger consequence (Khintchine's Theorem, see [9, p. 453]) is true for impulsive systems. A proof of the following lemma was given in [9, p. 451], for completeness we present here a short proof.

Lemma 3.2. *Let $\mathcal{D}$ be a measurable set with $\mathrm{vol}(\mathcal{D}) = 1$. If $\{E_i\}_{i=1}^{\infty}$ is an infinite sequence of measurable subsets in $\mathcal{D}$, all having a volume not less than $m \in (0,1)$, then for any $\lambda \in (0,1)$ there exist two sets E_j and E_k, $j \neq k$, such that $\mathrm{vol}(E_j \cap E_k) \geq \lambda m^2$.*

Proof. Suppose that $\mathrm{vol}(E_j \cap E_k) < \mu$ for any j and k. Let $F_1 = E_1$, $F_2 = E_2 - E_2 \cap F_1$, $F_3 = E_3 - E_3 \cap F_2 - E_3 \cap F_1$, $\ldots$, $F_n = E_n - E_n \cap F_{n-1} - \ldots - E_n \cap F_1$, then $\{F_i\}$ are pairwise disjoint and $F_i \subset E_i$ for each i. Consequently, $\mathrm{vol}(F_1) \geq m$, $\mathrm{vol}(F_2) \geq m - \mu$, $\ldots$, $\mathrm{vol}(F_n) \geq m - (n-1)\mu$. It follows that

$$\begin{aligned} 1 = \mathrm{vol}(\mathcal{D}) &\geq \mathrm{vol}(F_1 \cup \ldots \cup F_n) \\ &= \mathrm{vol}(F_1) + \ldots + \mathrm{vol}(F_n) \geq nm - \frac{1}{2}\, n(n-1)\mu. \end{aligned}$$

Now, choose n such that $m/\mu < n \leq m/\mu + 1$. Thus, we have

$$1 \geq \frac{m}{\mu}\, m - \frac{1}{2}\left(\frac{m}{\mu} + 1\right)\frac{m}{\mu}\, \mu = \frac{m^2}{2\mu} - \frac{m}{2},$$

which implies $\mu \geq m^2/(2+m)$. Since $m^2/(2+m) > m^2/4$, it is impossible that $\mathrm{vol}(E_j \cap E_k) < m^2/4$ for all j and k, otherwise, if we let $\mu = m^2/4$ we get a contradiction. Hence, there must be two sets E_j and E_k such that $\mathrm{vol}(E_j \cap E_k) \geq m^2/4$. Next, we consider the product space $\mathcal{D}^n$, in which we define a measure (or volume) such that the product of n measurable sets of $\mathcal{D}$ is measurable and has a measure that equals the product of the measures of its components. For $i \geq 1$, let $E_i^n = E_i \times \ldots \times E_i \subset \mathcal{D}^n$ be measurable. Then, applying to the sequence $\{E_i^n\}_{i \geq 1}$ the result we have just obtained, we find two sets E_j^n and E_k^n such that $\mathrm{vol}(E_j^n \cap E_k^n) \geq (m^n)^2/4$. Since $\mathrm{vol}(E_j^n \cap E_k^n) = [\mathrm{vol}(E_j \cap E_k)]^n$, it follows that $\mathrm{vol}(E_j \cap E_k) \geq (1/4)^{1/n} m^2$. Clearly, for any $\lambda \in (0,1)$, $(1/4)^{1/n} \geq \lambda$ holds as long as n is large enough. So, we have $\mathrm{vol}(E_j \cap E_k) \geq \lambda m^2$. □

We recall the notion of relative density. A set T of real numbers is called *relatively dense* if there exists a positive real number l such that every interval of length l contains at least one member of the set T.

Theorem 3.3. *Let $(\mathcal{D}, \widetilde{\varphi})$ be a volume-preserving impulsive dynamical system on the set $\mathcal{D}$ with $\mathrm{vol}(\mathcal{D}) = 1$. For any measurable set $E \subset \mathcal{D}$ with $\mathrm{vol}(E) > 0$ and any $\lambda < 1$, the inequality*

$$\mathrm{vol}(E \cap (E * t)) \geq \lambda(\mathrm{vol}(E))^2$$

holds for a set of values t that is relatively dense.

Proof. Suppose to the contrary that there exists a measurable set $E \subset \mathcal{D}$ and a real number $\lambda < 1$ such that

$$\mathrm{vol}(E \cap (E * t)) < \lambda(\mathrm{vol}(E))^2 \tag{3.1}$$

holds on arbitrarily large t-intervals. Let J_1 be a closed interval on which (3.1) is true, denote by l_1 its length and by c_1 its midpoint. Also, there exists an interval J_2 on which (3.1) is true and which has length $l_2 > 2|c_1|$ and $J_1 \cap J_2 = \emptyset$. Denote by c_2 the midpoint of J_2. Since $t = 0$ does not lie in J_2, we have

$|c_2| > |c_1|$. Then, inductively, for $n > 2$ let J_n be an interval with $J_n \cap J_i = \emptyset$, for $i < n$, on which (3.1) holds and which has length $l_n > 2|c_{n-1}|$, also denote by c_n the midpoint of J_n satisfying $|c_n| > |c_{n-1}|$. Thus the numbers $c_k - c_i$, $k > i$, belong to the intervals J_k. By the induction hypothesis, it follows that $\mathrm{vol}(E \cap E * (c_k - c_i)) < \lambda(\mathrm{vol}(E))^2$. Due to the invariance of the measure, we obtain $\mathrm{vol}(E * c_i \cap E * c_k) < \lambda(\mathrm{vol}(E))^2$. Now, the sequence $\{E * c_i\}$ satisfies (3.1), which contradicts Lemma 3.2. So, the theorem is proved. □

4. Examples

In this section, we present two examples to illustrate results established in the previous section.

Example 4.1. Consider the equations $\dot{x} = \alpha$, $\dot{y} = 0$ on the open set $U = \mathbb{R} \times (0,1) \subset \mathbb{R}^2$, where α is a positive real number. Let $M = \{1\} \times (0,1)$ be an impulsive set, and define the impulsive function by $I(1,y) = (0, 1-y)$ for $y \in (0,1)$. It is easy to see that conditions (1)–(3) hold. Thus, let $\mathcal{D} = [0,1) \times (0,1)$, we obtain an impulsive system $(\mathcal{D}, \widetilde{\varphi})$, which is a volume-preserving impulsive system. Actually, $\widetilde{\varphi}$ can be considered as a conjugate system of a flow on the Möbius strip. It is easy to see that each point in $\mathcal{D}$ is a periodic point, one periodic orbit is of order 1 and the other are of order 2. Of course, every point is nonwandering. Next, we replace I by a new impulsive function I_1, which is defined by $I_1(1,y) = (0, y(2-y))$ for $y \in (0,1)$. Thus we get a new system $(\mathcal{D}, \widetilde{\varphi_1})$. Now, conditions (1) and (2) hold, but (3) is not true. Clearly, each point in $\mathcal{D}$ is wandering for $\widetilde{\varphi_1}$. This illustrates that condition (3) is crucial.

Example 4.2. Let $X = \mathbb{S}^1 \times \mathbb{R}$ be a infinite cylinder. Consider the equations $\dot{\theta} = 1$ and $\dot{r} = 2\pi\alpha$ on X, where θ is an angular coordinate and $r \in \mathbb{R}$, α is a positive real number. Let $M = \mathbb{S}^1 \times \{1\}$ be the impulsive set, and define the impulsive function by $I(\theta, 1) = (\theta, 0)$ for $\theta \in [0, 2\pi)$. Clearly, conditions (1)–(3) hold. Thus, let $\mathcal{D} = \mathbb{S}^1 \times [0,1)$, we obtain a volume-preserving impulsive system $(\mathcal{D}, \widetilde{\varphi})$. In fact, $\widetilde{\varphi}$ can be considered as a conjugate system of a flow on the torus $\mathbb{S}^1 \times \mathbb{S}^1$. If α is a rational number, then each point in $\mathcal{D}$ is a periodic point. If α is an irrational number, then each orbit is dense in $\mathcal{D}$, i.e. $\mathcal{D}$ is a minimal set. In both cases, every point is nonwandering.

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