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UNIQUENESS OF POSITIVE AND COMPACTON-TYPE SOLUTIONS FOR A RESONANT QUASILINEAR PROBLEM

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Abstract. We study a one-dimensional p-Laplacian resonant problem with p-sublinear terms and depending on a positive parameter. By using quadrature methods we provide the exact number of positive solutions with respect to $\mu \in \,]0,+\infty[$. Specifically, we prove the existence of a critical value $\mu_1 > 0$ such that the problem under examination admits: no positive solutions and a continuum of nonnegative solutions compactly supported in $[0,1]$ for $\mu \in \,]0,\mu_1[$; a unique positive solution of compacton-type for $\mu = \mu_1$; a unique positive solution satisfying Hopf's boundary condition for $\mu \in \,]\mu_1,+\infty[$.

1. Introduction

We are concerned with the existence of positive solutions to the following two-point boundary value problem:

$$(\mathrm{P}_\mu) \qquad \begin{cases} -(|u'|^{p-2}u')' = \lambda_p u^{p-1} - \mu u^{r-1} + u^{s-1} & \text{in }]0,1[, \\ u(0) = u(1) = 0, \end{cases}$$

where $r, s, p \in \,]1,+\infty[$ with $r < s < p$, $\mu \in \,]0,+\infty[$ and

$$\lambda_p = (p-1)(2\pi)^p \left(p \sin\frac{\pi}{p}\right)^{-p}$$

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is the first eigenvalue of the one-dimensional p-Laplacian with Dirichlet boundary conditions, i.e. the least eigenvalue of the problem

$$\begin{cases} -(|u'|^{p-2}u')' = \lambda u^{p-1} & \text{in }]0,1[, \\ u(0) = u(1) = 0. \end{cases}$$

By a positive solution we mean any classical solution v to (P_μ) satisfying $v(0) = v(1) = 0$ and $v > 0$ in $]0,1[$. Among these solutions we are further interested in the so-called compactons, characterized by satisfying both Dirichlet and Neumann conditions at the endpoints of the interval.

Our approach relies on quadrature methods and was successfully adopted in the recent paper [1] to deal with a problem similar to (P_μ) with p-superlinear and p-sublinear terms. In that case the competition between the opposite trends of the nonlinearity resulted in the existence of (at least) three positive solutions and two distinct curves of compactons (Theorems 2.7 and 2.9 of [1], respectively). Here the situation is more delicate as the interaction occurs between a p-linear resonant term and two p-sublinear ones. The interesting fact we found out is that the presence of the resonance entails the uniqueness of positive solutions. Indeed one has the following exact description of the set of solutions of (P_μ) with respect to $\mu \in]0,+\infty[$ (see Theorem 2.3): there exists $\mu_1 > 0$ such that (P_μ) admits no positive solution for $\mu \in]0,\mu_1[$; a unique positive non-compacton solution for $\mu \in]\mu_1,+\infty[$ and a unique compacton for $\mu = \mu_1$. This represents the central result of the paper and is proved in the following section together with two crucial preliminary lemmas.

Before all, let us establish some notations and auxiliary results useful in the sequel. From now on we denote by F the primitive of the nonlinearity in (P_μ) vanishing at 0, i.e.

$$F(t) \stackrel{\mathrm{def}}{=} \frac{\lambda_p}{p}\, t^p - \frac{\mu}{r}\, t^r + \frac{1}{s}\, t^s \quad \text{for all } t \in [0,+\infty[.$$

For all $i \in \mathbb{N} \cup \{0\}$, $F^{(i)}$ stands for the i-th derivative of F where, as customarily, $F^{(0)} \stackrel{\mathrm{def}}{=} F$. Invoking Lemmas 2.1 and 2.2 of [1], one can easily prove the following properties.

Lemma 1.1. *There exist* $t_0, t_1, t_2 \in]0,+\infty[$*, with* $t_0 > t_1 > t_2$*, such that, for each* $i = 0,1,2$*, one has*

$$\begin{aligned} F^{(i)}(t) &< 0 \quad \text{if } t \in]0,t_i[, \\ F^{(i)}(t) &> 0 \quad \text{if } t \in]t_i,+\infty[. \end{aligned} \tag{1.1}$$

Lemma 1.2. *Let* t_0 *be as in Lemma* 1.1*. Then* $t_0 = t_0(\mu)$ *has the following properties*:

(a) t_0 *is increasing in* $]0,+\infty[$;

(b)

$$\lim_{\mu\to 0^+}\left(\frac{r}{s\mu}\right)^{1/(s-r)} t_0(\mu)=1,\qquad \lim_{\mu\to+\infty}\left(\frac{\lambda_p r}{p\mu}\right)^{1/(p-r)} t_0(\mu)=1. \tag{1.2}$$

The properties of the following function will be often used in the next computations (cf. [1, Lemma 2.3]).

Lemma 1.3. *Let $x,y\in\,]1,+\infty[$ with $x<y$. Then, the function*

$$h(t)\stackrel{\text{def}}{=}\frac{t^y-1}{t^x-1}$$

is strictly increasing in $]0,1[$, $h(0)=1$ and one has

$$\lim_{t\to 1} h(t)=\frac{y}{x}.$$

We recall that the classical *gamma* and *beta* functions are defined by

$$\Gamma(x)\stackrel{\text{def}}{=}\int_0^{+\infty} t^{x-1}e^{-t}\,dt \qquad \text{for all } x>0,$$

$$\beta(x,y)\stackrel{\text{def}}{=}\int_0^1 t^{x-1}(1-t)^{y-1}\,dt \quad \text{for all } x,y>0,$$

and satisfy the well-known relations

$$\beta(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \quad \text{for all } x,y>0, \tag{1.3}$$

$$\Gamma(x)\Gamma(1-x)=\frac{\pi}{\sin\pi x} \qquad \text{for all } 0<x<1. \tag{1.4}$$

We refer the reader to [5], [6], [7], [9], [11] and the references given there for several classes of two-point boundary value problems attacked through quadrature methods. Resonant problems involving the 1D p-Laplacian and investigated with different tools can be found in [2]–[4], [8], [10] and the references enclosed.

2. Results

Our approach relies on the careful analysis of some auxiliary functions.

Lemma 2.1. *Let $t_0\in\,]0,+\infty[$ be as in Lemma 1.1 and let $\psi\colon[t_0,+\infty[\,\to\mathbb{R}$ be the function defined by*

$$\psi(\sigma)\stackrel{\text{def}}{=}\int_0^\sigma (F(\sigma)-F(t))^{-1/p}\,dt \quad \textit{for each } \sigma\ge t_0. \tag{2.1}$$

Then, the following facts hold:

(a) $\lim_{\sigma\to+\infty}\psi(\sigma)=(p/(p-1))^{1/p}/2$;

(b) *the equation $\psi'(\sigma)=0$ admits a unique solution $\sigma_1\in\,]t_0,+\infty[$ and one has: $\psi'>0$ in $]\sigma_1,+\infty[$, $\psi'<0$ in $]t_0,\sigma_1[$, with $\lim_{\sigma\to t_0^+}\psi'(\sigma)=-\infty$.*

Proof. (a) A change of the variable of integration yields

$$\psi(\sigma)=\sigma\int_0^1(F(\sigma)-F(\sigma t))^{-1/p}\,dt=\sigma\varphi(\sigma)\quad\text{for each }\sigma\geq t_0,\tag{2.2}$$

where

$$\varphi(\sigma)\overset{\text{def}}{=}\int_0^1(F(\sigma)-F(\sigma t))^{-1/p}\,dt.$$

One has

$$\begin{aligned}\varphi(\sigma)&=\int_0^1\bigg(\frac{\lambda_p}{p}(1-t^p)\sigma^p-\frac{\mu}{r}(1-t^r)\sigma^r+\frac{1}{s}(1-t^s)\,\sigma^s\bigg)^{-1/p}dt\\&=\frac{1}{\sigma}\int_0^1\bigg(\frac{\lambda_p}{p}(1-t^p)-\frac{\mu}{r}(1-t^r)\sigma^{r-p}+\frac{1}{s}(1-t^s)\sigma^{s-p}\bigg)^{-1/p}dt\end{aligned}$$

and therefore

$$\begin{aligned}\lim_{\sigma\to+\infty}\psi(\sigma)\;&=\int_0^1\bigg(\frac{\lambda_p}{p}(1-t^p)\bigg)^{-1/p}dt=\frac{1}{p}\bigg(\frac{p}{\lambda_p}\bigg)^{1/p}\beta\Big(1-\frac{1}{p},\frac{1}{p}\Big)\\&=\frac{1}{p}\bigg(\frac{p}{\lambda_p}\bigg)^{1/p}\Gamma\Big(1-\frac{1}{p}\Big)\,\Gamma\Big(\frac{1}{p}\Big)=\frac{1}{2}\Big(\frac{p}{p-1}\Big)^{1/p},\end{aligned}$$

where we have also used (1.3) and (1.4).

(b) From (2.2) one has

$$\psi'(\sigma)=\sigma\varphi'(\sigma)+\varphi(\sigma)=\int_0^1(\sigma\theta_t'(\sigma)+\theta_t(\sigma))\,dt,\tag{2.3}$$

where

$$\theta_t(\sigma)\overset{\text{def}}{=}(F(\sigma)-F(\sigma t))^{-1/p},\tag{2.4}$$

for each $t\in[0,1]$ and $\sigma>t_0$. Since

$$\begin{aligned}\theta_t(\sigma)&=\bigg(\frac{\lambda_p}{p}(1-t^p)\sigma^p-\frac{\mu}{r}(1-t^r)\sigma^r+\frac{1}{s}(1-t^s)\sigma^s\bigg)^{-1/p},\\\sigma\theta_t'(\sigma)&=-\frac{1}{p}\,\theta_t(\sigma)^{p+1}\big(\lambda_p(1-t^p)\sigma^p-\mu(1-t^r)\sigma^r+(1-t^s)\sigma^s\big),\end{aligned}$$

we have

(2.5)

$$\sigma\theta_t'(\sigma)+\theta_t(\sigma)=-\frac{1}{p}(\theta_t(\sigma))^{p+1}(1-t^r)\bigg(\Big(1-\frac{p}{s}\Big)\frac{1-t^s}{1-t^r}\,\sigma^s-\mu\Big(1-\frac{p}{r}\Big)\,\sigma^r\bigg).$$

Moreover, in view of Lemma 1.3, we get

$$\Big(1-\frac{p}{s}\Big)\frac{1-t^s}{1-t^r}\,\sigma^s-\mu\Big(1-\frac{p}{r}\Big)\,\sigma^r\leq\Big(1-\frac{p}{s}\Big)\,\sigma^s-\mu\Big(1-\frac{p}{r}\Big)\,\sigma^r$$

and, consequently, there exists $\widetilde{\sigma}_1>t_0$ such that

$$\sigma\theta_t'(\sigma)+\theta_t(\sigma)>0\quad\text{for all }\sigma\geq\widetilde{\sigma}_1\text{ and }t\in[0,1[.$$

Hence, on account of (2.3), ψ' is positive in $]\widetilde{\sigma}_1,+\infty[$.

Now, let us exploit Lemma 1.3 again to obtain

$$\begin{aligned}\lambda_p(1-t^p)\sigma^p & -\mu(1-t^r)\sigma^r+(1-t^s)\sigma^s \\ &= (1-t^r)\Big(\lambda_p\,\frac{1-t^p}{1-t^r}\,\sigma^p-\mu\sigma^r+\frac{1-t^s}{1-t^r}\,\sigma^s\Big) \\ &\geq (1-t^r)(\lambda_p\sigma^p-\mu\sigma^r+\sigma^s)\geq M(1-t^r)>0,\end{aligned}$$

for each $\sigma > t_0$ and $t \in \,]0,1[$, where $M \stackrel{\mathrm{def}}{=} \lambda_p t_0^p - \mu t_0^r + t_0^s$ is a positive constant in view of Lemma 1.1. As a result,

$$\begin{aligned}\int_0^1 \sigma\theta_t'(\sigma)\,dt &\leq -\frac{M}{p}\int_0^1 (1-t^r)(\theta_t(\sigma))^{p+1}\,dt \\ &= -\frac{M}{p}\int_0^1 (1-t^r)(F(\sigma)-F(\sigma t))^{-(p+1)/p}\,dt.\end{aligned}$$

Moreover, note that

$$\begin{aligned}\lim_{\sigma\to t_0^+} & (F(\sigma)-F(\sigma t))^{-(p+1)/p} \\ &= \Big(-\frac{\lambda_p}{p}\,t_0^p t^p+\frac{\mu}{r}\,t_0^r t^r-\frac{1}{s}\,t_0^s t^s\Big)^{-(p+1)/p} \geq \Big(\frac{\mu}{r}\,t_0^r\Big)^{-(p+1)/p} t^{-r(p+1)/p}\end{aligned}$$

for each $t \in \,]0,1]$. Therefore, being $r(p+1)/p > 1$, by Fatou's Lemma, one has

$$\lim_{\sigma\to t_0^+}\sigma\varphi'(\sigma)=\lim_{\sigma\to t_0^+}\int_0^1\sigma\theta_t'(\sigma)\,dt=-\infty.$$

Recalling that

$$\lim_{\sigma\to t_0^+}\varphi(\sigma)=\frac{\psi(t_0)}{t_0},$$

we infer, using (2.3), that

$$\lim_{\sigma\to t_0^+}\psi'(\sigma)=\lim_{\sigma\to t_0^+}(\sigma\varphi'(\sigma)+\varphi(\sigma))=-\infty.$$

Therefore, from the continuity of ψ', there exists $\sigma_1 \in \,]t_0,+\infty[$ such that $\psi'(\sigma_1) = 0$. We now show that σ_1 is the unique zero of ψ. To this aim, let us write the function ψ as

$$\psi(\sigma)=\int_0^1(\eta_t(\sigma))^{-1/p}\,dt,$$

where

$$\eta_t(\sigma)\stackrel{\mathrm{def}}{=}\frac{\lambda_p}{p}\,(1-t^p)-\frac{\mu}{r}\,(1-t^r)\sigma^{r-p}+\frac{1}{s}\,(1-t^s)\sigma^{s-p},$$

for all $t \in [0,1]$ and for all $\sigma \in [t_0, +\infty[$. Then, the computation of the first two derivatives of ψ leads to

$$\begin{aligned}\psi'(\sigma) = &-\frac{1}{p}\int_0^1 (\eta_t(\sigma))^{-1/p-1} \\ &\cdot \bigg(\frac{s-p}{s}(1-t^s)\sigma^{s-p-1} - \mu\frac{r-p}{r}(1-t^r)\sigma^{r-p-1}\bigg)\,dt, \\ \psi''(\sigma) = &\frac{1}{p}\bigg(1+\frac{1}{p}\bigg)\int_0^1 (\eta_t(\sigma))^{-1/p-2} \\ &\cdot \bigg(\frac{s-p}{s}(1-t^s)\sigma^{s-p-1} - \mu\,\frac{r-p}{r}(1-t^r)\sigma^{r-p-1}\bigg)^2 dt \\ &-\frac{1}{p}\int_0^1 (\eta_t(\sigma))^{-1/p-1}\bigg(\frac{s-p}{s}(s-p-1)(1-t^s)\sigma^{s-p-2} \\ &-\mu\,\frac{r-p}{r}(r-p-1)(1-t^r)\sigma^{r-p-2}\bigg)\,dt.\end{aligned}$$

Now, fix $K \in \,]p+1-s, p+1-r[$ and put

$$\phi(\sigma) \overset{\text{def}}{=} \sigma\psi''(\sigma) + K\psi'(\sigma) \quad \text{for all } \sigma \in [t_0, +\infty[. \tag{2.6}$$

Let us show that ϕ is positive in $[t_0, +\infty[$. Indeed, if $\sigma \in [t_0, +\infty[$, we have

$$\begin{aligned}\phi(\sigma) > &-\frac{1}{p}\int_0^1 (\eta_t(\sigma))^{-1/p-1}\bigg(\frac{s-p}{s}(s-p-1)(1-t^s)\sigma^{s-p-1} \\ &-\mu\,\frac{r-p}{r}(r-p-1)(1-t^r)\sigma^{r-p-1}\bigg)\,dt \\ &-\frac{K}{p}\int_0^1 (\eta_t(\sigma))^{-1/p-1}\bigg(\frac{s-p}{s}(1-t^s)\sigma^{s-p-1} \\ &-\mu\,\frac{r-p}{r}(1-t^r)\sigma^{r-p-1}\bigg)\,dt \\ = &\frac{1}{p}\int_0^1 (\eta_t(\sigma))^{-1/p-1}\bigg(\frac{p-s}{s}(K-p-1+s)(1-t^s)\sigma^{s-p-1} \\ &-\mu\,\frac{p-r}{r}(K-p-1+r)(1-t^r)\sigma^{r-p-1}\bigg)\,dt\end{aligned}$$

and the choice of K forces $\phi(\sigma) > 0$. At this point, solving (2.6) for ψ', one has

$$\psi'(\sigma) = \frac{1}{\sigma^K}\bigg(\int_{t_0}^{\sigma} \tau^{K-1}\phi(\tau)\,d\tau + L\bigg)$$

for all $\sigma \in [t_0, +\infty[$ and for a suitable constant $L \in \mathbb{R}$. In view of the positivity of ϕ, ψ' admits exactly one zero in $]t_0, +\infty[$ and the conclusion is reached. □

The qualitative study of the composition of ψ with the function $\mu \mapsto t_0(\mu)$ is the content of the next lemma.

Lemma 2.2. *Let $t_0 \in \,]0,+\infty[$ be as in Lemma 1.1 and let $\Psi\colon]0,+\infty[\, \to \mathbb{R}$ be the function defined by*

$$\Psi(\mu) \overset{\text{def}}{=} \int_0^{t_0(\mu)} (-F(\mu,t))^{-1/p}\, dt \quad \textit{for each } \mu > 0. \tag{2.7}$$

Then one has:

(a) $\lim_{\mu\to 0^+} \Psi(\mu) = 0$;
(b) $\lim_{\mu\to +\infty} \Psi(\mu) = p(p/(p-1))^{1/p}/(2(p-r))$;
(c) Ψ *is increasing in* $]0,+\infty[$.

Proof. (a) By performing the change of variables $t = (s\mu/r)^{1/(s-r)}\tau$, we obtain

$$\begin{aligned} \Psi(\mu) &= \int_0^{t_0(\mu)} \left(\frac{\mu}{r}\, t^r - \frac{1}{s}\, t^s - \frac{\lambda_p}{p}\, t^p\right)^{-1/p} dt \\ &= s^{(p-r)/(p(s-r))} \left(\frac{\mu}{r}\right)^{(p-s)/(p(s-r))} \\ &\cdot \int_0^{(s\mu/r)^{1/(r-s)}t_0(\mu)} \left(\tau^r - \tau^s - \frac{\lambda_p}{p}\, s^{(p-r)/(s-r)} \left(\frac{\mu}{r}\right)^{(p-s)/(s-r)} \tau^p\right)^{-1/p} d\tau. \end{aligned} \tag{2.8}$$

Thanks to Lemma 1.2, taking the limit as $\mu \to 0^+$ the integral in (2.8) tends to the positive number

$$\int_0^1 (\tau^r - \tau^s)^{-1/p}\, d\tau$$

and hence the conclusion follows.

(b) After the change of the variable of integration $t = (p\mu/\lambda_p r)^{1/(p-r)}\tau$, the function Ψ turns into

$$\begin{aligned} \Psi(\mu) = \left(\frac{p}{\lambda_p}\right)^{1/p} \int_0^{(p\mu/\lambda_p r)^{1/(r-p)}t_0(\mu)} \bigg(\tau^r - \tau^p \\ - \frac{1}{s}\left(\frac{p}{\lambda_p}\right)^{(s-r)/(p-r)} \left(\frac{\mu}{r}\right)^{(s-p)/(p-r)} \tau^s\bigg)^{-1/p} d\tau. \end{aligned} \tag{2.9}$$

Then, taking again account of Lemma 1.2, we get

$$\begin{aligned} \lim_{\mu\to+\infty} \Psi(\mu) &= \left(\frac{p}{\lambda_p}\right)^{1/p} \int_0^1 (\tau^r - \tau^p)^{-1/p}\, d\tau \\ &= \frac{1}{p-r} \left(\frac{p}{\lambda_p}\right)^{1/p} \beta\left(1 - \frac{1}{p}, \frac{1}{p}\right) = \frac{p}{2(p-r)} \left(\frac{p}{p-1}\right)^{1/p}, \end{aligned}$$

as desired.

(c) The function $]0,+\infty[\, \ni \mu \mapsto t_0(\mu)$ is of class C^1, as it can easily be proved by the Implicit Function Theorem. From $F(\mu, t_0(\mu)) = 0$ one has

$$\frac{\lambda_p}{p} + \frac{1}{s}\, t_0(\mu)^{s-p} - \mu\, \frac{1}{r}\, t_0(\mu)^{r-p} = 0 \tag{2.10}$$

which implies

$$t_0(\mu)^{s-p} < \mu\,\frac{s}{r}\,t_0(\mu)^{r-p}$$

and, in turn,

$$\tag{2.11} \begin{aligned} &\frac{s-p}{s}\,t_0(\mu)^{s-p} - \mu\,\frac{r-p}{r}\,t_0(\mu)^{r-p} \\ &\quad > \frac{s-p}{s}\,\mu\,\frac{s}{r}\,t_0(\mu)^{r-p} - \mu\,\frac{r-p}{r}\,t_0(\mu)^{r-p} = \frac{\mu}{r}\,t_0(\mu)^{r-p}(s-r) > 0. \end{aligned}$$

Now, differentiating (2.10) with respect to μ, we get

$$\begin{aligned} 0 &= \frac{s-p}{s}\,t_0(\mu)^{s-p-1}t_0'(\mu) - \mu\,\frac{r-p}{r}\,t_0(\mu)^{r-p-1}t_0'(\mu) - \frac{1}{r}\,t_0(\mu)^{r-p} \\ &= \bigg(\frac{s-p}{s}\,t_0(\mu)^{s-p} - \mu\,\frac{r-p}{r}\,t_0(\mu)^{r-p}\bigg)\,t_0(\mu)^{-1}\,t_0'(\mu) - \frac{1}{r}\,t_0(\mu)^{r-p}. \end{aligned}$$

Due to (2.11), the above identity implies $t_0'(\mu) > 0$ and

$$\tag{2.12} \begin{aligned} \frac{d}{d\mu}&\bigg(\frac{F(\mu,t_0(\mu)\tau)}{(t_0(\mu)\tau)^p}\bigg) = \frac{d}{d\mu}\bigg(\frac{\lambda_p}{p} + \frac{1}{s}\,(t_0(\mu)\tau)^{s-p} - \frac{\mu}{r}\,(t_0(\mu)\tau)^{r-p}\bigg) \\ &= \frac{s-p}{s}\,t_0(\mu)^{s-p-1}\tau^{s-p}t_0'(\mu) \\ &\quad - \mu\,\frac{r-p}{r}\,t_0(\mu)^{r-p-1}\tau^{r-p}t_0'(\mu) - \frac{1}{r}\,t_0(\mu)^{r-p}\tau^{r-p} \\ &= \tau^{r-p}\bigg(\frac{s-p}{s}\,t_0(\mu)^{s-p-1}\tau^{s-r}t_0'(\mu) \\ &\quad - \mu\,\frac{r-p}{r}\,t_0(\mu)^{r-p-1}t_0'(\mu) - \frac{1}{r}\,t_0(\mu)^{r-p}\bigg) \\ &> \tau^{r-p}\bigg(\frac{s-p}{s}\,t_0(\mu)^{s-p-1}t_0'(\mu) \\ &\quad - \mu\,\frac{r-p}{r}\,t_0(\mu)^{r-p-1}t_0'(\mu) - \frac{1}{r}\,t_0(\mu)^{r-p}\bigg) = 0 \end{aligned}$$

for each $\tau \in\,]0,1[$. At this point, performing the change of variable $t = t_0(\mu)\tau$, we can write

$$\Psi(\mu) = \int_0^1 (-F(\mu,t_0(\mu)\tau))^{-1/p}t_0(\mu)\,d\tau = \int_0^1 \frac{1}{\tau}\bigg(-\frac{F(\mu,t_0(\mu)\tau)}{(t_0(\mu)\tau)^p}\bigg)^{-1/p}d\tau$$

and, differentiating and using (2.12), we get

$$\Psi'(\mu) = \int_0^1 -\frac{1}{p\tau}\bigg(-\frac{F(\mu,t_0(\mu)\tau)}{(t_0(\mu)\tau)^p}\bigg)^{-1/p-1}\bigg(-\frac{d}{d\mu}\bigg(\frac{F(\mu,t_0(\mu)\tau)}{(t_0(\mu)\tau)^p}\bigg)\bigg)\,d\tau > 0,$$

which concludes the proof. □

In the light of the previous two lemmas, we can state and prove our main result:

Theorem 2.3. *There exists $\mu_1 > 0$ such that*:

(a) *for* $0 < \mu < \mu_1$, (P_μ) *admits no positive solutions and a continuum of nonnegative solutions, compactly supported in* $[0,1]$;
(b) *for* $\mu = \mu_1$, (P_μ) *admits a unique positive compacton-type solution*;
(c) *for* $\mu > \mu_1$, (P_μ) *admits a unique positive solution satisfying Hopf's boundary condition.*

PROOF. It is a known matter that positive solutions u to (P_μ) are symmetric with respect to the midpoint of the interval of definition and attain there their unique global (and local) maximum; denote

$$\sigma \stackrel{\text{def}}{=} \|u\|_\infty = u\left(\frac{1}{2}\right).$$

By multiplying both sides of the equation in (P_μ) by u' and integrating by parts from $x \in]0,1/2[$ to $1/2$, we arrive at

$$\frac{p-1}{p}\,|u'(x)|^p = F(\sigma) - F(u(x)). \tag{2.13}$$

If we plug the value $x = 0$ in the above relation, we deduce that $F(\sigma) \geq 0$ and, on account of Lemma 1.1, that $\sigma \geq t_0$.

Now, being u increasing in $]0,1/2[$, from (2.13) one has

$$u'(x) = \left(\frac{p}{p-1}\right)^{1/p} (F(\sigma) - F(u(x)))^{1/p} \tag{2.14}$$

and therefore

$$\int_0^x u'(y)(F(\sigma) - F(u(y)))^{-1/p}\,dy = \left(\frac{p}{p-1}\right)^{1/p} x.$$

Putting $u(y) = t$, we get

$$\int_0^{u(x)} (F(\sigma) - F(t))^{-1/p}\,dt = \left(\frac{p}{p-1}\right)^{1/p} x, \tag{2.15}$$

which evaluated at $x = 1/2$ yields

$$\int_0^{\sigma} (F(\sigma) - F(t))^{-1/p}\,dt = \frac{1}{2}\left(\frac{p}{p-1}\right)^{1/p}. \tag{2.16}$$

In the light of these computations we deduce therefore the following fact: if $\sigma \in [t_0,+\infty[$ is a solution to (2.16), then the function u implicitly defined in $[0,1/2]$ by (2.15) and symmetrically extended to $[0,1]$ is a positive solution to (P_μ). As a result, problem (P_μ) admits as many positive solutions as the solutions to (2.16) (for the unknown σ) in $[t_0,+\infty[$.

Likewise, relation (2.13) tells us that the possible compacton-type solutions to (P_μ) correspond to the solutions (for μ) to the equation

$$\int_0^{t_0(\mu)} (-F(\mu,t))^{-1/p}\,dt = \frac{1}{2}\left(\frac{p}{p-1}\right)^{1/p}. \tag{2.17}$$

Taking all this into account, let us prove first item (b). Being $p > r$, one has

$$\frac{p}{2(p-r)}\left(\frac{p}{p-1}\right)^{1/p} > \frac{1}{2}\left(\frac{p}{p-1}\right)^{1/p}.$$

Then due to Lemma 2.2 and the continuity of Ψ, there exists a unique $\mu_1 > 0$ such that

$$\Psi(\mu_1) = \frac{1}{2}\left(\frac{p}{p-1}\right)^{1/p} \tag{2.18}$$

and so (P_μ) possesses a unique positive compacton-type solution for $\mu = \mu_1$.

Now let us pass to (c). On account of Lemma 2.2 and (2.18),

$$\Psi(\mu) > \frac{1}{2}\left(\frac{p}{p-1}\right)^{1/p} \quad \text{for any } \mu > \mu_1.$$

Then, by virtue of Lemma 2.1, for $\mu > \mu_1$ the equation

$$\psi(\sigma) = \frac{1}{2}\left(\frac{p}{p-1}\right)^{1/p}$$

will admit exactly one solution in $]t_0, +\infty[$ and, equivalently, (P_μ) will possess a unique positive solution. Moreover, named u_μ such a solution, one has $\sigma = \|u_\mu\|_\infty > t_0$ and so, by (2.14),

$$u'_\mu(0) = \left(\frac{p}{p-1}\,F(\sigma)\right)^{1/p} > 0 \quad \text{and, symmetrically,} \quad u'_\mu(1) < 0,$$

namely u_μ satisfies Hopf's condition on the boundary, as claimed.

Finally, let us show that (a) is verified as well. Since Ψ is increasing in $]0, +\infty[$ and $\lim_{\mu\to 0^+} \Psi(\mu) = 0$, then for each $\mu \in]0, \mu_1[$ there exists a continuum of nondegenerate compact intervals $[a, b] \subset [0, 1]$ such that

$$\Psi(\mu) = \frac{b-a}{2}\left(\frac{p}{p-1}\right)^{1/p}.$$

Thus, if $[a, b]$ is any of these intervals, in correspondence to μ we find a compacton solution $u_{a,b}$ to (P_μ) in the interval $[a, b]$. The zero-extension of $u_{a,b}$ to the whole $[0, 1]$ yields a nonnegative nonzero solution to (P_μ) in the interval $[0, 1]$ compactly supported when $[a, b] \subset]0, 1[$. Therefore, for $\mu \in]0, \mu_1[$ we can determine a continuum of non-negative non-zero solutions, which have compact support. Finally, observe that for $\mu \in]0, \mu_1[$, we have $\psi(\sigma) < (p/(p-1))^{1/p}/2$ for $\sigma \geq t_0$, and so no positive solution may exist. □

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