

**ON A CLASS OF QUASILINEAR ELLIPTIC PROBLEMS
WITH CRITICAL EXPONENTIAL GROWTH
ON THE WHOLE SPACE**

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ABSTRACT. In this paper we prove a kind of weighted Trudinger–Moser inequality which is employed to establish sufficient conditions for the existence of solutions to a large class of quasilinear elliptic differential equations with critical exponential growth. The class of operators considered includes, as particular cases, the Laplace, p -Laplace and k -Hessian operators when acting on radially symmetric functions.

1. Introduction

In this paper we deal with a general class of quasilinear operators in radial form which includes perturbations of p -Laplace and k -Hessian operators. Let us first consider the following p -Laplace equation:

$$(1.1) \quad \operatorname{div}(|\nabla u|^{p-2}\nabla u) + f(u) = 0 \quad \text{in } \Omega \subset \mathbb{R}^N, \quad u|_{\partial\Omega} = 0.$$

In the seminal work [20], Gidas, Ni and Nirenberg proved that all positive solutions $u \in C^2$ of the above problem are necessarily radially symmetric when $p = 2$, $f \in C^1$ and $\Omega = B_R$ is the open ball with center 0 and radius $R > 0$ in \mathbb{R}^N , $N \geq 2$. Also, in [21], they proved symmetry of solutions when $\Omega = \mathbb{R}^N$, $N \geq 3$, is the whole space. This kind of results for $p \neq 2$ was established by

2010 *Mathematics Subject Classification.* 35J92, 46E35, 26D10.

Key words and phrases. Trudinger–Moser inequality; quasilinear elliptic equation; critical growth; radial operators.

Felmer et al. in [19] and Damascelli et al. in [7], [8]. In view of this, if $\Omega = B_R$, for a wide class of nonlinearities f we can reduce problem (1.1) to the following:

$$(1.2) \quad r^{1-N}(r^{N-1}|u'|^{p-2}u')' + f(u) = 0 \quad \text{in } (0, R), \quad u'(0) = u(R) = 0.$$

Another interesting problem investigated in this paper concerns the k -Hessian equation

$$(1.3) \quad S_k(D^2u) + f(u) = 0 \quad \text{in } \Omega \subset \mathbb{R}^N, \quad u|_{\partial\Omega} = 0,$$

where $1 \leq k \leq N$ and $S_k(D^2u)$ is the the sum of all principal $k \times k$ minors of the Hessian matrix D^2u , see [28]. For instance, $S_1(D^2u) = \Delta u$ and $S_N(D^2u) = \det(D^2u)$ is the Monge–Ampère operator. As noted in [22], when $\Omega = B_R$ is an open ball in \mathbb{R}^N and f satisfies suitable conditions, the Alexandrov–Serrin moving plane method [27] used in [20] extends to (1.3) (see [11] for the Monge–Ampère case) reducing it to following equation:

$$(1.4) \quad r^{1-N}(r^{N-k}|u'|^{k-1}u')' + f(u) = 0 \quad \text{in } (0, R), \quad u'(0) = u(R) = 0.$$

Therefore, under the previous discussion, for a wide class of functions f all of the above problems are special cases of a more general family of problems

$$(1.5) \quad \begin{cases} r^{-\theta}(r^\alpha|u'|^{p-2}u')' + f(r, u) = 0 & \text{for } r \in (0, R), \\ u > 0 & \text{for } r \in (0, R), \\ u'(0) = u(R) = 0, \end{cases}$$

where certain conditions are to be imposed on the parameters α, p and θ . In recent years, several authors [5], [10], [17], [23], [24] have studied this class of problems under different conditions on parameters α, p and θ and on the nonlinearity f . In [5], de Figueiredo et al. introduced suitable function spaces to study problem (1.5) variationally. In particular, a critical exponent was found which allows to treat the Brezis–Nirenberg type problem [4]. More recently, in [17] the existence of non-trivial solution was established when f has critical exponential growth that represents the counterpart to [5].

All foregoing results on problem (1.5) were established for the bounded case $R < \infty$. The main goal of this article is to study the class of problems (1.5) for critical exponential growth on the whole space, that is, $R = +\infty$. In order to formulate our results, let us present the framework for the function space setting suitable to study these problems. Let $X_R^{1,p}(\alpha, \theta)$, or more simply X_R , be the weighted Sobolev spaces defined as follows: For $0 < R \leq \infty$ and $\theta \geq 0$, let $L_\theta^q = L_\theta^q(0, R)$ be the weighted Lebesgue space defined as the set of all measurable

functions u on $(0, R)$ for which

$$\|u\|_{L^q_\theta} = \begin{cases} \left(\omega_\theta \int_0^R r^\theta |u(r)|^q dr \right)^{1/q} < \infty & \text{if } 1 \leq q < \infty, \\ \text{ess sup}_{0 < r < R} |u(r)| < \infty & \text{if } q = \infty, \end{cases}$$

where ω_θ is a normalising constant defined by

$$(1.6) \quad \omega_\theta = \frac{2\pi^{(\theta+1)/2}}{\Gamma((\theta+1)/2)} \quad \text{with } \Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt.$$

Let $AC_{\text{loc}}(0, R)$ be the collection of locally absolutely continuous functions in $(0, R)$ and denote by $W_R^{1,p}(\alpha, \theta)$, with $p \geq 1$ and $\alpha, \theta \geq 0$, the space of all $u \in AC_{\text{loc}}(0, R)$ such that $u' \in L^p_\alpha$ and $u \in L^p_\theta$. The weighted Sobolev space $W_R^{1,p}$ has a Banach property equipped with the norm

$$(1.7) \quad \|u\|_{W_R^{1,p}} = (\|u'\|_{L^p_\alpha}^p + \|u\|_{L^p_\theta}^p)^{1/p}.$$

In this way, $X_R^{1,p}(\alpha, \theta)$ is the closure of the set

$$X = \{u \in W_R^{1,p}(\alpha, \theta) : \lim_{r \rightarrow R} u(r) = 0\},$$

under the norm (1.7). As noted in the previous papers [12], [13], [17], we can distinguish two cases with particular characteristics:

$$\alpha - p + 1 > 0 \quad (\text{Sobolev case}) \quad \text{and} \quad \alpha - p + 1 = 0 \quad (\text{Trudinger–Moser case}).$$

In this paper we are interested in the study of Trudinger–Moser case when $R = +\infty$. However, firstly we discuss (briefly) some results for the case $R > 0$ finite which will be used for our purpose. As a consequence of a Hardy-type inequality [5, Proposition 1.0], for the *Sobolev case*, we can prove continuity of the embeddings

$$(1.8) \quad X_R^{1,p}(\alpha, \theta) \hookrightarrow L^q_\nu \quad \text{if } q \in (1, p^*] \quad \text{and} \quad \min\{\theta, \nu\} \geq \alpha - p,$$

where $p^* := p^*(\alpha, p, \nu) = (\nu + 1)p/(\alpha - p + 1)$ is the Sobolev critical exponent for this class of spaces, see [5]. Also, the embeddings (1.8) are compact if $q < p^*$. On the other hand, in the *Trudinger–Moser case* we have compactness of the embeddings

$$(1.9) \quad X_R^{1,p}(\alpha, \theta) \hookrightarrow L^q_\nu \quad \text{for all } q \in (1, \infty) \quad \text{and} \quad \nu \geq 0.$$

For this case, $p^* \rightsquigarrow \infty$ (formally) which suggests that $X_R \subset L^\infty_\nu$, but it is not true (see [12]). Consequently, the question arises to determine the maximal growth for a function g such that $g(u) \in L^1_\nu$ whenever $u \in X_R$. To answer this question, in [12] there was proved a Trudinger–Moser type inequality (see, [26]) for spaces X_R which ensures that exponential growth is available. More precisely,

THEOREM 1.1. *Let $0 < R < \infty$, $\alpha \geq 1$, $\theta, \nu \geq 0$ be real numbers and $\alpha - p + 1 = 0$. Then, for any $\mu > 0$ and $u \in X_R^{1,p}(\alpha, \theta)$ we have $\exp(\mu|u|^{p'}) \in L_\nu^1$. Moreover, there exists $c > 0$ depending only on α, p, ν and R such that*

$$(1.10) \quad \sup_{\|u'\|_{L_\alpha^p} \leq 1} \int_0^R r^\nu e^{\mu|u|^{p'}} dr \begin{cases} \leq c & \text{if } \mu \leq \mu_{\alpha,\nu}, \\ = \infty & \text{if } \mu > \mu_{\alpha,\nu}, \end{cases}$$

where $\mu_{\alpha,\nu} = (\nu + 1)\omega_\alpha^{1/\alpha}$ and $p' = p/(p - 1)$.

In [17, Theorem 1.2], the authors offer a complete answer proving that the exponential growth is optimal. Moreover, in view of (1.10), we have the continuity of the embedding $X_R \hookrightarrow L_A(\nu)$, where $L_A(\nu)$ is the a weighted Orlicz space defined by the Young function $A(s) = \exp(|s|^{p'}) - 1$. However, this embedding is not compact, as showed in [17, Corollary 2.1]. So, in this sense the exponential growth is critical for the study of this class of problems (1.5).

At this point we turn our attention to the case $R = +\infty$. Note that $R > 0$ is finite in Theorem A and, thus in order to study problem (1.5) for critical exponential growth on the whole space, we must prove a corresponding Trudinger–Moser type inequality. For this purpose, let us denote

$$(1.11) \quad \varphi(s) = e^s - \sum_{k=0}^{k_0-2} \frac{s^k}{k!}, \quad \text{with } k_0 = \min \{z \in \mathbb{N} : z \geq p\}.$$

We shall prove the following weighted Trudinger–Moser inequality

THEOREM 1.2. *Suppose $\theta \geq 0$, $\alpha \geq 1$, $p = \alpha + 1$ and φ is defined in (1.11). Then, for all $u \in X_\infty^{1,p}(\alpha, \theta)$ and $\mu > 0$ we have $\varphi(\mu|u|^{p'}) \in L_\theta^1(0, \infty)$. Moreover, if $\|u\|_{L_\theta^p} \leq M$ and $\mu < \mu_{\alpha,\theta} = (\theta + 1)\omega_\alpha^{1/\alpha}$, then there exists $c > 0$, independent of u , such that*

$$\sup_{\|u'\|_{L_\alpha^p} \leq 1} \int_0^\infty r^\theta \varphi(\mu|u|^{p'}) dr \leq c.$$

We observe that this kind of Trudinger–Moser inequality has been investigated by several authors for the classical Sobolev spaces $W^{1,N}(\mathbb{R}^N)$. The pioneer work is due to Cao [6], for $N = 2$, which was extended by do Ó [16] to $N \geq 3$ and, more recently, in the presence of a singular term by de Souza [14].

In this note we apply the above result to establish existence of non-trivial weak solution for the class of quasilinear elliptic equations (1.5) when f has exponential growth on the whole space. More specifically, we shall consider the problem

$$(1.12) \quad \begin{cases} -r^{-\theta}(r^\alpha|u'|^{p-2}u')' + V(r)|u|^{p-2}u = f(r, u), & u > 0 \quad \text{in } (0, \infty), \\ \lim_{r \rightarrow \infty} u(r) = 0, \end{cases}$$

where we assume that the parameters satisfy

$$(1.13) \quad \alpha \geq 1, \quad \theta \geq 0 \quad \text{and} \quad \alpha - p + 1 = 0$$

and some conditions on the functions V and f are imposed. Namely, $V: [0, \infty) \rightarrow \mathbb{R}$ is a continuous function satisfying:

- (V₁) for some $V_0 > 0$, we have $V(r) \geq V_0$ for all $r \geq 0$,
- (V₂) $V(r) \rightarrow \infty$ as $r \rightarrow \infty$ (coercive);

and concerning $f: [0, \infty) \times \mathbb{R} \rightarrow \mathbb{R}$ we assume that

- (f₁) it is continuous and there exist positive constants μ_0, a_1 and a_2 such that

$$|f(r, u)| \leq a_1|u|^{p-1} + a_2\varphi(\mu_0|u|^{p'}),$$

where φ is given by (1.11),

- (f₂) there exists a constant $q > p$ such that, for all $r \in [0, \infty)$ and $u > 0$,

$$0 \leq qF(r, u) \leq uf(r, u), \quad \text{where } F(r, u) = \int_0^u f(r, s) ds,$$

- (f₃) there exist positive constants L and M_0 such that

$$0 < F(r, u) \leq M_0f(r, u) \quad \text{for all } r \in [0, \infty) \text{ and } u > L,$$

- (f₄) for Γ as in (1.6), there exists $\rho > 0$ such that

$$\lim_{u \rightarrow \infty} \frac{uf(r, u)}{e^{\mu_0|u|^{p'}}} > \frac{(\theta + 1)e^{\mathcal{S}_\rho}}{\rho^{\theta+1}} \left(\frac{\theta + 1}{\mu_0} \right)^{p-1} > 0,$$

$$\text{with } \mathcal{S}_\rho = \frac{\rho^{\theta+1}}{(\theta + 1)^p} \frac{\Gamma(p + 1)}{p - 1} \max_{s \in [0, \rho]} V(s).$$

uniformly on compact subsets of $[0, \infty)$.

Now, we consider the subspace $E \subset X_\infty^{1,p}(\alpha, \theta)$ given by

$$E = \left\{ u \in X_\infty^{1,p}(\alpha, \theta) : \int_0^\infty r^\theta V(r)|u|^p dr < \infty \right\}$$

endowed with the norm

$$\|u\|_E^p = \int_0^\infty r^\alpha |u'|^p dr + \int_0^\infty r^\theta V(r)|u|^p dr.$$

From (V₁), E is a Banach space continuously embedded in $X_\infty^{1,p}$ and, furthermore,

$$(1.14) \quad \Lambda_1 := \Lambda_1(\alpha, p, \theta, V) \\ = \inf_{u \in E \setminus \{0\}} \frac{\int_0^\infty r^\alpha |u'|^p dr + \int_0^\infty r^\theta V(r)|u|^p dr}{\int_0^\infty r^\theta |u|^p dr} \geq V_0 > 0.$$

Now, our existence result reads as

THEOREM 1.3. *Suppose (V_1) – (V_2) hold and f satisfies (f_1) – (f_4) . Furthermore, assume*

$$(1.15) \quad \limsup_{u \rightarrow 0^+} \frac{pF(r, u)}{|u|^p} < \Lambda_1 \quad \text{uniformly in } r \in [0, \infty).$$

Then, problem (1.12) has a non-trivial weak solution.

It is worth to emphasize that this type of existence result was established by do Ó [16] for the N -Laplace operator in the context of classical Sobolev spaces $W^{1,N}(\mathbb{R}^N)$. Some extensions of do Ó's result can be found in [1], [14], [18], [25] and references therein. Our results improve and complement those results by considering non-integer parameters α, θ and including the k -Hessian operator.

REMARK 1.4. In accordance with (1.2) and (1.4), our assumption on parameters (1.13) allows to include the p -Laplace operator for $p = N \geq 2$ and the k -Hessian operator for $k = N/2$.

We organize this paper as follows: In Section 2 we prove some preliminary material on the spaces $W_R^{1,p}$, X_R and E including extensions of the embeddings (1.8) and (1.9). In Section 3, we establish a Trudinger–Moser type inequality on the whole space, see Theorem 1.2. In Section 4, we give the variational formulation of problem (1.12) and show that the associated functional satisfies the conditions of the mountain-pass theorem due to Ambrosetti–Rabinowitz. Finally, the proof of existence result stated in Theorem 1.3 is given in Section 5.

2. Weighted Sobolev embedding

This section is devoted to proving some preliminary results on the weighted Sobolev spaces $W_R^{1,p}$, $X_R^{1,p}$ and E . Throughout this section we shall assume that the parameters α, θ, ν are non-negative and $p \geq 2$. We start by establishing the existence of an extension type operator.

LEMMA 2.1. *Let $0 < L \leq \infty$ satisfy $L > 2R$. Then there exists a linear extension operator*

$$T: W_R^{1,p}(\alpha, \theta) \rightarrow X_L^{1,p}(\alpha, \theta)$$

such that for any $u \in W_R^{1,p}(\alpha, \theta)$, we have $Tu = u$ in $(0, R)$ and $\text{supp } Tu \subset [0, 2R)$. Furthermore,

$$(2.1) \quad \|Tu\|_{W_L^{1,p}} \leq C \|u\|_{W_R^{1,p}},$$

where $C > 0$ depends only on α, p, R and θ .

PROOF. Fix an auxiliary function $\eta \in C^1[0, \infty)$, $0 \leq \eta \leq 1$, such that

$$\eta(r) = \begin{cases} 1 & \text{if } 0 \leq r < R/4, \\ 0 & \text{if } r > 3R/4. \end{cases}$$

Given $u \in W_R^{1,p}(\alpha, \theta)$, set $v_1, v_2: [0, L] \rightarrow \mathbb{R}$ be defined by

$$v_1(r) = \begin{cases} \eta u & \text{if } 0 \leq r \leq R, \\ 0 & \text{if } R < r \leq L, \end{cases}$$

and

$$v_2(r) = \begin{cases} (1 - \eta)u & \text{if } 0 \leq r \leq R, \\ (1 - \eta(2R - r))u(2R - r) & \text{if } R < r \leq 7R/4, \\ 0 & \text{if } 7R/4 \leq r \leq L. \end{cases}$$

Clearly v_1, v_2 are locally absolutely continuous functions on $[0, L]$ and $v_1(L) = v_2(L) = 0$.

Set $Tu = v_1 + v_2$. By construction, T is a linear operator and, obviously, $Tu = u$ on $(0, R)$ and $\text{supp } Tu \subset [0, 7R/4]$. It remains to prove (2.1). Since $0 \leq \eta \leq 1$ and $v_1 \equiv 0$ on $(R, L]$, we obtain

$$(2.2) \quad \int_0^L r^\theta |v_1|^p dr \leq \int_0^R r^\theta |u|^p dr.$$

Moreover,

$$(2.3) \quad \int_0^L r^\theta |v_2|^p dr \leq \int_0^R r^\theta |u|^p dr + \int_R^{7R/4} r^\theta |u(2R - r)|^p dr.$$

Since $s \mapsto (2R - s)^\theta / s^\theta$ is bounded on $(R/4, R)$, making the change $s = 2R - r$ we get

$$(2.4) \quad \begin{aligned} \int_R^{7R/4} r^\theta |u(2R - r)|^p dr &= \int_{R/4}^R s^{-\theta} (2R - s)^\theta s^\theta |u(s)|^p ds \leq C \int_0^R s^\theta |u|^p ds. \end{aligned}$$

Combining (2.4) and (2.3), we have

$$(2.5) \quad \int_0^L r^\theta |v_2|^p dr \leq C \int_0^R r^\theta |u|^p dr.$$

From (2.2) and (2.5),

$$(2.6) \quad \|Tu\|_{L^p_\theta(0,L)} \leq C \|u\|_{L^p_\theta(0,R)}.$$

On the other hand, using $|v'_1|^p \leq 2^p (|\eta' u|^p + |\eta u'|^p)$ on $(0, R)$ and $v'_1(r) = 0$ for $r \geq R$, we obtain

$$\begin{aligned} \int_0^L r^\alpha |v'_1|^p dr &\leq 2^p \int_0^R r^\alpha (|\eta' u|^p + |\eta u'|^p) dr \\ &= 2^p \left(\int_{R/4}^R r^{\alpha-\theta} |\eta'|^p r^\theta |u|^p dr + \int_0^R |\eta|^p r^\alpha |u'|^p dr \right). \end{aligned}$$

Thus, since $r \mapsto r^{\alpha-\theta}|\eta'|$ is bounded on $(R/4, R)$, we get

$$(2.7) \quad \|v'_1\|_{L^p_\alpha(0,L)} \leq C\|u\|_{W_R^{1,p}}.$$

Analogously, $|v'_2|^p \leq 2^p(|\eta'u|^p + |(1-\eta)u'|^p)$ on $(0, R)$, $v'_2 \equiv 0$ for $r \geq 7R/4$ and

$$|v'_2|^p \leq 2^p(|\eta'(2R-r)u(2R-r)|^p + |(1-\eta(2R-r))u'(2R-r)|^p)$$

on $(R, 7R/4)$. Therefore, arguing as in (2.4) and (2.7), we can write

$$(2.8) \quad \|v'_2\|_{L^p_\alpha(0,L)} \leq C\|u\|_{W_R^{1,p}}.$$

Finally, using the definition of T and combining (2.6)–(2.8), we obtain (2.1). \square

LEMMA 2.2. *Let $R > 0$ be finite. Then:*

- (a) *If $\alpha - p + 1 > 0$ and $\min\{\theta, \nu\} \geq \alpha - p$, we have the continuous embedding $W_R^{1,p}(\alpha, \theta) \hookrightarrow L^q_\nu(0, R)$ for any $q \in (1, p^*]$. Further, in the strict case $1 < q < p^*$ it is compact.*
- (b) *If $\alpha - p + 1 = 0$, we have the compact embedding $W_R^{1,p}(\alpha, \theta) \hookrightarrow L^q_\nu(0, R)$ for all $q \in (1, \infty)$.*

PROOF. Fix L such that $2R < L < \infty$. Let T be the linear extension operator given by Lemma 2.1. Using (1.8) and (1.9), under these assumptions in either case (a) or (b), we can consider the following chain of operators:

$$W_R^{1,p}(\alpha, \theta) \xrightarrow{T} X_L^{1,p}(\alpha, \theta) \xrightarrow{i} L^q_\nu(0, L),$$

where i is the inclusion operator. Since $i \circ T(u) = u$ on $(0, R)$, the result follows from (1.8) and (1.9) again. \square

LEMMA 2.3. *For any $u \in X_\infty^{1,p}(\alpha, \theta)$, we have*

$$|u(r)| \leq \frac{C}{r^\chi} \|u\|_{W_\infty^{1,p}} \quad \text{for all } r > 0,$$

where $\chi = (\alpha + (p-1)\theta)/p^2$ and $C > 0$ depends only on α, p and θ .

PROOF. By density, we can assume $u(r) \rightarrow 0$ as $r \rightarrow \infty$. Thus, for $p \geq 2$

$$|u(r)|^p = -p \int_r^\infty u'(s)|u(s)|^{p-2}u(s) ds.$$

Hence, from Young's inequality,

$$\begin{aligned} |u(r)|^p &\leq p \int_r^\infty s^{-\alpha/p-\theta/p'} (s^{\alpha/p}|u'(s)| \cdot s^{\theta/p'}|u(s)|^{p-1}) ds \\ &\leq pr^{-\alpha/p-\theta/p'} \int_r^\infty s^{\alpha/p}|u'(s)| \cdot s^{\theta/p'}|u(s)|^{p-1} ds \\ &\leq pr^{-\alpha/p-\theta/p'} \int_r^\infty \left(\frac{s^\alpha|u'(s)|^p}{p} + \frac{s^\theta|u(s)|^p}{p'} \right) ds \\ &\leq Cr^{-\alpha/p-\theta/p'} (\|u'\|_{L^p_\alpha}^p + \|u\|_{L^p_\theta}^p) \end{aligned}$$

which completes the proof. \square

LEMMA 2.4. *Suppose $R = \infty$. Then:*

- (a) *If $\alpha - p + 1 > 0$ and $\theta \geq \alpha - p$, we have the continuous embedding $X_\infty^{1,p}(\alpha, \theta) \hookrightarrow L_\theta^q$ for any $q \in [p, p^*]$. Moreover, in the strict case $p < q < p^*$ it is compact.*
- (b) *If $\alpha - p + 1 = 0$, we have the continuous embedding $X_\infty^{1,p}(\alpha, \theta) \hookrightarrow L_\theta^q$ for all $q \in [p, \infty)$. In the strict case $q > p$, this embedding is compact.*

PROOF. From Lemma (2.3), given $L > 0$ (finite), we can write for any $u \in X_\infty^{1,p}$

$$(2.9) \quad |u(r)| \leq \frac{C}{r^\chi} \|u\|_{W_\infty^{1,p}} \leq \frac{C}{L^\chi} \|u\|_{W_\infty^{1,p}} \quad \text{for all } r \geq L,$$

where $\chi > 0$. Thus, for all $q > p$,

$$\omega_\theta \int_L^\infty r^\theta |u|^q dr \leq \frac{C^{q-p}}{L^{(q-p)\chi}} \|u\|_{W_\infty^{1,p}}^{q-p} \left(\omega_\theta \int_L^\infty r^\theta |u|^p dr \right).$$

Hence, since $\omega_\theta \int_L^\infty r^\theta |u|^p dr \leq \|u\|_{W_\infty^{1,p}}^p$, it follows that

$$(2.10) \quad \omega_\theta \int_L^\infty r^\theta |u|^q dr \leq \frac{C^{q-p}}{L^{(q-p)\chi}} \|u\|_{W_\infty^{1,p}}^q$$

for all $q > p$. We proceed with showing continuity of the embeddings. Obviously, we have the continuous embedding $X_\infty^{1,p}(\alpha, \theta) \hookrightarrow L_\theta^p$. Thus, we can assume $q > p$. Using Lemma 2.2, there exists $C > 0$ such that

$$(2.11) \quad \|u\|_{L_\theta^q(0,L)}^q \leq C \|u\|_{W_L^{1,p}}^q,$$

where $1 < q \leq p^*$ if $\theta \geq \alpha - p$ and $\alpha - p + 1 > 0$, and for $1 < q < \infty$ when $\alpha - p + 1 = 0$. Under these conditions, combining (2.10) and (2.11), we obtain

$$\|u\|_{L_\theta^q(0,\infty)}^q \leq C \|u\|_{W_\infty^{1,p}}^q$$

for some $C > 0$. This completes the proof of continuity. In order to prove compactness we will show that, up to a subsequence, $u_n \rightarrow 0$ strongly in $L_\theta^q(0, \infty)$ whenever $u_n \rightharpoonup 0$ weakly in $X_\infty^{1,p}(\alpha, \theta)$. The weak convergence gives $\|u_n\|_{W_\infty^{1,p}} \leq c$ for some $c > 0$. Thus, fixed $q > p$, from (2.10), given $\varepsilon > 0$, we can take $L_0 > 0$ such that

$$(2.12) \quad \int_{L_0}^\infty r^\theta |u_n|^q dr \leq \frac{\varepsilon}{2} \quad \text{for all } n \in \mathbb{N}.$$

On the other hand, since that the restriction operator $u \mapsto u|_{(0,L_0)}$ is continuous from $X_\infty^{1,p}(\alpha, \theta)$ into $W_{L_0}^{1,p}(\alpha, \theta)$, we also have that $u_n \rightarrow 0$ in $W_{L_0}^{1,p}(\alpha, \theta)$. Therefore, due to compactness of the embeddings in Lemma 2.2, we can take n_0 for which

$$(2.13) \quad \int_0^{L_0} r^\theta |u_n|^q dr \leq \frac{\varepsilon}{2} \quad \text{for all } n \geq n_0,$$

where $1 < q < p^*$ if $\theta \geq \alpha - p$ and $\alpha - p + 1 > 0$, and $1 < q < \infty$ if $\alpha - p + 1 = 0$. Hence, combining (2.12) and (2.13), we get the result. \square

We observe that inequality (2.10) holds under the strict condition $q > p$. Thus, we cannot conclude that the embedding $X_\infty^{1,p}(\alpha, \theta) \hookrightarrow L_\theta^q$ is compact when $q = p$. However, for subspace $E \subset X_\infty^{1,p}(\alpha, \theta)$ the compactness holds for $p = q$. Namely,

LEMMA 2.5. *Suppose (V₁)–(V₂) hold. Then:*

- (a) *If $\alpha - p + 1 > 0$ and $\theta > \alpha - p$, we have the compact embedding $E \hookrightarrow L_\theta^q$ for any $q \in [p, p^*)$.*
- (b) *If $\alpha - p + 1 = 0$, we have the compact embedding $E \hookrightarrow L_\theta^q$ for all $q \in [p, \infty)$.*

PROOF. Due to continuity of the embeddings $E \hookrightarrow X_\infty^{1,p}(\alpha, \theta)$ and Lemma 2.4, we have the result for the strict case $q > p$. We will restrict our attention to the case $p = q$. Let (u_n) be a sequence in E so that $u_n \rightharpoonup 0$ weakly in E . It follows that $\|u_n\|_E \leq c$ for all $n \in \mathbb{N}$. From (V₂), given $\varepsilon > 0$, it is possible to choose $L_0 > 0$ such that $V(r) \geq 2c^p/\varepsilon$ for $r \geq L_0$. Therefore,

$$(2.14) \quad \int_{L_0}^\infty r^\theta |u_n|^p dr \leq \frac{\varepsilon}{2c^p} \int_{L_0}^\infty r^\theta V(r) |u_n|^p dr \leq \frac{\varepsilon}{2}.$$

Now, applying continuity of the embedding $E \hookrightarrow X_\infty^{1,p}(\alpha, \theta)$ and of the restriction operator $u \mapsto u|_{(0, L_0)}$ from $X_\infty^{1,p}(\alpha, \theta)$ into $W_{L_0}^{1,p}(\alpha, \theta)$, we get $u_n \rightharpoonup 0$ weakly in $W_{L_0}^{1,p}(\alpha, \theta)$. But, in both cases (a) and (b), compactness of the embeddings in Lemma 2.2 implies (note that, by definition, $\theta > \alpha - p$ gives $p < p^*$) that

$$(2.15) \quad \int_0^{L_0} r^\theta |u_n|^p dr \leq \frac{\varepsilon}{2} \quad \text{for all } n \geq n_0,$$

for some n_0 . Combining (2.14) and (2.15), we get the result. \square

3. A Trudinger–Moser type inequality

In this section, we prove the Trudinger–Moser type inequality stated in Theorem 1.2. We have divided the proof into two lemmas.

LEMMA 3.1. *Let $R > 0$ be finite. Then $e^{\mu|u|^{p'}} \in L_\theta^1(0, R)$ for any $u \in X_\infty^{1,p}(\alpha, \theta)$ and $\mu > 0$. Moreover, if $\mu < \mu_{\alpha, \theta}$ and $\|u'\|_{L_\alpha^p} \leq 1$ and $\|u\|_{L_\theta^p} \leq M$, then $\int_0^R r^\theta e^{\mu|u|^{p'}} dr \leq c$ for some $c > 0$ independent of u .*

PROOF. Let us begin by recalling the following two elementary inequalities: For $p \geq 2$, $\gamma > 0$ and p' and γ' such that $1/p + 1/p' = 1$ and $\gamma + \gamma' = 1$, we have

$$(3.1) \quad (s + t)^{p'} \leq s^{p'} + p' s^{1/(p-1)} t + t^{p'} \quad \text{for all } t, s \geq 0,$$

and for any $\varepsilon > 0$

$$(3.2) \quad s^\gamma t^{\gamma'} \leq \varepsilon s + \varepsilon^{-\gamma/\gamma'} t \quad \text{for all } t, s \geq 0.$$

Now, given $u \in X_\infty^{1,p}(\alpha, \theta)$ and $R > 0$, the function $v(r) = u_{|(0,R)}(r) - u(R)$ for $0 \leq r \leq R$ is such that $v \in X_R^{1,p}(\alpha, \theta)$. From (3.1), we get

$$|u(r)|^{p'} \leq |v(r)|^{p'} + p'|v(r)|^{1/(p-1)}|u(R)| + |u(R)|^{p'} \quad \text{for all } r \in (0, R),$$

and by (3.2)

$$\begin{aligned} |v(r)|^{1/(p-1)}|u(R)| &= \left(\frac{1}{p'} |v(r)|^{p'}\right)^{1/p} (p'^{1/(p-1)}|u(R)|^{p'})^{1/p'} \\ &\leq \frac{\varepsilon}{p'} |v(r)|^{p'} + \left(\frac{\varepsilon}{p'}\right)^{-1/(p-1)} |u(R)|^{p'}. \end{aligned}$$

It follows that

$$|u(r)|^{p'} \leq (1 + \varepsilon)|v(r)|^{p'} + c|u(R)|^{p'},$$

where $c = c(\varepsilon, p)$. Hence, from Lemma 2.3

$$(3.3) \quad |u(r)|^{p'} \leq (1 + \varepsilon)|v(r)|^{p'} + c\|u\|_{W_\infty^{1,p}}^{p'},$$

where c is independent of u . Hence, from Theorem 1.1 and (3.3)

$$(3.4) \quad \int_0^R r^\theta e^{\mu|u|^{p'}} dr \leq e^{c\|u\|_{W_\infty^{1,p}}^{p'}} \int_0^R r^\theta e^{\mu(1+\varepsilon)|v(r)|^{p'}} dr < \infty \quad \text{for all } \mu > 0.$$

If $\mu < \mu_{\alpha,\theta}$, we can take $\varepsilon > 0$ such that $\mu(1 + \varepsilon) \leq \mu_{\alpha,\theta}$. Moreover, since $v' = u'$ on $(0, R)$, we get $\|v'\|_{L_\alpha^p(0,R)} \leq \|u'\|_{L_\alpha^p(0,\infty)} \leq 1$. Therefore, using Theorem 1.1 and (3.4), for $\mu < \mu_{\alpha,\theta}$, $\|u'\|_{L_\alpha^p} \leq 1$ and $\|u\|_{L_\theta^p} \leq M$ we obtain $\int_0^R r^\theta e^{\mu|u|^{p'}} dr \leq c$ for some constant c independent of u . □

LEMMA 3.2. *Suppose $R > 0$ is finite. Then, for any $\mu > 0$ and $u \in X_\infty^{1,p}(\alpha, \theta)$ we have $\varphi(\mu|u|^{p'}) \in L_\theta^p(R, \infty)$. Moreover, if $\|u'\|_{L_\alpha^p} \leq 1$ and $\|u\|_{L_\theta^p} \leq M$, then $\int_R^\infty r^\theta \varphi(\mu|u|^{p'}) dr \leq c$, where c does not depend on u .*

PROOF. Fix an arbitrary element $u \in X_\infty^{1,p}(\alpha, \theta)$. From Lemma 2.4 (b), we have $u \in L_\theta^q$ for any $q \geq p$. Thus, the monotone convergence theorem implies

$$\begin{aligned} \int_R^\infty r^\theta \varphi(\mu|u|^{p'}) dr &= \sum_{k=k_0-1}^\infty \frac{\mu^k}{k!} \int_R^\infty r^\theta |u|^{kp/(p-1)} dr \\ &= \frac{\mu^{k_0-1}}{(k_0-1)!} \int_R^\infty r^\theta |u|^{\frac{(k_0-1)p}{p-1}} dr + \frac{\mu^{k_0}}{k_0!} \int_R^\infty r^\theta |u|^{k_0p/(p-1)} dr \\ &\quad + \sum_{k=k_0+1}^\infty \frac{\mu^k}{k!} \int_R^\infty r^\theta |u|^{kp/(p-1)} dr. \end{aligned}$$

Now, by definition of k_0 , we have $k_0 p' \geq (k_0 - 1)p' \geq p$. Then, the embeddings in (b), Lemma 2.4, imply

$$(3.5) \quad \int_R^\infty r^\theta \varphi(\mu|u|^{p'}) dr \leq c_1 \frac{\mu^{k_0-1}}{(k_0 - 1)!} \|u\|_{W_\infty^{1,p}}^{(k_0-1)p'} + c_2 \frac{\mu^{k_0}}{k_0!} \|u\|_{W_\infty^{1,p}}^{k_0 p'} + \sum_{k=k_0+1}^\infty \frac{\mu^k}{k!} \int_R^\infty r^\theta |u|^{kp/(p-1)} dr.$$

Also, from Lemma 2.3,

$$\int_R^\infty r^\theta |u(r)|^{kp/(p-1)} dr \leq (C\|u\|_{W_\infty^{1,p}})^{kp/(p-1)} \int_R^\infty r^{-\chi kp/(p-1)+\theta} dr.$$

Now, for any $k \geq k_0 + 1$ we have $k \geq p + 1$. Thus, by the definition of χ with $\alpha = p - 1$, we get

$$-\frac{kp\chi}{p-1} + \theta + 1 = (\theta + 1) \left(1 - \frac{k}{p}\right) < 0, \quad (\theta + 1) \left(\frac{k}{p} - 1\right) \geq (\theta + 1) \left(\frac{p+1}{p} - 1\right) > 0.$$

Hence, for $R > 0$,

$$\int_R^\infty r^{-\chi kp/(p-1)+\theta} dr = \frac{1}{(\theta + 1)(k/p - 1)} \frac{1}{R^{(\theta+1)(k/p-1)}} \leq \frac{C}{R^{(\theta+1)k/p}},$$

where C depends only on p, θ and R . Using (3.5), we get

$$(3.6) \quad \int_R^\infty r^\theta \varphi(\mu|u|^{p'}) dr \leq c_1 \frac{\mu^{k_0-1}}{(k_0 - 1)!} \|u\|_{W_\infty^{1,p}}^{(k_0-1)p'} + c_2 \frac{\mu^{k_0}}{k_0!} \|u\|_{W_\infty^{1,p}}^{k_0 p'} + c_3 \sum_{k=k_0+1}^\infty \frac{(\mu(C\|u\|_{W_\infty^{1,p}})^{p'} R^{-(\theta+1)/p})^k}{k!}$$

which proves the lemma. □

PROOF OF THEOREM 1.2. For any $u \in X_\infty^{1,p}(\alpha, \theta)$, $\mu > 0$ and $R > 0$, we have

$$\begin{aligned} \int_0^\infty r^\theta \varphi(\mu|u|^{p'}) dr &= \int_0^R r^\theta \varphi(\mu|u|^{p'}) dr + \int_R^\infty r^\theta \varphi(\mu|u|^{p'}) dr \\ &\leq \int_0^R r^\theta e^{\mu|u|^{p'}} dr + \int_R^\infty r^\theta \varphi(\mu|u|^{p'}) dr \end{aligned}$$

and the result follows directly from Lemmas 3.1 and 3.2. □

4. The variational formulation

This section is devoted to variational formulation of problem (1.12). In particular, we prove that the associated functional satisfies the geometry of mountain-pass theorem of Ambrosetti–Rabinowitz [2] and we estimate the mini-max-level. Firstly, since we are looking for non-negative solutions, it is convenient

to define $f(r, u) = 0$ when $u \leq 0$. From (f_1) , there are positive constants c and μ_1 for which

$$(4.1) \quad |F(r, u)| \leq c\varphi(\mu_1|u|^{p'}) \quad \text{for all } (r, u) \in [0, \infty) \times [0, \infty).$$

Therefore, by Theorem 1.2, we get $F(r, u) \in L^1_\theta(0, \infty)$ whenever $u \in X^{1,p}_\infty(\alpha, \theta)$. Thus, the functional $J: E \rightarrow \mathbb{R}$ given by

$$J(u) = \frac{\|u\|_E^p}{p} - \int_0^\infty r^\theta F(r, u) dr$$

is well defined. Moreover, using standard arguments (see [3, Theorem A.VI] and [18, Proposition 1]), we see that J is a C^1 functional on E and

$$J'(u)v = \int_0^\infty r^\alpha |u'|^{p-2} u' v' dr + \int_0^\infty r^\theta V(r) |u|^{p-2} uv dr - \int_0^\infty r^\theta f(r, u)v dr.$$

Consequently, critical points of the functional J are precisely the weak solutions of (1.12).

The next result concerns the mountain-pass geometry of J , it is a consequence of (V_1) , (f_1) – (f_3) and (1.15).

LEMMA 4.1. *The functional J satisfies the following conditions:*

- (a) *For any $u \in E \setminus \{0\}$ with compact support and $u \geq 0$, $J(tu) \rightarrow -\infty$ as $t \rightarrow \infty$.*
- (b) *There exist $\delta, \rho > 0$, such that $J(u) \geq \delta$, if $\|u\|_E = \rho$.*

PROOF. (a) From (f_2) and (f_3) , for $q > p$, there exist c_1 and c_2 for which $F(r, s) \geq c_1 s^q - c_2$, for any $r \in \text{supp } u$ and $s \in [0, \infty)$. Hence

$$J(tu) \leq \frac{t^p \|u\|_E^p}{p} - c_1 t^q \int_0^\infty r^\theta u^q dr + c_2 \int_{\text{supp } u} r^\theta dr$$

which gives the result.

(b) Firstly, we prove that given $\mu_2 > 0$ and $q > p$, there exists $c > 0$ depending only on α, p, θ and μ_2 such that

$$(4.2) \quad \int_0^\infty r^\theta |u|^q \varphi(\mu_2 |u|^{p/(p-1)}) dr \leq c \|u\|_E^q,$$

assuming that $\|u\|_E \leq M$ holds for M sufficiently small. Indeed, for any $R > 0$

$$(4.3) \quad \begin{aligned} & \int_0^\infty r^\theta |u|^q \varphi(\mu_2 |u|^{p/(p-1)}) dr \\ &= \int_0^R r^\theta |u|^q \varphi(\mu_2 |u|^{p/(p-1)}) dr + \int_R^\infty r^\theta |u|^q \varphi(\mu_2 |u|^{p/(p-1)}) dr. \end{aligned}$$

The Hölder inequality gives

$$\begin{aligned} \int_0^R r^\theta |u|^q \varphi(\mu_2 |u|^{p'}) \, dr &\leq \int_0^R r^\theta |u|^q e^{\mu_2 |u|^{p'}} \, dr \\ &\leq \left(\int_0^R r^\theta e^{\eta \mu_2 |u|^{p'}} \, dr \right)^{1/\eta} \left(\int_0^R r^\theta |u|^{q\eta'} \, dr \right)^{1/\eta'} \end{aligned}$$

with $\eta' = \eta/(\eta - 1)$, $\eta > 1$. Arguing as in (3.4), Lemma 3.1, we get

$$\int_0^R r^\theta e^{\eta \mu_2 |u|^{p'}} \, dr = \int_0^R r^\theta e^{\eta \mu_2 \|u'\|_{L^\alpha}^{p'}} (|u|/\|u'\|_{L^\alpha})^{p'} \, dr \leq c$$

for any $\|u\|_E \leq M$, if M is small such that $\eta \mu_2 M^{p'} \leq \mu_{\alpha,\theta}$. Thus, using the continuity of the embedding $E \hookrightarrow L_\theta^{\eta'}$ it follows that

$$(4.4) \quad \int_0^R r^\theta |u|^q \varphi(\mu_2 |u|^{p'}) \, dr \leq c \|u\|_E^q.$$

Moreover, for $k \geq k_0 - 1$, using Lemma 2.3 and the embedding $E \hookrightarrow X_\infty^{1,p}$, we get

$$\begin{aligned} \int_R^\infty r^\theta |u|^q |u|^{kp/(p-1)} \, dr &\leq (c \|u\|_E)^{kp/(p-1)} \int_R^\infty r^{-\chi kp/(p-1)} r^\theta |u|^q \, dr \\ &\leq \left(\frac{c \|u\|_E}{R^\chi} \right)^{kp/(p-1)} \left(\int_R^\infty r^\theta |u|^q \, dr \right). \end{aligned}$$

Thus, choosing $\|u\|_E \leq M$, with M such that $cM < 1$ and $R > 1$, the embedding $E \hookrightarrow L_\theta^q$ implies

$$\int_R^\infty r^\theta |u|^q |u|^{kp/(p-1)} \, dr \leq C \|u\|_E^q \quad \text{for all } k \geq k_0 - 1.$$

Hence

$$(4.5) \quad \begin{aligned} \int_R^\infty r^\theta |u|^q \varphi(\mu_2 |u|^{p'}) \, dr \\ = \sum_{k=k_0-1}^\infty \frac{\mu_2^k}{k!} \int_R^\infty r^\theta |u|^q |u|^{kp/(p-1)} \, dr \leq e^{\mu_2} C \|u\|_E^q. \end{aligned}$$

Combining (4.3)–(4.5), we obtain (4.2).

Now, by (f₁) and (1.15), there exist $\Lambda < \Lambda_1$ and $\mu_2, c > 0$ such that

$$F(r, u) \leq \frac{\Lambda}{p} |u|^p + c |u|^q \varphi(\mu_2 |u|^{p/(p-1)}) \quad \text{for all } (r, u) \in [0, \infty) \times \mathbb{R}.$$

Therefore, from (4.2) and using the definition (1.14), we can write

$$J(u) = \frac{\|u\|_E^p}{p} - \int_0^\infty r^\theta F(r, u) \, dr \geq \left(1 - \frac{\Lambda}{\Lambda_1}\right) \frac{\|u\|_E^p}{p} - c \|u\|_E^q.$$

Since $q > p$ and $\Lambda < \Lambda_1$, we may choose $\delta, \rho > 0$ such that $J(u) \geq \delta$ if $\|u\|_E = \rho$.

Now, in order to get a more precise information about the minimax level obtained by the mountain-pass theorem, we consider the Moser’s sequence

$$(4.6) \quad v_n(r) = \begin{cases} \log^{(p-1)/p} n & \text{if } r \leq \rho/n, \\ (\log(\rho/r))/(\log^{1/p} n) & \text{if } \rho/n \leq r \leq \rho, \\ 0 & \text{if } r \geq \rho. \end{cases}$$

We observe that in order to have a less cumbersome notation for the sequence (v_n) we have omitted in it its dependence on ρ . For any $\rho > 0$, we have $v_n \in X_\infty^{1,p}(\alpha, \theta)$ and $\int_0^\infty r^\alpha |v_n'|^p dr = 1$. Next, we summarize some useful properties of Moser’s sequence.

$$(i) \quad \int_0^\infty r^\theta |v_n|^p dr = \frac{\rho^{\theta+1}}{\theta+1} \frac{1}{\log n} \left(\frac{\Gamma(p+1)}{(\theta+1)^p} + o_n(1) \right).$$

Indeed, we have

$$\int_0^\infty r^\theta |v_n|^p dr = \frac{1}{\log n} \left(\frac{\log^p n}{n^{\theta+1}} \frac{\rho^{\theta+1}}{\theta+1} + \int_{\rho/n}^\rho r^\theta \left(\log \frac{\rho}{r} \right)^p dr \right)$$

and making the change of variable $s = (\theta + 1) \log(\rho/r)$, we get

$$\begin{aligned} \int_{\rho/n}^\rho r^\theta \left(\log \frac{\rho}{r} \right)^p dr &= \frac{\rho^{\theta+1}}{(\theta+1)^{p+1}} \int_0^{\log n^{\theta+1}} s^p e^{-s} ds \\ &= \frac{\rho^{\theta+1}}{(\theta+1)^{p+1}} \left(\Gamma(p+1) - \int_{\log n^{\theta+1}}^\infty s^p e^{-s} ds \right) \end{aligned}$$

which implies (i).

$$(ii) \quad \int_0^\infty r^\theta V(r) |v_n|^p dr \leq \frac{\rho^{\theta+1}}{\theta+1} \frac{\bar{V}}{\log n} \left(\frac{\Gamma(p+1)}{(\theta+1)^p} + o_n(1) \right),$$

$$\text{with } \bar{V} = \max_{r \in [0, \rho]} V(r).$$

$$(iii) \quad \|v_n\|_E^{-p/(p-1)} = 1 - \frac{1}{p-1} \int_0^\infty r^\theta V(r) |v_n|^p dr + o_n \left(\int_0^\infty r^\theta V(r) |v_n|^p dr \right).$$

We have $\|v_n\|_E^p = 1 + \int_0^\infty r^\theta V(r) |v_n|^p dr$ and from (ii) the last integral goes to 0 as $n \rightarrow \infty$. Thus, using that $g(t) = g(0) + g'(0)t + o(t)$, as $t \rightarrow 0$ for $g(t) = (1+t)^{-1/(p-1)}$ we get the result.

(iv) Let $w_n = v_n / \|v_n\|_E$. Then, there exists a sequence (d_n) satisfying

$$(4.7) \quad \begin{cases} w_n^{p/(p-1)}(r) = \log n + d_n & \text{for } r \leq \rho/n, \\ d_n / \log n \rightarrow 0 & \text{as } n \rightarrow \infty, \\ \liminf_n d_n \geq -S_\rho / (\theta + 1), \end{cases}$$

where

$$S_\rho = \frac{\rho^{\theta+1} \Gamma(p+1)}{(p-1)(\theta+1)^p} \bar{V}.$$

Combining (ii) and (iii) and (4.6) , we obtain (4.7).

(v) Set $\sigma_n = \|v_n\|_E$. Then, we have $\sigma_n \rightarrow 1$, as $n \rightarrow \infty$, and

$$(4.8) \quad \mathcal{L} = \lim_{n \rightarrow \infty} (\theta + 1)\sigma_n \log n \int_0^{\sigma_n^{-1}} e^{(s^{p/(p-1)} - \sigma_n s)(\theta+1) \log n} ds \geq 1.$$

Note that $\sigma_n^p = 1 + \int_0^\infty r^\theta V(r)|v_n|^p dr$. Thus, from (ii), $\sigma_n \rightarrow 1$, as $n \rightarrow \infty$. Moreover, for any n ,

$$\begin{aligned} (\theta + 1)\sigma_n \log n \int_0^{\sigma_n^{-1}} e^{(s^{p/(p-1)} - \sigma_n s)(\theta+1) \log n} ds \\ \geq (\theta + 1)\sigma_n \log n \int_0^{\sigma_n^{-1}} e^{-s\sigma_n(\theta+1) \log n} ds = 1 - \frac{1}{e^{(\theta+1) \log n}} \end{aligned}$$

from which there follows (4.8).

LEMMA 4.2. *Suppose that (V₁), (f₁)–(f₄) and (1.15) hold. Then for n large enough and suitable $\rho > 0$*

$$\max \{J(tw_n) : t \geq 0\} < \frac{1}{p} \left(\frac{\theta + 1}{\mu_0} \right)^{p-1},$$

where w_n is the normalized Moser’s sequence given by (iv).

PROOF. Choose $\rho > 0$ as in assumption (f₄) and b_0 such that

$$(4.9) \quad \lim_{u \rightarrow \infty} uf(r, u) e^{-\mu_0|u|^{p'}} \geq b_0 > \frac{(\theta + 1)e^{S_\rho}}{\rho^{\theta+1}} \left(\frac{\theta + 1}{\mu_0} \right)^{p-1}$$

uniformly on compact subsets of $[0, \infty)$. For v_n in (4.6) corresponding to the choice of ρ in (4.9), let $w_n = v_n/\|v_n\|_E$ be the normalized Moser’s sequence. Suppose, by contradiction, that for any $n \in \mathbb{N}$

$$\max \{J(tw_n) : t \geq 0\} \geq \frac{1}{p} \left(\frac{\theta + 1}{\mu_0} \right)^{p-1}.$$

From Lemma 4.1, for each w_n there exists a corresponding $t_n > 0$ such that $J(t_n w_n) = \max \{J(tw_n) : t \geq 0\}$. Thus, using that $F(r, u) \geq 0$ and the definition of J , we can write

$$(4.10) \quad t_n^p \geq \left(\frac{\theta + 1}{\mu_0} \right)^{p-1}.$$

On the other hand, using that t_n is a maximum point of the function $t \mapsto J(tw_n)$, we get

$$(4.11) \quad t_n^p = \int_0^\infty r^\theta t_n w_n f(r, t_n w_n) dr = \int_0^\rho r^\theta t_n w_n f(r, t_n w_n) dr.$$

From (4.9), given $\varepsilon > 0$, there exists $L_\varepsilon > 0$ such that for any $r \in [0, \rho]$

$$(4.12) \quad uf(r, u) \geq (b_0 - \varepsilon)e^{\mu_0|u|^{p/(p-1)}} \quad \text{for any } |u| \geq L_\varepsilon.$$

Therefore, for n large enough (4.11) and (4.7) imply

$$\begin{aligned}
 (4.13) \quad t_n^p &\geq (b_0 - \varepsilon) \int_0^{\rho/n} r^\theta e^{\mu_0 |t_n w_n|^{p/(p-1)}} dr \\
 &= (b_0 - \varepsilon) \int_0^{\rho/n} r^\theta e^{\mu_0 t_n^{p/(p-1)} (\log n + d_n)} dr \\
 &= (b_0 - \varepsilon) \frac{\rho^{\theta+1}}{\theta + 1} e^{\mu_0 t_n^{p/(p-1)} (\log n + d_n) - (\theta+1) \log n}.
 \end{aligned}$$

Hence, we have

$$(4.14) \quad 1 \geq (b_0 - \varepsilon) \frac{\rho^{\theta+1}}{\theta + 1} e^{H_n},$$

where

$$H_n = \mu_0 t_n^{p/(p-1)} (\log n + d_n) - (\theta + 1) \log n - p \log t_n.$$

Now, we conclude that (t_n) is bounded. Indeed, otherwise, up to a subsequence $H_n \rightarrow +\infty$ which contradicts (4.14). Moreover, using (4.13),

$$t_n^p \geq (b_0 - \varepsilon) \frac{\rho^{\theta+1}}{\theta + 1} e^{(\mu_0 t_n^{p'} - 1 - \theta) \log n + \mu_0 t_n^{p/(p-1)} d_n}$$

which implies (up to a subsequence)

$$(4.15) \quad t_n^p \rightarrow \left(\frac{\theta + 1}{\mu_0} \right)^{p-1}, \quad \text{as } n \rightarrow \infty.$$

Next, we shall estimate t_n^p by means of integral in (4.11). For this, let $A_n = \{r \in [0, \rho] : t_n w_n(r) \geq L_\varepsilon\}$ and $B_n = [0, \rho] \setminus A_n$. Combining (4.11) and (4.12), we can write

$$\begin{aligned}
 (4.16) \quad t_n^p &\geq (b_0 - \varepsilon) \int_0^\rho r^\theta e^{\mu_0 |t_n w_n|^{p'}} dr \\
 &\quad + \int_{B_n} r^\theta t_n w_n f(r, t_n w_n) dr - (b_0 - \varepsilon) \int_{B_n} r^\theta e^{\mu_0 |t_n w_n|^{p'}} dr.
 \end{aligned}$$

Notice that $w_n \rightarrow 0$ and the characteristic function $\mathbb{1}_{B_n}$ converges to 1 almost everywhere in $[0, \rho]$. Thus, the Lebesgue dominated convergence theorem implies

$$\begin{aligned}
 (4.17) \quad \lim_{n \rightarrow \infty} \int_{B_n} r^\theta t_n w_n f(r, t_n w_n) dr &= 0, \\
 \lim_{n \rightarrow \infty} \int_{B_n} r^\theta e^{\mu_0 |t_n w_n|^{p'}} dr &= \frac{\rho^{\theta+1}}{\theta + 1}.
 \end{aligned}$$

Also, using (4.7) and (4.10),

$$\begin{aligned}
 (4.18) \quad \int_0^\rho r^\theta e^{\mu_0 |t_n w_n|^{p'}} dr &\geq \int_0^\rho r^\theta e^{(\theta+1)|w_n|^{p'}} dr \\
 &= \int_0^{\rho/n} r^\theta e^{(\theta+1)(\log n + d_n)} dr + \int_{\rho/n}^\rho r^\theta e^{(\theta+1)|w_n|^{p'}} dr
 \end{aligned}$$

and again by (4.7), we obtain (up to a subsequence)

$$\int_0^{\rho/n} r^\theta e^{(\theta+1)(\log n + d_n)} dr = \frac{\rho^{\theta+1}}{\theta+1} e^{(\theta+1)d_n} \geq \frac{\rho^{\theta+1}}{\theta+1} e^{-S_\rho}.$$

Moreover, using the change of variable $s = (\log(\rho/r))/\sigma_n \log n$ we can write

$$\int_{\rho/n}^\rho r^\theta e^{(\theta+1)|w_n|^{p'}} dr = \frac{\rho^{\theta+1}}{\theta+1} (\theta+1)\sigma_n \log n \int_0^{\sigma_n^{-1}} e^{(s^{p/(p-1)} - \sigma_n s)(\theta+1) \log n} ds.$$

Hence,

$$\int_{\rho/n}^\rho r^\theta e^{(\theta+1)|w_n|^{p'}} dr \rightarrow \frac{\rho^{\theta+1}}{\theta+1} \mathcal{L}, \quad \text{as } n \rightarrow \infty,$$

where $\mathcal{L} \geq 1$ is given in (4.8). Letting $n \rightarrow \infty$ in (4.16) and using (4.15), (4.17) and (4.18), we obtain

$$\left(\frac{\theta+1}{\mu_0}\right)^{p-1} \geq (b_0 - \varepsilon) \frac{\rho^{\theta+1}}{\theta+1} (e^{-S_\rho} + \mathcal{L} - 1) \geq (b_0 - \varepsilon) \frac{\rho^{\theta+1}}{\theta+1} e^{-S_\rho}$$

which implies that

$$(4.19) \quad b_0 \leq \frac{(\theta+1)e^{S_\rho}}{\rho^{\theta+1}} \left(\frac{\theta+1}{\mu_0}\right)^{p-1}.$$

This contradicts to (4.9), and the proof is complete. □

5. The existence of solution to problem (1.12)

This section is devoted to the proof of Theorem 1.3. As mentioned earlier, the approach here is variational and relies on the mountain-pass theorem [2]. Indeed, in view of Lemma 4.1, we can apply the mountain-pass theorem to get a sequence $(u_n) \subset E$ verifying

$$(5.1) \quad J(u_n) \rightarrow c \quad \text{and} \quad J'(u_n) \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

where the level c is characterized by

$$c = \inf_{\gamma \in \Sigma} \max_{t \in [0,1]} J(\gamma(t)) \geq \delta$$

and $\Sigma = \{\gamma \in C([0, 1], E) : \gamma(0) = 0, J(\gamma(1)) < 0\}$. According to (5.1),

$$(5.2) \quad |J(u_n)| \leq C \quad \text{and} \quad |J'(u_n)v| \leq \varepsilon_n \|v\|_E \quad \text{for all } v \in E,$$

where $\varepsilon_n \rightarrow 0$ as $n \rightarrow \infty$. From (f₂) and (5.2), we have

$$\begin{aligned} C + \varepsilon_n \|u_n\|_E &\geq |J(u_n) - J'(u_n)u_n| \\ &\geq \left(\frac{q}{p} - 1\right) \|u_n\|_E^p + \int_0^\infty r^\theta [f(r, u_n)u_n - qF(r, u_n)] dr \geq \left(\frac{q}{p} - 1\right) \|u_n\|_E^p. \end{aligned}$$

Thus, since $q > p$, we get

$$\|u_n\|_E \leq C, \quad \int_0^\infty r^\theta f(r, u_n)u_n dr \leq C \quad \text{and} \quad \int_0^\infty r^\theta F(r, u_n) dr \leq C.$$

Now, using the uniform convexity of the space E and compactness of the embeddings in Lemma 2.5, we obtain

$$(5.3) \quad \begin{cases} u_n \rightharpoonup u & \text{weakly in } E, \\ u_n \rightarrow u & \text{in } L^p_\theta, \\ u_n(r) \rightarrow u(r) & \text{a.e. in } [0, \infty). \end{cases}$$

Moreover, arguing as in [17, Lemma 5.5] (see also, [9], [15]), for any $R > 0$ we have

$$(5.4) \quad \begin{cases} f(r, u_n) \rightarrow f(r, u) & \text{in } L^1_\theta(0, R), \\ u'_n(r) \rightarrow u'(r) & \text{a.e. in } (0, R), \\ |u'_n|^{p-2}u'_n \rightharpoonup |u'|^{p-2}u' & \text{weakly in } L^p_\alpha(0, R). \end{cases}$$

From (5.2), $J'(u_n)v \rightarrow 0$ as $n \rightarrow \infty$. Thus, (5.4) imply

$$\int_0^\infty r^\alpha |u'|^{p-2}u'v' dr + \int_0^\infty r^\theta V(r)|u|^{p-2}uv dr = \int_0^\infty r^\theta f(r, u)v dr$$

for all $v \in E$. Hence, u is a weak solution of (1.12). It remains to show that u is non-trivial. Assume, by contradiction, that $u \equiv 0$. From (5.4), there exists $g \in L^1_\theta(0, R)$ such that $|f(r, u_n)| \leq g$ almost everywhere in $(0, R)$ and (f₃) implies that $F(r, u_n) \leq M_1 + M_0f(r, u_n)$ almost every in $(0, R)$ for some $M_1 > 0$. Thus, by the generalized Lebesgue dominated convergence theorem, $F(r, u_n) \rightarrow 0$ in $L^1_\theta(0, R)$. Now, combining (f₁)–(f₂), we can write

$$(5.5) \quad \int_R^\infty r^\theta F(r, u_n) dr \leq c_1 \int_R^\infty r^\theta |u_n|^p dr + c_2 \int_R^\infty r^\theta |u_n| \varphi(\mu_0 |u_n|^{p'}) dr.$$

From (5.3), the first integral on the right-side goes to 0 as $n \rightarrow \infty$. Moreover, as in Lemma 3.2

$$(5.6) \quad \begin{aligned} \int_R^\infty r^\theta |u_n| \varphi(\mu_0 |u_n|^{p'}) dr &= \frac{\mu_0^{k_0-1}}{(k_0-1)!} \int_R^\infty r^\theta |u_n|^{p(k_0-1)/(p-1)+1} dr \\ &+ \frac{\mu_0^{k_0}}{k_0!} \int_R^\infty r^\theta |u_n|^{pk_0/(p-1)+1} dr + \sum_{k=k_0+1}^\infty \frac{\mu_0^k}{k!} \int_R^\infty r^\theta |u_n|^{pk/(p-1)+1} dr. \end{aligned}$$

Now, by definition of k_0 , we have $k_0p' + 1 \geq (k_0 - 1)p' + 1 \geq p + 1$. Then, compactness of the embeddings in Lemma 2.5 implies (up to a subsequence)

$$(5.7) \quad \lim_{n \rightarrow \infty} \int_R^\infty r^\theta |u_n|^{p(k_0-1)/(p-1)+1} dr = \lim_{n \rightarrow \infty} \int_R^\infty r^\theta |u_n|^{pk_0/(p-1)+1} dr = 0.$$

Notice that for any $k \geq k_0 + 1$ and $R > 1$

$$\begin{aligned} \int_R^\infty r^{\theta - (\theta+1)(p-1)(kp/(p-1)+1)/p^2} dr &= \frac{1}{(\theta + 1)[k/p + (p - 1)/p^2 - 1]} \frac{1}{R^{(\theta+1)[k/p+(p-1)/p^2-1]}} \\ &\leq \frac{1}{(\theta + 1)(2p - 1)/p^2} \frac{1}{R^{(\theta+1)(2p-1)/p^2}}. \end{aligned}$$

Finally, using Lemma 2.3 and the last estimate,

$$\begin{aligned} \sum_{k=k_0+1}^\infty \frac{\mu_0^k}{k!} \int_R^\infty r^\theta |u_n|^{pk/(p-1)+1} dr &\leq C \sum_{k=k_0+1}^\infty \frac{(\mu_0 C^{p/(p-1)})^k}{k!} \int_R^\infty r^{\theta - (\theta+1)(p-1)(kp/p-1+1)/p^2} dr \\ &\leq \frac{C_1}{R^{(\theta+1)(2p-1)/p^2}} \end{aligned}$$

for some $C_1 > 0$ depending only on p and θ . Hence,

$$(5.8) \quad \lim_{R \rightarrow \infty} \sum_{k=k_0+1}^\infty \frac{\mu_0^k}{k!} \int_R^\infty r^\theta |u_n|^{pk/(p-1)+1} dr = 0$$

uniformly on n . Thus, combining (5.5)–(5.8), we conclude that $F(r, u_n) \rightarrow 0$ in $L^1_\theta(0, \infty)$. This together with (5.1) imply

$$(5.9) \quad \lim_{n \rightarrow \infty} \|u_n\|_E^p = pc > 0.$$

Hence, it follows from Lemma 4.2 that

$$\|u'_n\|_{L^p_\alpha}^p \leq \omega_\alpha \limsup_{n \rightarrow \infty} \|u_n\|_E^p = \omega_\alpha pc < \omega_\alpha \left(\frac{\theta + 1}{\mu_0}\right)^{p-1}.$$

Thus, for $\eta > 1$ sufficiently close to 1, we have that $\eta\mu_0\|u'_n\|_{L^p_\alpha}^{p'} < \mu_{\alpha,\theta}$ and Theorem 1.2 implies

$$\int_0^\infty r^\theta \varphi(\mu_0 \eta |u_n|^{p'}) dr = \int_0^\infty r^\theta \varphi\left(\mu_0 \eta \|u'_n\|_{L^p_\alpha}^{p'} \left(\frac{|u_n|}{\|u'_n\|_{L^p_\alpha}}\right)^{p'}\right) dr \leq c.$$

Moreover, since $\varphi^\eta(s) \leq c_\eta \varphi(\eta s)$ for $s \geq 0$, from the Hölder inequality

$$\begin{aligned} \int_0^\infty r^\theta \varphi(\mu_0 |u_n|^{p'}) |u_n| dr &\leq c_\eta \left(\int_0^\infty r^\theta \varphi(\mu_0 \eta |u_n|^{p'}) dr\right)^{1/\eta} \left(\int_0^\infty r^\theta |u_n|^{\eta'} dr\right)^{1/\eta'} \leq C \|u_n\|_{L^{\eta'}_\theta}. \end{aligned}$$

Choosing $\eta > 1$ such that $\eta\mu_0\|u'_n\|_{L^\alpha_\alpha}^{p'} < \mu_{\alpha,\theta}$ and $\eta' = \eta/(\eta - 1) \geq p$, the above estimate and compactness of the embedding $E \hookrightarrow L^{\eta'}$ imply

$$\lim_{n \rightarrow \infty} \int_0^\infty r^\theta \varphi(\mu_0|u_n|^{p'})|u_n| \, dr = 0$$

which in combination with (f₁) implies that

$$\lim_{n \rightarrow \infty} \int_0^\infty r^\theta f(r, u_n)u_n \, dr = 0.$$

Therefore, from (5.2) with $v = u_n$ we obtain $\|u_n\|_E \rightarrow 0$ as $n \rightarrow \infty$, which contradicts (5.9). Thus, $u \neq 0$ and we conclude the proof. \square

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Manuscript received February 3, 2016

accepted March 10, 2016

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