Topological **M**ethods in **N**onlinear **A**nalysis
Volume 49, No. 2, 2017, 445–461
DOI: 10.12775/TMNA.2016.081

MULTIPLICITY RESULTS FOR NONLOCAL FRACTIONAL p-KIRCHHOFF EQUATIONS VIA MORSE THEORY

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Abstract. In this paper, we apply Morse theory and local linking to study the existence of nontrivial solutions for Kirchhoff type equations involving the nonlocal fractional p-Laplacian with homogeneous Dirichlet boundary conditions:

$$\begin{cases} \displaystyle\bigg[M\bigg(\iint_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dx\,dy\bigg)\bigg]^{p-1}(-\Delta)^s_p u(x)=f(x,u) & \text{in } \Omega,\\ u=0 & \text{in } \mathbb{R}^N\setminus\Omega,\end{cases}$$

where Ω is a smooth bounded domain of $\mathbb{R}^N$, $(-\Delta)^s_p$ is the fractional p-Laplace operator with $0<s<1<p<\infty$ with $sp<N$, $M\colon \mathbb{R}^+_0\to\mathbb{R}^+$ is a continuous and positive function not necessarily satisfying the increasing condition and f is a Carathéodory function satisfying some extra assumptions.

1. Introduction

In this paper we are interested in the following fractional p-Laplacian equation of Kirchhoff type:

$$\tag{1.1} \begin{cases} \displaystyle\bigg[M\bigg(\iint_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,dx\,dy\bigg)\bigg]^{p-1}(-\Delta)^s_p u(x)=f(x,u) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{in } \Omega,\\ u=0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{in } \mathbb{R}^N\setminus\Omega,\end{cases}$$

2010 *Mathematics Subject Classification.* 35R11, 35A15, 35J60, 58E05.

Key words and phrases. Fractional p-Laplacian; Kirchhoff type equations; Morse theory; local linking.

where $0 < s < 1 < p < \infty$ with $sp < N$, $(-\Delta)^s_p$ is the fractional p-Laplace operator which (up to normalization factors) may be defined along a function $\varphi \in C_0^\infty(\mathbb{R}^N)$ as

$$(-\Delta)^s_p \varphi(x) = 2 \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{|\varphi(x) - \varphi(y)|^{p-2}(\varphi(x) - \varphi(y))}{|x-y|^{N+ps}}\, dy$$

for $x \in \mathbb{R}^N$, where $B_\varepsilon(x) := \{y \in \mathbb{R}^N : |x-y| < \varepsilon\}$, see [15], [18], [19], [22], [48] and the references therein for further details on the fractional p-Laplacian. In particular, the operator $(-\Delta)^s_p$ can be reduced to the fractional operator $(-\Delta)^s$ when $p = 2$. An intrinsic feature of such operator is the nonlocality, that is to say, the operator $(-\Delta)^s$ cares for the entire space $\mathbb{R}^N$, instead of the boundary $\partial\Omega$.

When $p = 2$ and $M \equiv 1$, problem (1.1) conduces to the following fractional Laplacian equation:

$$\begin{cases} (-\Delta)^s u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{1.2}$$

In recent years, a great attention has been focused on the study of the fractional Laplacian equation (1.2). Indeed, the fractional and nonlocal operators of elliptic type arise in a quite natural way in many different applications, such as, continuum mechanics, phase transition phenomena, population dynamics, minimal surfaces and game theory, as they are the typical outcome of stochastically stabilization of Lévy processes, see for example [2], [8]. In the context of fractional quantum mechanics, nonlinear fractional Schrödinger equation has been proposed by Laskin in [23], [24] as a result of expanding the Feynman path integral, from the Brownian-like to the Lévy-like quantum mechanical paths. The literature on fractional and nonlocal operators and on their applications is quite large, for example, we refer the reader to [3], [7], [31]–[36], for some recent results in this direction and to [6], [13] for some recent results about another fractional operators. For the differences between two fractional operators, the reader is referred to [43]. For the basic properties of fractional Sobolev spaces, the reader is referred to [14].

When $p = 2$ and $s \to 1^-$, problem (1.1) becomes the elliptic equation of Kirchhoff type

$$-M\bigg(\int_\Omega |\nabla u|^2\, dx\bigg)\Delta u = f(x,u) \quad \text{in } \Omega, \tag{1.3}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth domain, u satisfies some boundary conditions, see for example [1], [30] for more information about equation (1.3). Note that equation

(1.3) is related to the stationary analogue of the Kirchhoff equation

$$u_{tt} - M\bigg(\int_\Omega |\nabla u|^2\,dx\bigg)\Delta u = f(x,u), \tag{1.4}$$

where $M(t) = a + bt$ for all $t \geq 0$, here $a, b > 0$, see for instance [44], [45] for recent results. It was proposed by Kirchhoff in 1883 as a generalization of the well-known D'Alembert wave equation

$$\rho\,\frac{\partial^2 u}{\partial t^2} - \bigg(\frac{p_0}{\lambda} + \frac{E}{2L}\int_0^L \bigg|\frac{\partial u}{\partial x}\bigg|^2\,dx\bigg)\,\frac{\partial^2 u}{\partial x^2} = f(x,u)$$

for free vibrations of elastic strings, see [21]. Kirchhoff's model takes into account the changes in length of the string produced by transverse vibrations. Here, L is the length of the string, h is the area of the cross section, E is the Young modulus of the material, ρ is the mass density and p_0 is the initial tension. It is worth pointing out that problem (1.4) received much attention only after Lions [25] proposed an abstract framework to the problem. It was pointed out in [1] that equation (1.4) models several physical systems, where u describes a process which depends on the average of itself. Nonlocal effect also finds its applications in biological systems.

In [17], Fiscella and Valdinoci first proposed a stationary Kirchhoff variational model in bounded regular domains of $\mathbb{R}^N$, which takes into account the nonlocal aspect of the tension arising from nonlocal measurements of the fractional length of the string. In [37], Nyamoradi studied a class of Kirchhoff nonlocal fractional equations in a bounded domain Ω and obtained three solutions by using three critical point theorem. Recently, Xiang, Zhang and Ferrara in [46] investigated the existence of weak solutions for a Kirchhoff type problem driven by a nonlocal integro-differential operator involving the fractional p-Laplacian by using variational methods. See [4] for some recent results in a bounded domain. Pucci and Saldi in [38] established the existence and multiplicity of nontrivial solutions for a Kirchhoff type eigenvalue problem in $\mathbb{R}^N$ involving a critical nonlinearity and the nonlocal fractional Laplacian. We refer also to [3] for some recent development in the whole space.

Note that as $s \to 1^-$, problem (1.1) reduces to the class of nonlocal problems of p-Kirchhoff type

$$\begin{cases} -\bigg[M\bigg(\displaystyle\int_\Omega |\nabla u|^p\,dx\bigg)\bigg]^{p-1}\Delta_p u(x) = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$ and $1 < p < N$. In [9], Corrêa and Figueiredo investigated the existence of positive solutions to problem (1.5) via variational methods. In [10], they dealt with the multiplicity of solutions to problem (1.5) via Krasnoselskiĭ genus. In [27], Liu and Zhao showed the existence of nontrivial solutions to problem (1.5) via Morse theory, see also [26] for related discussions. For the applications of Morse theory to the existence of solutions to the fractional Laplacian equations, we refer the reader to [16], [18] and the references therein.

Motivated by the above works, we would like to investigate problem (1.1) via Morse theory and local linking. To the best of our knowledge, there are no papers dealing with the Kirchhoff problem using Morse theory in the fractional Laplacian setting. According to the original meaning of the Kirchhoff function M in (1.4), it is natural to assume that M is a continuous and increasing function. However, in this paper we will drop the increasing assumption. More precisely, we just assume the following:

(M) $M\colon \mathbb{R}_0^+ \to \mathbb{R}^+$ is a continuous function and there exists a constant $a_0 > 0$ such that $M(t) \geq a_0$ for all $t \geq 0$.

Obviously, a typical example for M is given by $M(t) = a + bt^m$ with $m > 0$, $a > 0$, $b \geq 0$ for all $t \geq 0$. When M is of this type, problem (1.1) is said to be non-degenerate if $a > 0$ and $b \geq 0$, while it is called degenerate if $a = 0$ and $b > 0$. For example, we refer to [9], [39], [46] for non-degenerate Kirchhoff type problems and [11], [12], [40], [47] for degenerate Kirchhoff type problems.

Now we require that f is a Carathéodory function and satisfies the following conditions:

(f_1) There exist $C > 0$ and $q \in [1, p_s^*)$ such that $|f(x,t)| \leq C(1 + |t|^{q-1})$ for almost every $x \in \Omega$ and all $t \in \mathbb{R}$, where p_s^* is the fractional Sobolev critical exponent defined by $p_s^* = Np/(N - sp)$.

(f_2) There exist $r > 0$ small and $\overline{\lambda} \in (\lambda_1, \widehat{\lambda})$ such that $a_1^{p-1}\lambda_1 < a_0^{p-1}\widehat{\lambda}$ and

$$a_1^{p-1}\lambda_1|t|^p \leq pF(x,t) = p\int_0^t f(x,\xi)\,d\xi \leq a_0^{p-1}\overline{\lambda}|t|^p,$$

for $t \in \mathbb{R}$ with $|t| \leq r$, almost every $x \in \Omega$, where $a_1 = \max\limits_{0\leq t\leq 1} M(t)$.

(f_3) $\limsup\limits_{|t|\to\infty} pF(x,t)/|t|^p < a_0^{p-1}\lambda_1$ uniformly in $x \in \Omega$.

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(f_5) $\lim\limits_{|t|\to\infty} (pF(x,t) - a_0^{p-1}\lambda_1|t|^p) = -\infty$ uniformly in $x \in \Omega$.

Here $\lambda_1 > 0$ is the first eigenvalue of $(-\Delta)_p^s$, see Section 2 for more details. For the number $\widehat{\lambda}$, see Remark 2.3 for more information. Note that condition (f_5) is weaker than the following condition:

(f_5') $\lim\limits_{|t|\to\infty} (pF(x,t) - f(x,t)t) = -\infty$ uniformly in $x \in \Omega$.

See [29, Lemma 3.2] for a direct proof.

Definition 1.1. We say that $u \in W_0$ is a (weak) *solution* of problem (1.1), if

$$\begin{aligned}\bigg[M\bigg(\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dx\,dy\bigg)\bigg]^{p-1} \\ \cdot \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+ps}}\,dxdy \\ = \int_{\mathbb{R}^N} f(x,u)\varphi(x)\,dx\end{aligned}$$

for any $\varphi \in W_0$, where the space W_0 will be introduced in Section 2.

Now we are in a position to state our main results as follows.

Theorem 1.2. *Let* (M) *and* (f_1)–(f_3) *hold. Then problem* (1.1) *admits at least two nontrivial solutions in* W_0.

Theorem 1.3. *Let* (M), (f_1), (f_2), (f_4) *and* (f_5) *hold. Then problem* (1.1) *admits at least two nontrivial solutions in* W_0.

Remark 1.4. In order to obtain the existence of solutions, many authors often assumed that M is increasing, see for example [26] for the Laplacian setting and [17], [38] for the fractional setting. To extend M to a larger class, the authors in [1, 9] assumed that $\widehat{M}(t) = \int_0^t M(s)\,ds \geq M(t)t$ for $t \geq 0$ as $p = 2$, which is to ensure the boundedness of the Palais–Smale sequences of the energy functional. However, this assumption is far away from the original physical motivation of the Kirchhoff problem. To the best of our knowledge, there have been few papers dealing with the Kirchhoff problem without the increasing assumption in the fractional context. Therefore, this paper will make some contribution in this direction.

Remark 1.5. (a) Clearly, if $M \equiv 1$, then problem (1.1) becomes the usual fractional p-Laplace problem. In addition to $p = 2$, problem (1.1) becomes the usual fractional Laplace problem, and hence Theorem 1.2 extends Theorem 1.4 in [16]. Furthermore, we also assume that $s \to 1^-$, then Theorem 1.3 becomes Theorem 1.2 in [20].

(b) In [27], Liu and Zhao considered the version of Theorems 1.2 and 1.3 in the setting of p-Laplacian under the bounded assumption on M. Moreover, they used condition (f_5') instead of condition (f_5). Thus, our results and context are more general than those of Liu and Zhao in [27].

This paper is organized as follows. In Section 2, we give some related definitions and fundamental properties of the space W_0. In Section 3, we verify the compactness conditions for our main results. In Section 4, we give the proofs of Theorems 1.2 and 1.3.

2. Preliminaries

We first give some basic results for our working space W_0 which will be used later.

The Gagliardo seminorm is defined for all measurable functions $u\colon \mathbb{R}^N \to \mathbb{R}$ by

$$[u]_{s,p} = \bigg(\iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x-y|^{N+sp}}\, dx\, dy \bigg)^{1/p}.$$

The fractional Sobolev space $W^{s,p}(\mathbb{R}^N)$ is defined as

$$W^{s,p}(\mathbb{R}^N) = \{u \in L^p(\mathbb{R}^N) : [u]_{s,p} < \infty\},$$

endowed with the norm

$$\|u\|_{s,p} = \big(\|u\|^p_{L^p(\mathbb{R}^N)} + [u]^p_{s,p}\big)^{1/p}.$$

For a detailed account on the properties of $W^{s,p}(\mathbb{R}^N)$, we refer to [14]. We shall work in the closed linear subspace

$$W_0 = \big\{u \in W^{s,p}(\mathbb{R}^N) : u(x) = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega\big\}.$$

From Theorem 6.5 in [14], it follows that $\|u\|_{W_0}$ can be equivalently renormed by $[u]_{s,p}$. Thus W_0 is a uniformly convex Banach space, especially a reflexive Banach space, see for example [19] for a simple proof or [46] for another proof. Furthermore, the embedding $W_0 \hookrightarrow L^\nu(\Omega)$ is continuous for $\nu \in [1, p^*_s]$ and compact for $\nu \in [1, p^*_s)$, see [14, Theorem 6.5 and Corollary 7.2]. As $p = 2$, we refer the interested readers to [35], [31], [41], [42] for some recent results in a bounded domain.

Next we consider the eigenvalue of the operator $(-\Delta)^s_p$ with homogeneous Dirichlet boundary data, namely the eigenvalue of the problem

$$\begin{cases} (-\Delta)^s_p u = \lambda |u|^{p-2} u & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases} \tag{2.1}$$

More precisely, the following weak formulation of (2.1) was discussed: there is a function $u \in W_0$ such that

$$\iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))(v(x) - v(y))}{|x-y|^{N+ps}}\, dx\, dy = \lambda \int_\Omega |u(x)|^{p-2} u(x) \varphi(x)\, dx, \tag{2.2}$$

for any $\varphi \in W_0$. We recall that $\lambda \in \mathbb{R}$ is an eigenvalue of $(-\Delta)^s_p$ if there exists a nontrivial solution $u \in W_0$ of problem (2.1) or its weak formulation (2.2), and any solution will be called an eigenfunction corresponding to the eigenvalue λ. The set of eigenvalues is referred to as the spectrum of (2.1) in W_0 and denoted by

$\sigma(s,p)$, for a detailed discussion about higher eigenvalue we refer to [15] and [22]. Moreover, it easily follows from continuous embedding that the Rayleigh quotient

$$\lambda_1 = \inf_{u \in W_0 \setminus \{0\}} \frac{\|u\|_{W_0}^p}{\|u\|_{L^p(\Omega)}^p} \in (0,\infty). \tag{2.3}$$

Here we list some related results which will use in the sequel, see for example [15], [19], [22].

Lemma 2.1. *The eigenvalues and eigenfunctions of* (2.1) *have the following properties*:

(a) $\lambda_1 = \min \sigma(s,p)$ *is an isolated point of* $\sigma(s,p)$;
(b) λ_1 *is simple, i.e. all* λ_1*-eigenfunctions are proportional*;
(c) *if* u *is a* λ_1*-eigenfunction, then either* $u(x) > 0$ *or* $u(x) < 0$ *almost everywhere in* Ω;
(d) *all eigenfunctions are in* $L^\infty(\Omega)$.

Remark 2.2. From Lemma 2.1 (b)–(c) we know that $V = \operatorname{span}\{\phi\}$ is an one-dimensional eigenspace associated with λ_1, where $\phi > 0$ in Ω and $\|\phi\|_{W_0} = 1$. Taking one subspace $Z \subset W_0$ completing V such that $W_0 = V \oplus Z$. By Lemma 2.1 (a), the number

$$\widehat{\lambda} = \inf_{u \in Z \setminus \{0\}} \frac{\|u\|_{W_0}^p}{\|u\|_{L^p(\Omega)}^p}$$

exists and satisfies $\widehat{\lambda} > \lambda_1$. Obviously, we have also $\|u\|_{W_0}^p \geq \widehat{\lambda}\|u\|_{L^p(\Omega)}^p$ for $u \in Z$.

3. Compactness conditions

For $u \in W_0$, we define

$$J(u) = \frac{1}{p}\widetilde{M}(\|u\|_{W_0}^p), \qquad H(u) = \int_\Omega F(x,u)\,dx,$$

where $\widetilde{M}(t) = \int_0^t [M(y)]^{p-1}\,dy$. Let $I(u) = J(u) - H(u)$. Obviously, the energy functional $I\colon W_0 \to \mathbb{R}$ associated with problem (1.1) is well defined.

Lemma 3.1. *Let* (M) *holds. Then* $J\colon W_0 \to \mathbb{R}$ *is of class* $C^1(W_0,\mathbb{R})$ *and*

$$\langle J'(u), v\rangle = \big[M(\|u\|_{W_0}^p)\big]^{p-1} \cdot \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+ps}}\,dx\,dy \tag{3.1}$$

for all $u, v \in W_0$. *Moreover,* J *is weakly lower semi-continuous in* W_0.

Proof. It is easy to see that J is Gâteaux-differentiable in W_0 and that (3.1) holds for all $u, v \in W_0$. Now, let $\{u_n\}_n \subset W_0$ and $u \in W_0$ satisfy $u_n \to u$

strongly in W_0 as $n \to \infty$. Without loss of generality, we assume that $u_n \to u$ almost everywhere in $\mathbb{R}^N$. Then the sequence

$$\left\{ \frac{|u_n(x) - u_n(y)|^{p-2}(u_n(x) - u_n(y))}{|x-y|^{(N+ps)/p'}} \right\}_n$$

is bounded in $L^{p'}(\mathbb{R}^{2N})$, as well as

$$\frac{|u_n(x) - u_n(y)|^{p-2}(u_n(x) - u_n(y))}{|x-y|^{(N+ps)/p'}} \to \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x-y|^{(N+ps)(p')}}$$

almost everywhere in $\mathbb{R}^{2N}$. Thus, the Brézis–Lieb Lemma implies

$$\begin{aligned} \lim_{n\to\infty} \iint_{\mathbb{R}^{2N}} \bigg| & \frac{|u_n(x) - u_n(y)|^{p-2}(u_n(x) - u_n(y))}{|x-y|^{(N+ps)/p'}} \\ & - \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x-y|^{(N+ps)/p'}} \bigg|^{p'} \, dx\, dy \\ = \lim_{n\to\infty} & \iint_{\mathbb{R}^{2N}} \left(\frac{|u_n(x) - u_n(y)|^p}{|x-y|^{N+ps}} - \frac{|u(x) - u(y)|^p}{|x-y|^{N+ps}} \right) dx\, dy. \end{aligned} \tag{3.2}$$

The fact that $u_n \to u$ strongly in W_0 yields that

$$\lim_{n\to\infty} \iint_{\mathbb{R}^{2N}} \left(\frac{|u_n(x) - u_n(y)|^p}{|x-y|^{N+ps}} - \frac{|u(x) - u(y)|^p}{|x-y|^{N+ps}} \right) dx\, dy = 0.$$

Moreover, the continuity of M implies that

$$\lim_{n\to\infty} [M(\|u_n\|_{W_0}^p)]^{p-1} = [M(\|u\|_{W_0}^p)]^{p-1}. \tag{3.3}$$

From (3.2) it follows that

$$\begin{aligned} \lim_{n\to\infty} \iint_{\mathbb{R}^{2N}} \bigg| & \frac{|u_n(x) - u_n(y)|^{p-2}(u_n(x) - u_n(y))}{|x-y|^{(N+ps)/p'}} \\ & - \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x-y|^{(N+ps)/p'}} \bigg|^{p'} \, dx\, dy = 0. \end{aligned} \tag{3.4}$$

Combining (3.3)–(3.4) with the Hölder inequality, we have

$$\|J'(u_n) - J'(u)\|_{W_0'} = \sup_{\varphi \in W_0,\, \|\varphi\|_{W_0} = 1} |\langle J'(u_n) - J'(u), \varphi\rangle| \to 0$$

as $n \to \infty$, where W_0' is the dual space of W_0. Hence, $J \in C^1(W_0, \mathbb{R})$. Finally, notice that the map $v \mapsto \|v\|_{W_0}^p$ is lower semi-continuous in the weak topology of W_0 and $\widetilde{M}(t)$ is nondecreasing and continuous in $\mathbb{R}^+$, so that $v \mapsto \widetilde{M}(\|v\|_{W_0}^p)$ is lower semi-continuous in the weak topology of W_0. Indeed, we define a functional $\psi\colon W_0 \to \mathbb{R}$ as

$$\psi(v) = \iint_Q |v(x) - v(y)|^p |x-y|^{-(N+sp)} \, dx\, dy.$$

It is easy to see that $\psi \in C^1(W_0)$ and ψ is a convex functional in W_0. By Corollary 3.8 in [5], we obtain $\psi(v) \le \liminf_{n\to\infty} \psi(v_n)$, and hence the desired claim follows. $\square$

LEMMA 3.2. *Let* (f_1) *holds. Then the functional H is of class $C^1(W_0,\mathbb{R})$ and for any fixed $u \in W_0$*

$$\langle H'(u), v\rangle = \int_{\mathbb{R}^N} f(x,u(x))v(x)\,dx \quad \textit{for all } v \in W_0,$$

and $H'(u) \in W_0'$. Furthermore, if $v_n \rightharpoonup v$ weakly in W_0, then $\langle H'(u), v_n\rangle \to \langle H'(u), v\rangle$ and hence the functional H is weakly continuous in W_0.

PROOF. We only need to prove that H is weakly continuous in W_0 and is of class C^1. Let $\{u_n\}_n \subset W_0$ and $u \in W_0$ satisfy $u_n \rightharpoonup u$ weakly in W_0 as $n \to \infty$. Without loss of generality, we assume that $u_n \to u$ strongly in $L^q(\Omega)$ for $1 \le q < p_s^*$ and almost everywhere in Ω. Then it easily follows from (f_1) and the continuity of Nemytskiĭ operator that

$$\lim_{n\to\infty} \int_\Omega |f(x,u_n) - f(x,u)|^{q'}\,dx = 0.$$

Thus it is easy to verify that H is weakly continuous in W_0 and also of class C^1.$\square$

Combining Lemmas 3.1 and 3.2, we get that $I \in C^1(W_0,\mathbb{R})$ and I is weakly lower semi-continuous in W_0.

DEFINITION 3.3. We say that I satisfies the (PS) condition in W_0, if any (PS) sequence $\{u_n\}_n \subset W_0$, i.e. $\{I(u_n)\}_n$ is bounded and $I'(u_n) \to 0$ as $n \to \infty$, admits a strongly convergent subsequence in W_0.

LEMMA 3.4. *Let* (M) *holds. Then any bounded sequence $\{u_n\}_n \subset W_0$ such that $I'(u_n) \to 0$ as $n \to \infty$ has a strongly convergent subsequence in W_0.*

PROOF. Assume that $\{u_n\}_n$ is bounded in W_0. Going if necessary to a subsequence, we have

$$\begin{aligned} u_n &\rightharpoonup u \quad \text{in } W_0,\\ u_n &\to u \quad \text{in } L^q(\Omega),\\ u_n &\to u \quad \text{a.e. in } \Omega. \end{aligned} \tag{3.5}$$

To show that $\{u_n\}_n$ converges strongly to u in W_0, we first introduce a simple notation. Let $\varphi \in W_0$ be fixed and denote by B_φ the linear functional on W_0 defined by

$$B_\varphi(v) = \iint_{\mathbb{R}^{2N}} \frac{|\varphi(x)-\varphi(y)|^{p-2}(\varphi(x)-\varphi(y))}{|x-y|^{N+ps}}\,(v(x)-v(y))\,dx\,dy$$

for all $v \in W_0$. Clearly, by the Hölder inequality, B_φ is also continuous, being

$$|B_\varphi(v)| \le \|\varphi\|_{W_0}^{p-1}\|v\|_{W_0} \quad \text{for all } v \in W_0.$$

Hence, (3.5) and the boundedness of M on closed interval give that

$$\lim_{n\to\infty}\big([M(\|u_n\|^p_{W_0})]^{p-1}-[M(\|u\|^p_{W_0})]^{p-1}\big)B_u(u_n-u)=0. \tag{3.6}$$

Now, by (f_1) and the Hölder inequality, we get

$$\begin{aligned}\int_\Omega &|(f(x,u_n)-f(x,u))(u_n-u)|\,dx\\ &\leq\int_\Omega[C+C(|u_n|^{q-1}+|u|^{q-1})]|u_n-u|\,dx\\ &\leq C|\Omega|^{(p-1)/p}\|u_n-u\|_{L^p(\Omega)}+C(\|u_n\|^{q-1}_{L^q(\Omega)}+\|u\|^{q-1}_{L^q(\Omega)})\|u_n-u\|_{L^q(\Omega)}.\end{aligned}$$

Then (3.5) implies that

$$\lim_{n\to\infty}\int_\Omega(f(x,u_n)-f(x,u))(u_n-u)\,dx=0. \tag{3.7}$$

Obviously, $\langle I'(u_n)-I'(u),u_n-u\rangle\to0$ as $n\to\infty$, since $u_n\rightharpoonup u$ in W_0 and $I'(u_n)\to0$ in $W_0^{'}$. Hence, (3.5)–(3.7) give as $n\to\infty$

$$\begin{aligned}o(1)&=\langle I'(u_n)-I'(u),u_n-u\rangle\\ &=[M(\|u_n\|^p_{W_0})]^{p-1}B_{u_n}(u_n-u)-[M(\|u_n\|^p_{W_0})]^{p-1}B_u(u_n-u)\\ &\quad+\big([M(\|u_n\|_{W_0})]^{p-1}-[M(\|u\|^p_{W_0})]^{p-1}\big)B_u(u_n-u)\\ &\quad-\int_\Omega(f(x,u_n)-f(x,u))(u_n-u)\,dx\\ &=[M(\|u_n\|^p_{W_0})]^{p-1}[B_{u_n}(u_n-u)-B_u(u_n-u)]+o(1),\end{aligned}$$

that is

$$\lim_{n\to\infty}[M(\|u\|^p_{W_0})]^{p-1}[B_{u_n}(u_n-u)-B_u(u_n-u)]=0.$$

Since $[M(\|u_n\|^p_{W_0})]^{p-1}[B_{u_n}(u_n-u)-B_u(u_n-u)]\geq0$ for all $n\in\mathbb{N}$ by convexity and (M), we have in particular

$$\lim_{n\to\infty}[B_{u_n}(u_n-u)-B_u(u_n-u)]=0. \tag{3.8}$$

Let us now recall the well-known Simon inequalities,

$$|\xi-\eta|^p\leq\begin{cases}C_p(|\xi|^{p-2}\xi-|\eta|^{p-2}\eta)(\xi-\eta) & \text{for } p\geq2,\\ C_p^{'}[(|\xi|^{p-2}\xi-|\eta|^{p-2}\eta)(\xi-\eta)]^{p/2}(|\xi|^p+|\eta|^p)^{(2-p)/2} & \text{for } 1<p<2,\end{cases}$$

for all $\xi,\eta\in\mathbb{R}^N$, where C_p and $C_p^{'}$ are positive constants depending only on p.

Assume first that $p \geq 2$. Then by the Simon inequality and (3.8) as $n \to \infty$

$$\begin{aligned}
[u_n - u]_{s,p}^p &= \iint_{\mathbb{R}^{2N}} |u_n(x) - u_n(y) - u(x) + u(y))|^p |x-y|^{-(N+ps)}\,dx\,dy \\
&\leq C_p \iint_{\mathbb{R}^{2N}} \big[|u_n(x) - u_n(y)|^{p-2}(u_n(x) - u_n(y)) - |u(x)-u(y)|^{p-2}(u(x)-u(y))\big] \\
&\quad \times (u_n(x) - u(x) - u_n(y) + u(y))|x-y|^{-(N+ps)}\,dx\,dy \\
&= C_p\,[B_{u_n}(u_n - u) - B_u(u_n - u)] = o(1).
\end{aligned}$$

In conclusion, $\|u_n - u\|_{W_0} \to 0$ as $n \to \infty$, as required.

Finally, it remains to consider the case when $1 < p < 2$. By (3.5) there exists $\kappa > 0$ such that $[u_n]_{s,p} \leq \kappa$ for all $n \in \mathbb{N}$. Now by the Simon inequality, the Hölder inequality and (3.8) as $n \to \infty$

$$\begin{aligned}
[u_n - u]_{s,p}^p &\leq C_p'\,[B_{u_n}(u_n - u) - B_u(u_n - u)]^{p/2}([u_n]_{s,p}^p + [u]_{s,p}^p)^{(2-p)/2} \\
&\leq C_p'\,[B_{u_n}(u_n - u) - B_u(u_n - u)]^{p/2}([u_n]_{s,p}^{p(2-p)/2} + [u]_{s,p}^{p(2-p)/2}) \\
&\leq C_p''\,[B_{u_n}(u_n - u) - B_u(u_n - u)]^{p/2} = o(1),
\end{aligned} \tag{3.9}$$

where $C_p'' = 2C_p'\kappa^{p(2-p)/2}$ and where we have applied the following elementary inequality:

$$(a+b)^{(2-p)/2} \leq a^{(2-p)/2} + b^{(2-p)/2} \quad \text{for all } a, b \geq 0 \text{ and } 1 < p < 2.$$

Hence, $\|u_n - u\|_{W_0} \to 0$ as $n \to \infty$ also in this second case. □

THEOREM 3.5. *Suppose that* (M), (f_1) *and* (f_3) *are fulfilled. Then the functional I is coercive and then satisfies the* (PS) *condition.*

PROOF. From Lemma 3.4, it suffices to verify the boundedness of (PS) sequences. It follows from (f_1) and (f_3) that for some $\delta > 0$ small, there exists a constant $C_\delta > 0$ such that

$$|F(x,t)| \leq \frac{{a_0}^{p-1}}{p}\,(\lambda_1 - \delta)|t|^p + C_\delta \quad \text{for all } t \in \mathbb{R}, \text{ a.e. } x \in \Omega.$$

Thus, by (M) we obtain for $u \in W_0$

$$\begin{aligned}
I(u) &= \frac{1}{p}\,\widetilde{M}(\|u\|_{W_0}^p) - \int_\Omega F(x,u(x))\,dx \\
&\geq \frac{{a_0}^{p-1}}{p}\,\|u\|_{W_0}^p - \frac{{a_0}^{p-1}}{p}\,(\lambda_1 - \delta)\|u\|_{L^p(\Omega)}^p - C_\delta|\Omega| \\
&\geq \frac{{a_0}^{p-1}}{p}\left(1 - \frac{\lambda_1 - \delta}{\lambda_1}\right)\|u\|_{W_0}^p - C_\delta|\Omega| \to +\infty
\end{aligned}$$

as $\|u\|_{W_0} \to \infty$. That is to say, I is coercive on W_0. Hence, we get the desired assertion. □

THEOREM 3.6. *Suppose* (M), (f_1), (f_4) *and* (f_5) *are fulfilled. Then the functional I is coercive and then satisfies the* (PS) *condition.*

Proof. Similar to the proof of Proposition 3.2 in [20], and for the reader's convenience, we will give the detailed treatment of the proof. We will show that the functional I is coercive on W_0. Hence the (PS) sequence of I must be bounded. Denote

$$\mathcal{A}(x,t) = \frac{a_0^{p-1}}{p}\,\lambda_1|t|^p - F(x,t).$$

Then (f_5) implies that

$$\lim_{|t|\to\infty} \mathcal{A}(x,t) = +\infty, \quad \text{uniformly in } x \in \Omega. \tag{3.10}$$

Assume I is not coercive on W_0, then there exist a sequence $\{u_n\} \subset W_0$ and a constant $C_0 > 0$ such that

$$\|u_n\|_{W_0} \to \infty, \quad \text{as } n \to \infty,\ \text{but } I(u_n) \le C_0. \tag{3.11}$$

Let $v_n = \|u_n\|_{W_0}^{-1} u_n$, then $||v_n||_{W_0} = 1$. Up to a subsequence, we have

$$v_n \rightharpoonup v \text{ in } W_0,\ v_n \to v \text{ in } L^q(\Omega),\ v_n \to v \ \text{ a.e. } x \in \Omega. \tag{3.12}$$

Therefore, from (M), (3.10)–(3.12) we obtain

$$\begin{aligned} \frac{C_0}{||u_n||_{W_0}^p} \ge \frac{I(u_n)}{||u_n||_{W_0}^p} &= \frac{1}{p||u_n||_{W_0}^p}\,\widetilde{M}(\|u_n\|_{W_0}^p) \\ &\quad - \frac{a_0^{p-1}}{p}\,\lambda_1 \int_\Omega |v_n|^p\,dx + \frac{1}{||u_n||_{W_0}^p}\int_\Omega \mathcal{A}(x,u_n)\,dx \\ &\ge \frac{a_0^{p-1}}{p}\bigg(\|v_n\|_{W_0}^p - \lambda_1\int_\Omega |v_n|^p\,dx\bigg) - \frac{C_1}{||u_n||_{W_0}^p} \end{aligned} \tag{3.13}$$

for some $C_1 > 0$. It follows from (3.11)–(3.13) that

$$\limsup_{n\to\infty} \|v_n\|_{W_0}^p \le \lambda_1 ||v||_{L^p(\Omega)}^p. \tag{3.14}$$

On the other hand, by the lower semicontinuity of the norm, we have

$$\lambda_1||v||_{L^p(\Omega)}^p \le \|v_n\|_{W_0}^p \le \liminf_{n\to\infty} \|v_n\|_{W_0}^p. \tag{3.15}$$

In view of (3.14), (3.15) and the uniform convexity of W_0, we obtain (see [5, Proposition 3.32])

$$v_n \to v \quad \text{in } W_0, \qquad \|v\|_{W_0}^p = \lambda_1||v||_{L^p(\Omega)}^p. \tag{3.16}$$

Thus $||v||_{W_0} = 1$ and hence $v = \pm\phi_1$, where $\phi_1 > 0$ is the first eigenfunction for the fractional p-Laplacian operator $(-\Delta)_p^s$ with homogeneous Dirichlet boundary data by Lemma 2.1 (c). Take $v = \phi_1$, then $u_n \to +\infty$ almost everywhere in Ω.

By the variational characterization of λ_1, (M), (3.10) and the Fatou Lemma we have

$$\begin{aligned} C_0 \geq I(u_n) &= \frac{1}{p}\,\widetilde{M}(\|u_n\|_{W_0}^p) - \frac{1}{p}\,\lambda_1 \int_\Omega |u_n|^p\,dx + \int_\Omega \mathcal{A}(x,u_n)\,dx \\ &\geq \frac{a_0^{p-1}}{p}\left(\|u_n\|_{W_0}^p - \lambda_1 \int_\Omega |u_n|^p\,dx\right) + \int_\Omega \mathcal{A}(x,u_n)\,dx \\ &\geq \int_\Omega \mathcal{A}(x,u_n)\,dx \to +\infty, \quad \text{as} n \to \infty, \end{aligned}$$

which is a contradiction. Therefore, I is coercive on W_0 and hence satisfies the (PS) condition by Lemma 3.4. □

4. Proofs of main results

In this section, we first recall some related concepts and results about critical groups.

Let Y be a real Banach space and $\mathcal{J} \in C^1(Y,\mathbb{R})$, $\mathcal{K} = \{u \in Y : \mathcal{J}'(u) = 0\}$, then the qth critical group of $\mathcal{J}$ at an isolated critical point $u \in \mathcal{K}$ with $\mathcal{J}(u) = c$ is defined by

$$C_q(\mathcal{J},u) := H_q(\mathcal{J}^c \cap U, \mathcal{J}^c \cap U \setminus \{u\}),\ q \in \mathbb{N} := \{0,1,\ldots\},$$

where $\mathcal{J}^c = \{u \in Y : \mathcal{J}(u) \leq c\}$, U is any neighbourhood of u, containing the unique critical point, H_q is the qth singular relative homology with integer coefficients in an Abelian group G. Finally, let Θ be the trivial homological group.

We say that $u \in \mathcal{K}$ is a homologically nontrivial critical point of $\mathcal{J}$ if at least one of its critical groups is nontrivial. Now we first present the following critical point theorem which will be used later.

PROPOSITION 4.1 (see [29, Theorem 2.1]). *Let X be a real Banach space and let $\mathcal{J} \in C^1(X,\mathbb{R})$ satisfy the* (PS) *condition and be bounded from below. If $\mathcal{J}$ has a critical point that is homologically nontrivial and is not a minimizer of $\mathcal{J}$, then $\mathcal{J}$ has at least three critical points.*

For the proofs of our theorems, in what follows we may assume that Φ has only finitely many critical points. In applications one often needs to computer the critical groups $C_q(I,0)$ in order to find nontrivial critical points. Liu [28] showed that under some local conditions near zero, zero is homologically nontrivial. In the following, let us present this result which will be used in the sequel.

LEMMA 4.2 (see [29, Proposition 2.1]). *Assume that $\mathcal{J}$ has a critical point $u = 0$ with $\mathcal{J}(0) = 0$. Suppose that $\mathcal{J}$ has a local linking at* 0 *with respect to*

$Y = V \oplus Z$, $k = \dim V < \infty$, *that is, there exists* $\rho > 0$ *small such that*

$$\begin{cases} \mathcal{J}(u) \le 0 & \text{for } u \in V,\ ||u|| \le \rho, \\ \mathcal{J}(u) > 0 & \text{for } u \in Z,\ 0 < ||u|| \le \rho. \end{cases}$$

Then $C_k(\mathcal{J}, 0) \ne \Theta$. *Hence,* 0 *is a homologically nontrivial critical point of* $\mathcal{J}$.

Theorem 4.3. *If* (M), (f_1) *and* (f_2) *are satisfied, then* $C_1(\mathcal{J}, 0) \ne \Theta$.

Proof. In view of Lemma 4.2, it suffices to verify that the functional I has a local linking at 0 with respect to $W_0 = V \oplus Z$, where V and Z are implied in Remark 2.2.

(a) Since V is one-dimensional, we have that for given $r > 0$, there exists $\rho \in (0, 1]$ small such that for $u \in V$, $||u||_{W_0} \le \rho$ implies $|u(x)| \le r$ for almost every $x \in \Omega$. Let $u \in V$. Then by (M) and (f_2), we obtain that for $u \in V$ with $\|u\|_{W_0} \le \rho$,

$$\begin{aligned} I(u) &= \frac{1}{p}\,\widetilde{M}(\|u\|^p_{W_0}) - \int_\Omega F(x, u(x))\,dx \\ &= \frac{1}{p}\,\widetilde{M}(\|u\|^p_{W_0}) - \int_{\{|u(x)| \le r\}} F(x, u(x))\,dx \\ &\le \frac{1}{p}\Big[\max_{0 \le t \le \rho} M(t)\Big]^{p-1} \|u\|^p_{W_0} - \int_{\{|u(x)| \le r\}} F(x, u(x))\,dx \\ &\le \int_{\{|u(x)| \le r\}} \bigg(\frac{1}{p}\,a_1^{p-1}\lambda_1 |u|^p - F(x, u(x))\bigg)\,dx \le 0. \end{aligned}$$

(b) For $u \in Z$, by (M), (f_1) and (f_2), together with the Sobolev embedding theorem, it follows that

$$\begin{aligned} I(u) &= \frac{1}{p}\,\widetilde{M}(\|u\|^p_{W_0}) - \int_\Omega F(x, u(x))\,dx \\ &\ge \frac{a_0^{p-1}}{p}\,(\|u\|^p_{W_0} - \overline{\lambda}\|u\|^p_{L^p(\Omega)}) \\ &\quad - \bigg(\int_{\{|u(x)| \le r\}} + \int_{\{|u(x)| > r\}}\bigg)\bigg[F(x, u(x)) - \frac{a_0^{p-1}}{p}\,\overline{\lambda}|u|^p\bigg]\,dx \\ &\ge \frac{a_0^{p-1}}{p}\bigg(1 - \frac{\overline{\lambda}}{\overline{\overline{\lambda}}}\bigg)\|u\|^p_{W_0} - C_1 \int_{\{|u(x)| > r\}} |u|^q\,dx \\ &\ge \frac{a_0^{p-1}}{p}\bigg(1 - \frac{\overline{\lambda}}{\overline{\overline{\lambda}}}\bigg)\|u\|^p_{W_0} - C_2 \int_{\{|u(x)| > r\}} |u|^m\,dx \\ &\ge \frac{a_0^{p-1}}{p}\bigg(1 - \frac{\overline{\lambda}}{\overline{\overline{\lambda}}}\bigg)\|u\|^p_{W_0} - C_3\|u\|^m_{W_0}, \end{aligned}$$

where $\max\{q, p\} < m < p_s^*$, C_2 and C_3 are positive constants. From which we can deduce that $I(u) > 0$ as $u \in Z$ and $0 < \|u\|_{W_0} < \rho$ with $\rho > 0$ small enough. $\square$

PROOF OF THEOREM 1.2. Due to Theorem 3.5, we know that I satisfies the (PS) condition. From Lemmas 3.1 and 3.2 we know that I is weak lower semi-continuous, and hence I is bounded from below because of Theorem 3.5. In view of Theorem 4.3, it follows that 0 is not a minimizer of I and is homologically nontrivial. Hence the desired conclusion follows from Proposition 4.1. $\square$

PROOF OF THEOREM 1.3. Due to Theorem 3.6, we know that I satisfies the (PS) condition. From Lemmas 3.1 and 3.2 we know that I is weak lower semi-continuous, and hence I is bounded from below because of Theorem 3.6. In terms of Theorem 4.3, it follows that 0 is not a minimizer of I and is homologically nontrivial. Hence the desired conclusion follows from Proposition 4.1. $\square$

Acknowlegements. Z. Binlin was supported by the Natural Science Foundation of Heilongjiang Province of China (No. A201306) and the Research Foundation of Heilongjiang Educational Committee (No. 12541667) and the Doctoral Research Foundation of Heilongjiang Institute of Technology (No. 2013BJ15). M. Xiang was supported by the Fundamental Research Funds for the Central Universities (No. 3122015L014) and National Natural Science Foundation of China (No. 11601515).

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Manuscript received November 13, 2015

accepted January 19, 2016

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