RIGOROUS NUMERICS FOR FAST-SLOW SYSTEMS WITH ONE-DIMENSIONAL SLOW VARIABLE: TOPOLOGICAL SHADOWING APPROACH

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ABSTRACT. We provide a rigorous numerical computational method to validate periodic, homoclinic and heteroclinic orbits as the continuation of singular limit orbits for the fast-slow system

$$\begin{cases} x' = f(x, y, \varepsilon), \\ y' = \varepsilon g(x, y, \varepsilon) \end{cases}$$

with one-dimensional slow variable y. Our validation procedure is based on topological tools called isolating blocks, cone conditions and covering relations. Such tools provide us with existence theorems of global orbits which shadow singular orbits in terms of a new concept, the covering-exchange. Additional techniques called slow shadowing and m-cones are also developed. These techniques give us not only generalized topological verification theorems, but also easy implementations for validating trajectories near slow manifolds in a wide range, via rigorous numerics. Our procedure is available to validate global orbits not only for sufficiently small $\varepsilon > 0$ but all ε in a given half-open interval $(0, \varepsilon_0]$. Several sample verification examples are shown as a demonstration of applicability.

1. Introduction

1.1. Background of problems and our aims. In this paper, we consider the dynamical system in $\mathbb{R}^n \times \mathbb{R}$ of the following form:

$$(1.1) \qquad \begin{cases} x' = f(x, y, \varepsilon), \\ y' = \varepsilon g(x, y, \varepsilon), \end{cases}$$

where $' = d/dt$ is the time derivative and f, g are C^r-functions with $r \geq 1$. The factor ε is a nonnegative but sufficiently small real number. We shall write (1.1) as $(1.1)_\varepsilon$ if we explicitly represent the ε-dependence of the system. System (1.1) can be reformulated with a change of time-scale variable as

$$(1.2) \qquad \begin{cases} \varepsilon \dot{x} = f(x, y, \varepsilon), \\ \dot{y} = g(x, y, \varepsilon), \end{cases}$$

where $\dot{} = d/d\tau$ and $\tau = t/\varepsilon$. One tries to analyze the dynamics of (1.1), equivalently (1.2), by suitably combining the dynamics of the *layer problem*

$$(1.3) \qquad \begin{cases} x' = f(x, y, 0), \\ y' = 0, \end{cases}$$

and the dynamics of the *reduced problem*

$$(1.4) \qquad \begin{cases} 0 = f(x, y, 0), \\ \dot{y} = g(x, y, 0), \end{cases}$$

which are the limiting problems for $\varepsilon = 0$ on the fast and the slow time scale, respectively. Notice that (1.4) makes sense *only on* $f(x, y, 0) = 0$, while (1.3) makes sense in the whole $\mathbb{R}^{n+1}$ as the y-parameter family of x-systems. The meaning of the "$\varepsilon \to 0$-limit" is thus different for (1.1) and (1.2). This is why (1.1) or (1.2) is a kind of *singular perturbation problems*. In particular, (1.1) or (1.2) is known as *fast-slow systems* (or *slow-fast systems*), where x dominates the behavior in the fast time scale and y dominates the behavior in the slow time scale.

When we study the dynamical system of the form (1.1), we often consider limit systems (1.3) and (1.4) independently at first. Then one tries to match them in an appropriate way to obtain trajectories for the full system (1.1). One of major methods for completely solving singularly perturbed systems like (1.1) is the *geometric singular perturbation theory* formulated by Fenichel [11], Jones–Kopell [21], Szmolyan [33] and many researchers. A series of theories are established so that formally constructed singular limit orbits of (1.3) and (1.4) can perturb to true orbits of (1.1) for sufficiently small $\varepsilon > 0$. In geometric singular perturbation theory, there are mainly two key points to consider. One

is the description of slow dynamics for sufficiently small ε near the nullcline $\{(x, y) \mid f(x, y, 0) = 0\}$. The other is the matching of fast and slow dynamics. As for the former, Fenichel [11] provided the *Invariant Manifold Theorem* for describing the dynamics on and around locally invariant manifolds, called slow manifolds, for sufficiently small $\varepsilon > 0$. Such manifolds can be realized as the perturbation of normally hyperbolic invariant manifolds at $\varepsilon = 0$, which are often given by submanifolds of nullcline in (1.3). As for the latter, Jones and Kopell [21] originally formulated the geometric answer for the matching problem deriving the *Exchange Lemma*. This lemma informs that the manifold configuration upon exit from the neighbourhood of slow manifolds under the assumption of transversal intersection between tracking invariant manifolds and the stable manifold of slow manifolds. Afterwards, the Exchange Lemma has been extended in various directions, see e.g. [20], [23], [34]. Combining these terminologies, one can prove the existence of homoclinic or heteroclinic orbits of invariant sets near singular orbits for sufficiently small $\varepsilon > 0$. There are also topological ways to prove their existence for sufficiently small ε provided by, say, Carpenter [4], Gardner–Smoller [13], etc., by using algebraic-topological concepts such as the mapping degree or the Conley index [7], [26]. Such topological approaches also mention the existence of periodic orbits near singular orbits.

On the other hand, all such mathematical results do not give us *how large such sufficiently small ε is*. In other words, it remains an open problem whether there exist global orbits given by the continuation of singular orbits for a given ε. This intrinsic problem has been mentioned in many discussions (e.g. [18]). From the viewpoint of numerical computations, if $\varepsilon > 0$ is sufficiently small, (1.1) becomes the stiff problem and numerically unstable. Although the effective method for computing slow manifolds is provided by Guchenheimer and Kuehn [17], computations for extremely small ε (e.g. close to machine epsilon) is still hard to operate correctly. These circumstances show that there are gaps between mathematical results (i.e. sufficiently small ε) and numerical observations (i.e. given ε) for completely understanding dynamics of fast-slow systems. These are mainly because there are no estimations to measure mathematically rigorous consequences not only quantitatively but also qualitatively. The construction of procedures which bridge mathematical results and numerical observations is necessary to completely understand phenomena in concrete dynamical systems, which is also the case of singular perturbation problems.

Our main aim in this paper is to provide implementations for validating the continuation of various global orbits of (1.1) for all $\varepsilon \in [0, \varepsilon_0]$ rigorously, where $\varepsilon_0 > 0$ is a *given* number. In other words, we provide a method to validate

(M1) the singular limit orbit H_0 for (1.1) with $\varepsilon = 0$, as well as

(M2) global orbits H_ε near H_0 for all $\varepsilon \in (0, \varepsilon_0]$ for a given $\varepsilon_0 > 0$.

Singular limit orbit means the union of several heteroclinic orbits in (1.3) and the submanifolds of the nullcline $\{f(x, y, 0) = 0\}$. Global orbits mean homoclinic, heteroclinic orbits of invariant sets and periodic orbits.

To this end, we provide the notion of *covering-exchange*, which is a topological analogue of the Exchange Lemma. This concept consists of the following topological notions with suitable assumptions: (i) isolating blocks, (ii) cone conditions and (iii) covering relations. These three notions have been already applied to validations of global orbits in dynamical systems very well, as stated in Section 1.2. The covering-exchange is constructed involving these notions, and informs us about

- the existence of slow manifolds for (1.1) as well as normally hyperbolic invariant manifolds for (1.3), and
- the existence of trajectories which not only converge to invariant sets on slow manifolds but which also exist neighbourhoods of slow manifolds after time $T = O(1/\varepsilon)$.

These properties and the general consequence of covering relations yield the existence of global orbits. We also generalize the covering-exchange by introducing the additional notion of *slow shadowing*, which guarantees the local existence of trajectories which shadow slow manifolds with nonlinear structure. This notion enables us to trace trajectories which not only tend to slow manifolds but also stay near slow manifolds for time $O(1/\varepsilon)$ from topological viewpoint. This concept is also very compatible with numerical computations, in particular, for validating trajectories near slow manifolds.

The other main tool to establish our procedure is the assistance of *rigorous numerics*, namely, computations of enclosures where mathematically correct objects are contained in the phase space. All such computations can be realized by *interval arithmetics* and mathematical error estimates. Combination of the covering-exchange, slow shadowing and rigorous numerics in reasonable processes provide us with a method proving (M1) and (M2) simultaneously.

Note that there are two approaches to consider singular perturbation problems; one is the continuation of structures from the singular limit systems ($\varepsilon = 0$) to the full systems (i.e. $\varepsilon > 0$), and the other is the consideration of full systems to the singular limit $\varepsilon \to 0$. Our attitude is the former. In particular, we consider our problems via topological approach on the basis of geometric singular perturbation theory.

This paper is organized as follows. In Section 2, we briefly review Fenichel's invariant manifold theorem and topological notions called covering relations and isolating blocks. A systematic procedure of isolating blocks for validations of invariant manifolds with computer assistance (e.g. [24], [43]) is also discussed.

In Section 3, we show how slow manifolds can be validated in given regions with an explicit range $[0, \varepsilon_0]$ of ε. One sees that our fundamental arguments are basically followed by the proof in Jones' article [19]. Such arguments can be validated via the construction of isolating blocks and singular perturbation problems' version of *cone conditions* (cf. [40]) and the Lyapunov condition [24].

In Section 4, we discuss treatments of slow dynamics. First we introduce the new notion called the *covering-exchange* for describing the behavior of trajectories around slow manifolds (Section 4.1). This concept is a topological analogue of the Exchange Lemma so that we can reasonably validate tracking invariant manifolds near slow manifolds in a suitable sense. This concept also solves the matching problem between fast and slow dynamics. We also provide a generalization of the covering-exchange; a collection of local behavior near slow manifolds called the *slow shadowing*, *drop* and *jump* (Section 4.3). These concepts enable us to construct true trajectories in full system which shadow ones on slow manifolds in reasonable way via rigorous numerics. Furthermore, we provide a slight extension of cones, called *m-cones*, which enables us to sharpen enclosures of stable and unstable manifolds of normally hyperbolic invariant manifolds (Section 4.4). The main idea itself is just a slight modification of cone conditions stated in Section 3. But this technique gives us a lot of benefits in many scenes incorporating with rigorous numerics. On the other hand, dynamics on slow manifolds should be considered when slow manifolds exhibit the nontrivial dynamics such as equilibria, periodic orbits, homoclinic orbits, etc. As an example, we discuss validations of equilibria on slow manifolds (Section 4.5). In the end of Section 4, we discuss unstable manifolds of invariant sets on slow manifolds (Section 4.6). To deal with these manifolds, we discuss the invariant foliations of slow manifolds, and translate this fiber bundle structure into the terms of cones and covering relations. This is one of key considerations of heteroclinic orbits in $(1.1)_\varepsilon$.

In Section 5, the existence theorems for periodic and heteroclinic orbits near singular limit orbits with an explicit range $[0, \varepsilon_0]$ of ε are presented.

As a demonstration of our proposing implementations, we study the FitzHugh–Nagumo equation:

$$(1.5) \qquad \begin{cases} u' = v, \\ v' = \delta^{-1}(cv - f(u) + w), \\ w' = \varepsilon c^{-1}(u - \gamma w), \end{cases}$$

where $f(u) = u(u - a)(1 - u)$, $a \in (0, 1/2)$ and $c, \gamma, \delta > 0$.

The existence of global orbits such as periodic or homoclinic orbits for sufficiently small ε are widely discussed by many authors (e.g. [4], [13]). Note that,

as we mentioned before, the existence of these orbits *for a given* ε remains an open question. We propose ideas which suggest a road to answer this question.

COMPUTER ASSISTED RESULT 1.1 (Existence of homoclinic orbits). Consider (1.5) with $a = 0.3$, $\gamma = 10.0$ and $\delta = 9.0$. Then for all $c \in [0.799, 0.801]$, there exist the following two kinds of trajectories:

- At $\varepsilon = 0$, a singular heteroclinic chain H_0 consisting of
  - a heteroclinic orbit from the equilibrium p_0 near $(0, 0, 0) \in \mathbb{R}^3$ to the equilibrium q_1 near $(1, 0, 0) \in \mathbb{R}^3$,
  - a heteroclinic orbit from the equilibrium p_1 near

    $$(0.870020061, 0, 0.06362) \in \mathbb{R}^3$$

    to the equilibrium q_0 near $(-0.12966517, 0, 0.06335) \in \mathbb{R}^3$, and
  - two branches of nullcline $\{(u, v, w) \mid v = 0, f(u) = w\}$ connecting two heteroclinic orbits.
- For all $\varepsilon \in (0, 5.0 \times 10^{-5}]$, a homoclinic orbit H_ε of $p^\varepsilon \approx p_0$ near H_0.

The precise statements and other sample validation results are shown in Section 6. Throughout the rest of this paper we make the following assumption, which is essential to our whole discussions herein.

ASSUMPTION 1.2. Vector fields f and g have the following form:

$$f(x, y, \varepsilon) = f_0(x, y) + \sum_{i=1}^{m} \varepsilon^i f_i(x, y) + o(\varepsilon^m), \quad f_0 \not\equiv 0,$$

$$g(x, y, \varepsilon) = g_0(x, y) + \sum_{i=1}^{m} \varepsilon^i g_i(x, y) + o(\varepsilon^m), \quad g_0 \not\equiv 0.$$

General dynamical systems depend on parameters. For example, the Fitz-Hugh–Nagumo system (1.5) contains a, c, γ, δ as parameters. Throughout this paper we do not care about parameter dependence of dynamical systems unless otherwise specified.

1.2. Several preceding works related to global trajectories and singular perturbation problems with rigorous numerics. There are many preceding works for the existence of global orbits with rigorous numerics for *regular* dynamical systems, namely, the $g \equiv 0$ case. For example, Wilczak and Zgliczyński [38] apply the topological tool called covering relations to the existence of various type of trajectories in dynamical systems, such as periodic orbits, homoclinic orbits and heteroclinic orbits. The essence of covering relations is to describe the behavior of rectangular-like sets called h-sets and apply the mapping degree to the existence of solutions. One of powerful properties of covering relations is that every h-set can be used as a joint of trajectories and that we can validate various complicating behaviors of dynamical systems.

Indeed, for example, [36] and [37] by Wilczak validate various types of complex trajectories such as Shil'nikov homoclinic solutions, heteroclinic solutions and infinitely many periodic solutions in concrete systems (e.g. Michelson system or Rössler system).

On the other hand, van der Berg et al. [35] produce the other approach of rigorous numerical computations of connecting orbits using the *radii polynomials* and the *parametrization* technique. Their main idea is to reduce the original problem to a projected boundary value problem in an infinite dimensional functional space via a fixed point argument. Their formulation involves a higher order parametrization of invariant manifolds near equilibria for describing stable and unstable manifolds. Their approach is free from integrations of vector fields. Hence one can validate various additional properties of invariant manifolds without any knowledge of existence of trajectories [5].

As for rigorous numerics for singular perturbation problems, Gameiro et al. [12] provide a validation method combined with the algebraic-topological singular perturbation analysis. Such analysis is known as the Conley index theory [7], [26]. They actually apply its singular perturbation version [15], [14] to the singularly perturbed predetor-prey model with two slow variables. As a result, they prove the existence of topological horseshoe, in particular, infinite number of periodic orbits with computer assistance. Their results show, however, the existence of solutions for only sufficiently small ε and hence the bound of $\varepsilon > 0$ where the existence result holds is not given. When we apply the Conley index technique, it is necessary to provide an appropriate neighbourhood of desired orbits whose boundary *transversally* intersects the vector field defined by the full system (1.1).

On the other hand, Guckenheimer et al. [16] discuss rigorous enclosures of slow manifolds with computer assistance within explicit ranges of ε. They introduce a concept of computational slow manifolds related to slow manifolds in geometric singular perturbation theory and succeed with validations of slow manifolds in their settings with explicit ranges of ε, while the validated ranges of ε are bounded away from $\varepsilon = 0$, say, $\varepsilon \in [10^{-6}, 10^{-2}]$. Note that [16] also discusses validations of tangential bifurcations of slow manifolds for all $\varepsilon \in (0, \varepsilon_0]$, where ε_0 is a given number, say, 10^{-3}. One of the other works is a very recent one by Arioli and Koch [1], which studies the existence and stability of traveling pulse solutions for the (singularly perturbed) FitzHugh–Nagumo system. This study, however, focuses only on the parameter range which is *not so small*, say, $\varepsilon \approx 0.1$ or 0.001. In other words, the singular perturbation structure is ignored.

In the case of singular perturbation problems, direct computations of global orbits without any ideas are not practical in various scenes both in the rigorous and in the non-rigorous sense.

Direct applications of the mentioned works without any modifications would yield the failure of operations if we try to cover not only ε which is not so small (e.g. $\varepsilon = 10^{-5}$ or 10^{-6}) but also extremely small ε, possibly smaller than machine epsilon. This failure is due either to the stiffness of problems or to the effect of fast dynamics. Even if it succeeds, there would be huge computation costs due to very slow behavior around slow manifolds. Of course, there is still a matching problem connecting fast and slow behavior, which typically arises in singular perturbation problems. If we can overcome all such difficulties as simply as possible, the scope of applications of the preceding concepts will extend dramatically.

2. Preliminaries

When we consider a fast-slow system (1.1) from the viewpoint of geometric singular perturbation theory following Fenichel (e.g. [11]), the central issue is *normally hyperbolic invariant manifolds*. Fenichel's theory provides us with very rich structure of normally hyperbolic invariant manifolds consisting of equilibria and their small perturbations. Such structures can be fully applicable to our arguments. In the beginning of this section, we review several results on normally hyperbolic invariant manifolds for fast-slow systems.

Our main methodologies to consider fast-slow systems are well-known topological tools called *covering relations*, *isolating blocks* and *cone conditions*. These tools well describe the behavior of solution sets as well as their asymptotic behavior. In successive sections we will see that these tools work well even for fast-slow systems. In this section, we also review two of such tools. Moreover, we provide a procedure of isolating blocks suitable for fast-slow systems so that they validate slow manifolds, which are available to various systems with computer assistance. Cone conditions for singular perturbation problems are stated later. Note that all results in Sections 2 and 3 are valid for (1.1)–(1.4) as systems for $(x, y) \in \mathbb{R}^n \times \mathbb{R}^l$ with $l \geq 1$.

Finally note that readers who are familiar with the mentioned topics can skip this section.

2.1. Fenichel's Invariant Manifold Theorems: review. Here we briefly review *Fenichel's Invariant Manifold Theorems* (e.g. [11]), following arguments in [19]. The central goal of these results is the description of a flow near the set $S_0 = \{(x, y, 0) \mid f(x, y, 0) = 0\}$ with manifold structures: the critical manifold. The critical manifold S_0 can be considered as the y-parameter family of equilibria of the layer problem (1.3). Under appropriate hypotheses, S_0 can be represented by the graph of a function $x = h(y)$ for $y \in K$, where $K \subset \mathbb{R}^l$ is a compact, simply connected set.

A central assumption in Fenichel's theory is the *normal hyperbolicity* and the graph representation of S_0.

(F) The set S_0 is given by the graph of the C^∞-function $h^0(y)$ for $y \in K$, where the set K is a compact, simply connected domain whose boundary is an $(l-1)$-dimensional C^∞-submanifold. Moreover, we assume that S_0 is normally hyperbolic.

REMARK 2.1 (cf. [19]). Recall that the manifold S_0 is *normally hyperbolic* if the linearization of (1.3) at each point in S_0 has exactly l 0-eigenvalues.

A set M is said to be *locally invariant under the flow generated by* (1.1) if it has a neighbourhood V of M so that no trajectory can leave M without also leaving V. In other words, it is locally invariant if for all $x \in M$, $\varphi([0,t],x) \subset V$ implies that $\varphi([0,t],x) \subset M$, similarly with $[0,t]$ replaced by $[t,0]$ when $t < 0$, where φ is a flow.

Without loss of generality, we can assume that $h^0(y) = 0$ for all $y \in K$. Then, there exist stable and unstable eigenspaces, $S(y)$ and $U(y)$, such that $\dim S(y) = s$ and $\dim U(y) = u$ hold for all $y \in K$. With this in mind, we take the transformation $x(\in \mathbb{R}^n) \mapsto (a,b) \in \mathbb{R}^{u+s}$ so that (1.1) is expressed by

$$(2.1) \quad \begin{cases} a' = A(y)a + F_1(x,y,\varepsilon), \\ b' = B(y)b + F_2(x,y,\varepsilon), \\ y' = \varepsilon g(x,y,\varepsilon). \end{cases}$$

Here $A(y)$ denotes a $u \times u$ matrix which all eigenvalues have positive real part and $B(y)$ denotes an $s \times s$ matrix which all eigenvalues have negative real part. F_1 and F_2 denote higher order terms which admit a positive number $\gamma = \gamma(K) > 0$ satisfying

$$|F_i| \le \gamma(|x| + \varepsilon) \quad \text{as } |x| \to 0.$$

More precisely, assumptions for $A(y)$ and $B(y)$ are as follows: there exist quantities $\lambda_A > 0$ and $\mu_B < 0$ such that

$$(2.2) \qquad \lambda_A < \operatorname{Re}\lambda \quad \text{for all } \lambda \in \operatorname{Spec}(A(y)) \text{ and } y \in K,$$

$$(2.3) \qquad \mu_B > \operatorname{Re}\lambda \quad \text{for all } \lambda \in \operatorname{Spec}(B(y)) \text{ and } y \in K.$$

The key consideration of slow manifolds is the following Fenichel's invariant manifold theorem in terms of graph representations.

PROPOSITION 2.2 (Persistence of invariant manifolds, cf. [19]). *Under assumption* (F), *for sufficiently small* $\varepsilon > 0$, *there is a function* $x = h^\varepsilon(y)$ *defined on* K. *The graph* $S_\varepsilon = \{(x,y) \mid x = h^\varepsilon(y)\}$ *is locally invariant under* (1.1). *Moreover,* h^ε *is* C^r, *for any* $r < +\infty$, *jointly in* y *and* ε.

Proposition 2.2 is just a consequence of the second Invariant Manifold Theorem which states

PROPOSITION 2.3 (Stable and unstable manifolds, cf. [19]). *Under assumption* (F), *for sufficiently small* $\varepsilon > 0$, *for some* $\Delta > 0$:

(a) *There is a function* $a = h_{\mathrm{s}}(b, y, \varepsilon)$ *defined on* $\{(b, y, \varepsilon) \mid |b| \leq \Delta, y \in K\}$, *such that the graph* $W^{\mathrm{s}}(S_\varepsilon) = \{(a, b, y) \mid a = h_{\mathrm{s}}(b, y, \varepsilon)\}$ *is locally invariant under* (1.1). *Moreover,* $a = h_{\mathrm{s}}(b, y, \varepsilon)$ *is* C^r, *for any* $r < +\infty$, *jointly in* y *and* ε.

(b) *There is a function* $b = h_{\mathrm{u}}(s, y, \varepsilon)$ *defined on* $\{(a, y, \varepsilon) \mid |a| \leq \Delta, y \in K\}$, *such that the graph* $W^{\mathrm{u}}(S_\varepsilon) = \{(a, b, y) \mid b = h_{\mathrm{u}}(a, y, \varepsilon)\}$ *is locally invariant under* (1.1). *Moreover,* $b = h_{\mathrm{u}}(a, y, \varepsilon)$ *is* C^r, *for any* $r < +\infty$, *jointly in* y *and* ε.

Fenichel's theorems, Propositions 2.2 and 2.3, state that normally hyperbolic invariant manifolds as well as their stable and unstable manifolds persist to locally invariant manifolds in the full system (1.1) for sufficiently small $\varepsilon > 0$. The perturbed manifold S_ε for $\varepsilon > 0$ is called *slow manifold.* In other words, slow manifolds can be realized as the ε-continuation of normally hyperbolic critical manifolds in the layer problem (1.3). One of strategies for constructing such manifolds is the construction of a family of *isolating blocks* and *moving cones* (see [19]). Isolating blocks describe the behavior of vector fields which are transversal to their boundaries, which are widely discussed in the Conley index theory [7], [26]. Isolating blocks are reviewed in Section 2.3.

Fenichel's invariant manifold theory informs us not only about the existence of perturbed slow manifolds but also about invariant foliations of $W^{\mathrm{s}}(S_\varepsilon)$ and $W^{\mathrm{u}}(S_\varepsilon)$. More precisely, the following result (the third Invariant Manifold Theorem) is known.

PROPOSITION 2.4 (Fenichel fibering, cf. Theorems 6, 7 in [19]). *Under assumption* (F), *for sufficiently small* $\varepsilon > 0$, *the following statements hold. For each* $v = v_\varepsilon = (\widehat{y}, \varepsilon) \in S_\varepsilon$:

(a) *There is a function* $(a, y) = h_{\mathrm{s}}^v(b)$ *for* $|b| \leq \Delta$ *sufficiently small so that the graph* $W^{\mathrm{s}}(v) = \{(a, b, y, \varepsilon) \mid (a, y) = h_{\mathrm{s}}^v(b)\} \subset W^{\mathrm{s}}(S_\varepsilon)$ *forms a locally invariant manifold in the sense that* $\varphi_{\varepsilon,N}(t, W^{\mathrm{s}}(v)) \subset W^{\mathrm{s}}(\varphi_\varepsilon(t, v))$ *holds if* $\varphi_\varepsilon(s, v) \in N$ *for all* $s \in [0, t]$ *with* $t > 0$. *Here the set* $\varphi_{\varepsilon,N}(t, D)$ *denotes the forward evolution of a set* D *restricted to* N *given by* $\varphi_{\varepsilon,N}(t, D) = \{\varphi_\varepsilon(t, u) \mid u \in D \text{ and } \varphi_\varepsilon([0, t], u) \subset N\}$. *Moreover,* $(a, y) = h_{\mathrm{s}}^v(b)$ *is* C^r *in* v *and* ε *jointly for any* $r < \infty$.

(b) *There is a function* $(b, y) = h_{\mathrm{u}}^v(a)$ *for* $|a| \leq \Delta$ *sufficiently small so that the graph* $W^{\mathrm{u}}(v) = \{(a, b, y, \varepsilon) \mid (b, y) = h_{\mathrm{u}}^v(a)\} \subset W^{\mathrm{u}}(S_\varepsilon)$ *forms a locally invariant manifold in the sense that* $\varphi_{\varepsilon,N}(t, W^{\mathrm{u}}(v)) \subset W^{\mathrm{u}}(\varphi_\varepsilon(t, v))$ *holds*

if $\varphi_\varepsilon(s, v) \in N$ for all $s \in [t, 0]$ with $t < 0$. Moreover, $(b, y) = h_u^v(a)$ is C^r in v and ε jointly for any $r < \infty$.

These invariant foliations are sometimes referred to as *Feniciel fiberings*. These fiberings ensure the following representations:

$$W^u(D) = \bigcup_{v \in D} W^u(v), \qquad W^s(D) = \bigcup_{v \in D} W^s(v),$$

where D is a subset of slow manifolds.

2.2. Covering relations: review. Our main approach for tracking solution orbits is a topological tool called the *covering relations*. Covering relations describe topological transversality of rectangular-like domains called *h-sets* relatively to continuous maps, and there are various studies not only of the mathematical viewpoint but also of applications with rigorous numerics (e.g. [3], [36], [38], [40], [42]). In the present study, we apply this topological methodology to singular perturbation problems. In this section, we summarize notions of the theory of covering relations. For a given norm on $\mathbb{R}^m$, let $B_m(c, r)$ be the open ball of radius r centered at $c \in \mathbb{R}^m$. For simplicity, also let $B_m = B_m(0, 1)$. We set $\mathbb{R}^0 = \{0\}$, $B_0(0, r) = \{0\}$ and $\partial B_0(0, r) = \emptyset$.

DEFINITION 2.5 (*h*-set, cf. [40], [42]). An *h-set* consists of the following set, integers and a map:

   (a) A compact subset $N \subset \mathbb{R}^m$.
   (b) Nonnegative integers $u(N)$ and $s(N)$ such that $u(N) + s(N) = n$ with $n \leq m$.
   (c) A homeomorphism $c_N \colon \mathbb{R}^n \to \mathbb{R}^{u(N)} \times \mathbb{R}^{s(N)}$ satisfying

$$c_N(N) = \overline{B_{u(N)}} \times \overline{B_{s(N)}}.$$

Finally define the *dimension* of an *h*-set N by $\dim N := n$.

We shall write an *h*-set $(N, u(N), s(N), c_N)$ simply as N if no confusion arises. Let

$$\begin{aligned}
N_c &:= \overline{B_{u(N)}} \times \overline{B_{s(N)}}, \\
N_c^- &:= \partial \overline{B_{u(N)}} \times \overline{B_{s(N)}}, \qquad N_c^+ := \overline{B_{u(N)}} \times \partial \overline{B_{s(N)}}, \\
N^- &:= c_N^{-1}(N_c^-), \qquad\qquad\quad N^+ := c_N^{-1}(N_c^+).
\end{aligned}$$

The following notion describes the topological transversality between two *h*-sets relatively to continuous maps.

DEFINITION 2.6 (Covering relations, cf. [40], [42]). Let N, M be *h*-sets with $u(N) = u(M) = u$. $f \colon N \to \mathbb{R}^{\dim M}$ denotes a continuous map and $f_c := c_M \circ f \circ c_N^{-1} \colon N_c \to \mathbb{R}^u \times \mathbb{R}^{s(M)}$. We say N *f-covers* M ($N \stackrel{f}{\Longrightarrow} M$) if the following statements hold:

(a) There exists a continuous homotopy $h\colon [0,1] \times N_{\mathrm{c}} \to \mathbb{R}^u \times \mathbb{R}^{s(M)}$ satisfying

$$h_0 = f_c, \qquad h([0,1], N_{\mathrm{c}}^-) \cap M_c = \emptyset, \qquad h([0,1], N_{\mathrm{c}}) \cap M_c^+ = \emptyset,$$

where $h_\lambda = h(\lambda, \cdot)$, $\lambda \in [0,1]$.

(b) There exists a map $A\colon \mathbb{R}^u \to \mathbb{R}^u$ such that

$$(2.4) \qquad \begin{cases} h_1(p,q) = (A(p), 0), \\ A(\partial B_u(0,1)) \subset \mathbb{R}^u \setminus \overline{B_u}(0,1), \\ \deg(A, \overline{B_u}, 0) \neq 0, \end{cases}$$

holds for $p \in \overline{B_u}(0,1)$, $q \in \overline{B_s}(0,1)$.

REMARK 2.7. In the definition of covering relation between N and M, the disagreement between $\dim N$ and $\dim M$ is not essential. On the contrary, the equality $u(N) = u(M) = u$ is essential because the mapping degree of the u-dimensional map A should be derived.

The following propositions give us useful sufficient conditions for detecting covering relations in practical situations.

PROPOSITION 2.8 (Finding covering relations, Theorem 15 in [42]). *Let N, M be two h-sets in $\mathbb{R}^n$ such that $u(N) = u(M) = u$ and $s(N) = s(M) = s$. Let $f\colon N \to \mathbb{R}^n$ be continuous. Let $f_c = c_M \circ f \circ c_N^{-1}\colon N_{\mathrm{c}} \to \mathbb{R}^u \times \mathbb{R}^s$. Assume that there exists $q_0 \in \overline{B_s}$ such that following conditions are satisfied*:

(a) *Setting $S(M)_c^- = \{(p,q) \in \mathbb{R}^u \times \mathbb{R}^s \mid \|p\| > 1\}$,*

$$f_c(\overline{B_u} \times \{q_0\}) \subset \mathrm{int}\,(S(M)_c^- \cup M_c), \qquad f_c(N_{\mathrm{c}}^-) \cap M_c = \emptyset, \qquad f_c(N_{\mathrm{c}}) \cap M_c^+ = \emptyset.$$

(b) *Define a map $A_{q_0}\colon \mathbb{R}^u \to \mathbb{R}^u$ by $A_{q_0}(p) := \pi_u(f_c(p, q_0))$, where $\pi_u\colon \mathbb{R}^u \times \mathbb{R}^s \to \mathbb{R}^u$ is the orthogonal projection onto $\mathbb{R}^u$, $\pi_u(p,q) = p$. Assume that $A_{q_0}(\partial B_u) \subset \mathbb{R}^u \setminus \overline{B_u}$ and $\deg(A_{q_0}, \overline{B_u}, 0) \neq 0$.*

Then $N \overset{f}{\Longrightarrow} M$.

PROPOSITION 2.9 (Covering relation in the case $u = 1$, Definition 10 in [41]). *Let N, M be h-sets with $u(N) = u(M) = 1$. Let $f\colon N \to \mathbb{R}^{\dim M}$ be continuous. Set*

$$N_{\mathrm{c}}^{\mathrm{L}} := \{-1\} \times \overline{B_{s(N)}}, \qquad\qquad N_{\mathrm{c}}^{\mathrm{R}} := \{+1\} \times \overline{B_{s(N)}},$$
$$S(N)_c^{\mathrm{L}} := (-\infty, -1) \times \mathbb{R}^{s(N)}, \qquad S(N)_c^{\mathrm{R}} := (+1, +\infty) \times \mathbb{R}^{s(N)}.$$

Assume that there exists $q_0 \in \overline{B_{s(N)}}$ such that

$$f(c_N([-1,1] \times \{q_0\})) \subset \mathrm{int}\,(S(M)^{\mathrm{L}} \cup M \cup S(M)^{\mathrm{R}}), \qquad f(N) \cap M^+ = \emptyset$$

and either of the following conditions holds:

$$f(N^{\mathrm{L}}) \subset S(M)^{\mathrm{L}} \quad and \quad f(N^{\mathrm{L}}) \subset S(M)^{\mathrm{R}},$$

$$f(N^{\mathrm{L}}) \subset S(M)^{\mathrm{R}} \quad and \quad f(N^{\mathrm{L}}) \subset S(M)^{\mathrm{L}}.$$

Then $N \overset{f}{\Longrightarrow} M$.

We also consider covering relations with respect to the inverse of continuous maps.

DEFINITION 2.10 (Back-covering relation, Definitions 3 and 4 in [42]). Let N be an h-set. Define the h-set N^{T} as follows:

(a) The compact subset of the quadruple N^{T} is N itself.
(b) $u(N^{\mathrm{T}}) = s(N)$, $s(N^{\mathrm{T}}) = u(N)$.
(c) The homeomorphism $c_{N^{\mathrm{T}}} \colon \mathbb{R}^n \to \mathbb{R}^n = \mathbb{R}^{u(N^{\mathrm{T}})} \times \mathbb{R}^{s(N^{\mathrm{T}})}$ is given by

$$c_{N^{\mathrm{T}}}(x) = j(c_N(x)),$$

where $j \colon \mathbb{R}^{u(N)} \times \mathbb{R}^{s(N)} \to \mathbb{R}^{s(N)} \times \mathbb{R}^{u(N)}$ is given by $j(p, q) = (q, p)$.
Notice that $N^{\mathrm{T},+} = N^-$ and $N^{\mathrm{T},-} = N^+$.

Next, let N, M be h-sets such that $u(N) = u(M)$. Let $g \colon \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$. Assume that $g^{-1} \colon M \to \mathbb{R}^n$ is well defined and continuous. Then we say N g-back-covers M ($N \overset{g}{\Longleftarrow} M$) if $M^{\mathrm{T}} \overset{g^{-1}}{\Longrightarrow} N^{\mathrm{T}}$ holds.

A fundamental result in the theory of covering relations is the following proposition.

PROPOSITION 2.11 (Theorem 4 in [42]). *Let N_i, $i = 0, \ldots, k$, be h-sets such that $u(N_i) = u$ for $i = 0, \ldots, k$ and let $f_i \colon N_i \to \mathbb{R}^{\dim(N_{i+1})}$, $i = 0, \ldots, k-1$, be continuous. Assume that, for all $i = 0, \ldots, k-1$, either of the following holds:*

$$N_i \overset{f_i}{\Longrightarrow} N_{i+1} \quad or \quad N_i \subset D(f_i^{-1}) \quad and \quad N_i \overset{f_i}{\Longleftarrow} N_{i+1}.$$

Then there is a point $p \in \operatorname{int} N_0$ such that

$$f_i \circ \ldots \circ f_0(p) \in \operatorname{int} N_i \quad for \ all \ i = 0, \ldots, k-1.$$

If we additionally assume $N_k \overset{f_k}{\Longrightarrow} N_0$, then the point $p \in \operatorname{int} N_0$ can be chosen so that $f_k \circ \ldots \circ f_0(p) = p$.

We sometimes consider an infinite sequence of covering relations. To deal with such a situation, we define the following concept.

DEFINITION 2.12 (Admissibility, cf. Definition 2.4 in [36]). Let $\{M_i\}_{i=1}^k$ be h-sets in $\mathbb{R}^m$ and $f \colon \bigcup_{i=1}^m M_i \to \mathbb{R}^m$ be a continuous map. We say that the index sequence $\{i_j\}_{j \in \mathbb{Z}} \subset \{1, \ldots, k\}^{\mathbb{Z}}$ is *admissible with respect to* f if $M_{i_j} \overset{f}{\Longrightarrow} M_{i_{j+1}}$ holds for all $j \in \mathbb{Z}$. Similarly, we say that the index sequence $\{i_j\}_{j \in \mathbb{Z}} \subset$

$\{1, \ldots, k\}^{\mathbb{Z}}$ is *back-admissible with respect to* f if $M_{i_j} \overset{f}{\Longleftarrow} M_{i_{j+1}}$ holds for all $j \in \mathbb{Z}$. In this case, f^{-1} is assumed to be well defined in a neighbourhood of $\bigcup\limits_{i=1}^{m} M_i$ and continuous.

Recall that the second Invariant Manifold Theorem, Proposition 2.3, claims that the stable and unstable manifolds of normally hyperbolic invariant manifolds can be described by graphs of smooth functions. The concepts of *horizontal* and *vertical disks* are useful for description of asymptotic trajectories in terms of covering relations for realizing these situations.

DEFINITION 2.13 (Horizontal and vertical disks, e.g. [36], [40]). Let N be an h-set. Let $b_s \colon \overline{B_{s(N)}} \to N$ be continuous and let $(b_s)_c = c_N \circ b_s$. We say that b_s is a *vertical disk in* N if there exists a homotopy $h \colon [0,1] \times \overline{B_{s(N)}} \to N_c$ such that

$$h_0 = (b_s)_c,$$
$$h_1(x) = (0, x), \quad \text{for all } x \in \overline{B_{s(N)}},$$
$$h(t, x) \in N_c^+, \quad \text{for all } t \in [0,1] \text{ and } x \in \partial B_{s(N)}.$$

Let $b_u \colon \overline{B_{u(N)}} \to N$ be continuous and let $(b_u)_c = c_N \circ b_u$. We say that b_u is a *horizontal disk in* N if there exists a homotopy $h \colon [0,1] \times \overline{B_{u(N)}} \to N_c$ such that

$$h_0 = (b_u)_c,$$
$$h_1(x) = (x, 0), \quad \text{for all } x \in \overline{B_{u(N)}},$$
$$h(t, x) \in N_c^-, \quad \text{for all } t \in [0,1] \text{ and } x \in \partial B_{u(N)}.$$

Combining these concepts with covering relations, we obtain the following result, which is often applied to the existence of homoclinic and heteroclinic orbits.

PROPOSITION 2.14 (Theorem 3 in [38], Theorem 3.9 in [36]). *Let* N_i, $i = 0, \ldots, k$, *be* h-*sets such that* $u(N_i) = u$ *for* $i = 0, \ldots, k$, *and let* $f_i \colon N_i \to \mathbb{R}^{\dim(N_{i+1})}$, $i = 0, \ldots, k-1$, *be continuous. Let* $b \colon \overline{B_u} \to N_0$ *be a horizontal disk in* N_0 *and let* $v \colon \overline{B_{s(N_k)}} \to N_k$ *be a vertical disk in* N_k. *If* $N_i \overset{f_i}{\Longrightarrow} N_{i+1}$ *holds for* $i = 0, \ldots, k-1$, *then there exists* $\tau \in \overline{B_u}$ *such that*

$$(f_i \circ \ldots \circ f_0)(b(\tau)) \in N_{i+1}, \quad \text{for } i = 0, \ldots, k-2,$$
$$(f_{k-1} \circ \ldots \circ f_0)(b(\tau)) \in v(\overline{B_{s(N_k)}}).$$

2.3. Isolating blocks: review and applications to fast-slow systems. A concept of *isolating blocks* is typically discussed in the Conley index theory (e.g. [7], [26]), which studies the structure of isolated invariant sets from the algebraic-topological viewpoint. Central notions in the Conley index theory are *isolating neighbourhoods* or *index pairs*, but we concentrate our attention on *isolating blocks* defined as follows. In our case, the blocks can be considered

very flexible from the viewpoint of not only covering relations but also rigorous numerics. Moreover, isolating blocks play a central role for the existence of slow manifolds (cf. [19] and Section 3). In this section, we firstly review the definition of isolating blocks. Secondly, we show a procedure of isolating blocks around equilibria and their alternatives for fast-slow systems with computer assistance, following [43].

2.3.1. Definition.

DEFINITION 2.15 (Isolating block). Let $N \subset \mathbb{R}^m$ be a compact set. We say N is an *isolating neighbourhood* if $\mathrm{Inv}(N) \subset \mathrm{int}\,(N)$ holds, where

$$\mathrm{Inv}(N) := \{x \in N \mid \varphi(\mathbb{R}, x) \subset N\}$$

for a flow $\varphi\colon \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^m$ on $\mathbb{R}^m$. Next let $B \subset \mathbb{R}^m$ be a compact set and $x \in \partial B$. We say x is an *exit* (resp. *entrance*) *point* of B, if for every solution $\sigma\colon [-\delta_1, \delta_2] \to \mathbb{R}^m$ through $x = \sigma(0)$, with $\delta_1 \geq 0$ and $\delta_2 > 0$ there are $0 \leq \varepsilon_1 \leq \delta_1$ and $0 < \varepsilon_2 \leq \delta_2$ such that, for $0 < t \leq \varepsilon_2$,

$$\sigma(t) \notin B \quad (\text{resp. } \sigma(t) \in \mathrm{int}\,(B)),$$

and, for $-\varepsilon_1 \leq t < 0$,

$$\sigma(t) \notin \partial B \quad (\text{resp. } \sigma(t) \notin B)$$

hold. B^{exit} (resp. B^{ent}) denote the set of all exit (resp. entrance) points of the closed set B. We call B^{exit} and B^{ent} the *exit* and the *entrance* of B, respectively. Finally B is called an *isolating block* if $\partial B = B^{\mathrm{exit}} \cup B^{\mathrm{ent}}$ holds and B^{exit} is closed in ∂B.

Obviously, an isolating block is also an isolating neighbourhood.

2.3.2. Construction around equilibria via rigorous numerics: a basic form.
For the preceding work on the systematic construction of isolating blocks around equilibria, see [43]. Here we briefly review the method therein keeping the fast-slow system (1.1) in mind. The first part is the review of [43]. As the second part, we discuss the analogue of arguments to fast-slow systems. One will see that such procedures are very suitable for analyzing dynamics around invariant manifolds.

Let $K \subset \mathbb{R}^l$ be a compact, connected and simply connected set. Consider first the differential equation of the following abstract form:

$$(2.5) \qquad x' = f(x, \lambda), \quad x \in \mathbb{R}^n, \quad \lambda \in K, \quad f\colon \mathbb{R}^n \times K \to \mathbb{R}^n,$$

which corresponds to the layer problem (1.3). For simplicity, assume that f is C^∞. Our purpose here is to construct an isolating block which contains an equilibrium of (2.5).

Let p_0 be a numerical equilibrium of (2.5) at $\lambda_0 \in K$ and rewrite (2.5) as a series around (p_0, λ_0):

$$(2.6) \qquad\qquad x' = f_x(p_0, \lambda_0)(x - p_0) + \widehat{f}(x, \lambda),$$

where $f_x(p_0, \lambda_0)$ is the Fréchet differential of f with respect to x-variable at (p_0, λ_0). $\widehat{f}(x, \lambda)$ denotes the higher order term of f with $O(|x - p_0|^2 + |\lambda - \lambda_0|)$. This term may in general contain an additional term arising from the numerical error $f(p_0, \lambda_0) \approx 0$.

Here assume that the $n \times n$-matrix $f_x(p_0, \lambda_0)$ is nonsingular. Diagonalizing $f_x(p_0, \lambda_0)$, which is generically possible, (2.6) is further rewritten as the following perturbed diagonal system around (p_0, λ_0):

$$(2.7) \qquad\qquad z_j' = \mu_j z_j + \widetilde{f}_j(z, \lambda), \quad j = 1, \dots, n.$$

Here $\mu_j \in \mathbb{C}$, $z = (z_1, \dots, z_n)$ and $\widetilde{f}(z, \lambda) = (\widetilde{f}_1(z, \lambda), \dots, \widetilde{f}_n(z, \lambda))^{\mathrm{T}}$ are defined by $x = Pz + p_0$ and $\widetilde{f}(x, \lambda) = P^{-1}(\widehat{f}(z, \lambda))$, where $P = (P_{ij})_{i,j=1,\dots,n}$ is a nonsingular matrix diagonalizing $f_x(p_0, \lambda_0)$ and $*^{\mathrm{T}}$ is the transpose.

Let $N \subset \mathbb{R}^n$ be a compact set containing p_0. Assume that each $\widetilde{f}_j(z, \lambda)$ has a bound $[\delta_j^-, \delta_j^+]$ in $N \times K$, namely,

$$\{\widetilde{f}_j(z, \lambda) \mid x = Pz + p_0 \in N, \lambda \in K\} \subsetneq [\delta_j^-, \delta_j^+].$$

Then z_j' must satisfy

$$\mu_j\left(z_j + \frac{\delta_j^-}{\mu_j}\right) < z_j' < \mu_j\left(z_j + \frac{\delta_j^+}{\mu_j}\right),$$

for all z with $x = Pz + p_0 \in N$, for all $\lambda \in K$. For simplicity we assume that each μ_j is real. We then obtain the candidate of an isolating block B in the z-coordinate given by the following:

$$(2.8) \qquad B := \prod_{j=1}^n B_j, \quad B_j = [z_j^-, z_j^+] := \left[-\frac{\delta_j^+}{\mu_j}, -\frac{\delta_j^-}{\mu_j}\right] \quad \text{if } \mu_j > 0,$$

$$(2.9) \qquad\qquad\qquad B_j = [z_j^-, z_j^+] := \left[-\frac{\delta_j^-}{\mu_j}, -\frac{\delta_j^+}{\mu_j}\right] \quad \text{if } \mu_j < 0.$$

REMARK 2.16. In the case that μ_j is complex-valued for some i, $f_x(p_0, \lambda_0)$ contains the complex conjugate of μ_j as the other eigenvalue. Without loss of generality, we may assume $\mu_j = \alpha_j + \sqrt{-1}\,\beta_j$, $\mu_{j+1} = \overline{\mu}_j = \alpha_j - \sqrt{-1}\,\beta_j$, $\beta_j > 0$. To be simplified, we further assume that μ_j and μ_{j+1} are the only complex pair of eigenvalues of $f_x(p_0, \lambda_0)$. The general case can be handled in the same manner. The dynamics for z_j and z_{j+1} is formally written as

$$z_j' = \mu_j z_j + \widetilde{f}_j(z, \lambda), \qquad z_{j+1}' = \mu_{j+1} z_{j+1} + \widetilde{f}_{j+1}(z, \lambda).$$

Now we would like to consider real dynamical systems. To do this we transform the above form into

$$w'_j = \alpha_j w_j + \beta_j w_{j+1} + \overline{f}_j(w, \lambda), \qquad w'_{j+1} = -\beta_j w_j + \alpha_j w_{j+1} + \overline{f}_{j+1}(w, \lambda)$$

via $Q = \left(\begin{smallmatrix} 1 & 1 \\ \sqrt{-1} & -\sqrt{-1} \end{smallmatrix} \right)$, $(w_j, w_{j+1})^{\mathrm{T}} = Q(y_j, y_{j+1})^{\mathrm{T}}$, $(\overline{f}_j, \overline{f}_{j+1})^{\mathrm{T}} = Q(\widetilde{f}_j, \widetilde{f}_{j+1})^{\mathrm{T}}$, where $w = (w_1, \ldots, w_n)$ is the new coordinate satisfying $w_i = y_i$ for $i \neq j, j+1$. Let $r_j(w, \lambda) := \sqrt{\overline{f}_j(w, \lambda)^2 + \overline{f}_{j+1}(w, \lambda)^2}$ and assume that $r_j(w, \lambda)$ is bounded from above by a positive number $\overline{r}_j$ uniformly on $N \times K$. Our aim here is to construct a candidate for an isolating block and hence we assume that

- the scalar product of the vector field and the coordinate vector

$$(w_j, w_{j+1}) \cdot \left(\alpha_j w_j + \beta_j w_{j+1} + \overline{f}_j(w, \lambda), -\beta_j w_j + \alpha_j w_{j+1} + \overline{f}_{j+1}(w, \lambda) \right)$$

has the identical sign and the above function never attains 0 on

$$\left\{ (w_j, w_{j+1}) \; \middle| \; \sqrt{w_j^2 + w_{j+1}^2} \leq b_j \right\}, \quad b_j > 0.$$

With this assumption in mind, we set as the candidate for the isolating block in i- and $i+1$-th coordinate

$$B_{j,j+1} := \left\{ (w_j, w_{j+1}) \; \middle| \; \sqrt{w_j^2 + w_{j+1}^2} \leq \frac{\overline{r}_j}{|\alpha_j|} \right\}.$$

Its boundary becomes exit if $\alpha_j > 0$ and entrance if $\alpha_j < 0$. Finally, replace $B_j \times B_{j+1}$ in the definition of B ((2.8) and (2.9)) by $B_{j,j+1}$.

A series of estimates for error terms involves N and it makes sense only if it is self-consistent, namely, $\{p_0\} + PB \subset N$. If it is the case, then B is the desired isolating block for (2.5). Indeed, if $\mu_j > 0$, then

$$z'_j|_{z_j = z_j^-} < 0 \quad \text{and} \quad z'_j|_{z_j = z_j^+} > 0$$

hold. Namely, the set $\{z \in B \mid z_j = z_j^{\pm}\}$ is contained in the exit. Similarly if $\mu_j < 0$ then

$$z'_j|_{z_j = z_j^-} > 0 \quad \text{and} \quad z'_j|_{z_j = z_j^+} < 0$$

hold. So, the set $\{z \in B \mid z_j = z_j^{\pm}\}$ is contained in the entrance. Obviously, ∂B is the union of the closed exit and the entrance, which shows that B is an isolating block.

Once such an isolating block B is constructed, one obtains an equilibrium in B.

PROPOSITION 2.17 (cf. [43]). *Let B be an isolating block constructed as above. Then B contains an equilibrium of (2.5) for all $\lambda \in K$.*

This proposition is a consequence of the general theory of the Conley index ([25]). Note that the construction of isolating blocks described in Proposition 2.17 around points which are *not necessarily equilibria* implies the existence of *rigorous* equilibria inside blocks. With an additional property such as uniqueness or hyperbolicity of equilibria, this procedure will provide with a smooth λ-parameter family of equilibria, which is stated in Section 3.

When we apply these ideas to the fast-slow system (1.1), we consider only the fast system $x' = f(x, y, \varepsilon)$. Let $K \subset \mathbb{R}^l$ be as above, $\varepsilon_0 > 0$ and $x = p_0$ be a numerical zero of $f(x, y_0, 0)$ at $y = y_0 \in K$. In this case we set $\varepsilon = 0$ for computing numerical zeros. Via the above procedure, the fast system $x' = f(x, y, \varepsilon)$ can be generically written in the following form:

$$(2.10) \qquad a' = Aa + F_1(a, b, y, \varepsilon), \qquad b' = Bb + F_2(a, b, y, \varepsilon).$$

Here A is a $u \times u$ dimensional diagonal matrix each of whose eigenvalues has positive real part. Similarly, B is an $s \times s$ dimensional $(u+s = n)$ diagonal matrix each of whose eigenvalues has negative real part. F_1 and F_2 are higher order terms depending on p_0 and y_0. Equivalently, writing (2.10) component-wise,

$$a_j' = \mu_j^a a_j + F_{1,j}(x, y, \varepsilon), \quad \operatorname{Re} \mu_j^a > 0, \quad j = 1, \ldots, u,$$
$$b_j' = \mu_j^b b_j + F_{2,j}(x, y, \varepsilon), \quad \operatorname{Re} \mu_j^b < 0, \quad j = 1, \ldots, s.$$

Let $N \subset \mathbb{R}^n$ be a compact set containing p_0. As before, assume that every μ_j^a and μ_j^b are real and that each F_{i,j_i}, $i = 1, 2$, $j_1 = 1, \ldots, u$, $j_2 = 1, \ldots, s$, admits the following enclosure with respect to $N \times K \times [0, \varepsilon_0]$:

$$(2.11) \quad \left\{ F_{i,j_i}(x, y, \varepsilon) \mid x = P(a, b) + p_0 \in N, \ y \in K, \ \varepsilon \in [0, \varepsilon_0] \right\} \subsetneq [\delta_{i,j_i}^-, \delta_{i,j_i}^+].$$

Define the set $D_c \subset \mathbb{R}^{n+l}$ as follows:

$$(2.12)$$
$$D_c := \prod_{j=1}^{u} [a_j^-, a_j^+] \times \prod_{j=1}^{s} [b_j^-, b_j^+] \times K,$$

$$[a_j^-, a_j^+] := \left[-\frac{\delta_{1,j}^+}{\mu_j^a}, -\frac{\delta_{1,j}^-}{\mu_j^a} \right], \qquad [b_j^-, b_j^+] := \left[-\frac{\delta_{2,j}^-}{\mu_j^b}, -\frac{\delta_{2,j}^+}{\mu_j^b} \right].$$

If $\{(p_0, y_0)\} + \widetilde{P}D_c \subset N \times K$ holds with the linear transform $\widetilde{P} \equiv P \times \mathrm{id}_l$ on $\mathbb{R}^{n+l}$, then this procedure is self-consistent. Due to this self-consistence, we immediately know that

$$a_j' > 0 \quad \text{for all } (a, b, y, \varepsilon) \in D_c \times [0, \varepsilon_0] \text{ with } a_j = a_j^+,$$
$$a_j' < 0 \quad \text{for all } (a, b, y, \varepsilon) \in D_c \times [0, \varepsilon_0] \text{ with } a_j = a_j^-,$$
$$b_j' < 0 \quad \text{for all } (a, b, y, \varepsilon) \in D_c \times [0, \varepsilon_0] \text{ with } b_j = b_j^+,$$
$$b_j' > 0 \quad \text{for all } (a, b, y, \varepsilon) \in D_c \times [0, \varepsilon_0] \text{ with } b_j = b_j^-.$$

Remark that these inequalities hold for all $\varepsilon \in [0, \varepsilon_0]$. This observation is the key point of the construction not only of limiting critical manifolds but of slow manifolds for $\varepsilon \in (0, \varepsilon_0]$, which is provided in Section 3.

DEFINITION 2.18 (Fast-saddle-type block). Let $D_c \subset \mathbb{R}^{n+l}$ be constructed by (2.12). Assume that $K = \overline{B_l} \subset \mathbb{R}^l$. We call D_c, equivalently $D := c_D^{-1}(D)$ via a homeomorphism c_D, a *fast-saddle-type block*. Moreover, set

$$D_c^{f,-} := \{(a, b, y) \in D_c \mid a_j = a_j^{\pm}, \ j = 1, \ldots, u\},$$
$$D_c^{f,+} := \{(a, b, y) \in D_c \mid b_j = b_j^{\pm}, \ j = 1, \ldots, s\},$$
$$D_c^{s} := \{(a, b, y) \in D_c \mid y \in \partial B_l\},$$
$$D^{f,-} := c_D^{-1}(D_c^{f,-}), \quad D^{f,+} := c_D^{-1}(D_c^{f,+}), \quad D^{s} := c_D^{-1}(D_c^{s}).$$

We call $D_c^{f,-}$ (equivalently $D^{f,-}$) the *fast-exit of D* and $D_c^{f,+}$ (equivalently $D^{f,+}$) the *fast-entrance of D*.

REMARK 2.19. Obviously, D is an h-set, but the integers $u(D)$ and $s(D)$ in Definition 2.5 are not necessarily equal to u and s, respectively. We do not pose any assumptions on D^s in the above definition. Indeed, D is not necessarily an isolating block in the sense of Definition 2.15. That is why we omit the word "isolating" in the definition of D.

This construction can be slightly extended. Let $\{\eta_{i,j_i}^{\pm}\}_{j_1=1,\ldots,u,\, j_2=1,\ldots,s}^{i=1,2}$ be a sequence of positive numbers. Defining

$$\widehat{D}_c := \prod_{j=1}^{u} [\widehat{a}_j^-, \widehat{a}_j^+] \times \prod_{j=1}^{s} [\widehat{b}_j^-, \widehat{b}_j^+] \times K,$$

(2.13)
$$[\widehat{a}_j^-, \widehat{a}_j^+] := \left[-\frac{\delta_{1,j}^+}{\mu_j^a} - \eta_{1,j_1}^-, -\frac{\delta_{1,j}^-}{\mu_j^a} + \eta_{1,j_1}^+ \right],$$

$$[\widehat{b}_j^-, \widehat{b}_j^+] := \left[-\frac{\delta_{2,j}^-}{\mu_j^b} - \eta_{2,j_2}^-, -\frac{\delta_{2,j}^+}{\mu_j^b} + \eta_{2,j_2}^+ \right],$$

we can prove that $\widehat{D}_c$ is also a fast-saddle-type block if $\widetilde{P}\widehat{D}_c + \{(p_0, y_0)\} \subset N \times K$ holds. We further know that

$$a_j' > 0 \quad \text{for all } (a, b, y, \varepsilon) \in \widehat{D}_c \times [0, \varepsilon_0] \text{ with } a_j \in [a_j^+, \widehat{a}_j^+],$$
$$a_j' < 0 \quad \text{for all } (a, b, y, \varepsilon) \in \widehat{D}_c \times [0, \varepsilon_0] \text{ with } a_j \in [\widehat{a}_j^-, a_j^-],$$
$$b_j' < 0 \quad \text{for all } (a, b, y, \varepsilon) \in \widehat{D}_c \times [0, \varepsilon_0] \text{ with } b_j \in [b_j^+, \widehat{b}_j^+],$$
$$b_j' > 0 \quad \text{for all } (a, b, y, \varepsilon) \in \widehat{D}_c \times [0, \varepsilon_0] \text{ with } b_j \in [\widehat{b}_j^-, b_j^-].$$

This extension leads to the explicit lower bound estimate of distance between $\widehat{D}^{f,\pm}$ and slow manifolds, which is stated in Section 3.

2.3.3. Construction around equilibria via rigorous numerics: the predictor-corrector approach. Here we provide another approach for validating fast-saddle-type blocks. In the previous subsection, fast-saddle-type blocks are constructed centered at $\{(\overline{x}, y) \mid y \in K\}$, where $K \subset \mathbb{R}^l$ is a small compact neighbourhood of $\overline{y} \in K$. All transformations concerning eigenpairs are done at a point $(\overline{x}, \overline{y})$. Figure 1 (a) briefly shows this situation. On the other hand, we can reselect the center of a candidate of blocks so that blocks can be chosen smaller. As continuation of equilibria with respect to parameters, the *predictor-corrector approach* is one of effective ones. We now revisit the construction of fast-saddle-type blocks with the predictor-corrector approach.

Let $(\overline{x}, \overline{y})$ be a (numerical) equilibrium for (1.3), i.e. $f(\overline{x}, \overline{y}, 0) \approx 0$, such that $f_x(\overline{x}, \overline{y}, 0)$ is invertible. Let K be a compact neighbourhood of $\overline{y}$. The central idea is to choose the center as follows instead of $(\overline{x}, \overline{y})$:

$$(2.14) \qquad \left(\overline{x} + \frac{dx}{dy}(\overline{y})(y - \overline{y}), y \right) \equiv \left(\overline{x} - f_x(\overline{x}, \overline{y}, 0)^{-1} f_y(\overline{x}, \overline{y}, 0)(y - \overline{y}), y \right),$$

where $x = x(y)$ is the parametrization of x with respect to y such that $\overline{x} = x(\overline{y})$ and that $f(x(y), y, 0) = 0$, which is actually realized in a small neighbourhood of $\overline{y}$ in $\mathbb{R}^l$ since $f_x(\overline{x}, \overline{y}, 0)$ is invertible. See Figure 1 (b). Obviously, the identification in (2.14) makes sense thanks to the Implicit Function Theorem.
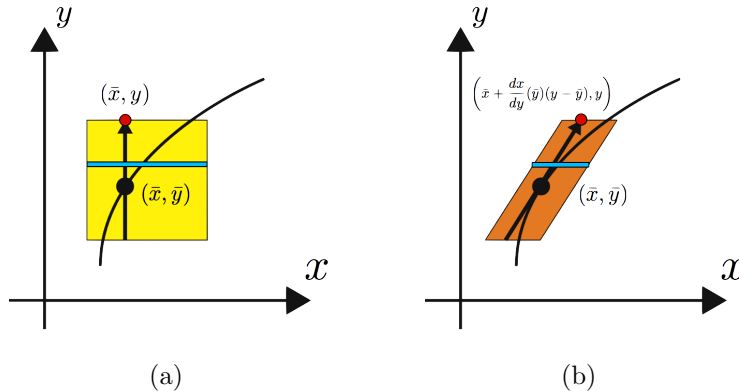

(a)   (b)

Figure 1. Fast-saddle-type blocks with (a) basic form, and (b) predictor-corrector form. (a) The center point $\bar{x}$ of blocks in the x-coordinate is fixed. The blue rectangle shows a fast-saddle-type block at $y \in K$. This procedure can be realized just by extending error bounds in (2.11) to y-directions. However, the "higher order" term $F_{i, j_i}(x, y, \varepsilon)$ may contain the linear term with respect to y. This may cause the increase of error bounds. (b) The center point of blocks is moved along the tangent line (or plane for higher dimensional systems) at $(\overline{x}, \overline{y})$. The blue rectangle shows a fast-saddle-type block at $y \in K$. In principle, sizes of blocks will be smaller than (a) since the linear term with respect to y in error term is approximately dropped.

Around the new center, we define the new affine transformation $T\colon (z,w) \mapsto (x,y)$ as

$$(2.15) \qquad (x,y) = T(z,w) := \big(Pz + \overline{x} - f_x(\overline{x},\overline{y},0)^{-1} f_y(\overline{x},\overline{y},0)w, \; w + \overline{y}\big),$$

where P is a nonsingular matrix diagonalizing $f_x(\overline{x},\overline{y},0)$. Over the new (z,w)-coordinate, the fast system $x' = f(x,y,\varepsilon)$ is transformed into the following:

$$
\begin{aligned}
z' &= P^{-1}(x' + \overline{f_x}^{-1}\overline{f_y}w') \\
&= P^{-1}(f(x,y,\varepsilon) + \varepsilon\overline{f_x}^{-1}\overline{f_y}g(x,y,\varepsilon)) \\
&= P^{-1}\big(\overline{f_x}(Pz - \overline{f_x}^{-1}\overline{f_y}w) + \widehat{f}(z,w,\varepsilon) + \varepsilon\overline{f_x}^{-1}\overline{f_y}g(x,y,\varepsilon)\big) \\
&= \Lambda z + P^{-1}\big(-\overline{f_y}w + \widehat{f}(z,w,\varepsilon) + \varepsilon\overline{f_x}^{-1}\overline{f_y}g(Pz + \overline{x} - \overline{f_x}^{-1}\overline{f_y}w, \; w + \overline{y},\varepsilon)\big) \\
&\equiv \Lambda z + F(z,w,\varepsilon),
\end{aligned}
$$

where $\overline{f_x} = f_x(\overline{x},\overline{y},0)$, $\overline{D_y f} = f_y(\overline{x},\overline{y},0)$, and $\Lambda = \operatorname{diag}(\mu_1^a,\ldots,\mu_u^a,\mu_1^b,\ldots,\mu_s^b)$. The function $\widehat{f}(z,w,\varepsilon)$ denotes the higher order term of f with $O(|z|^2+|w|+|\varepsilon|)$. Dividing z into (a,b) corresponding to eigenvalues with positive real parts and negative real parts, respectively, as in (2.10), we can construct a candidate set of fast-saddle-type blocks as in (2.12). By similar implementations to Section 2.3.2, we can construct fast-saddle-type blocks centered at (2.14) for $y \in K$.

Note that the higher order term $\widehat{f}(z,w,\varepsilon)$ contains the linear term of w as $\overline{f_y}w$ with small errors in a sufficiently small neighbourhood K of $\overline{y}$. This fact indicates that, in principle, sizes of blocks become smaller than those in Section 2.3.2, as seen in Figure 1 (b). This benefit is also useful for the future arguments.

3. Slow manifold validations

In this section, we provide a verification theorem for slow manifolds as well as their stable and unstable manifolds. Our goal here is to provide sufficient conditions to validate not only the critical manifold S_0 but also the perturbed slow manifold S_ε of $(1.1)_\varepsilon$ *for all $\varepsilon \in (0,\varepsilon_0]$ in given regions*.

Recall that Fenichel's results (Propositions 2.2, 2.3) assume normal hyperbolicity and graph representation of the critical manifold S_0. These assumptions are nontrivial, but very essential to prove the persistence. Our verification theorem contains the verification of normal hyperbolicity and graph representation of S_0.

The main idea is based on discussions in [19]. For technical reasons, we use a multiple of ε as the new auxiliary variable. We set $\varepsilon = \eta\sigma$ and $\sigma := \varepsilon_0 > 0$, where ε_0 is a given positive number. We add the equation $\eta' = 0$ to (2.1).

Furthermore, we consider the following system instead of (2.1) for simplicity:

$$(3.1) \quad \begin{cases} a' = Aa + F_1(x, y, \varepsilon), \\ b' = Bb + F_2(x, y, \varepsilon), \\ y' = \varepsilon g(x, y, \varepsilon), \\ \eta' = 0. \end{cases}$$

Here A denotes the $u \times u$ matrix which all eigenvalues have positive real part and B denotes the $s \times s$ matrix which all eigenvalues have negative real part. Note that matrices A and B are (locally) independent of y. This formulation is natural when the construction of fast-saddle-type blocks stated in Section 2.3 is taken into account.

Let M be a fast-saddle-type block for (3.1). Section 2.3 implies that the coordinate representation, M_c, is given by (2.12) (or (2.13)), which is directly obtained from system (3.1). A fast-saddle-type block M has the form (2.12), which has the a-, b- and y-coordinates following (3.1). With this in mind, we make following notations.

NOTATIONS 3.1. Let $\pi_a, \pi_b, \pi_y, \pi_{a,b}, \pi_{a,y}$ and $\pi_{b,y}$ be the projections onto the a-, b-, y-, (a, b)-, (a, y)- and (b, y)-coordinates in M, respectively. If no confusion arises, we drop the phrase "in M" in the notations.

We identify nonlinear terms $F_1(x, y, \varepsilon), F_2(x, y, \varepsilon), g(x, y, \varepsilon)$ with $F_1(a, b, y, \varepsilon)$, $F_2(a, b, y, \varepsilon)$ and $g(a, b, y, \varepsilon)$, respectively, via an affine transform $x(\in \mathbb{R}^n) \mapsto (a, b) \in \mathbb{R}^{u+s}$.

For a squared matrix A with $\operatorname{Spec}(A) \subset \{\lambda \in \mathbb{C} \mid \operatorname{Re} \lambda > 0\}$, $\lambda_A > 0$ denotes a positive number such that

$$(3.2) \quad \lambda_A < \operatorname{Re} \lambda, \quad \text{for all } \lambda \in \operatorname{Spec}(A).$$

Similarly, for a squared matrix B with $\operatorname{Spec}(B) \subset \{\lambda \in \mathbb{C} \mid \operatorname{Re} \lambda < 0\}$, $\mu_B < 0$ denotes a negative number such that

$$(3.3) \quad \mu_B > \operatorname{Re} \mu, \quad \text{for all } \mu \in \operatorname{Spec}(B).$$

Finally, let $\operatorname{dist}(\,\cdot\,, \,\cdot\,)$ be the distance between A and B, $A, B \subset \mathbb{R}^{n+l}$, given by $\operatorname{dist}(A, B) = \inf_{x \in A, \, y \in B} |x - y|$.

The following assumptions are other keys of our verification theorem, the so-called *cone conditions*.

ASSUMPTION 3.2. Consider (3.1). Let $N \subset \mathbb{R}^{n+l}$ be a fast-saddle-type block for all $\varepsilon = \eta \sigma \in [0, \varepsilon_0]$, such that the coordinate representation N_c is actually given by (2.12), and $z = (x, y, \varepsilon)$.

Define $\sigma_{\mathbb{A}_1^u} = \sigma_{\mathbb{A}_1^u}(z)$, $\sigma_{\mathbb{A}_2^u} = \sigma_{\mathbb{A}_2^u}(z)$, $\sigma_{\mathbb{B}_1^u} = \sigma_{\mathbb{B}_1^u}(z)$, $\sigma_{\mathbb{B}_2^u} = \sigma_{\mathbb{B}_2^u}(z)$, $\sigma_{g_1^u} = \sigma_{g_1^u}(z)$ and $\sigma_{g_2^u} = \sigma_{g_2^u}(z)$ be maximal singular values of the following matrices

at z, respectively:

$$\sigma_{\mathbb{A}_1^u} : \mathbb{A}_1^u(z) = \left(\frac{\partial F_1}{\partial a}(z)\right), \qquad\qquad u \times u\text{-matrix},$$

$$\sigma_{\mathbb{A}_2^u} : \mathbb{A}_2^u(z) = \left(\frac{\partial F_1}{\partial b}(z) \quad \frac{\partial F_1}{\partial y}(z) \quad \frac{\partial F_1}{\partial \eta}(z)\right), \qquad u \times (s+l+1)\text{-matrix},$$

$$\sigma_{\mathbb{B}_1^u} : \mathbb{B}_1^u(z) = \left(\frac{\partial F_2}{\partial a}(z)\right), \qquad\qquad s \times u\text{-matrix},$$

$$\sigma_{\mathbb{B}_2^u} : \mathbb{B}_2^u(z) = \left(\frac{\partial F_2}{\partial b}(z) \quad \frac{\partial F_2}{\partial y}(z) \quad \frac{\partial F_2}{\partial \eta}(z)\right), \qquad s \times (s+l+1)\text{-matrix},$$

$$\sigma_{g_1^u} : g_1^u(z) = \left(\frac{\partial g}{\partial a}(z)\right), \qquad\qquad l \times u\text{-matrix},$$

$$\sigma_{g_2^u} : g_2^u(z) = \left(\eta\frac{\partial g}{\partial b}(z) \quad \eta\frac{\partial g}{\partial y}(z) \quad g(z)+\eta\frac{\partial g}{\partial \eta}(z)\right), \quad l \times (s+l+1)\text{-matrix}.$$

Assume that the following inequalities hold:

$$(3.4) \qquad\qquad \lambda_A - (\sup\sigma_{\mathbb{A}_1^u} + \sup\sigma_{\mathbb{A}_2^u}) > 0,$$

$$(3.5) \quad \lambda_A - \mu_B - \Big\{\sup\sigma_{\mathbb{A}_1^u} + \sup\sigma_{\mathbb{A}_2^u} + \sup\sigma_{\mathbb{B}_1^u} + \sup\sigma_{\mathbb{B}_2^u}$$
$$+ \sigma(\sup\sigma_{g_1^u} + \sup\sigma_{g_2^u})\Big\} > 0,$$

where λ_A and μ_B are real numbers satisfying (3.2) and (3.3), respectively, and the notation " sup" means the supremum on $N \times [0, \varepsilon_0]$.

ASSUMPTION 3.3. Consider (3.1). Let $N \subset \mathbb{R}^{n+l}$ be a fast-saddle-type block for all $\varepsilon = \eta\sigma \in [0, \varepsilon_0]$, such that the coordinate representation N_c is actually given by (2.12), and $z = (x, y, \varepsilon)$.

Define $\sigma_{\mathbb{A}_1^s} = \sigma_{\mathbb{A}_1^s}(z)$, $\sigma_{\mathbb{A}_2^s} = \sigma_{\mathbb{A}_2^s}(z)$, $\sigma_{\mathbb{B}_1^s} = \sigma_{\mathbb{B}_1^s}(z)$, $\sigma_{\mathbb{B}_2^s} = \sigma_{\mathbb{B}_2^s}(z)$, $\sigma_{g_1^s} = \sigma_{g_1^s}(z)$ and $\sigma_{g_2^s} = \sigma_{g_2^s}(z)$ be maximal singular values of the following matrices at z, respectively:

$$\sigma_{\mathbb{A}_1^s} : \mathbb{A}_1^s(z) = \left(\frac{\partial F_1}{\partial b}(z)\right), \qquad\qquad u \times s\text{-matrix},$$

$$\sigma_{\mathbb{A}_2^s} : \mathbb{A}_2^s(z) = \left(\frac{\partial F_1}{\partial a}(z) \quad \frac{\partial F_1}{\partial y}(z) \quad \frac{\partial F_1}{\partial \eta}(z)\right), \qquad u \times (u+l+1)\text{-matrix},$$

$$\sigma_{\mathbb{B}_1^s} : \mathbb{B}_1^s(z) = \left(\frac{\partial F_2}{\partial b}(z)\right), \qquad\qquad s \times s\text{-matrix},$$

$$\sigma_{\mathbb{B}_2^s} : \mathbb{B}_2^s(z) = \left(\frac{\partial F_2}{\partial a}(z) \quad \frac{\partial F_2}{\partial y}(z) \quad \frac{\partial F_2}{\partial \eta}(z)\right), \qquad s \times (u+l+1)\text{-matrix},$$

$$\sigma_{g_1^s} : g_1^s(z) = \left(\frac{\partial g}{\partial b}(z)\right), \qquad\qquad l \times s\text{-matrix},$$

$$\sigma_{g_2^s} : g_2^s(z) = \left(\eta\frac{\partial g}{\partial a}(z) \quad \eta\frac{\partial g}{\partial y}(z) \quad g(z)+\eta\frac{\partial g}{\partial \eta}(z)\right), \quad l \times (u+l+1)\text{-matrix}.$$

Assume that the following inequalities hold:

$$(3.6) \qquad\qquad \mu_B + (\sup \sigma_{\mathbb{B}_1^s} + \sup \sigma_{\mathbb{B}_2^s}) < 0,$$

$$(3.7) \qquad \lambda_A - \mu_B - \big\{ \sup \sigma_{\mathbb{A}_1^s} + \sup \sigma_{\mathbb{A}_2^s} + \sup \sigma_{\mathbb{B}_1^s} + \sup \sigma_{\mathbb{B}_2^s}$$
$$+ \sigma(\sup \sigma_{g_1^s} + \sup \sigma_{g_2^s}) \big\} > 0,$$

where λ_A and μ_B are real numbers satisfying (3.2) and (3.3), respectively, and the notation "sup" means the supremum on $N \times [0, \varepsilon_0]$.

DEFINITION 3.4 (Cone conditions). We say a fast-saddle-type block N satisfies the *unstable cone condition* for $(1.1)_\varepsilon$ if Assumption 3.2 holds. Similarly, we say a fast-saddle-type block N satisfies the *stable cone condition* for $(1.1)_\varepsilon$ if Assumption 3.3 holds. They make sense only when $\varepsilon \in [0, \varepsilon_0]$.

By following the proof of Theorem 4 in [19], we obtain the next result, which is the main result in this section.

THEOREM 3.5. *Consider* (3.1). *Let* $N \subset \mathbb{R}^{n+l}$ *be a fast-saddle-type block such that the coordinate representation* N_c *is actually given by* (2.12) *with* $\pi_y N = K \subset \mathbb{R}^l$. *Then:*

(a) *If we assume*

$$(3.8) \qquad \begin{aligned} \lambda_A - \left[\sup \sigma_{\mathbb{A}_1^u} + \frac{\sup \sigma_{\mathbb{A}_2^u} + \sup \sigma_{\mathbb{B}_2^s}}{2} \right] &> 0, \\ |\mu_B| - \left[\sup \sigma_{\mathbb{B}_1^s} + \frac{\sup \sigma_{\mathbb{A}_2^u} + \sup \sigma_{\mathbb{B}_2^s}}{2} \right] &> 0, \end{aligned}$$

there is an l-dimensional normally hyperbolic invariant manifold $S_0 \cap N$ in $\mathbb{R}^{n+l}$ for the limit system (1.3) *such that $S_0 \cap N$ is the graph of a smooth function depending on y.*

(b) *In addition to assumptions in* (a), *if the unstable cone condition is satisfied, there exists a Lipschitz function $a = h_s(b, y, \varepsilon)$ defined in $\overline{B_s} \times \overline{B_l} \times [0, \varepsilon_0]$ such that the graph*

$$W^s(S_\varepsilon) := \{ (a, b, y, \varepsilon) \mid a = h_s(b, y, \varepsilon) \}$$

is locally invariant with respect to (3.1).

(c) *In addition to assumptions in* (a), *if the stable cone condition is satisfied, there exists a Lipschitz function $b = h_u(a, y, \varepsilon)$ defined in $\overline{B_u} \times \overline{B_l} \times [0, \varepsilon_0]$ such that the graph*

$$W^u(S_\varepsilon) := \{ (a, b, y, \varepsilon) \mid b = h_u(a, y, \varepsilon) \}$$

is locally invariant with respect to (3.1).

As a consequence, under stable and unstable cone conditions, $S_\varepsilon := W^s(S_\varepsilon) \cap W^u(S_\varepsilon)$ can be defined by the locally invariant l-dimensional submanifold of $\mathbb{R}^{n+l}$ inside N for any $\varepsilon \in [0, \varepsilon_0]$.

Proof. Our proof is based on a slight modification of the proof of Theorem 4 in [19] for our current setting. The proof consists of three parts:

(1) normal hyperbolicity and graph representation of S_0,
(2) existence of the graph representation of $W^s(S_\varepsilon)$, and
(3) coincidence of the graph of such a derived function with the whole $W^s(S_\varepsilon)$.

We shall prove them by tracing discussions in [19]. Readers, who are not familiar with the Invariant Manifold Theorem in singular perturbation problems very well, are referred to Chapter 2 in [19]. Here we shall derive discussions only for the stable manifold $W^s(S_\varepsilon)$, while the case of the unstable manifold can be derived replacing matrices in Assumption 3.2 by those in Assumption 3.3 and time reversal. Remark that an assumption for N_c implies that all discussions are done in terms of (2.12). Without loss of generality, we may assume $N = N_c$.

(1) *Normal hyperbolicity and the graph representation of S_0.* First of all, consider the special case $\varepsilon = 0$. In this case, the slow variable y is just a parameter. Inequalities in (3.8) yield the existence of local Lyapunov functions, which is a consequence of arguments in [24] with a little modifications as follows. Fix $y \in K$. Define a function $L_y \colon \mathbb{R}^n \to \mathbb{R}$ by

$$L_y(a(t), b(t)) := -|a(t) - a_y|^2 + |b(t) - b_y|^2,$$

where (a_y, b_y, y) is an equilibrium for (3.1) with $\varepsilon = 0$ in N at y. The existence of (a_y, b_y, y) follows from Proposition 2.17. Then

$$
\begin{aligned}
\frac{dL_y}{dt}(a(t), b(t))\bigg|_{t=0} &= -\frac{d}{dt}(a - a_y)^{\mathrm{T}} \cdot (a - a_y) - (a - a_y)^{\mathrm{T}} \cdot \frac{d}{dt}(a - a_y) \\
&\quad + \frac{d}{dt}(b - b_y)^{\mathrm{T}} \cdot (b - b_y) + (b - b_y)^{\mathrm{T}} \cdot \frac{d}{dt}(b - b_y) \\
&= -(Aa + F_1(a, b, y, 0))^{\mathrm{T}} \cdot (a - a_y) - (a - a_y)^{\mathrm{T}} \cdot (Aa + F_1(a, b, y, 0)) \\
&\quad + (Bb + F_1(a, b, y, 0))^{\mathrm{T}} \cdot (b - b_y) + (b - b_y)^{\mathrm{T}} \cdot (Bb + F_1(a, b, y, 0)).
\end{aligned}
$$

Let 0_k be the k-dimensional k-vector and O_{k_1, k_2} be the (k_1, k_2)-zero matrix. Now we obtain

$$-(Aa + F_1(a, b, y, 0))^{\mathrm{T}} \cdot (a - a_y)$$

$$
= -\begin{pmatrix} a - a_y \\ b - b_y \\ 0_l \\ 0 \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} A + \dfrac{\partial F_1}{\partial a}(\widetilde{z}_1) & \dfrac{\partial F_1}{\partial b}(\widetilde{z}_1) & \dfrac{\partial F_1}{\partial y}(\widetilde{z}_1) & \dfrac{\partial F_1}{\partial \eta}(\widetilde{z}_1) \\ O_{u,s} & O_{s,s} & O_{l,s} & O_{1,s} \\ O_{u,l} & O_{s,l} & O_{l,l} & O_{1,l} \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} a - a_y \\ b - b_y \\ 0_l \end{pmatrix}
$$

$$\leq -\lambda_A |a - a_y|^2 + \sigma_{\mathbb{A}_1^u}(\widetilde{z}_1) |a - a_y|^2 + \sigma_{\mathbb{A}_2^u}(\widetilde{z}_1) |a - a_y||b - b_y|$$

$$\leq -\lambda_A |a - a_y|^2 + \sigma_{\mathbb{A}_1^u}(\widetilde{z}_1) |a - a_y|^2 + \frac{\sigma_{\mathbb{A}_2^u}(\widetilde{z}_1)}{2}(|a - a_y|^2 + |b - b_y|^2)$$

via the Mean Value Theorem, where $\widetilde{z}_1 = (\widetilde{a}_1, \widetilde{b}_1, y, 0) \in N \times [0, \varepsilon_0]$. The same inequalities hold for $(a - a_y)^{\mathrm{T}} \cdot (Aa + F_1(a, b, y, 0))$. Similarly, we obtain

$$(Bb + F_2(a, b, y, 0))^{\mathrm{T}} \cdot (b - b_y)$$

$$= \begin{pmatrix} a - a_y \\ b - b_y \\ 0_l \\ 0 \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} O_{u,u} & O_{s,u} & O_{l,u} & O_{1,u} \\ \dfrac{\partial F_2}{\partial a}(\widetilde{z}_2) & B + \dfrac{\partial F_2}{\partial b}(\widetilde{z}_2) & \dfrac{\partial F_2}{\partial y}(\widetilde{z}_2) & \dfrac{\partial F_2}{\partial \eta}(\widetilde{z}_2) \\ O_{u,l} & O_{s,l} & O_{l,l} & O_{1,l} \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} a - a_y \\ b - b_y \\ 0_l \end{pmatrix}$$

$$\leq \mu_B |b - b_y|^2 + \sigma_{\mathbb{B}_1^s}(\widetilde{z}_2)|b - b_y|^2 + \sigma_{\mathbb{B}_2^s}(\widetilde{z}_2)|a - a_y||b - b_y|$$

$$\leq \mu_B |b - b_y|^2 + \sigma_{\mathbb{B}_1^s}(\widetilde{z}_2)|b - b_y|^2 + \frac{\sigma_{\mathbb{B}_2^s}(\widetilde{z}_2)}{2}\left(|a - a_y|^2 + |b - b_y|^2\right)$$

via the Mean Value Theorem, where $\widetilde{z}_2 = (\widetilde{a}_2, \widetilde{b}_2, y, 0) \in N \times [0, \varepsilon_0]$. The same inequalities hold for $(b - b_y)^{\mathrm{T}} \cdot (Bb + F_2(a, b, y, 0))$. Summarizing these inequalities, we obtain

$$\frac{1}{2}\frac{dL_y}{dt}(a(t), b(t))\bigg|_{t=0}$$

$$\leq \left(-\lambda_A + \sigma_{\mathbb{A}_1^u}(\widetilde{z}_1) + \frac{\sigma_{\mathbb{A}_2^u}(\widetilde{z}_1) + \sigma_{\mathbb{B}_2^s}(\widetilde{z}_2)}{2}\right)|a - a_y|^2$$

$$+ \left(\mu_B + \sigma_{\mathbb{B}_1^s}(\widetilde{z}_2) + \frac{\sigma_{\mathbb{A}_2^u}(\widetilde{z}_1) + \sigma_{\mathbb{B}_2^s}(\widetilde{z}_2)}{2}\right)|b - b_y|^2$$

$$\leq \left(-\lambda_A + \sup_{N \times [0, \varepsilon_0]} \sigma_{\mathbb{A}_1^u} + \frac{\sup_{N \times [0, \varepsilon_0]} \sigma_{\mathbb{A}_2^u} + \sup_{N \times [0, \varepsilon_0]} \sigma_{\mathbb{B}_2^s}}{2}\right)|a - a_y|^2$$

$$+ \left(\mu_B + \sup_{N \times [0, \varepsilon_0]} \sigma_{\mathbb{B}_1^s} + \frac{\sup_{N \times [0, \varepsilon_0]} \sigma_{\mathbb{A}_2^u} + \sup_{N \times [0, \varepsilon_0]} \sigma_{\mathbb{B}_2^s}}{2}\right)|b - b_y|^2.$$

The right-hand side is strictly negative unless $(a, b, y) = (a_y, b_y, y)$, which follows from (3.4) and (3.6). Obviously, $dL_y/dt = 0$ if and only if $(a, b, y) = (a_y, b_y, y)$. Therefore, L_y is a Lyapunov function. This observation leads to the following facts:

- $(a, b, y) = (a_y, b_y, y)$ is an equilibrium for (1.3) which is unique in N with fixed y.
- The linearized matrix

$$\begin{pmatrix} -A - \dfrac{\partial F_1}{\partial a}(\widetilde{z}_1) & -\dfrac{\partial F_1}{\partial b}(\widetilde{z}_1) \\ \dfrac{\partial F_2}{\partial a}(\widetilde{z}_2) & B + \dfrac{\partial F_2}{\partial b}(\widetilde{z}_2) \end{pmatrix}$$

  is strictly negative definite for all $\widetilde{z}_1, \widetilde{z}_2 \in N$. This implies that $(a, b, y) = (a_y, b_y, y)$ is a hyperbolic equilibrium for the layer problem (1.3).

These observations hold for arbitrary $y \in K$. Thanks to the Implicit Function Theorem, we can construct the graph of a smooth y-parameter family of such

hyperbolic equilibria, which is S_0. In particular, S_0 has the structure of normally hyperbolic invariant manifold with graph representation.

(2) *The graph representation of $W^{\mathrm{s}}(S_\varepsilon)$.* We perform the modification to deal with slow directions. We choose a set $\widehat{K}$ whose interior contains K so that its boundary is given by the condition $\widehat{\nu}(y) = 0$ for some C^∞-function $\widehat{\nu}(y)$ and that $\widehat{\nu}(y)$ satisfies $\nabla\widehat{\nu}(y) \neq 0$ for all $y \in \partial\widehat{K}$. The function $\widehat{\nu}(y)$ is assumed to be normalized so that $\nabla\widehat{\nu}(y) = n_y$ is a unit *inward* ([1]) normal vector for $\partial\widehat{K}$. Let $\rho(y)$ be a C^∞-function that has the following values:

$$\rho(y) = \begin{cases} 1 & \text{if } y \in \widehat{K}^{\mathrm{c}}, \\ 0 & \text{if } y \in K. \end{cases}$$

We now modify our system (3.1) by adding the term $\delta\rho(y)n_y$, where δ is a positive number which remains to be chosen.

We add an equation for the small parameter ε. Following Jones [19], we use a multiple of ε as the new auxiliary variable, $\varepsilon = \eta\sigma$, and append the equation $\eta' = 0$ to (3.1), as noted in the beginning of this section. We then obtain the system ([2])

$$(3.9) \qquad \begin{cases} a' = Aa + F_1(x, y, \varepsilon), \\ b' = Bb + F_2(x, y, \varepsilon), \\ y' = \eta\sigma g(x, y, \varepsilon) + \delta\rho(y)n_y, \\ \eta' = 0. \end{cases}$$

As in (3.1) it is understood that x is a function of a and b, which is already realized in Section 2.3, and ε is a function of η. If Theorem 3.5 is restated with ε replaced by η and proved in that formulation, its original version can easily be recaptured by substituting ε back in.

We define the new family of sets $\widehat{N}$ by $\widehat{N} := \pi_{a,b}N \times \widehat{K} \times [0, 1]$. Define the set $\Gamma_{\mathrm{s}} := \{(a, b, y, \eta) \mid \widetilde{\varphi}_\varepsilon(t, (a, b, y, \eta)) \in \widehat{N} \text{ for all } t \geq 0\}$, where $\widetilde{\varphi}_\varepsilon$ is the flow

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([1]) In the lecture note by Jones [19], n_y denotes the *outward* normal vector. We should remark that it is wrong. We shall determine the immediate exit and entrance of a block later. Our claim here is that the exit is $\{(a, \zeta \mid a \in \partial(\pi_a N))\}$ and does not contain $\partial\widehat{K}$. To this end, n_y should be the inward normal vector. If we prove the existence of $W^{\mathrm{u}}(S_\varepsilon)$, n_y is chosen as the outward normal vector.

([2]) This modification guarantees that the vector field has the *inflowing property* with respect to $(\widehat{N}, \partial\widehat{K})$ (e.g. [6], [11]). Namely, at any point $p \in \partial\widehat{K}$, the vector field at p goes inside $\widehat{N}$. This property plays an important role to prove the existence of center-stable manifolds via the graph representation. In the case of $W^{\mathrm{u}}(S_\varepsilon)$ or generally center-unstable manifolds, we modify the vector field so that the vector field has the *overflowing property* with respect to $(\widehat{N}, \partial\widehat{K})$. Namely, at any point $p \in \partial\widehat{K}$, the vector field at p goes outside $\widehat{N}$ or tangently to $\partial\widehat{N}$. In our case, it is sufficient to choose the unit vector n_y as the unit *outward* normal vector to $\partial\widehat{K}$ and to choose ν in the similar manner to $W^{\mathrm{s}}(S_\varepsilon)$.

generated by (3.9). In this part we shall prove that Γ_s is the graph of a function of (b, y, η), say, $a = h_\mathrm{s}(b, y, \eta)$.

Set $\zeta = (b, y, \eta)$ and $\widehat{N}_{\widehat{\zeta}}$ denote the crossing section $\widehat{N}_{\widehat{\zeta}} := \{(a, \widehat{\zeta}) \in \widehat{N}\}$ of $\widehat{N}$ at $\zeta = \widehat{\zeta}$. We shall show that there is at least one point $(a, \widehat{\zeta}) \in \widehat{N}$ for which $\widetilde{\varphi}_\varepsilon(t, (a, \widehat{\zeta})) \in \widehat{N}$ for all $t \geq 0$. To achieve this, we use the Ważewski Principle. If the exit $\widehat{N}^{\mathrm{exit}}$ is closed in $\partial\widehat{N}$ and $\widehat{N}$ is an isolating block for (3.9), then the map $W \colon \widehat{N} \to \widehat{N}^{\mathrm{exit}}$ given by

$$W(z) = \widetilde{\varphi}_\varepsilon(\tau^-(z), z), \quad z = (a, b, y, \eta),$$
$$\tau^-(z) = \sup\{t \geq 0 \mid \widetilde{\varphi}_\varepsilon([0, t], z) \cap N^{\mathrm{exit}} = \emptyset\},$$

is continuous. It is the general consequence of isolating blocks (see [32, Chapter 22]; in [19], such $\widehat{N}$ is called the Ważewski set). Since N is of saddle-type, the flow is repelling on the fast-exit $\widehat{N}^{f,-}$ for all $\eta \in [0, 1]$ and is attracting on the fast-entrance $\widehat{N}^{f,+}$ for all $\eta \in [0, 1]$. It is actually satisfied by choosing $\widehat{K} \supset K$ sufficiently small, if necessary. The rest are the y-direction and the η-direction. Note that an appropriate choice of $\widehat{K}$ enables us to construct local Lyapunov functions L_y in (1) for all $y \in \widehat{K}$, which is due to continuous dependence of estimates on y.

For points on $\widehat{N}$ with $y \in \partial\widehat{K}$, we know

$$\langle y', n_y\rangle = \varepsilon\langle g(x, y, \varepsilon), n_y\rangle + \delta\langle n_y, n_y\rangle$$

since $\rho = 1$ holds on $\partial\widehat{K}$, where $\langle \cdot, \cdot \rangle$ denotes an inner product on appropriate vector spaces. Setting $M_{\widehat{N}} := \sup_{\widehat{N}}\{|g|, |Dg|\}$, the above can be estimated as

$$\langle y', n_y\rangle \geq \delta - \varepsilon_0 M_{\widehat{N}} > 0,$$

if $\delta > \varepsilon_0 M_{\widehat{N}}$. Since we can choose δ arbitrarily, then it can be achieved. Therefore, $\partial\widehat{K} \cap \widehat{N}$ is a part of entrance. Finally, in the case $\eta = 0$ or 1, both of these sets are invariant since $\eta' = 0$ holds everywhere and thus render neither the entrance nor the exit.

The exit $\widehat{N}^{\mathrm{exit}}$ is then seen to be $\{(a, \zeta) \mid a \in \partial(\pi_a N)\}$. By our construction of N, for any $\widehat{\zeta}$, the set $\widehat{N}_{\widehat{\zeta}}$ is a ball of dimension u. Suppose that $\Gamma_\mathrm{s} \cap \widehat{N}_{\widehat{\zeta}} = \emptyset$. All points in $\widehat{N}_{\widehat{\zeta}}$ then go to $\widehat{N}^{\mathrm{exit}}$ in finite time and hence, restricting the Ważewski map to the crossing section, we obtain a continuous map $W \colon \widehat{N}_{\widehat{\zeta}} \to \widehat{N}^{\mathrm{exit}}$. If we follow this by the projection $\pi_a(a, \zeta) = a$, we see that $\pi_a \circ W$ maps a u-dimensional ball onto its boundary, while keeping the boundary fixed. This contradicts the well-known No-Retract Theorem. There is thus a point in $\Gamma_\mathrm{s} \cap \widehat{N}_{\widehat{\zeta}}$. Since $\widehat{\zeta}$ is arbitrary, this gives at least one value for a as a function of (b, y, η). We shall put it $h_\mathrm{s}(b, y, \eta)$.

(3) *Uniqueness of $W^\mathrm{s}(S_\varepsilon)$ as a graph of $a = h_\mathrm{s}(b, y, \eta)$.* In the sequel we show that the graph of the above derived function is all of Γ_s. At the same time, it

will be shown that the function is, in fact, Lipschitz continuous with Lipschitz constant 1. A comparison between the growth rates in different directions will be derived in the following proposition in our setting. Let $(a_i(t), \zeta_i(t))$, $i = 1, 2$, be two solutions of (3.9), set $\Delta a := a_1(t) - a_2(t)$ and $\Delta \zeta := \zeta_2(t) - \zeta_1(t)$. Further we define

$$M(t) := |\Delta a(t)|^2 - |\Delta \zeta(t)|^2.$$

PROPOSITION 3.6 (cf. Lemma 2 in [19]). *Under Assumption 3.2, if $M(t) = 0$ then $M'(t) > 0$ holds as long as the two solutions stay in $\widehat{N}$, unless $\Delta a = 0$.*

PROOF OF PROPOSITION 3.6. Discussions in our proof are based on the proof of Lemma 2 in [19] with arrangements in our setting. We will estimate each of the quantities $\langle \Delta a, \Delta a' \rangle$, etc. The equation for Δa is

$$\Delta a' = A(a_2 - a_1) + F_1(a_2, b_2, y_2, \eta_2) - F_1(a_1, b_1, y_1, \eta_1),$$

which we rewrite as

$$(3.10) \qquad \Delta a' = A \Delta a + \Delta F_1,$$

where $\Delta F_1 = F_1(a_2, b_2, y_2, \eta_2) - F_1(a_1, b_1, y_1, \eta_1)$. Since $\widehat{N}$ is the product of h-sets associated with an affine transformation, thanks to the Mean Value Theorem, one can derive the following equality:

$$\Delta F_1 = \frac{\partial F_1}{\partial a}(\widetilde{a}_1, \widetilde{b}_1, \widetilde{y}_1, \widetilde{\eta}_1)\Delta a + \frac{\partial F_1}{\partial b}(\widetilde{a}_1, \widetilde{b}_1, \widetilde{y}_1, \widetilde{\eta}_1)\Delta b$$
$$+ \frac{\partial F_1}{\partial y}(\widetilde{a}_1, \widetilde{b}_1, \widetilde{y}_1, \widetilde{\eta}_1)\Delta y + \frac{\partial F_1}{\partial \eta}(\widetilde{a}_1, \widetilde{b}_1, \widetilde{y}_1, \widetilde{\eta}_1)\Delta \eta$$

for some points $(\widetilde{a}_1, \widetilde{b}_1, \widetilde{y}_1, \widetilde{\eta}_1) =: \widetilde{z}_1 \in \widehat{N}$. Such an estimate makes sense since $\widehat{N}$ is convex.

We can estimate $\langle \Delta a, \Delta a' \rangle = 2\langle \Delta a, \Delta a' \rangle$ by taking the inner product of (3.10) with Δa. As a result, we obtain

$$(3.11) \quad \langle \Delta a, \Delta a' \rangle = (\Delta a)^{\mathrm{T}} \cdot \left\{ A\Delta a + \frac{\partial F_1}{\partial a}(\widetilde{z}_1)\Delta a + \frac{\partial F_1}{\partial b}(\widetilde{z}_1)\Delta b \right.$$
$$\left. + \frac{\partial F_1}{\partial y}(\widetilde{z}_1)\Delta y + \frac{\partial F_1}{\partial \eta}(\widetilde{z}_1)\Delta \eta \right\}.$$

We also obtain the following equalities in the same manner:

$$(3.12) \quad \langle \Delta b, \Delta b' \rangle = (\Delta b)^{\mathrm{T}} \cdot \left\{ B\Delta b + \frac{\partial F_2}{\partial a}(\widetilde{z}_2)\Delta a + \frac{\partial F_2}{\partial b}(\widetilde{z}_2)\Delta b \right.$$
$$\left. + \frac{\partial F_2}{\partial y}(\widetilde{z}_2)\Delta y + \frac{\partial F_2}{\partial \eta}(\widetilde{z}_2)\Delta \eta \right\}$$

as well as

$$(3.13) \quad \langle \Delta y, \Delta y' \rangle = \sigma(\Delta y)^{\mathrm{T}} \cdot \left\{ \eta \frac{\partial g}{\partial a}(\widetilde{z}_3) \Delta a + \eta \frac{\partial g}{\partial b}(\widetilde{z}_3) \Delta b \right.$$
$$\left. + \eta \frac{\partial g}{\partial y}(\widetilde{z}_3) \Delta y + \left(g(\widetilde{z}_3) + \frac{\partial g}{\partial \eta}(\widetilde{z}_3) \right) \Delta \eta \right\}$$

for some $\widetilde{z}_i \in \widehat{N}$, $i = 2, 3$. Obviously $\langle \Delta \eta, \Delta \eta' \rangle = 0$ since $\eta' = 0$.

Recall that our final objective is the estimate of

$$M'(t) = 2\{ \langle \Delta a, \Delta a' \rangle - (\langle \Delta b, \Delta b' \rangle + \langle \Delta y, \Delta y' \rangle) \}.$$

To this end, we estimate (3.11)–(3.13) as the quadratic form associated with non-square matrices. For example, equality (3.11) can be rewritten is the following matrix form:

$$\langle \Delta a, \Delta a' \rangle = (\Delta a)^{\mathrm{T}} A \Delta a + (\Delta a)^{\mathrm{T}} \mathbb{A}^u \begin{pmatrix} \Delta a \\ \Delta b \\ \Delta y \\ \Delta \eta \end{pmatrix}$$
$$:= (\Delta a)^{\mathrm{T}} A \Delta a + (\Delta a)^{\mathrm{T}} \mathbb{A}_1^u \Delta a + (\Delta a)^{\mathrm{T}} \mathbb{A}_2^u \Delta \zeta,$$

where $\mathbb{A}_1^u$ is the $u \times u$-matrix given by $(\partial F_1 / \partial a)(\widetilde{z}_1)$ and $\mathbb{A}_2^u$ is the $u \times (s+l+1)$-matrix given by

$$\mathbb{A}_2^u = \mathbb{A}_2^u(\widetilde{z}_1) := \left(\frac{\partial F_1}{\partial b}(\widetilde{z}_1) \quad \frac{\partial F_1}{\partial y}(\widetilde{z}_1) \quad \frac{\partial F_1}{\partial \eta}(\widetilde{z}_1) \right).$$

Denoting by $\| \cdot \|_2$ the (matrix) 2-norm, general theory of linear algebra yields

$$(\Delta a)^{\mathrm{T}} \mathbb{A}^u \begin{pmatrix} \Delta a \\ \Delta b \\ \Delta y \\ \Delta \eta \end{pmatrix} \leq |\Delta a| (\|\mathbb{A}_1^u\|_2 |\Delta a| + \|\mathbb{A}_2^u\|_2 |\Delta \zeta|)$$
$$= \sigma_{\mathbb{A}_1^u}(\widetilde{z}_1) |\Delta a|^2 + \sigma_{\mathbb{A}_2^u}(\widetilde{z}_1) |\Delta a| |\Delta \zeta|,$$

where $\sigma_{\mathbb{A}_i^u} = \sigma_{\mathbb{A}_i^u}(\widetilde{z}_1)$, $i = 1, 2$, is the maximal singular value of $\mathbb{A}_i^u$ at $\widetilde{z}_1 \in \widehat{N}$ stated in Assumption 3.2. We obtain the following estimate of (3.11):

$$(3.14) \qquad \langle \Delta a, \Delta a' \rangle \geq \left(\lambda_A - \sup_{\widehat{N}} \sigma_{\mathbb{A}_1^u} \right) |\Delta a|^2 - \sup_{\widehat{N}} \sigma_{\mathbb{A}_2^u} |\Delta a| |\Delta \zeta|,$$

where λ_A is a positive number satisfying (3.2). In a similar manner we also obtain the estimate of $\langle \Delta b, \Delta b' \rangle$ and $\langle \Delta y, \Delta y' \rangle$ by

$$\langle \Delta b, \Delta b' \rangle \leq \mu_B |\Delta b|^2 + |\Delta b| \left(\sup_{\widehat{N}} \sigma_{\mathbb{B}_1^u} |\Delta a| + \sup_{\widehat{N}} \sigma_{\mathbb{B}_2^u} |\Delta \zeta| \right),$$

$$\langle \Delta y, \Delta y' \rangle \leq \sigma |\Delta y| \left(\sup_{\widehat{N}} \sigma_{g_1^u} |\Delta a| + \sup_{\widehat{N}} \sigma_{g_2^u} |\Delta \zeta| \right),$$

where μ_B denotes a negative number satisfying (3.3). Functions $\sigma_{\mathbb{B}_1^u} = \sigma_{\mathbb{B}_1^u}(\widetilde{z})$, $\sigma_{\mathbb{B}_2^u} = \sigma_{\mathbb{B}_2^u}(\widetilde{z})$, $\sigma_{g_1^u} = \sigma_{g_1^u}(\widetilde{z})$ and $\sigma_{g_2^u} = \sigma_{g_2^u}(\widetilde{z})$ are maximal singular values of the matrix-valued functions defined in Assumption 3.2 at $\widetilde{z}$, respectively. In particular, at a point satisfying $M(t) = 0$, we further obtain

$$\langle \Delta a, \Delta a' \rangle \geq \left(\lambda_A - \sup_{\widehat{N}} \sigma_{\mathbb{A}_1^u} - \sup_{\widehat{N}} \sigma_{\mathbb{A}_2^u} \right) |\Delta a|^2,$$

$$\langle \Delta b, \Delta b' \rangle \leq \left(\mu_B + \sup_{\widehat{N}} \sigma_{\mathbb{B}_1^u} + \sup_{\widehat{N}} \sigma_{\mathbb{B}_2^u} \right) |\Delta a|^2,$$

$$\langle \Delta y, \Delta y' \rangle \leq \sigma |\Delta y| \left(\sup_{\widehat{N}} \sigma_{g_1^u} |\Delta a| + \sup_{\widehat{N}} \sigma_{g_2^u} |\Delta \zeta| \right)$$

to show

$$M'(t) \geq 2 \left[\lambda_A - \mu_B - \left\{ \sup_{\widehat{N}} \sigma_{\mathbb{A}_1^u} + \sup_{\widehat{N}} \sigma_{\mathbb{A}_2^u} + \sup_{\widehat{N}} \sigma_{\mathbb{B}_1^u} \right. \right.$$
$$\left. \left. + \sup_{\widehat{N}} \sigma_{\mathbb{B}_2^u} + \sigma \left(\sup_{\widehat{N}} \sigma_{g_1^u} + \sup_{\widehat{N}} \sigma_{g_2^u} \right) \right\} \right] |\Delta a|^2,$$

which proves our statement. Note that the coefficient of $|\Delta a|^2$ can be positive if $\widehat{K} \supset K$ is sufficiently small, thanks to cone conditions. $\qquad \square$

We go back to the proof in (3). We have already shown that the set Γ_{s} contains the graph of a function denoted by $a = h_{\mathrm{s}}(b, y, \eta)$. Suppose that Γ_{s} contains more than one point with the same value of b, y, η. There would then be a_1 and a_2 so that both (a_1, b, y, η) and (a_2, b, y, η) lie in Γ_{s}. At $t = 0$, we have $|\Delta a| \geq |\Delta \zeta|$. By Proposition 3.6, $|\Delta a(t)| \geq |\Delta \zeta(t)|$ holds for all $t \geq 0$. In estimate (3.14) we can then replace $|\Delta \zeta|$ by $|\Delta a|$ to obtain $\{|\Delta a|^2\}' \geq 2(\lambda_A - \beta)|\Delta a|^2$. A positive number β can be chosen as $\sup_{\widehat{N}} \sigma_{\mathbb{A}_1^u} + \sup_{\widehat{N}} \sigma_{\mathbb{A}_2^u}$. See (3.4). From which it can be easily concluded that Δa grows exponentially, which contradicts the hypothesis that both points stay in $\widehat{N}$ for all $t \geq 0$.

The same argument can be used to show that h_{s} is also Lipschitz. If (a_1, ζ_1) and (a_2, ζ_2) are both in Γ_{s} and $|a_2 - a_1| \geq |\zeta_2 - \zeta_1|$, then $|a_2 - a_1|$ grows exponentially, which contradicts the hypothesis again that both points lie in Γ_{s}. We then have shown that the set Γ_{s} is the graph of a Lipschitz function. In particular, the Lipschitz function $a = h_{\mathrm{s}}(b, y, \varepsilon)$ is well defined. This manifold is $W^{\mathrm{s}}(S_\varepsilon)$ when $y \in \widehat{K}$ is restricted to K, in which the modified equation (3.9) agrees with the original one (3.1). $\qquad \square$

A direct consequence of graph representations of $W^{\mathrm{s}}(S_\varepsilon)$ as well as $W^{\mathrm{u}}(S_\varepsilon)$ is that they are homotopic to flat hyperplanes inside h-sets. In terms of the theory of covering relations (Section 2.2), $W^{\mathrm{s}}(S_\varepsilon)$ is a vertical disc and $W^{\mathrm{u}}(S_\varepsilon)$ is a horizontal disc in a given fast-saddle-type block.

COROLLARY 3.7. *Let $N \in \mathbb{R}^{n+l}$ be a fast-saddle-type block for* (3.1) *such that the coordinate representation N_c is actually of the form* (2.12). *Assume that N satisfies* (3.4) *and* (3.5). *Let $b = h_u(a, y, \varepsilon)$ be a function defining $W^u(S_\varepsilon)$ in N_c. Then there is a homotopy $H \colon [0, 1] \times N_c \times [0, \varepsilon_0] \to N_c \times [0, \varepsilon_0]$ satisfying*

$$H_0(a, b, y, \varepsilon) = H(0, a, b, y, \varepsilon) = (a, h_u(a, y, \varepsilon), y, \varepsilon), \quad H_1(a, b, y, \varepsilon) = (a, 0, y, \varepsilon).$$

Similarly, if N_c satisfies (3.6) *and* (3.7) *and $a = h_s(b, y, \varepsilon)$ denotes a function defining $W^s(S_\varepsilon)$ in N, then there is a homotopy $H \colon [0, 1] \times N_c \times [0, \varepsilon_0] \to N_c \times [0, \varepsilon_0]$ satisfying*

$$H_0(a, b, y, \varepsilon) = H(0, a, b, y, \varepsilon) = (h_s(b, y, \varepsilon), b, y, \varepsilon), \quad H_1(a, b, y, \varepsilon) = (0, b, y, \varepsilon).$$

PROOF. When N_c satisfies (3.4) and (3.5), define a map H on $[0, 1] \times N_c \times [0, \varepsilon_0] \to \mathbb{R}^{n+l+1}$ by $H(\lambda, a, b, y, \varepsilon) := (a, (1-\lambda)h_u(a, y, \varepsilon), y, \varepsilon)$. By the construction of h_u, H is continuous. Since N_c is convex, then H maps $[0, 1] \times N_c \times [0, \varepsilon_0]$ into $N_c \times [0, \varepsilon_0]$. Thus H is our desired homotopy. The proof of the case that N_c satisfies (3.6) and (3.7) is similar.   □

REMARK 3.8. We do not need to assume the normal hyperbolicity of the critical manifold S_0 and the graph representation of S_0 in Theorem 3.5, while the Fenichel's classical theory needs to assume them. Indeed, part (a) in the proof of the theorem shows that this assumption can be validated by computable estimates (3.8).

Let D and $\widehat{D}$ be fast-saddle-type blocks given by (2.12) and (2.13), respectively. Theorem 3.5 says that, under cone conditions, the slow manifold S_ε is contained in D. Obviously, S_ε is also contained in $\widehat{D}$ since $D \subset \widehat{D}$. Moreover, S_ε is uniquely determined in $\widehat{D}$. If $d_0 \le \mathrm{dist}(\partial(\pi_{a,b}\widehat{D}), \pi_{a,b}D)$, then our observations imply that the distance between S_ε and $\partial\widehat{D}$ in fast components is greater than d_0. For example, the distance of $\partial\widehat{D}_c$ and S_ε in the a_1-component is greater than $\min\{\eta_{1,1}^\pm\}$. Summarizing these arguments we have the following result, which is essential to Section 4.3.

COROLLARY 3.9 (Fast-saddle-type blocks with spaces). *Consider* (2.10). *Let $D, \widehat{D} \subset \mathbb{R}^{n+l}$ be fast-saddle-type blocks for* (2.10) *such that the coordinate representations D_c and $\widehat{D}_c$ are actually given by* (2.12) *and* (2.13), *respectively, for a given sequence of positive numbers $\{\eta_{i,j_i}^\pm\}_{j_1=1,\dots,u,\, j_2=1,\dots,s}^{i=1,2}$. Assume that stable and unstable cone conditions hold in $\widehat{D}_c$. Then the same statement as Theorem 3.5 holds in $\widehat{D}_c$. Moreover, the distance between $\partial\widehat{D}_c$ and the validated slow manifold S_ε is estimated by*

$$\mathrm{dist}(\pi_a(\partial\widehat{D}_c), \pi_a(W^s(S_\varepsilon))) \ge \min_{j=1,\dots,u} \{\eta_{1,j}^\pm\},$$

$$\mathrm{dist}(\pi_b(\partial\widehat{D}_c), \pi_b(W^u(S_\varepsilon))) \ge \min_{j=1,\dots,s} \{\eta_{2,j}^\pm\} \quad \text{for } \varepsilon \in [0, \varepsilon_0].$$

The main feature of our present results is that slow manifolds as well as their stable and unstable manifolds in Fenichel's theorems are validated in given blocks with an explicit range $\varepsilon \in [0, \varepsilon_0]$. Our criteria can be explicitly validated with rigorous numerics, as many preceding works (e.g. [36], [38]). We end this section noting that all results in this section still hold for $y \in \mathbb{R}^l$ and $\mathbb{R}$-valued functions g with appropriate smoothness and $l > 1$.

4. Covering-exchange and dynamics around slow manifolds

In Section 3 we have discussed the validation of slow manifolds. In this section, we move to the next stage; the behavior near and on validated slow manifolds. There are mainly two cases of the exhibition of slow dynamics. One is monotone and the other is nontrivial in the sense that dynamics on slow manifolds admits nontrivial invariant sets such as equilibria or periodic orbits.

First we consider the dynamics near slow manifolds. If, for sufficiently small $\varepsilon > 0$, a point $q_\varepsilon \in \mathbb{R}^{n+1}$ is sufficiently close to slow manifolds, the trajectory through q_ε moves near slow manifolds spending a long time. However, the precise description of trajectories off slow manifolds is not easy since they clearly have an influence of fast dynamics $\dot{x} = f(x, y, \varepsilon)$. In order to describe such behavior as simply as possible, we introduce an extension of covering relations: the *covering-exchange* (Section 4.1), so that it can be appropriately applied to singular perturbation problems. This extension is a topological counterpart of the (C^0-version of) well-known *Exchange Lemma*. In the successive subsections, we provide a generalization of covering-exchange: the *slow shadowing* and *m-cones* (Sections 4.2–4.4). These concepts enable us to validate slow manifolds with nonlinear structures as well as their stable and unstable manifolds in reasonable ways via rigorous numerics, keeping the essence of covering-exchange property.

Next, we consider the nontrivial dynamics on slow manifolds, namely, the presence of nontrivial invariant sets. We can use ideas discussed in Section 2.3 to validate nontrivial invariant sets even on slow manifolds.

Finally, we discuss the unstable manifold of invariant sets on slow manifolds. The invariant foliation structure with respect to slow manifolds is essentially applied to validating unstable manifolds of invariant sets in terms of covering relations.

4.1. Covering-exchange with one-dimensional slow variable. First we consider the case that the vector field near slow manifolds is monotone. In such a case, geometric singular perturbation theory has an answer which describes dynamics around slow manifolds in terms of locally invariant manifolds; the *Exchange Lemma* (e.g. [19], [20], [21], [34]). The Exchange Lemma solves matching problems between fast dynamics and slow dynamics. More precisely, if a family of tracking invariant manifolds $\{\Sigma_\varepsilon\}_{\varepsilon \geq 0}$ has transversal intersection

$\Sigma_0 \cap_T W^s(S_0)$ at $\varepsilon = 0$, then a trajectory $\varphi_\varepsilon([0, T_\varepsilon], q_\varepsilon)$ for a point $q_\varepsilon \in \Sigma_\varepsilon$ describes the match of fast and slow trajectories near the slow manifold in the full system $(1.1)_\varepsilon$ for sufficiently small $\varepsilon > 0$. Moreover, the Exchange Lemma also states that the manifold $\varphi_\varepsilon(T_\varepsilon, \Sigma_\varepsilon)$ is $O(\varepsilon)$-close to the unstable manifold $W^u(J)$ of a segment $J \subset S_0$ in a neighbourhood of $\varphi_\varepsilon(T_\varepsilon, q_\varepsilon)$. However, the Exchange Lemma requires the assumption of transversality between a tracking (locally invariant) manifold and the stable manifold of a slow manifold, which is not easy to validate in rigorous numerics. Alternatively, we consider the topological analogue of statements in the $(C^0$-)Exchange Lemma.

Every trajectories in fast-saddle-type blocks leaves them through exits. Note that our desired trajectories are the continuation of chains consisting of critical manifolds and heteroclinic orbits in the limit system (1.3). One thus expects that singularly perturbed global trajectories typically leave fast-saddle-type blocks through *fast-exits*. To describe our expectations precisely, we define the following notions. For our purpose, we restrict $u = u(M)$ for h-sets to $u = 1$ unless otherwise noted.

DEFINITION 4.1 (Fast-exit face). Let M be an $(n+1)$-dimensional h-set with the following expression via a homeomorphism $c_M \colon \mathbb{R}^{n+1} \to \mathbb{R}^{u(M)+s(M)}$:

$$M_c = c_M(M) = \overline{B_u} \times \overline{B_s} \times [0, 1]$$

with $u = u(M) = 1$ and $s(M) = s + 1$. We call an h-set M^a a *fast-exit face* of M if $u(M^a) = u$, $s(M^a) = s$ and there exist an element $a \in \partial B_u$ and a compact interval $K_0 \subset (0, 1)$ such that $c_M(M^a) = \{a\} \times \overline{B_s} \times K_0$. See also Figure 2.

DEFINITION 4.2 (Covering-exchange). Consider $(1.1)_\varepsilon$ with fixed $\varepsilon \geq 0$. Let $N \subset \mathbb{R}^{n+1}$ be an h-set with $u(N) = u = 1$ and M be an $(n + 1)$-dimensional h-set in $\mathbb{R}^{n+1}$ with $u(M) = 1$. We say N satisfies the *covering-exchange property for $(1.1)_\varepsilon$ with respect to M* if the following statements are satisfied:

(CE1) M is a fast-saddle-type block with the coordinate system

$$M_c = \overline{B_1} \times \overline{B_s} \times [0, 1], \quad a \in \overline{B_1}, \quad (b_1, \dots, b_s) \in \overline{B_s}, \quad y \in [0, 1].$$

(CE2) Stable and unstable cone conditions (Definition 3.4) for $(1.1)_\varepsilon$ are satisfied in M.

(CE3) For $q \in \{\pm 1\}$, $q \cdot g(x, y, \varepsilon) > 0$ holds in M. We shall call q the *slow direction number*.

(CE4) Letting φ_ε be the flow of $(1.1)_\varepsilon$, there exists $T > 0$ such that $N \overset{\varphi_\varepsilon(T, \cdot)}{\Longrightarrow} M$.

(CE5) M possesses a fast-exit face $M^a \subset M^{f,-}$ with the expression

$$c_M(M^a) = \{a\} \times \overline{B_s} \times [y^-, y^+], \quad a \in \partial B_1, \quad [y^-, y^+] \subset (0, 1)$$

such that

$$
\begin{cases}
\sup\left(\pi_y \varphi_\varepsilon(T, N) \cap M\right) < y^- & \text{if } q = +1, \\
\inf\left(\pi_y \varphi_\varepsilon(T, N) \cap M\right) > y^+ & \text{if } q = -1,
\end{cases}
$$

holds.

In this situation we shall call the pair (N, M) the *covering-exchange pair* for system $(1.1)_\varepsilon$.

A brief illustration of the covering-exchange property is shown in Figure 2. (CE2) implies the existence of a slow manifold S_ε as well as a limiting normally hyperbolic critical manifold S_0 at $\varepsilon = 0$, as shown in Theorem 3.5. Combining with (CE1), one sees that M is repelling in the a-direction and attracting in the b-direction. (CE3) means that slow dynamics in M is monotone. Remark that the monotonicity is assumed not only on S_ε but also in the whole region M. (CE4) topologically describes the transversality of the stable manifold $W^s(S_\varepsilon)$ of the slow manifold S_ε and the h-set $\varphi_\varepsilon(T, N)$.

For a fast-saddle-type block M of the form in Definition 4.2 with $q = +1$, we call $M_c^{s,-} \equiv \overline{B_1} \times \overline{B_s} \times \{1\}$ and $M_c^{s,+} \equiv \overline{B_1} \times \overline{B_s} \times \{0\}$ (equivalently $M^{s,-}$ and $M^{s,+}$) the *slow exit* and *slow entrance* of M, respectively. Similarly, in the case of $q = -1$, $M_c^{s,-} \equiv \overline{B_1} \times \overline{B_s} \times \{0\}$ and $M_c^{s,+} \equiv \overline{B_1} \times \overline{B_s} \times \{1\}$ are called the slow exit and slow entrance of M, respectively.
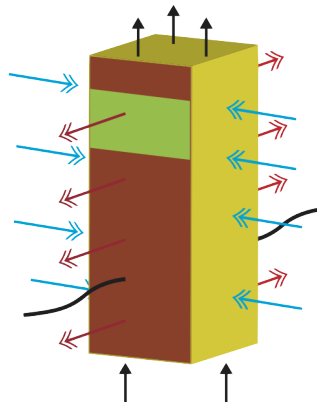

FIGURE 2. Illustrations of the covering-exchange pair and a fast-exit face. The rectangular parallelepiped M is an h-set with $u(M) = 1$ and $s(M) = 2$. Arrows colored red, blue and black show the fast unstable direction, fast stable direction and slow direction of (3.1) at corresponding boundaries, respectively. The black curve is an image $\varphi_\varepsilon(T, N)$ of another h-set N for some $T > 0$. This figure describes $N \overset{\varphi_\varepsilon(T, \cdot)}{\Longrightarrow} M$. Since the y-component of the vector field in M is monotone and the fast component of the vector field on ∂M is already validated concretely, then $\varphi_\varepsilon(T, N)$ keeps the covering relation $N \overset{\varphi_\varepsilon(t, \cdot)}{\Longrightarrow} M$ for all $t \geq T$ until N exits M from its top. A fast-exit face M^a is colored green, in which case $u = 1$, $s = 1$, $l = 1$.

Define the Poincaré map $P_\varepsilon^M \colon M \to \partial M$ in M by

$$P_\varepsilon^M(z) := \varphi_\varepsilon(t_\varepsilon(z), z), \qquad t_\varepsilon(z) := \sup\{t \mid \varphi_\varepsilon([0, t], z) \subset M\}.$$

If no confusion arises, we use this notation for representing the Poincaré map in a set M.

NOTATIONS 4.3. For $\widehat{y} \in [0, 1]$ and any set $A \subset \mathbb{R}^{n+1}$, let $A_{\widehat{y}} := A \cap \{y = \widehat{y}\}$, $A_{\leq \widehat{y}} := A \cap \{y \leq \widehat{y}\}$ and $A_{\geq \widehat{y}} := A \cap \{y \geq \widehat{y}\}$.

LEMMA 4.4. *Fix $\varepsilon > 0$. Let M be an h-set satisfying* (CE1)–(CE3) *in Definition 4.2 with the slow direction number $q = +1$, and M^{exit} be the fast-exit face with $(M^{\mathrm{exit}})^+ \subset M^{f,+}$, where*

$$c_M(M^{\mathrm{exit}}) = \{1\} \times \overline{B_s} \times [y^-, y^+], \quad [y^-, y^+] \subset (0, 1).$$

Then, the restriction of the Poincaré map P_ε^M to $((P_\varepsilon^M)^{-1}(M^{\mathrm{exit}}))_{\widehat{y}}$ is a homeomorphism for each $\widehat{y} \in [0, y^-)$. Moreover, there exists an h-set $\widetilde{M} \subset M$ such that $\widetilde{M} \overset{P_\varepsilon^M}{\Longrightarrow} M^{\mathrm{exit}}$.

PROOF. Via a homeomorphism c_M, we may assume that $M = \overline{B_1} \times \overline{B_s} \times [0, 1]$. Since M is an isolating block for φ_ε, then P_ε^M is continuous on M, which is the consequence of standard theory of isolating blocks [7], [19], [32]. Consider the backward Poincaré map $P_\varepsilon^{M,-} \colon M \to \partial M$ given by

$$P_\varepsilon^{M,-}(z) := \varphi_\varepsilon(\tau(z), z), \qquad \tau(z) := \inf\{t \leq 0 \mid \varphi_\varepsilon([t, 0], z) \subset M\}.$$

This is obviously well defined via the property of flows. Since M is also an isolating block for the backward flow, then the exit of M in backward time is closed and hence $P_\varepsilon^{M,-}$ is also continuous. The correspondence between points on $(P_\varepsilon^M)^{-1}(M^{\mathrm{exit}}) \cap \{y = \widehat{y}\}$ and their image is one-to-one and onto under P_ε^M. This property holds since $M \cap \{y = \widehat{y}\}$ is a crossing section from (CE3). By restricting M to $M_{\geq \widehat{y}}$ and redefining P_ε^M and $P_\varepsilon^{M,-}$ for $M_{\geq \widehat{y}}$, their continuity yields that $(P_\varepsilon^M)^{-1}(M^{\mathrm{exit}}) \cap \{y = \widehat{y}\}$ is compact. Therefore, the restriction of P_ε^M to $(P_\varepsilon^M)^{-1}(M^{\mathrm{exit}}) \cap \{y = \widehat{y}\}$ is a homeomorphism.

Next consider $P_\varepsilon^{M_{\geq \widehat{y}}}$ instead of P_ε^M. Since $M \subset \mathbb{R}^{n+1}$ is a fast-saddle-type block with $u(M) = 1$ satisfying stable and unstable cone conditions, the stable manifold $W^s(S_\varepsilon)$ divides M into two components, where S_ε is the slow manifold in M. Let M_+ be the component of $M \setminus W^s(S_\varepsilon)$ containing M^{exit}. Now we can construct an h-set with the desired covering relation as follows.

Consider a section $M_{+,\widehat{b}} = \{(a, \widehat{b}, \widehat{y}) \mid (a, \widehat{b}, \widehat{y}) \in N\}$. Let $a_{\widehat{b}} \in \overline{B_1}$ be such that $W^s(S_\varepsilon) \cap \{(b, y) = (\widehat{b}, \widehat{y})\} = \{(a_{\widehat{b}}, \widehat{b}, \widehat{y})\}$. Since $P_\varepsilon^{M_{\geq \widehat{y}}} |_{M_+} \colon M_+ \to P_\varepsilon^{M_{\geq \widehat{y}}}(M_+) \subset \partial M$ is homeomorphic, $g(x, y, \varepsilon) > 0$ holds and

$$P_\varepsilon^{M_{\geq \widehat{y}}}(1, \widehat{b}, \widehat{y}) = (1, \widehat{b}, \widehat{y}) \in M^{f,-}, \qquad P_\varepsilon^{M_{\geq \widehat{y}}}(a_{\widehat{b}}, \widehat{b}, \widehat{y}) \in M^{s,-}$$

hold, Proposition 3.6 implies that $z_0, z_1, z_2 \in M$ with $z_0 \in W^{\mathrm{s}}(S_\varepsilon)$, $z_1, z_2 \in$ $C^{\mathrm{u}}(z_0)$, $|\pi_a(z_1 - z_0)| > |\pi_a(z_2 - z_0)|$ and $\pi_y(z_1) = \pi_y(z_2)$ satisfy $\pi_y(P_\varepsilon^{M \geq \widehat{y}}(z_1)) <$ $\pi_y(P_\varepsilon^{M \geq \widehat{y}}(z_2))$. Here $C^{\mathrm{u}}(z)$ denotes the unstable cone with the vertex z:

$$C^{\mathrm{u}}(z) = \left\{ (a, b, y) \mid |a - \pi_a(z)|^2 \geq |b - \pi_b(z)|^2 + |y - \pi_y(z)|^2 \right\}.$$

Thus there exist points $(a_{\widehat{b}}^{\pm}, \widehat{b}, \widehat{y}) \in M_{+, \widehat{b}}$ such that $P_\varepsilon^{M \geq \widehat{y}}(a_{\widehat{b}}^{\pm}, \widehat{b}, \widehat{y}) \in \partial M \cap \{ y = y^{\pm} \}$ with $a_{\widehat{b}}^- < a_{\widehat{b}}^+ < a_{\widehat{b}}^-$.

Let $\psi^{\pm} \colon \overline{B_s} \to \overline{B_1}$ be given by $\psi^{\pm}(\widehat{b}) = a_{\widehat{b}}^{\pm}$. Thanks to the continuity of $P_\varepsilon^{M \geq \widehat{y}}$, both $\psi^{\pm}$ are continuous. Finally, define $\widetilde{M_{\widehat{y}}}$ by

$$\widetilde{M_{\widehat{y}}} := \{ (a, b, \widehat{y}) \mid \psi^+(b) \leq a \leq \psi^-(b),\, b \in \overline{B_s} \}.$$

It is indeed an h-set with

(4.1)
$$\begin{aligned}
\widetilde{M_{\widehat{y}}}^- &:= \{ (a, b, \widehat{y}) \mid a = \psi^{\pm}(b),\, b \in \overline{B_s} \}, \\
\widetilde{M_{\widehat{y}}}^+ &= \{ (a, b, \widehat{y}) \mid \psi^+(b) \leq a \leq \psi^-(b),\, b \in \partial B_s \}.
\end{aligned}$$

Slight modifications of $\psi^{\pm}$ yield the correspondence

$$\pi_y\left(P_\varepsilon^{M \geq \widehat{y}}(\psi^-(b), b, \widehat{y}) \right) \in (\widehat{y}, y^-), \qquad \pi_y\left(P_\varepsilon^{M \geq \widehat{y}}(\psi^+(b), b, \widehat{y}) \right) \in (y^+, 1)$$

for $b \in \overline{B_s}$. We rewrite the corresponding h-set as $\widetilde{M_{\widehat{y}}}$ again.

Note that $P_\varepsilon^{M \geq \widehat{y}}(\widetilde{M_{\widehat{y}}}^-) \cap M^{\mathrm{exit}} = \emptyset$ by definition of $\psi^{\pm}$. Also, $P_\varepsilon^{M \geq \widehat{y}}(\widetilde{M_y}) \cap (M^{\mathrm{exit}})^+ = \emptyset$ holds since $(M^{\mathrm{exit}})^+$ is on $M^{f,+}$. The arbitrary choice of $q_0 \in \overline{B_s}$ leads to the statement in Proposition 2.8. In particular, $\widetilde{M_{\widehat{y}}} \overset{P_\varepsilon^{M \geq \widehat{y}}}{\Longrightarrow} M^{\mathrm{exit}}$ holds. Of course, $\widetilde{M_{\widehat{y}}} \overset{P_\varepsilon^{M}}{\Longrightarrow} M^{\mathrm{exit}}$ also holds since $P_\varepsilon^{M \geq \widehat{y}}$ is just a restriction of P_ε^M in $M_{\geq \widehat{y}}$. Moreover, the union $\widetilde{M} := \bigcup_{y \in [0, \widehat{y}]} \widetilde{M_{\widehat{y}}}$ also satisfies $\widetilde{M} \overset{P_\varepsilon^M}{\Longrightarrow} M^{\mathrm{exit}}$. The proof of the case $q = -1$ is similar. $\qquad \square$

The following proposition is the core of the covering-exchange property.

PROPOSITION 4.5. *Let (N, M) be a covering-exchange pair for $(1.1)_\varepsilon$ with a fast-exit face M^{exit}. Then there exists an h-set $\widetilde{M} \subset M$ such that*

$$N \overset{\varphi_\varepsilon(T, \cdot)}{\Longrightarrow} \widetilde{M} \overset{P_\varepsilon^M}{\Longrightarrow} M^{\mathrm{exit}}.$$

PROOF. Without loss of generality, we may assume that $M = \overline{B_1} \times \overline{B_s} \times [0, 1]$ and $M^{\mathrm{exit}} = \{1\} \times \overline{B_s} \times [y^-, y^+]$ via a homomorphism c_M, where $[y^-, y^+] \subset (0, 1)$. Also, let $q = +1$.

By Lemma 4.4, we can construct an h-set depending on $\widehat{y} \in [0, y^-)$ which P_ε^M-covers M^{exit}. Let $\widetilde{M_{\widehat{y}}}$ be such an h-set. Note that such an h-set can be constructed for all $\widehat{y} \in [0, y^-)$. Now choose $\widehat{y}_{\pm} \in [0, y^-)$ so that

$$\widehat{y}_- < \inf \pi_y(\varphi(T, N) \cap M) < \sup \pi_y(\varphi(T, N) \cap M) < \widehat{y}_+,$$

and define a set $\widetilde{M}$ by $\widetilde{M} := \bigcup\limits_{y \in [\widehat{y}_-, \widehat{y}_+]} \widetilde{M}_y$. The monotonicity of g and homeomorphic correspondence of the flow imply that $\widetilde{M}$ is actually an h-set with $\widetilde{M}^\pm = \bigcup\limits_{y \in [\widehat{y}_-, \widehat{y}_+]} \widetilde{M}_y^\pm$, where $\widetilde{M}_y^\pm$ are given by (4.1). The covering relation $\widetilde{M} \overset{P_\varepsilon^M}{\Longrightarrow} M^{\text{exit}}$ immediately holds from the proof of Lemma 4.4. Moreover, $N \overset{\varphi_\varepsilon(T, \cdot)}{\Longrightarrow} \widetilde{M}$ also holds by the construction of $\widetilde{M}$ and the assumption $N \overset{\varphi_\varepsilon(T, \cdot)}{\Longrightarrow} M$. $\qquad \square$

Proposition 4.5 shows that the covering-exchange property enables us to track solution orbits near slow manifolds. This is just a consequence of Proposition 2.11.

REMARK 4.6. Our implementations of the covering-exchange property automatically solve the matching problem in singular perturbation problems. Indeed, let (N, M) be a covering-exchange pair with a fast-exit face M^{exit}. In most cases, N is a family of tracking invariant manifolds whose behavior is mainly dominated by fast dynamics. On the other hand, the behavior of points in $\widetilde{M}$ obtained in Proposition 4.5 is mainly dominated by the slow dynamics until it leaves M. If such a point is also on $\varphi_\varepsilon(T, N)$, then we can prove the existence of a point $q_\varepsilon \in N$ which goes to M along the fast dynamics in the first stage, stays M dominated by the slow dynamics until it leaves M through M^{exit} in the second stage and goes outside M along the fast dynamics again in the third stage. The trajectory through q_ε is actually dominated by the full system $(1.1)_\varepsilon$. Such a behavior is thus considered as the match of dynamics in different time scales.

One of the other key points of the covering-exchange is that the choice of u-dimensional variables is changed during time evolutions around slow manifolds. Indeed, the covering relation $N \overset{\varphi_\varepsilon(T, \cdot)}{\Longrightarrow} \widetilde{M}$ typically corresponds to validation of codimension p connecting orbits between saddle-type equilibria in the limit system. In such a case, the $u = u(N) = p$-dimensional variables are chosen from y-variables, while the u-dimensional variables of $\widetilde{M}$ are chosen from a-variables: fast-unstable directions, which is natural from the viewpoint of codimension p connecting orbits. Finally, during the covering relation $\widetilde{M} \overset{P_\varepsilon^M}{\Longrightarrow} M^{\text{exit}}$, the u-dimensional variables return to the part of y-variables, since M^{exit} is regarded as the initial data of other codimension p connecting orbits in the limit system. Consequently, we observe the "exchange" of u-dimensional variables during the covering relation $\widetilde{M} \overset{P_\varepsilon^M}{\Longrightarrow} M^{\text{exit}}$ in Proposition 4.5. In this sense, we call this phenomenon the covering-"*exchange*". Needless to say, one can describe such matching property without solving any differential equations in M.

4.2. Expanding and contracting rate of disks. Assumptions of the covering-exchange in the previous subsection require that each branch of slow

manifolds should be validated by one fast-saddle-type block. However, slow manifolds may have nontrivial curvature in general. It is not thus realistic to validate slow manifolds by one block with computer assistance without any knowledge of vector fields. To overcome such computational difficulties, we provide several techniques for generalizing conditions in the covering-exchange.

As a preliminary to next subsections, we consider the expansion and contraction rate of disks in a fast-saddle-type block. Consider system (3.1). Note that A in (3.1) is the $u \times u$-diagonal matrix such that all eigenvalues of A have positive real part, and that B in (3.1) is the $s \times s$-diagonal matrix such that all eigenvalues of B have negative real part. Let $N \subset \mathbb{R}^{n+1}$ be a fast-saddle-type block satisfying stable and unstable cone conditions for fixed $\varepsilon > 0$. Without loss of generality, via homomorphism c_N, we may assume that

$$N = \overline{B_u} \times \overline{B_s} \times [0,1].$$

Here we also assume that $g(x, y, \varepsilon) > 0$ holds in N. Two cross sections $\overline{B_u} \times \overline{B_s} \times \{0, 1\}$ then become the slow entrance $N^{\mathrm{s},+}$ and the slow exit $N^{\mathrm{s},-}$ of φ_ε, respectively.

DEFINITION 4.7. An α-*Lipschitz unstable disk* is the graph of a Lipschitz function $\psi \colon \overline{B_u} \to \overline{B_s} \times [0,1]$ with $\mathrm{Lip}(\psi) \leq \alpha$. Similarly, an α-*Lipschitz stable disk* is the graph of a Lipschitz function $\psi \colon \overline{B_s} \to \overline{B_u} \times [0,1]$ with $\mathrm{Lip}(\psi) \leq \alpha$.

Next we consider the expanding and contracting rate of stable and unstable disks.

LEMMA 4.8. *Let $N \subset \mathbb{R}^{n+1}$ be a fast-saddle-type block satisfying stable and unstable cone conditions. Let $D^{\mathrm{u}}(r)$ and $D^{\mathrm{s}}(r)$ be a 1-Lipschitz unstable disk and a stable disk of the diameter r, respectively, contained in N. Let $T > 0$ be fixed.*

(a) *Assume that $\varphi_\varepsilon([0,T], D^{\mathrm{u}}(r)) \subset N$. Then, for all $z_1, z_2 \in D^{\mathrm{u}}(r)$, the following inequality holds:*

$$|\pi_a(z_1(t) - z_2(t))| \geq e^{\lambda_{\min} t}|\pi_a(z_1 - z_2)| \quad \text{for } t \in [0,T],$$

where $\lambda_{\min} = \lambda_A - (\sup \sigma_{\mathbb{A}_1^u} + \sup \sigma_{\mathbb{A}_2^u}) > 0$ is given by (3.4). Here λ_A denotes a positive number satisfying (3.2).

(b) *Assume that $\varphi_\varepsilon([-T,0], D^{\mathrm{s}}(r)) \subset N$. Then, for all $z_1, z_2 \in D^{\mathrm{s}}(r)$, the following inequality holds:*

$$|\pi_b(z_1(-t) - z_2(-t))| \geq e^{-\mu_{\min} t}|\pi_b(z_1 - z_2)| \quad \text{for } t \in [0,T],$$

where $\mu_{\min} = \mu_B + (\sup \sigma_{\mathbb{B}_1^s} + \sup \sigma_{\mathbb{B}_2^s}) < 0$ is given by (3.6). Here μ_B denotes a negative number satisfying (3.3).

The above inequalities imply that the expanding rate of $D^u(r)$ is uniformly bounded below by $\lambda_{\min}$ and that the contracting rate of $D^s(r)$ is uniformly bounded above by $\mu_{\min}$.

Proof. (a) Assume first that $z_1, z_2 \in D^u(r)$. The unstable cone condition with our assumption implies that $z_2 \in C^u(z_1)$ and $z_2(t) \in C^u(z_1(t))$ hold for all $t \in [0, T]$. In particular, $|\pi_a(z_1(t) - z_2(t))| \geq |\pi_{b,y}(z_1(t) - z_2(t))|$ holds for all $t \in [0, T]$. Let $\Delta a(t) := \pi_a(z_1(t) - z_2(t))$, $\Delta b(t) := \pi_b(z_1(t) - z_2(t))$ and $\Delta y(t) := \pi_y(z_1(t) - z_2(t))$. The first variation equation in the a-direction yields

$$\langle \Delta a', \Delta a \rangle \geq \lambda_A |\Delta a|^2 - (\sup \sigma_{\mathbb{A}_1^u} |\Delta a|^2 + \sup \sigma_{\mathbb{A}_2^u} |\Delta a|^2),$$

which is obtained by arguments in the proof of Proposition 3.6. Here we used the fact that $|\Delta a(t)| \geq (|\Delta b(t)|^2 + |\Delta y(t)|)^{1/2}$ holds in $C^u(z_1(t))$ for all $t \in [0, T]$. This inequality leads to $(|\Delta a|^2)' \geq 2\lambda_{\min} |\Delta a|^2$, and hence $|\Delta a(t)| \geq \exp(\lambda_{\min} t)|\Delta a(0)|$.

(b) Next assume that $z_1, z_2 \in D^s(r)$. The stable cone condition with our assumption implies that $z_2 \in C^s(z_1)$, $z_2(-t) \in C^s(z_1(-t))$ hold for all $t \in [0, T]$, where $C^s(z) = \{(a, b, y) \mid |b - \pi_b(z)|^2 \geq |a - \pi_a(z)|^2 + |y - \pi_y(z)|^2\}$. In particular, $|\pi_b(z_1(-t) - z_2(-t))| \geq |\pi_{a,y}(z_1(-t) - z_2(-t))|$ holds for all $t \in [0, T]$.

The backward first variation equation in the b-direction yields

$$\frac{1}{2} \frac{d}{d\widetilde{t}} |\Delta b|^2 \geq |\mu_B| |\Delta b|^2 - (\sup \sigma_{\mathbb{B}_1^s} |\Delta b|^2 + \sup \sigma_{\mathbb{B}_2^s} |\Delta b|^2), \quad \text{where } \widetilde{t} = -t.$$

Here we used the fact that $|\Delta b(-t)| \geq (|\Delta a(-t)|^2 + |\Delta y(-t)|)^{1/2}$ holds in $C^s(z_1(-t))$ for all $t \in [0, T]$. This inequality leads to $(|\Delta b|^2)' \geq 2|\mu_{\min}||\Delta b|^2$, and hence $|\Delta b(-t)| \geq \exp(-\mu_{\min} t)|\Delta b(0)|$. □

Note that Lemma 4.8 holds even when $u \neq 1$.

4.3. Slow shadowing, drop and jump. Under suitable assumptions, slow manifolds as well as limiting critical manifolds are represented by graphs of nonlinear functions. In such cases, it is not easy to validate slow manifolds by *one* block of fast-saddle-type. It is thus natural to construct enclosure of such manifolds by the finite sequence of fast-saddle-type blocks, which will be easier than validation by one h-set. However, the union of finite fast-saddle-type blocks is not generally convex and hence we cannot apply arguments of the covering-exchange directly. In particular, we have to consider the effect of slow exits and slow entrances of the union. Nevertheless, the behavior in the y-direction is much slower than the behavior in the (a, b)-direction for sufficiently small ε. It is thus natural to consider that slow exits and entrances cause no effect on validation of trajectories sufficiently close to slow manifolds for (1.1). Here we construct a sufficient condition to guarantee such expectation. The key feature consists of two parts: (i) the comparison of expansion and contraction rate with the

speed of slow vector field, and (ii) the abstract construction of covering relations around slow manifolds. The proposing concept, named the *slow shadowing*, expresses both features. The slow shadowing can incorporate with the covering relations in Proposition 4.5; the *drop* corresponding to $N \overset{\varphi_\varepsilon(T, \cdot)}{\Longrightarrow} \widetilde{M}$ and the *jump* corresponding to $\widetilde{M} \overset{P_\varepsilon^M}{\Longrightarrow} M^{\text{exit}}$. As a consequence, the concept of covering-exchange is generalized to the finite union of fast-saddle-type blocks.

The center of our considerations is a sequence of fast-saddle-type blocks $\{N_j\}$ satisfying all assumptions in Theorem 3.5. In this case, Theorem 3.5 indicates that the stable manifold $W^s(S_\varepsilon)_j$ is given by the graph of a Lipschitz function h_s^j in each N_j. Cone conditions also yield that these manifolds are patched globally in $\bigcup N_j$.

LEMMA 4.9. *Let $\{N_j\}_{j=0}^{m_j}$ be a sequence of fast-saddle-type blocks satisfying all assumptions in Theorem 3.5. Assume that, for all j, $N_j \cap N_{j+1} \neq \emptyset$ and that each section $(N_j \cap N_{j+1})_{\overline{y}}$ contains a unique point of the critical manifold S_0. Then, for all $\varepsilon \in [0, \varepsilon_0]$, validated stable manifolds $W^s(S_\varepsilon)_j$ and $W^s(S_\varepsilon)_{j+1}$ in blocks N_j and N_{j+1}, respectively, coincide with each other in the intersection $N_j \cap N_{j+1}$ for $j = 0, \dots, m_j - 1$. A similar result holds for $W^u(S_\varepsilon)$.*

PROOF. The same arguments as in Theorem 3.5 for $(a_1, b, y) \in N_j \cap N_{j+1} \cap W^s(S_\varepsilon)_j$ and $(a_2, b, y) \in N_j \cap N_{j+1} \cap W^s(S_\varepsilon)_{j+1}$ with a fixed (b, y) yield the result. □

Consider two fast-saddle-type blocks N_1 and N_2. Assume that each N_i is constructed in the local coordinate $((a_i, b_i), y_i)$ whose origin corresponds to $(\overline{x}_i, \overline{y}_i)$ such that (1.1) locally has the form (3.1). Two coordinate systems $\{((a_i, b_i), y_i) \mid i = 1, 2\}$ are related to each other by the following commutative diagram:

$$(4.2) \qquad \begin{array}{ccc} ((a_1, b_1), y_1) & \xrightarrow{\ (T_{12})_c\ } & ((a_2, b_2), y_2) \\ {\scriptstyle P_1 \times I_1 = c_{N_1}^{-1}} \downarrow & & \downarrow {\scriptstyle P_2 \times I_1 = c_{N_2}^{-1}} \\ (x - \overline{x}_1, \overline{y}_1) & \xrightarrow[\text{translation}]{} & (x - \overline{x}_2, \overline{y}_2) \end{array}$$

where $P_1, P_2 \colon \mathbb{R}^n \to \mathbb{R}^n$ are nonsingular matrices determining the approximate diagonal form (3.1) around $(\overline{x}_1, \overline{y}_1)$ and $(\overline{x}_2, \overline{y}_2)$, respectively, and $I_m \colon \mathbb{R}^m \to \mathbb{R}^m$ denotes the identity map on $\mathbb{R}^m$. The map $T_{12} \equiv c_{N_2}^{-1} \circ (T_{12})_c \circ c_{N_1}$ denotes the composition map given by

$$((a_2, b_2), \overline{y}_2) = (T_{12})_c((a_1, b_1), \overline{y}_1) \equiv ((T_{x,12})_c(a_1, b_1), (T_{y,12})_c \overline{y}_1)$$
$$:= (P_2 \times I_1)^{-1}\{(P_1(a_1, b_1), \overline{y}_1) + (\overline{x}_1 - \overline{x}_2, \overline{y}_1 - \overline{y}_2)\}.$$

We assume

(SS1) $u(N_1) = u(N_2) = u$.

(SS2) $N_1 \cap N_2 \neq \emptyset$. For given sequences of positive numbers $\{\eta_{i,j_i}^{k,\pm}\}_{j_1=1,j_2=1,\ldots,s}^{i=1,2}$, $k = 1, 2$, $(N_k)_c$ is constructed by (2.13) with $\{\eta_{i,j_i}^{k,\pm}\}_{j_1=1,\ldots,u,\,j_2=1,\ldots,s}^{i=1,2}$. The y-component of N_k is given by $\pi_y(N_k) = [y_k^-, y_k^+]$ with

$$ y_k^- < y_k^+ \quad (k = 1, 2), \qquad y_2^- \in (y_1^-, y_1^+], \qquad y_1^+ \in [y_2^-, y_2^+), $$

where π_y is the projection onto the slow variable component.

(SS3) All assumptions in Theorem 3.5 are satisfied in N_1 and N_2. Moreover, $q \cdot \varepsilon(x, y, \varepsilon) > 0$ holds with the slow direction number $q \in \{\pm 1\}$ in $N_1 \cup N_2$.

(SS4) For $k = 1, 2$, let A_k and B_k be diagonal matrices representing (2.10) in N_k, and α_k and β_k be real numbers satisfying (3.2) for A_k and (3.3) for B_k, respectively. Also, let

$$ r_{a,k} := \min_{j=1,\cdots,u} \eta_{1,j}^{k,\pm}, \quad r_a := \min\{r_{a,1}, r_{a,2}\}, \quad \overline{r}_{a,k} := \mathrm{diam}(\pi_a(N_k)), $$

$$ r_{b,k} := \min_{j=1,\cdots,s} \eta_{2,j}^{k,\pm}, \quad r_b := \min\{r_{b,1}, r_{b,2}\}, \quad \overline{r}_{b,k} := \mathrm{diam}(\pi_b(N_k)), $$

$$ \overline{h} \in \left(0, \min_{k=1,2}\{y_k^+ - y_k^-\}\right), $$

(4.3)
$$ \varepsilon_k^{\mathrm{up}} := \sup_{N_k \times [0,\varepsilon_0]} \varepsilon g(x, y, \varepsilon), \qquad \varepsilon_k^{\mathrm{low}} := \inf_{N_k \times [0,\varepsilon_0]} \varepsilon g(x, y, \varepsilon), $$
$$ \overline{\varepsilon}_k := \max\{|\varepsilon_k^{\mathrm{up}}|, |\varepsilon_k^{\mathrm{low}}|\} $$

$$ \lambda_k := \alpha_k - \left(\sup_{N_k \times [0,\varepsilon_0]} \sigma_{\mathbb{A}_1^u} + \sup_{N_k \times [0,\varepsilon_0]} \sigma_{\mathbb{A}_2^u}\right), $$

$$ \mu_k := \beta_k + \left(\sup_{N_k \times [0,\varepsilon_0]} \sigma_{\mathbb{B}_1^s} + \sup_{N_k \times [0,\varepsilon_0]} \sigma_{\mathbb{B}_2^s}\right). $$

See Assumptions 3.2 and 3.3 for the definition of $\sigma_{\mathbb{A}_1^u}, \sigma_{\mathbb{A}_2^u}, \sigma_{\mathbb{B}_1^s}$ and $\sigma_{\mathbb{B}_2^s}$.

(SS5) Fix positive numbers $d_a, d_b \in (0, 1)$ arbitrarily. Let $D_1^{\mathrm{u}} \subset N_1$ and $D_2^{\mathrm{s}} \subset N_2$ be families of disks given by

$$ (D_1^{\mathrm{u}})_c = \left(\bigcup_{q \in W^{\mathrm{u}}(S_\varepsilon)_c} \overline{B_s(q, d_b r_b)}\right) \cap (N_1)_c, $$

$$ (D_2^{\mathrm{s}})_c = \left(\bigcup_{q \in W^s(S_\varepsilon)_c} \overline{B_u(q, d_a r_a)}\right) \cap (N_2)_c $$

via c_{N_1} and c_{N_2}, respectively. Then

(4.4)
$$ (D_1^{\mathrm{u}})_y \overset{T_{x,12}}{\Longrightarrow} (D_2^{\mathrm{s}})_y $$

holds for all $y \in \pi_y(N_1) \cap \pi_y(N_2)$.

Since both P_1 and P_2 are linear maps, the covering relation (4.4) is just a transversality of rectangular domains relative to an affine map.

DEFINITION 4.10 (Slow shadowing pair). Let $\chi \in (0, 1]$ be a fixed number. Assume (SS1)–(SS5). We say the pair $\{N_1, N_2\}$ satisfies the *slow shadowing condition (with the ratio χ and the slow direction number q)* if the following inequality holds:

$$(4.5) \quad \max\left\{ \frac{1}{\lambda_k} \log\left(\frac{\overline{r}_{a,k} - r_a}{d_a r_a} \right), \frac{1}{|\mu_k|} \log\left(\frac{\overline{r}_{b,k} - r_b}{d_b r_b} \right) \right\} < \chi \frac{\overline{h}}{\overline{\varepsilon}_k}, \quad k = 1, 2.$$

We shall call the pair $\{N_1, N_2\}$ the *slow shadowing pair (with χ and the slow direction number q)* if $\{N_1, N_2\}$ satisfies the slow shadowing condition (with q). We call the number χ the *slow shadowing ratio*.

In practical computations, we set sequences of positive numbers $\{\eta_{1,j}^{k,\pm}\}_{j=1}^u$ and $\{\eta_{2,j}^{k,\pm}\}_{j=1}^s$, $k = 1, 2$, as identical positive numbers, which make settings in practical computations simple. Assumption (SS5) admits a sufficient condition for validations in terms of cones. We revisit the condition in the end of Section 4.4.

The slow shadowing condition can be generalized to a sequence of fast-saddle-type blocks as follows.

DEFINITION 4.11 (Slow shadowing sequence). Consider a finite sequence of fast-saddle-type blocks $\{N_i\}_{i=0}^m$. We shall say the sequence $\{N_i, \chi_i\}_{i=0}^m$ of blocks and positive numbers satisfies the *slow shadowing condition (with $\{\chi_i\}_{i=0}^m$ and $q \in \{\pm 1\}$)* if, for $i = 0, \cdots, m-1$ and $\overline{h}_i$ satisfying (SS4), each pair $\{N_i, N_{i+1}\}$ is a slow shadowing pair with an identical slow direction number q in Definition 4.10, where χ_i is the slow shadowing ratio for $\{N_i, N_{i+1}\}$. We shall call such a sequence $\{N_i\}_{i=0}^m$ the *slow shadowing sequence (with q)*.

The slow shadowing ratio χ gives us a benefit in practical computations. We mention this point concretely in Section 6.2.

The core of the slow shadowing is that all disks transversal to stable and unstable manifolds of slow manifolds rapidly expand and contract, respectively, so that the covering relation on a crossing section can be derived. In what follows we only consider the case $q = +1$. The corresponding results for $q = -1$ can be shown in a similar manner.

PROPOSITION 4.12 (Slow shadowing). *Let $\{N_1, N_2\}$ be a slow shadowing pair with the ratio χ. Then there exist h-sets $M_1 \subset (N_1)_{\overline{y}} = N_1 \cap \{y = \overline{y}\}$ with $\overline{y} \in [y_1^-, y_1^+ - \overline{h}] \cap [y_2^- - \chi\overline{h}, y_2^+ - \overline{h}]$ and $M_2 \subset (N_2)_{\overline{y}+\chi\overline{h}} = N_2 \cap \{y = \overline{y} + \chi\overline{h}\}$ such that*

$$M_1 \overset{P_\varepsilon^{(N_1)_{\leq \overline{y}+\chi\overline{h}}}}{\Longrightarrow} M_2,$$

where $P_\varepsilon^{(N_1)_{\leq \overline{y}+\chi\overline{h}}} : (N_1)_{\leq \overline{y}+\chi\overline{h}} \to \partial(N_1)_{\leq \overline{y}+\chi\overline{h}}$ is the Poincaré map.

Proof. For simplicity, we assume $\chi = 1$. All arguments below are valid for general χ.

Let S_ε be the slow manifold validated in $N_1 \cup N_2$. For simplicity, we write $\overline{N}_1 \equiv (N_1)_{\overline{y}+\overline{h}}$, $\overline{N}_{1,\le} \equiv (N_1)_{\le \overline{y}+\overline{h}}$ and $\overline{P}_\varepsilon^1 \equiv P_\varepsilon^{(N_1)_{\le \overline{y}+\overline{h}}}$.

First note that $\overline{\varepsilon}_k$ is an upper bound of the absolute speed in the y-direction. This implies that any point in $(N_1)_{\overline{y}}$ which arrives at $\overline{N}_1$ through the orbit in N_1 takes at least time $\overline{h}/\overline{\varepsilon}_1$. Also note that $W^{\mathrm{s}}(S_\varepsilon)$ and $W^{\mathrm{u}}(S_\varepsilon)$ can be represented by families of 1-Lipschitz stable and unstable disks, respectively, which follows from cone conditions.

Let $M_{\overline{y}}(\delta_1) \subset (N_1)_{\overline{y}}$ be the δ_1-neighbourhood of $W^{\mathrm{s}}(S_\varepsilon) \cap \{y = \overline{y}\}$ in $(N_1)_{\overline{y}}$, namely,

$$M_{\overline{y}}(\delta_1) = \big\{ z = (a, b, \overline{y}) \in N_1 \mid \mathrm{dist}(z, W^{\mathrm{s}}(S_\varepsilon)) < \delta_1 \big\}.$$

Also, let $M_{\overline{y};\widehat{b}}(\delta_1)$ be a section of $M_{\overline{y}}(\delta_1)$ at $(b, y) = (\widehat{b}, \overline{y})$, namely,

$$M_{\overline{y};\widehat{b}}(\delta_1) = \{ z = (a, \widehat{b}, \overline{y}) \in M_{\overline{y}}(\delta) \}.$$

Obviously, all points in $M_{\overline{y};\widehat{b}}$ are contained in a unstable cone $C^{\mathrm{u}}(z_0(\widehat{b}, \overline{y}))$, where $z_0(\widehat{b}, \overline{y})$ is the unique point in $W^{\mathrm{s}}(S_\varepsilon) \cap \{(b, y) = (\widehat{b}, \overline{y})\}$. The local positive invariance of the unstable cone implies that $z_1(t) \in C^{\mathrm{u}}(z_0(t; \widehat{b}, \overline{y}))$, where $\{z_0(t; \widehat{b}, \overline{y})\}$ is the solution orbit with $z_0(0; \widehat{b}, \overline{y}) = z_0(\widehat{b}, \overline{y})$, and $z_1(t)$ is the solution orbit with $z_1(0) \in M_{\overline{y};\widehat{b}}(\delta_1)$. This relationship holds for all $t \ge 0$ until $z_1(t)$ arrives at ∂N_1. Lemma 4.8 (a) then yields

$$|\pi_a(z_1(t) - z_0(t; \widehat{b}, \overline{y}))| \ge e^{\lambda_1 t} |\pi_a(z_1(0) - z_0(0; \widehat{b}, \overline{y}))|, \quad t \ge 0.$$

The slow shadowing condition implies that the unstable disk $M_{\overline{y};\widehat{b}}(\delta_1)$ expands through the flow, and all points on the boundary arrive at $N_1^{f,-}$ before time $\overline{T} = \overline{h}/\overline{\varepsilon}_1$ if $\delta_1 \ge d_a r_a$. In this case, $P_\varepsilon^{N_1}(M_{\overline{y}}(\delta_1))$ has an intersection with $N_1^{f,-} \cap \{a = \widehat{a}\}$ for all $\widehat{a} \in \partial B_u$. This observation holds for arbitrary $\widehat{b} \in \overline{B}_s$.

The preimage $M_1 \equiv (\overline{P}_\varepsilon^1)^{-1}(\overline{P}_\varepsilon^1(M_{\overline{y}}(\delta_1)) \cap \overline{N}_{1,\le}^{\mathrm{s},-}) \cap (N_1)_{\overline{y}}$ thus satisfies

$$\sup_{z \in M_1} \mathrm{dist}(\pi_a(z), \pi_a W^{\mathrm{s}}(S_\varepsilon)) < d_a r_a.$$

Note that $\overline{N}_{1,\le}^{\mathrm{s},-}$ is equal to $\overline{N}_1 \equiv (N_1)_{\overline{y}+\overline{h}}$.

Lemma 4.13. *The set M_1 is an h-set.*

Proof. We may assume that $N_1 = \overline{B}_u \times \overline{B}_s \times [0, 1]$ via a homeomorphism c_{N_1}. M_1 is contained in $M_{\overline{y}}(\delta)$ for some $\delta_1 \in (0, d_a r_a)$. For each $\widehat{b} \in \overline{B}_s$, consider the section $M_{\overline{y};\widehat{b}}(\delta_1) = \{(a, \widehat{b}, \overline{y}) \mid a \in \overline{B}_u\}$. Let $a_{\widehat{b},\overline{y}}$ be such that $(a_{\widehat{b},\overline{y}}, \widehat{b}, \overline{y}) = z_0(\widehat{b}, \overline{y})$, which is uniquely determined for each $\widehat{b} \in \overline{B}_s$. Note that, for each $\widehat{b} \in \overline{B}_s$, $(a_{\widehat{b},\overline{y}}, \widehat{b}, \overline{y}) = z_0(\widehat{b}, \overline{y})$ is contained in M_1.

Since $\overline{P}^1_\varepsilon|_{(N_1)_{\overline{y}}} : (N_1)_{\overline{y}} \to \overline{P}^1_\varepsilon((N_1)_{\overline{y}}) \subset \partial \overline{N}_{1,\leq}$ is homeomorphic, $g(x, y, \varepsilon) > 0$ holds in N_1 and

$$\overline{P}^1_\varepsilon(\widehat{a}, \widehat{b}, \overline{y}) = (\widehat{a}, \widehat{b}, \overline{y}) \in \overline{N}^{f,-}_{1,\leq} \quad \text{for } \widehat{a} \in \partial B_u,$$
$$\overline{P}^1_\varepsilon(a_{\widehat{b},\overline{y}}, \widehat{b}, \overline{y}) \in \overline{N}_1,$$

then for all $\theta \in \partial B_u \cong S^{u-1}$, there exists $a_{\widehat{b},\overline{y}}(\theta) \in B_u$ such that $\overline{P}^1_\varepsilon(a_{\widehat{b},\overline{y}}(\theta), \widehat{b}, \overline{y}) \in \overline{N}_1 \cap \overline{N}^{f,-}_{1,\leq} \cap \{a = \theta\}$, which is uniquely determined by the property of flows. Notice that $|\pi_a(a_{\widehat{b},\overline{y}}(\theta), \widehat{b}, \overline{y}) - \pi_a(a_{\widehat{b},\overline{y}}, \widehat{b}, \overline{y})|$ monotonically increases along the flow for all $\theta \in \partial B_u$. Since $\overline{P}^1_\varepsilon$ is continuous, then $a_{\widehat{b},\overline{y}}(\theta)$ depends continuously on $\widehat{b} \in \overline{B_s}$ and $\theta \in S^{u-1}$. As a result, we have a continuous graph $\psi_{1,\overline{y}} : \overline{B_s} \times S^{u-1} \to \overline{B_u}$ given by $\psi_{1,\overline{y}}(b, \theta) = a_{b,\overline{y}}(\theta)$. Then M_1 is given by

$$(4.6) \quad M_1 = \big\{(a, b, \overline{y}) \mid a = (1 - \lambda)a_{b,\overline{y}} + \lambda\psi_{1,\overline{y}}(b, \theta),$$
$$b \in \overline{B_s},\, \theta \in S^{u-1},\, \lambda \in [0,1]\big\},$$

which is an h-set. $\qquad \square$

We go back to the proof of Proposition 4.12. The next interest is

$$\delta_2 = \sup_{z \in \overline{P}^1_\varepsilon(M_1)} \text{dist}(\pi_b(z), \pi_b(W^u(S_\varepsilon))).$$

Consider the behavior of sections

$$\overline{M}_{1;\widehat{a}} := \{z = (\widehat{a}, b, \overline{y} + \overline{h}) \in \overline{N}_1 \mid b \in \overline{B_s}\} \cap \overline{P}^1_\varepsilon(M_1)$$

in the backward flow. Since each $\overline{M}_{1;\widehat{a}}$ is a stable disk with Lipschitz constant less than 1, then each point $z_1 \in \overline{M}_{1;\widehat{a}}$ is contained in $C^s(z_0(\widehat{a}, \overline{y} + \overline{h}))$, where $z_0(\widehat{a}, \overline{y} + \overline{h})$ is the unique point in $W^u(S_\varepsilon) \cap \overline{M}_{1;\widehat{a}}$. Lemma 4.8 (b) yields

$$|\pi_b(z_1(t) - z_0(t; \widehat{a}, \overline{y} + \overline{h}))| \geq e^{\mu_1 t}|\pi_b(z_1(0) - z_0(0; \widehat{a}, \overline{y} + \overline{h}))| \quad \text{for } t \leq 0.$$

The slow shadowing condition implies that the stable disk $\overline{M}_{1;\widehat{a}}$ expands in the b-direction through the backward flow, and its boundary arrives at $N^{f,+}_1$ before time $\overline{T} = \overline{h}/\overline{\varepsilon}_1$ if $\delta_2 \geq dr_b$. Note that $M_1 \subset (N_1)_{\overline{y}}$ is an h-set given by (4.6), which implies that there exists some $t \in [-\overline{T}, 0)$ such that the image $\varphi_\varepsilon(t, \overline{M}_{1,\widehat{a}})$ must have intersections with $N_1 \cap \{b = \overline{b}\}$ for all $\overline{b} \in \partial B_s$. This observation holds for arbitrary $\widehat{a}$ with $(\widehat{a}, b, \overline{y} + \overline{h}) \in N_1$. Therefore, δ_2 should be less than $d_b r_b$.

The same arguments enable us to construct an h-set $M_2 \subset (N_2)_{\overline{y}+\overline{h}}$ in the same way as M_1 under assumptions of (4.5). In this case, our assumptions and the definition of the slow shadowing imply

$$\sup_{z \in M_2} \text{dist}(\pi_a(z), \pi_a W^s(S_\varepsilon)) < d_a r_a.$$

h-sets of M_1, M_2 and their boundaries are given by

$$M_1 = \big\{(a, b, \overline{y}) \mid a = (1 - \lambda)a_{b,\overline{y}} + \lambda\psi_{1,\overline{y}}(b, \theta),\ b \in \overline{B_s},\ \theta \in S^{u-1},\ \lambda \in [0,1]\big\},$$
$$M_2 = \big\{(a, b, \overline{y} + \overline{h}) \mid a = (1 - \lambda)a_{b,\overline{y}+\overline{h}} + \lambda\psi_{2,\overline{y}+\overline{h}}(b, \theta),$$
$$b \in \overline{B_s},\ \theta \in S^{u-1},\ \lambda \in [0,1]\big\},$$
$$M_1^- = \big\{(a, b, \overline{y}) \mid a = \psi_{1,\overline{y}}(b, \theta),\ b \in \overline{B_s}\big\}, \qquad M_1^+ = M_1 \cap N_1^{f,+},$$
$$M_2^- = \big\{(a, b, \overline{y} + \overline{h}) \mid a = \psi_{2,\overline{y}+\overline{h}}(b),\ b \in \overline{B_s}\big\}, \quad M_2^+ = M_2 \cap N_2^{f,+},$$

where $\psi_{2,\overline{y}+\overline{h}}\colon \overline{B_s} \times S^{u-1} \to \overline{B_u}$ is the map associated with M_2, which is constructed in the same way as $\psi_{1,\overline{y}}$.

Now we check if all assumptions in Proposition 2.8 hold with $f = \overline{P}_\varepsilon^1$.

The estimate $\sup_{z \in M_2} \mathrm{dist}(\pi_a(z), \pi_a W^{\mathrm{s}}(S_\varepsilon)) < d_a r_a$ implies $M_2 \subset D_2^{\mathrm{s}} \cap (N_2)_{\overline{y}+\overline{h}}$. Similarly, the estimate $\sup_{z \in \overline{P}_\varepsilon^1(M_1)} \mathrm{dist}(\pi_b(z), \pi_b W^{\mathrm{u}}(S_\varepsilon)) < d_b r_b$ implies $\overline{P}_\varepsilon^1(M_1) \subset (T_{x,12} \times I_1)D_1^{\mathrm{u}} \cap (N_1)_{\overline{y}+\overline{h}}$. From our constructions of M_1 and M_2 as well as Lemma 4.13, $\overline{P}_\varepsilon^1(M_1)$ and M_2 can be regarded as families of horizontal and vertical disks in D_1^{u} and D_2^{s}, respectively. In particular, from (SS5), $M_2 \cap (D_1^{\mathrm{u}})^{f,-} = \emptyset$, where $(D_1^{\mathrm{u}})^{f,-} = D_1^{\mathrm{u}} \cap N_1^{f,-}$. This disjointness yields $\overline{P}_\varepsilon^1(M_1^-) \cap M_2 = \emptyset$. Similarly, $\overline{P}_\varepsilon^1(M_1) \cap (D_2^{\mathrm{s}})^{f,+} = \emptyset$, which yields $\overline{P}_\varepsilon^1(M_1) \cap M_2^+ = \emptyset$, where $(D_2^{\mathrm{s}})^{f,+} = D_2^{\mathrm{s}} \cap N_2^{f,+}$. The rest of assumptions obviously holds if we choose $q_0 \in \overline{B_s}$ so that $(a, q_0, \overline{y}) \in S_\varepsilon$ for some $a \in \overline{B_u}$. The property of degree obviously holds since $\overline{P}_\varepsilon\colon (N_1)_{\overline{y}} \to (N_1)_{\overline{y}+\overline{h}}$ is just a composite of uniformly contracting and expanding maps in corresponding directions. $\square$

Proposition 4.12 can be generalized to a slow shadowing sequence $\{N_i\}_{i=1}^m$, which is straightforward.

The same arguments as in Proposition 4.5 yield the following result

PROPOSITION 4.14 (Covering-Exchange: Drop). *Let $\{N_1, N_2\}$ be a slow shadowing pair with the ratio χ. Assume that there is an h-set N such that $N \overset{\varphi_\varepsilon(T, \cdot)}{\Longrightarrow} (N_1)_{\leq \overline{y}}$ holds for some $T > 0$, where $\overline{y} \in (y_1^-, y_1^+ - \overline{h}] \cap [y_2^- - \chi\overline{h}, y_2^+ - \overline{h}]$. Then there are h-sets $\widetilde{M} \subset (N_1)_{\leq \overline{y}}$ and $M_2 \subset (N_2)_{\overline{y}+\chi\overline{h}}$ such that*

$$N \overset{\varphi_\varepsilon(T, \cdot)}{\Longrightarrow} \widetilde{M} \overset{P_\varepsilon^{(N_1)_{\leq \overline{y}+\chi\overline{h}}}}{\Longrightarrow} M_2.$$

PROOF. As in the proof of Proposition 4.12, we may assume $\chi = 1$.

Let $M_1 \subset (N_1)_{\overline{y}}$ and $M_2 \subset (N_2)_{\overline{y}+\overline{h}}$ be as in Proposition 4.12 and $\widetilde{M} := (P_\varepsilon^{(N_1)_{\leq \overline{y}}})^{-1}(M_1)$. Obviously, $N \overset{\varphi_\varepsilon(T, \cdot)}{\Longrightarrow} \widetilde{M}$ holds since $N \overset{\varphi_\varepsilon(T, \cdot)}{\Longrightarrow} (N_1)_{\leq \overline{y}}$. By the construction of $\widetilde{M}$, the covering relation $\widetilde{M} \overset{P_\varepsilon^{(N_1)_{\leq \overline{y}+\overline{h}}}}{\Longrightarrow} M_2$ also holds. $\square$

We provide the slow shadowing when a fast-saddle-type block admits a fast-exit face, which corresponds to the covering relation $\widetilde{M} \overset{P_\varepsilon^M}{\Longrightarrow} M^{\mathrm{exit}}$ in Proposition 4.5: the *covering-exchange*: *jump*. We restrict the unstable dimension u to 1 in the current considerations.

PROPOSITION 4.15 (Covering-Exchange: Jump). *Let* $\{N_1, N_2\}$ *be a slow shadowing pair with* $u = 1$. *Also, let* $N_2^{\mathrm{exit}} \subset N_2^{f,-}$ *be the fast-exit face of* N_2 *and* $\overline{y} \in [y_1^-, y_1^+ - \overline{h}] \cap [y_2^- - \chi\overline{h}, y_2^+ - \overline{h}]$. *Assume that* $\mathrm{dist}(N_2^{\mathrm{exit}}, \{y = \overline{y} + \chi\overline{h}\}) \geq \chi\overline{h}$. *Then there are h-sets* $M_1 \subset (N_1)_{\overline{y}}$ *and* $M_2 \subset (N_2)_{\overline{y}+\chi\overline{h}}$ *such that*

$$M_1 \overset{P_\varepsilon^{(N_1)_{\leq \overline{y}+\chi\overline{h}}}}{\Longrightarrow} M_2 \overset{P_\varepsilon^{N_2}}{\Longrightarrow} N_2^{\mathrm{exit}}.$$

PROOF. As in the proof of Proposition 4.12, we may assume $\chi = 1$. The proof consists of two parts: (i) $M_2 \overset{P_\varepsilon^{N_2}}{\Longrightarrow} N_2^{\mathrm{exit}}$ and (ii) $M_1 \overset{P_\varepsilon^{(N_1)_{\leq \overline{y}+\overline{h}}}}{\Longrightarrow} M_2$. Part (i) is a consequence of Lemma 4.4 with additional property of $M_2 \subset (N_2)_{\overline{y}+\overline{h}}$. The assumption $\mathrm{dist}(N_2^{\mathrm{exit}}, \{y = \overline{y}+\overline{h}\}) > \overline{h}$ and slow shadowing condition allow M_2 to satisfy

$$\sup_{x \in M_2} \mathrm{dist}(z, W^s(S_\varepsilon) \cap \{y = \overline{y} + \overline{h}\}) < d_a r_a.$$

Therefore, the same arguments as in Proposition 4.12 yield $M_1 \overset{P_\varepsilon^{(N_1)_{\leq \overline{y}+\overline{h}}}}{\Longrightarrow} M_2$, the statement (ii). Note that $W^s(S_\varepsilon) \cap M_2 = \emptyset$. See also Figure 3 (e). $\square$

For the convenience and correspondence to Definition 4.2, we introduce the following notion.

DEFINITION 4.16 (Covering-exchange sequence). Let $N \subset \mathbb{R}^{n+1}$ be an h-set and $\{N_\varepsilon^j\}_{j=0}^{j_M}$ be a sequence of fast-saddle-type blocks. Assume that:

(a) (Slow shadowing) $\{N_\varepsilon^j\}_{j=0}^{j_M - 1}$ is a slow shadowing sequence with $u = 1$.

(b) (Drop) $N \overset{\varphi_\varepsilon(T, \cdot)}{\Longrightarrow} (N_\varepsilon^0)_{\leq \overline{y}}$ holds for some $T > 0$, where $\overline{y}$ is given in Proposition 4.14.

(c) (Jump) $\{N_\varepsilon^{j_N - 1}, N_\varepsilon^{j_N}\}$ is a slow shadowing pair with a fast-exit face $N_\varepsilon^{\mathrm{exit}}$ of $N_\varepsilon^{j_N}$ satisfying assumptions in Proposition 4.15.

Then we call the triple $(N, \{N_\varepsilon^j\}_{j=0}^{j_N}, N_\varepsilon^{\mathrm{exit}})$ the *covering-exchange sequence*.

Obviously, the case $j_N = 1$ is nothing but the notion of the covering-exchange pair. Remark that, in the current setting, covering-exchange sequences are always assumed to be defined with $u = 1$.

Throughout the rest of this paper, the bold-style phrases *Drop* and *Jump* denote the corresponding descriptions stated in Definition 4.16.
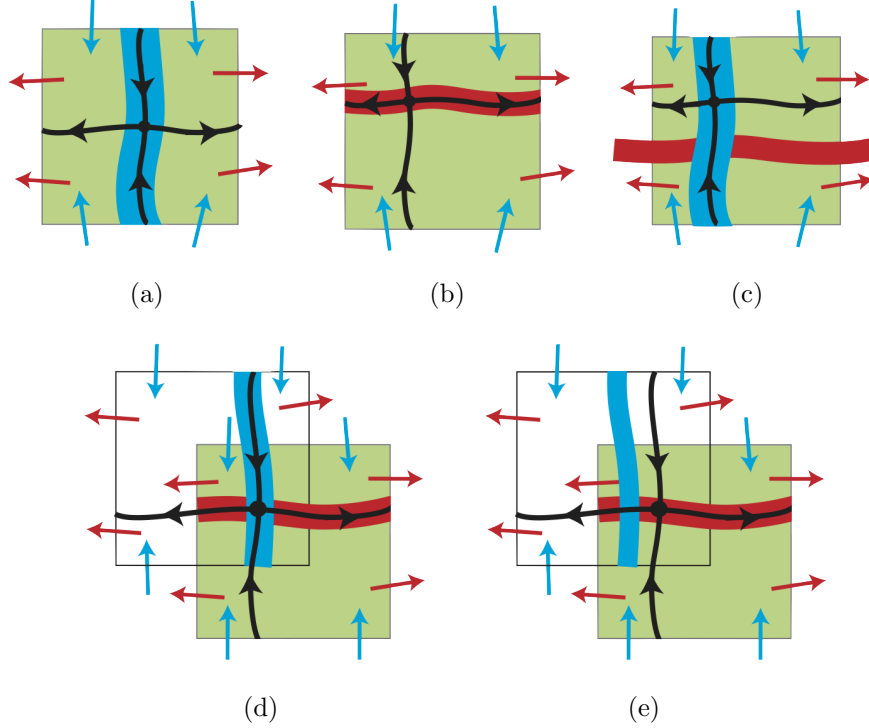
Figure 3. Slow shadowing, drop and jump: schematic illustrations. Note that all figures here show the projection onto the (a, b)-space.

Slow shadowing. The covering relation $M_1 \overset{P_\varepsilon^{(N_1)\leq\overline{y}+\overline{h}}}{\Longrightarrow} M_2$ in Proposition 4.12 is described by the process "(a) → (b) → (d)" where $P_\varepsilon^N : N \to \partial N$ is the Poincaré map in N. A set colored blue in (a) denotes M_1. The Poincaré map $P_\varepsilon^{(N_1)\leq\overline{y}+\overline{h}}$ maps M_1 to $P_\varepsilon^{(N_1)\leq\overline{y}+\overline{h}}(M_1) \subset (N_1)_{\overline{y}+\overline{h}}$ described by the red set in (b) and (d). The slow shadowing condition admits the choice of M_2 in $(N_2)_{\overline{y}+\overline{h}}$ drawn by the blue set in (d).

Covering-Exchange: *Drop.* The covering relation $N \overset{\varphi_\varepsilon(T,\,\cdot\,)}{\Longrightarrow} \widetilde{M} \overset{P_\varepsilon^{(N_1)\leq\overline{y}+\overline{h}}}{\Longrightarrow} M_2$ in Proposition 4.14 is described by the process "(c) → (b) → (d). In (c), the set colored red denotes $\varphi_\varepsilon(T, N)$ and the blue one denotes $\widetilde{M}$. The Poincaré map $P_\varepsilon^{(N_1)\leq\overline{y}+\overline{h}}$ maps $\widetilde{M}$ to $P_\varepsilon^{(N_1)\leq\overline{y}+\overline{h}}(\widetilde{M}) \subset (N_1)_{\overline{y}+\overline{h}}$ described by the red set in (b) and (d). The slow shadowing condition admits the choice of M_2 in $(N_2)_{\overline{y}+\overline{h}}$ drawn by the blue set in (d).

Covering-Exchange: *Jump.* The covering relation $M_1 \overset{P_\varepsilon^{(N_1)\leq\overline{y}+\overline{h}}}{\Longrightarrow} M_2 \overset{P_\varepsilon^{N_2}}{\Longrightarrow} N_2^{\mathrm{exit}}$ in Proposition 4.15 is described by the process "(a) → (b) → (e)". A set colored blue in (a) denotes M_1. The Poincaré map $P_\varepsilon^{(N_1)\leq\overline{y}+\overline{h}}$ maps M_1 to $P_\varepsilon^{(N_1)\leq\overline{y}+\overline{h}}(M_1) \subset (N_1)_{\overline{y}+\overline{h}}$ described by the red set in (b) and (e). The slow shadowing condition admits the choice of M_2 in $(N_2)_{\overline{y}+\overline{h}}$ drawn by the blue set in (e). The h-set M_2 $P_\varepsilon^{N_2}$-covers $N_2^{\mathrm{exit}} \subset N_2^{f,-}$. The fast exit $N_2^{f,-}$ is drawn by the edge of the white rectangle which admits horizontal red arrows.

4.4. m**-cones.** We have discussed in Section 3 that it is systematically possible to construct fast-saddle-type blocks as well as cone conditions. However, such blocks are generally too small compared with validation enclosures of trajectories, if we try to validate covering-exchange sequences. Moreover, when we solve differential equations with a fast-exit face as initial data in this situation, solution orbits will hardly move in the early stage because the vector field is close to zero. This phenomenon causes accumulation of computational errors (e.g. wrapping effect) and extra computational costs (e.g. memory or time). In particular, there is little hope to validate covering-exchange sequences. Such difficulties can be avoided if we find a large fast-saddle-type block directly. A direct approach would be finding crossing sections which form boundaries of a large fast-saddle-type block. It is not realistic to find such sections via interval arithmetics because vector fields are nonlinear and we have to consider the effect of slow dynamics. In many cases, direct search of blocks would be based on trial and error, which is not systematic. Our aim in this subsection is to provide an appropriate method to overcome difficulties with respect to solving differential equations.

In Section 3, we have constructed cones of the form $C^{\mathrm{u}} = \{|a - a_0| > |\zeta - \zeta_0|\}$ and $C^{\mathrm{s}} = \{|b - b_0| > |\nu - \nu_0|\}$ with a vertex $(a, b, y) = (a_0, b_0, y_0)$ and fixed ε, where $\zeta = (b, y)$ and $\nu = (a, y)$. See Figure 4 (a) for the illustration of cones and the unstable manifold of a saddle equilibrium. One expects that, in a neighbourhood of C^{u}, for example, C^{u} can be extended to $\widetilde{C}$ like in Figure 4 (b).

On the other hand, if the unstable manifold is sufficiently regular, one also expects that such an extended cone can be sharper. More precisely, C^{u} can be extended to the union of C^{u} and a collection of m-cones $C_m^{\mathrm{u}} = \{|a - a_0| > m|\zeta - \zeta_0|\}$ for some $m > 1$ keeping isolation, which is shown in Figure 4 (c). The same expectation is valid for C^{s}.

Proposition 3.6 means that the difference of two solutions in a fast-saddle-type block $\widehat{N}$ along dynamics is restricted by the moving cone $C^{\mathrm{u}} = \{(a, b, y, \eta) \mid M(t) > 0\}$. The analogue of this argument in the case of sharper cones is derived below.

Assumption 4.17. Consider (3.1). Let $N \subset \mathbb{R}^{n+1}$ be an h-set, $z = (x, y, \varepsilon)$ and fix $m > 0$. Define $\sigma_{\mathbb{A}_1^u}^m = \sigma_{\mathbb{A}_1^u}^m(z)$, $\sigma_{\mathbb{A}_2^u}^m = \sigma_{\mathbb{A}_2^u}^m(z)$, $\sigma_{\mathbb{B}_1^u}^m = \sigma_{\mathbb{B}_1^u}^m(z)$, $\sigma_{\mathbb{B}_2^u}^m = \sigma_{\mathbb{B}_2^u}^m(z)$, $\sigma_{g_1^u}^m = \sigma_{g_1^u}^m(z)$ and $\sigma_{g_2^u}^m = \sigma_{g_2^u}^m(z)$ be maximal singular values of the following matrices at z, respectively:

$$\sigma_{\mathbb{A}_1^u}^m : \mathbb{A}_1(z) = \left(\frac{\partial F_1}{\partial a}(z) \right), \qquad\qquad u \times u\text{-matrix},$$

$$\sigma_{\mathbb{A}_2^u}^m : \mathbb{A}_2(z) = m^{-1} \left(\frac{\partial F_1}{\partial b}(z) \quad \frac{\partial F_1}{\partial y}(z) \quad \frac{\partial F_1}{\partial \eta}(z) \right), \qquad u \times (s + 1 + 1)\text{-matrix},$$

$$\sigma^m_{\mathbb{B}^u_1}: \mathbb{B}_1(z) = m\left(\frac{\partial F_2}{\partial a}(z)\right), \qquad\qquad s \times u\text{-matrix},$$

$$\sigma^m_{\mathbb{B}^u_2}: \mathbb{B}_2(z) = \left(\frac{\partial F_2}{\partial b}(z) \quad \frac{\partial F_2}{\partial y}(z) \quad \frac{\partial F_2}{\partial \eta}(z)\right), \qquad s \times (s+1+1)\text{-matrix},$$

$$\sigma^m_{g^u_1}: g_1(z) = m\left(\frac{\partial g}{\partial a}(z)\right), \qquad\qquad 1 \times u\text{-matrix},$$

$$\sigma^m_{g^u_2}: g_2(z) = \left(\eta\frac{\partial g}{\partial b}(z) \quad \eta\frac{\partial g}{\partial y}(z) \quad g(z) + \eta\frac{\partial g}{\partial \eta}(z)\right), \quad 1 \times (s+1+1)\text{-matrix}.$$

Assume that the following inequalities hold:

$$(4.7) \qquad\qquad \lambda_A - \left(\sup \sigma^m_{\mathbb{A}^u_1} + \sup \sigma^m_{\mathbb{A}^u_2}\right) > 0,$$

$$(4.8) \quad \lambda_A - \mu_B - \left\{ \sup \sigma^m_{\mathbb{A}^u_1} + \sup \sigma^m_{\mathbb{A}^u_2} + \sup \sigma^m_{\mathbb{B}^u_1} + \sup \sigma^m_{\mathbb{B}^u_2} \right.$$
$$\left. + \sigma\left(\sup \sigma^m_{g^u_1} + \sup \sigma^m_{g^u_2}\right)\right\} > 0,$$

where λ_A and μ_B are real numbers satisfying (3.2) and (3.3), respectively, and the notation " sup" means the supremum on $N \times [0, \varepsilon_0]$.

PROPOSITION 4.18 (Unstable m-cone). *Consider* (3.1). *Let* $N \subset \mathbb{R}^{n+1}$ *be an h-set and fix $m > 0$. Assume that inequalities* (4.7) *and* (4.8) *in Assumption 4.17 hold. Then, letting $M^{u,m}(t) := |\Delta a(t)|^2 - m^2|\Delta\zeta(t)|^2$, $M^{u,m\prime}(t) > 0$ holds for all points in N satisfying $M^{u,m}(t) = 0$ unless $\Delta a = 0$, where $\zeta = (b, y, \eta)$.*

PROOF. Follow the same arguments as in the proof of Proposition 3.6, replacing $M(t)$ by $M^{u,m}(t)$. $\square$

ASSUMPTION 4.19. Consider (3.1). Let $N \subset \mathbb{R}^{n+1}$ be an h-set, $z = (x, y, \varepsilon)$ and fix $m > 0$. Define $\sigma^m_{\mathbb{A}^s_1} = \sigma^m_{\mathbb{A}^s_1}(z)$, $\sigma^m_{\mathbb{A}^s_2} = \sigma^m_{\mathbb{A}^s_2}(z)$, $\sigma^m_{\mathbb{B}^s_1} = \sigma^m_{\mathbb{B}^s_1}(z)$, $\sigma^m_{\mathbb{B}^s_2} = \sigma^m_{\mathbb{B}^s_2}(z)$, $\sigma^m_{g^s_1} = \sigma^m_{g^s_1}(z)$ and $\sigma^m_{g^s_1} = \sigma^m_{g^s_1}(z)$ be maximal singular values of the following matrices at z, respectively:

$$\sigma^m_{\mathbb{A}^s_1}: \mathbb{A}_1(z) = m\left(\frac{\partial F_1}{\partial b}(z)\right), \qquad\qquad u \times s\text{-matrix},$$

$$\sigma^m_{\mathbb{A}^s_2}: \mathbb{A}_2(z) = \left(\frac{\partial F_1}{\partial a}(z) \quad \frac{\partial F_1}{\partial y}(z) \quad \frac{\partial F_1}{\partial \eta}(z)\right), \qquad u \times (u+1+1)\text{-matrix},$$

$$\sigma^m_{\mathbb{B}^s_1}: \mathbb{B}_1(z) = \left(\frac{\partial F_2}{\partial b}(z)\right), \qquad\qquad s \times s\text{-matrix},$$

$$\sigma^m_{\mathbb{B}^s_2}: \mathbb{B}_2(z) = m^{-1}\left(\frac{\partial F_2}{\partial a}(z) \quad \frac{\partial F_2}{\partial y}(z) \quad \frac{\partial F_2}{\partial \eta}(z)\right), \qquad s \times (u+1+1)\text{-matrix},$$

$$\sigma^m_{g^s_1}: g_1(z) = m\left(\frac{\partial g}{\partial b}(z)\right), \qquad\qquad 1 \times s\text{-matrix},$$

$$\sigma^m_{g^s_2}: g_2(z) = \left(\eta\frac{\partial g}{\partial a}(z) \quad \eta\frac{\partial g}{\partial y}(z) \quad g(z) + \eta\frac{\partial g}{\partial \eta}(z)\right), \quad 1 \times (u+1+1)\text{-matrix}.$$

Assume that the following inequalities hold:

$$(4.9) \qquad \mu_B + \left(\sup \sigma^m_{\mathbb{B}^s_1} + \sup \sigma^m_{\mathbb{B}^s_2} \right) < 0,$$

$$(4.10) \quad \lambda_A - \mu_B - \Big\{ \sup \sigma^m_{\mathbb{A}^s_1} + \sup \sigma^m_{\mathbb{A}^s_2} + \sup \sigma^m_{\mathbb{B}^s_1} + \sup \sigma^m_{\mathbb{B}^s_2}$$
$$+ \sigma \left(\sup \sigma^m_{g^s_1} + \sup \sigma^m_{g^s_2} \right) \Big\} > 0,$$

where λ_A and μ_B are real numbers satisfying (3.2) and (3.3), respectively, and the notation "sup" means the supremum on $N \times [0, \varepsilon_0]$.

PROPOSITION 4.20 (Stable m-cone). *Consider* (3.1). *Let $N \in \mathbb{R}^{n+1}$ be an h-set and fix $m > 0$. Assume that inequalities* (4.9) *and* (4.10) *in Assumption 4.17 hold. Then, defining a function $M^{s,m}(t) := |\Delta b(t)|^2 - m^2|\Delta \nu(t)|^2$, $M^{s,m\prime}(t) < 0$ holds for all points on N satisfying $M^{s,m}(t) = 0$ unless $\Delta b = 0$, where $\nu = (a, y, \eta)$.*

PROOF. Follow the same arguments as in the proof of Proposition 3.6, replacing $M(t)$ by $M^{s,m}(t)$. □

DEFINITION 4.21 (m-cone conditions). We shall call inequalities (4.7) and (4.8) in Assumption 4.17 the *unstable m-cone condition in N*. Similarly, we shall call inequalities (4.9) and (4.10) in Assumption 4.19 the *stable m-cone condition in N*. When these conditions are satisfied, *the unstable m-cone and the stable m-cone with the vertex $z = (a_0, b_0, y_0)$* (in the (a, b, y)-coordinate) are given as follows, respectively:

$$C^u_m(z) := \{(a, b, y) \mid |a - a_0|^2 \geq m^2(|b - b_0|^2 + |y - y_0|^2)\},$$
$$C^s_m(z) := \{(a, b, y) \mid |b - b_0|^2 \geq m^2(|a - a_0|^2 + |y - y_0|^2)\}.$$

Validations of m-cones themselves are in fact independent of the construction of fast-saddle-type blocks discussed in Section 2.3. Moreover, the choice of m can be arbitrary as long as the corresponding cone conditions hold.

An implementation of m-cones in the unstable direction is the following.

(1) Prepare a fast-saddle-type block N such that N_c is given by (2.13) satisfying the unstable cone condition. Via a homeomorphism c_N we may assume that N is represented by

$$N = \prod_{j=1}^{u} [a^-_j, a^+_j] \times \prod_{j=1}^{s} [b^-_j, b^+_j] \times [0, 1].$$

(2) Choose a fast-exit face N^{exit}. For example, set

$$N^{\mathrm{exit}} = \prod_{j=1}^{j_0-1} [a^-_j, a^+_j] \times \{a^*_{j_0}\} \times \prod_{j=j_0-1}^{u} [a^-_j, a^+_j] \times \prod_{j=1}^{s} [b^-_j, b^+_j] \times [y^-, y^+], \quad * \in \{\pm\},$$

where $[y^-, y^+] \subset [0,1]$. Let $\ell > 0$ is a given number and V_m^{u,j_0} be an h-set given by

$$(4.11) \quad V_m^{u,j_0} = \prod_{j=1}^{u} [\widetilde{a}_j^-, \widetilde{a}_j^+] \times \prod_{j=1}^{s} \left[b_j^- - \frac{\ell}{m}, b_j^+ + \frac{\ell}{m} \right] \times \left[y^- - \frac{\ell}{m}, y^+ + \frac{\ell}{m} \right],$$

$$[\widetilde{a}_j^-, \widetilde{a}_j^+] = \begin{cases} [a_j^-, a_j^+ + \ell] & \text{if } j = j_0 \text{ and } * = +, \\ [a_j^- - \ell, a_j^+] & \text{if } j = j_0 \text{ and } * = -, \\ [a_j^-, a_j^+] & \text{otherwise.} \end{cases}$$

See Figure 4 (c).

(3) Verify the unstable m-cone condition, (4.7) and (4.8), in V_m^{u,j_0}.

The following lemma is a consequence of discussions in Lemma 4.18 and arguments in Theorem 3.5.

Lemma 4.22. *Let N be a fast-saddle-type block satisfying the stable cone condition, and V_m^{u,j_0} be given by* (4.11). *Assume that the m-unstable cone condition is satisfied in V_m^{u,j_0}. Then any point on N^{exit} leaves the set $N \cup C_m^{u,j_0}$, where*

$$C_m^{u,j_0} := \left\{ (a, \zeta) = (a, b, y) \in V_m^{u,j_0} \mid |a - a_0| \geq m|\zeta - \zeta_0|, (a_0, \zeta_0) \in N^{\mathrm{exit}} \right\}$$

under the flow through $(C_m^{u,j_0})^* := C_m^{u,j_0} \cap \{a = \widetilde{a}_{j_0}^*\}$, $* \in \{\pm\}$. The sign $*$ is exactly the location of N^{exit}.

m-cones in the stable direction are constructed in a similar manner.

(1) Prepare a fast-saddle-type block N such that N_{c} is given by (2.13) satisfying the stable cone condition. Via a homeomorphism c_N we may assume that N is represented by

$$N = \prod_{j=1}^{u} [a_j^-, a_j^+] \times \prod_{j=1}^{s} [b_j^-, b_j^+] \times [0,1].$$

(2) Choose a face of the fast-entrance N^{ent}. As an example, set

$$N^{\mathrm{ent}} = \prod_{j=1}^{u} [a_j^-, a_j^+] \times \prod_{j=1}^{j_0-1} [b_j^-, b_j^+] \times \{b_{j_0}^*\} \times \prod_{j=j_0-1}^{s} [b_j^-, b_j^+] \times [y^-, y^+], \quad * \in \{\pm\},$$

where $[y^-, y^+] \subset [0,1]$. Let $\ell > 0$ be a given number and V_m^{s,j_0} be an h-set given by

$$(4.12) \quad V_m^{s,j_0} = \prod_{j=1}^{u} \left[a_j^- - \frac{\ell}{m}, a_j^+ + \frac{\ell}{m} \right] \times \prod_{j=1}^{s} [\widetilde{b}_j^-, \widetilde{b}_j^+] \times \left[y^- - \frac{\ell}{m}, y^+ + \frac{\ell}{m} \right],$$

$$[\widetilde{b}_j^-, \widetilde{b}_j^+] = \begin{cases} [b_j^-, b_j^+ + \ell] & \text{if } j = j_0 \text{ and } * = +, \\ [b_j^- - \ell, b_j^+] & \text{if } j = j_0 \text{ and } * = -, \\ [b_j^-, b_j^+] & \text{otherwise.} \end{cases}$$

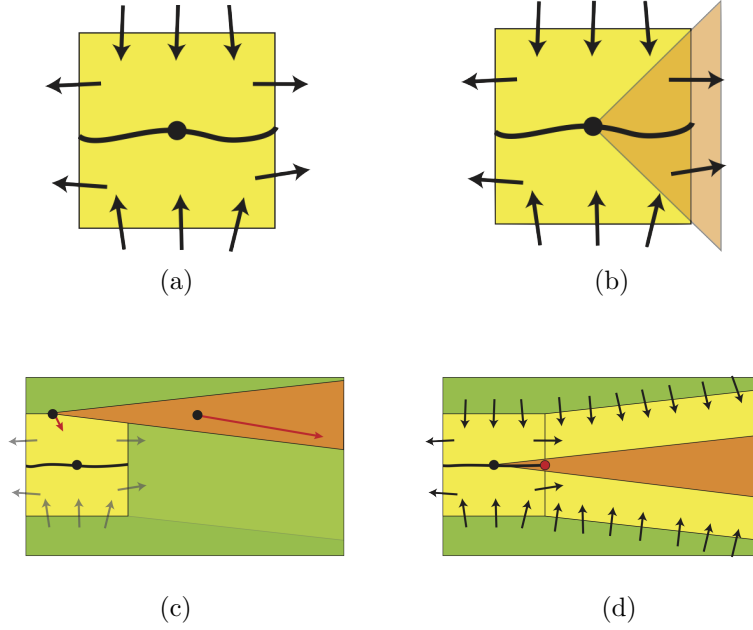(3) Verify the stable m-cone condition, (4.9) and (4.10), in V_m^{s,j_0}.

(a)   (b)

(c)   (d)

FIGURE 4. Unstable m-cones. (a) A fast-saddle-type block N containing the slow manifold S_ε validated in Theorem 3.5. In this figure, a black ball corresponds to the slow manifold S_ε and a black curve corresponds to its unstable manifold $W^{\mathrm{u}}(S_\varepsilon)$. The accuracy of the stable (resp. unstable) manifold $W^{\mathrm{s}}(S_\varepsilon)$ (resp. $W^{\mathrm{u}}(S_\varepsilon)$ is measured by the size of the fast-entrance $N^{f,+}$ (resp. the fast-exit $N^{f,-}$). In general, blocks are small and the flow stays near these blocks for small t, what causes the accumulation of various computational errors. (b) A candidate for extended cones. One expects that our validated cones stated in Theorem 3.5 can be locally extended. However, one can imagine that it is quite too large for the enclosure of $W^{\mathrm{u}}(S_\varepsilon)$ if $W^{\mathrm{u}}(S_\varepsilon)$ is sufficiently smooth. The extension of cones drawn here is thus quite coarse for smooth manifolds. (c) Validation of the m-cone condition. It is sufficient to verify the m-cone condition on a rectangular domain V_m^u given by (4.11). The set V_m^u is colored green. The orange region is the m-cone with a vertex in N. The m-cone conditions imply that all points in an m-cone stay in the m-cone until they leave V_m^*. (d) An unstable m-cone C_m^{u} of N, which is the union of m-cones with vertices on N^{exit}. The union $N \cup C_m^{\mathrm{u}}$ is colored yellow. All trajectories through N^{exit} leave V_m^u through $C_m^{\mathrm{u}} \cap \partial V_m^u$. In general, $C_m^{\mathrm{u}} \cap \partial V_m^u$ is far from slow manifolds. This fact helps us with validations and reasonable computational steps and accuracy. See also Section 6.3.

The following lemma is a consequence of similar arguments to Lemma 4.22.

LEMMA 4.23. *Let N be a fast-saddle-type block satisfying the stable cone condition, and V_m^{s,j_0} be given by (4.12). Assume that the m-stable cone condition is satisfied in V_m^{s,j_0}. Then any point on N^{ent} leaves $N \cup C_m^{s,j_0}$, where*

$$C_m^{s,j_0} := \left\{ (b,\nu) \in V_m^{s,j_0} \mid \nu = (a,y), |b - b_0| \geq m|\nu - \nu_0|, (b_0, \nu_0) \in N^{\mathrm{ent}} \right\}$$

under the backward flow through $(C_m^{s,j_0})^* := C_m^{s,j_0} \cap \{b = \widetilde{b}_{j_0}^*\}$, $* \in \{\pm\}$. *The sign $*$ is exactly the location of N^{ent}.*

DEFINITION 4.24 (m-cones). We call the set of the form C_m^{u,j_0} an *unstable m-cone of N with the length ℓ*. Similarly, we call the set of the form C_m^{s,j_0} a *stable m-cone of N with the length ℓ*.

Using m-cones, we can extend an h-set after construction of a small fast-saddle-type block. Such an h-set keeps isolation in radial direction, thanks to m-cone conditions. Moreover, we can obtain a priori estimates of trajectories *far from equilibria*, keeping their accuracy as well as possible by adjustments of m and ℓ. Of course, we do not need to solve differential equations to obtain such estimates. This technique reduces extra computational costs mentioned at the beginning of this subsection. Furthermore, in layer problems (1.3), m-cones immediately yield continuation of stable and unstable manifolds of critical invariant manifolds. Indeed, the unions $N \cup C_m^{u,j_0}$ and $N \cup C_m^{s,j_0}$ are h-sets satisfying m-(un)stable cone conditions. The unstable (resp. stable) manifold is hence represented by a horizontal (resp. vertical) disk in $N \cup C_m^{u,j_0}$ (resp. $N \cup C_m^{s,j_0}$). See [40] for details.

REMARK 4.25. Arguments involving Propositions 4.18 and 4.20 still hold replacing $y \in \mathbb{R}$ and an $\mathbb{R}$-valued function g by $y \in \mathbb{R}^l$ and an $\mathbb{R}^l$-valued function g, respectively.

In what follows we show applications of m-cones to fast-slow systems. In the case of fast-slow systems, we have to care about movements of trajectories in the slow direction. With the help of the slow shadowing condition, we can construct a sequence of covering relations for m-cones. For technical reasons we restrict u and s to 1, in particular, $n = 2$. This is exactly the case in our verification examples, Section 6. Consider (3.1) again.

DEFINITION 4.26 (Departure time, arrival time). Let N be a fast-saddle-type block such N_c is given by (2.13) with $u = s = 1$. Also, let V_m^u be a set of the form (4.11) with $u = s = 1$ satisfying the unstable m-cone condition. Define the *departure time T_{dep} in V_m^u* by

$$(4.13) \qquad T_{\mathrm{dep}} = T_{\mathrm{dep}}(V_m^u) := \frac{1}{2} \int_{(d_a r_a)^2}^{(\overline{r}_a + \ell)^2} \frac{1}{\lambda_{\min}(a)} \frac{d(a^2)}{a^2} + \delta,$$

where d_a is a given number in (SS5), $\overline{r}_a = \mathrm{diam}(\pi_a(N))$, r_a is given in (4.3), $\delta > 0$ is a sufficiently small number,

$$\lambda_{\min}(\widetilde{a}) := \lambda_A - \left(\sup_{z \in V_m^u \cap \{a \le \widetilde{a}\}} \sigma_{\mathbb{A}_1^u}^m(z) + \sup_{z \in V_m^u \cap \{a \le \widetilde{a}\}} \sigma_{\mathbb{A}_2^u}^m(z) \right)$$

and $\lambda_A > 0$ is a real number satisfying (3.2). Note that the unstable m-cone condition implies that $\lambda_{\min}(\widetilde{a}) > 0$ for $\widetilde{a} \in (d_a r_a, \overline{r}_a + \ell)$.

Similarly, let V_m^s be a set of the form (4.12) with $u = s = 1$ satisfying the stable m-cone condition. Define the *arrival time* T_{arr} in V_m^s by

$$(4.14) \qquad T_{\mathrm{arr}} = T_{\mathrm{arr}}(V_m^s) := -\frac{1}{2} \int_{(d_b r_b)^2}^{(\bar{r}_b + \ell)^2} \frac{1}{\mu_{\min}(b)} \frac{d(b^2)}{b^2} + \delta,$$

where d_b is a given number in (SS5), $\bar{r}_b = \mathrm{diam}(\pi_b(N))$, r_b is given in (4.3), $\delta > 0$ is a sufficiently small number,

$$\mu_{\min}(\widetilde{b}) := \mu_B + \left(\sup_{z \in V_m^s \cap \{b \leq \widetilde{b}\}} \sigma_{\mathbb{B}_1^s}^m(z) + \sup_{z \in V_m^s \cap \{b \leq \widetilde{b}\}} \sigma_{\mathbb{B}_2^s}^m(z) \right)$$

and $\mu_B < 0$ is a real number satisfying (3.3). Note that the stable m-cone condition implies that $\mu_{\min}(\widetilde{b}) < 0$ for $\widetilde{b} \in (d_b r_b, \bar{r}_b + \ell)$.

REMARK 4.27. In practical computations, we use the following upper bounds of departure and arrival times:

$$\frac{1}{2} \int_{(d_a r_a)^2}^{(\bar{r}_a + \ell)^2} \frac{1}{\lambda_{\min}(a)} \frac{d(a^2)}{a^2}$$

$$\leq \frac{1}{2} \sum_{j=0}^{T-1} \lambda_{\min}\left(d_a r_a + \frac{(j+1)L_a}{T} \right)^{-1} \int_{(d_a r_a + jL_a/T)^2}^{(d_a r_a + (j+1)L_a/T)^2} \frac{d(a^2)}{a^2}$$

$$= \sum_{j=0}^{T-1} \lambda_{\min}\left(d_a r_a + \frac{(j+1)L_a}{T} \right)^{-1} \log\left(\frac{T d_a r_a + (j+1)L_a}{T d_a r_a + jL_a} \right),$$

$$-\frac{1}{2} \int_{(d_b r_b)^2}^{(\bar{r}_b + \ell)^2} \frac{1}{\mu_{\min}(b)} \frac{d(b^2)}{b^2}$$

$$\leq -\frac{1}{2} \sum_{j=0}^{T-1} \mu_{\min}\left(d_b r_b + \frac{(j+1)L_b}{T} \right)^{-1} \int_{(d_b r_b + jL_b/T)^2}^{(d_b r_b + (j+1)L_b/T)^2} \frac{d(b^2)}{b^2}$$

$$= -\sum_{j=0}^{T-1} \mu_{\min}\left(d_b r_b + \frac{(j+1)L_b}{T} \right)^{-1} \log\left(\frac{T d_b r_b + (j+1)L_b}{T d_b r_b + jL_b} \right),$$

where $L_a = \bar{r}_a + \ell - d_a r_a$ and $L_b = \bar{r}_b + \ell - d_b r_b$. Usually, the bigger the number of partitions T is, the smaller right-hand sides of above inequalities are. We may set the difference between summations with different T's as δ in (4.13) and (4.14).

The following results show that m-cones generate additional covering relations to validate global orbits for $(1.1)_\varepsilon$.

PROPOSITION 4.28. *Let N be a fast-saddle-type block for (3.1) in $\mathbb{R}^3$ with $u = s = 1$ which forms $N = [a^-, a^+] \times [b^-, b^+] \times [y_0^-, y_0^+]$ and actually given by (2.13). Also, let V_m^u be a set of the form (4.11) with $u = s = 1$ which*

contains N. Assume that the unstable m-cone condition is satisfied in V_m^u. Let $T_{\mathrm{dep}} = T_{\mathrm{dep}}(V_m^u)$ be the departure time in V_m^u. Define

$$\varepsilon^+ := \sup_{V_m^u \times [0, \varepsilon_0]} \varepsilon g(x, y, \varepsilon), \qquad \varepsilon^- := \inf_{V_m^u \times [0, \varepsilon_0]} \varepsilon g(x, y, \varepsilon)$$

$$N^{\mathrm{exit}} := \{a^*\} \times [b^-, b^+] \times [y^-, y^+]$$

$$\text{with } [y^-, y^+] \cup [y^- + \varepsilon^- T_{\mathrm{dep}}, y^+ + \varepsilon^+ T_{\mathrm{dep}}] \subset [y_0^-, y_0^+],$$

$$(4.15) \qquad (C_m^u)^{\mathrm{exit}} := C_m^u \cap \left(\{\widetilde{a}^*\} \times \left[b^- - \frac{\ell}{m}, b^+ + \frac{\ell}{m} \right] \right.$$

$$\left. \times [y^- + \varepsilon^+ T_{\mathrm{dep}}, y^+ + \varepsilon^- T_{\mathrm{dep}}] \right),$$

where $* \in \{\pm\}$ is chosen so that either $\widetilde{a}^- = a^- - \ell$ or $\widetilde{a}^+ = a^+ + \ell$ holds. Then $N^{\mathrm{exit}} \overset{P_\varepsilon^{C_m^u}}{\Longrightarrow} (C_m^u)^{\mathrm{exit}}$ holds.

PROOF. Let $z_1 = (a_1, \zeta_1) \in W^{\mathrm{s}}(S_\varepsilon) \subset N$ and $z_2 = (a_2, \zeta_2) \in N^{\mathrm{exit}}$ be such that $\zeta_1 = \zeta_2$ and that $\pi_y(z_1) = \pi_y(z_2) \in [y^-, y^+]$. The unstable m-cone condition and Lemma 4.18 imply that $z_2(t) \in C_m^u(z_1(t))$ for all $t \geq 0$ until $z_2(t)$ arrives at $(C_m^u)^*$, where $(C_m^u)^*$ is given in Lemma 4.22. The difference $\Delta a \equiv a_2(t) - a_1(t) = \pi_a(z_2(t)) - \pi_a(z_1(t))$ satisfies

$$(|\Delta a|^2)' \geq 2\lambda_{\min}(a_2)|\Delta a|^2,$$

which follows from the same argument as the proof of Lemma 4.8. This implies that the solution orbit $z_2(t)$ with $z_2(0) = z_2$ arrives at $(C_m^u)^*$ at time $t < T_{\mathrm{dep}}$ and that

$$\pi_y(P_\varepsilon(N^{\mathrm{exit}} \cap \{y = y^+\})) \subsetneq (y^+ + \varepsilon^- T_{\mathrm{dep}}, y^+ + \varepsilon^+ T_{\mathrm{dep}}),$$
$$\pi_y(P_\varepsilon(N^{\mathrm{exit}} \cap \{y = y^-\})) \subsetneq (y^- + \varepsilon^- T_{\mathrm{dep}}, y^- + \varepsilon^+ T_{\mathrm{dep}}).$$

Thus $P_\varepsilon(N^{\mathrm{exit}} \cap \{y = y^\pm\}) \cap (C_m^u)^{\mathrm{exit}} = \emptyset$ holds. Note that $\partial(C_m^u)^{\mathrm{exit}}$ consists of

$$(C_m^u)^{\mathrm{exit}, -} = C_m^u \cap \left(\{\widetilde{a}^*\} \times \left[b^- - \frac{\ell}{m}, b^+ + \frac{\ell}{m} \right] \times \{y^\pm + \varepsilon^\mp T_{\mathrm{dep}}\} \right),$$

$$(C_m^u)^{\mathrm{exit}, +} = \overline{\partial C_m^u \setminus \{a = a^*, \widetilde{a}^*\}}$$

$$\cap \left(\{\widetilde{a}^*\} \times \left[b^- - \frac{\ell}{m}, b^+ + \frac{\ell}{m} \right] \times [y^- + \varepsilon^+ T_{\mathrm{dep}}, y^+ + \varepsilon^- T_{\mathrm{dep}}] \right).$$

Moreover, $(C_m^u)^{\mathrm{exit}, +} \cap P_\varepsilon(N^{\mathrm{exit}}) = \emptyset$ holds by Proposition 4.18. Then $N^{\mathrm{exit}} \overset{P_\varepsilon^{C^u}}{\Longrightarrow} (C_m^u)^{\mathrm{exit}}$ holds by Proposition 2.8 with $q_0 \in N^{\mathrm{exit}} \cap W^{\mathrm{u}}(S_\varepsilon)$. $\qquad \square$

DEFINITION 4.29. We shall call $(C_m^u)^{\mathrm{exit}}$ given in (4.15) the *fast-exit face* of C_m^u.

The following statements also hold due to the same arguments under the backward flow.

PROPOSITION 4.30. *Let N_1 and N_2 be fast-saddle-type blocks for (3.1) in $\mathbb{R}^3$ with $u = s = 1$ which form*

$$N_1 = [a_1^-, a_1^+] \times [b_1^-, b_1^+] \times [y_1^-, y_1^+], \qquad N_2 = [a_2^-, a_2^+] \times [b_2^-, b_2^+] \times [y_2^-, y_2^+]$$

such that $\{N_1, N_2\}$ is a slow shadowing pair with the slow direction number $q = +1$ and $\overline{h}$ given below. Also, let V_m^s be a set of the form (4.12) with $u = s = 1$ containing N_1. Assume that the stable m-cone condition is satisfied in V_m^s. Let $T_{\mathrm{arr}} = T_{\mathrm{arr}}(V_m^s)$ be the arrival time in V_m^s. Define

$$\varepsilon^+ := \sup_{V_m^s \times [0, \varepsilon_0]} \varepsilon g(x, y, \varepsilon), \qquad \varepsilon^- := \inf_{V_m^u \times [0, \varepsilon_0]} \varepsilon g(x, y, \varepsilon),$$

$$N_1^{\mathrm{ent}} := [a_1^-, a_1^+] \times \{b_1^+\} \times [y^-, \overline{y}] \quad \text{with } [y^-, \overline{y}] \subset [y_1^-, y_1^+].$$

Assume that there is an h-set N_0 and $T_0 > 0$ such that $N_0 \overset{\varphi_\varepsilon(T_0, \cdot)}{\Longrightarrow} C_m^s \cap \{y \in [y^-, \overline{y}]\}$ and that

$$(4.16) \quad y^- < \inf \pi_y(\varphi_\varepsilon(T_0, N)) + \varepsilon^- T_{\mathrm{arr}}, \qquad \sup \pi_y(\varphi_\varepsilon(T_0, N)) + \varepsilon^+ T_{\mathrm{arr}} < \overline{y}.$$

Let $\overline{h} > 0$ be such that $\overline{y} + \overline{h} < y_1^+$. Then there exist h-sets $\widetilde{M}_1 \subset (N_1 \cup C_m^s)_{\leq \overline{y}}$ and $M_2 \subset (N_2)_{\overline{y} + \overline{h}}$ such that

$$N_0 \overset{\varphi_\varepsilon(T_0, \cdot)}{\Longrightarrow} \widetilde{M}_1 \overset{(N_1 \cup C_m^s)_{\overline{y} + \overline{h}}}{\underset{P_\varepsilon}{\Longrightarrow}} M_2.$$

PROOF. First let S_ε be the slow manifold validated in $N_1 \cup N_2$. Also, let $M_1 \subset (N_1)_{\overline{y}}$ and $M_2 \subset (N_2)_{\overline{y} + \overline{h}}$ be h-sets such that $M_1 \overset{(N_1)_{\leq \overline{y} + \overline{h}}}{\underset{P_\varepsilon}{\Longrightarrow}} M_2$, which are constructed in Proposition 4.12. Note that M_1 contains $W^s(S_\varepsilon) \cap (N_1)_{\overline{y}}$. Let $\widetilde{M} := (P_\varepsilon^{(N_1) \leq \overline{y}})^{-1}(M_1) \subset N_1$ and $\widetilde{M}^{\mathrm{ent}} := \widetilde{M} \cap N_1^{\mathrm{ent}}$. Clearly the set $\widetilde{M}$ contains $W^s(S_\varepsilon) \cap (N_1)_{\leq \overline{y}}$.

Next, let $w_1 = (b_1, \nu_1) \in \widetilde{M}^{\mathrm{ent}}$ and $w_2 \in W^u(S_\varepsilon)$ be such that $w_2 \in C_m^s(w_1)$. The stable m-cone condition and Lemma 4.23 imply that $w_2(-t) \in C_m^s(w_1(-t))$ for all $t \geq 0$ until $w_2(-t)$ arrives at ∂C_m^s under the backward flow. The difference $\Delta b \equiv b_2(-t) - b_1(-t) = \pi_b(w_2(-t)) - \pi_b(w_1(-t))$ satisfies

$$\frac{d}{d\widetilde{t}}(|\Delta b|^2) \geq 2\mu_{\min}(b_2)|\Delta b|^2$$

by the same argument as in the proof of Lemma 4.8, where $\widetilde{t} = -t$. This inequality implies that the backward solution orbit $w_2(-t)$ with $w_2(0) = w_2$ arrives at ∂C_m^s at time $t < T_{\mathrm{arr}}$ and that

$$\pi_y\left((P_\varepsilon^{C_m^s})^{-1}(\widetilde{M}^{\mathrm{ent}} \cap \{y = \overline{y}\})\right) \subsetneq (\overline{y} - \varepsilon^+ T_{\mathrm{arr}}, \overline{y} - \varepsilon^- T_{\mathrm{arr}}),$$

$$\pi_y\left((P_\varepsilon^{C_m^s})^{-1}(\widetilde{M}^{\mathrm{ent}} \cap \{y = y^-\})\right) \subsetneq (y^- - \varepsilon^+ T_{\mathrm{arr}}, y^- - \varepsilon^- T_{\mathrm{arr}}).$$

Define an h-set $\widetilde{M}_1 := \widetilde{M} \cup (P_\varepsilon^{C_m^{\mathrm{s}}})^{-1}(\widetilde{M}^{\mathrm{ent}})$, where

$$\widetilde{M}_1^+ = \overline{(\widetilde{M}_1 \cap \{b = \widetilde{b}^\pm\})} \cup \overline{(\widetilde{M} \cap \{y = \overline{y}, y^-\})} \cup (P_\varepsilon^{C_m^{\mathrm{s}}})^{-1}(\widetilde{M}^{\mathrm{ent}} \cap \{y = \overline{y}, y^-\}),$$

$$\widetilde{M}_1^- = \overline{\partial \widetilde{M}_1 \setminus \widetilde{M}_1^+}.$$

Note that $\varphi_\varepsilon(T_0, N_0) \cap \widetilde{M}_1^+ = \emptyset$ follows from (4.16) and $N_0 \overset{\varphi_\varepsilon(T_0, \cdot)}{\Longrightarrow} C_m^{\mathrm{s}} \cap \{y \in [y^-, \overline{y}]\}$. The relationship $\varphi_\varepsilon(T_0, N_0^-) \cap \widetilde{M}_1 = \emptyset$ immediately follows from $N_0 \overset{\varphi_\varepsilon(T_0, \cdot)}{\Longrightarrow} C_m^{\mathrm{s}} \cap \{y \in [y^-, \overline{y}]\}$. Then the covering relation $N_0 \overset{\varphi_\varepsilon(T_0, \cdot)}{\Longrightarrow} \widetilde{M}_1$ also follows from $N_0 \overset{\varphi_\varepsilon(T_0, \cdot)}{\Longrightarrow} C_m^{\mathrm{s}} \cap \{y \in [y^-, \overline{y}]\}$. □

The slow shadowing sequence with extended m-cones determines the generalized sequence of covering-exchange sequence in Definition 4.16.

DEFINITION 4.31 (Covering-exchange sequence with extended cones). Consider $(1.1)_\varepsilon$ in $\mathbb{R}^3$. Let $N \subset \mathbb{R}^3$ be an h-set with $u(N) = 1$ and $\{N_\varepsilon^j\}_{j=0}^{j_N}$ be a sequence such that

(a) $\{N_\varepsilon^j\}_{j=0}^{j_N}$ is a slow shadowing sequence for $(1.1)_\varepsilon$ with $u = s = 1$;
(b) $\{N_\varepsilon^{j_N-1}, N_\varepsilon^{j_N}\}$ is a slow shadowing pair for $(1.1)_\varepsilon$ with the exit $N_\varepsilon^{\mathrm{exit}} \subset (N_\varepsilon^{j_M})^{f,-}$.

Also, let $C_{m^u}^{\mathrm{u}}$ be the unstable m^u-cone of $N_\varepsilon^{j_N}$ and $C_{m^s}^{\mathrm{s}}$ be the stable m^s-cone of N_ε^0. Assume that

(c) all assumptions in Proposition 4.28 are satisfied with $N_\varepsilon^{\mathrm{exit}}$ and $C_m^{\mathrm{u}} = C_{m^u}^{\mathrm{u}}$;
(d) all assumptions in Proposition 4.30 are satisfied with $N_0 = N$, $N_i = N_\varepsilon^{i-1}$ and $C_m^{\mathrm{s}} = C_{m^s}^{\mathrm{s}}$.

Then we call the collection $(N, \{N_\varepsilon^j\}_{j=1}^{j_M}, N_\varepsilon^{\mathrm{exit}}, C_{m^u}^{\mathrm{u}}, C_{m^s}^{\mathrm{s}})$ the covering-exchange sequence with extended cones $C_{m^u}^{\mathrm{u}}, C_{m^s}^{\mathrm{s}}$.

Another benefit of m-cones is that we can validate assumption (SS5) in the slow shadowing in terms of m-cones. In the following proposition, we do not pose any restrictions on u and s.

PROPOSITION 4.32 (A sufficient condition for (SS5)). *Consider two fast-saddle-type blocks N_1 and N_2 constructed in the local coordinates $((a_i, b_i), y_i)$, where (1.1) locally forms (3.1) in each coordinate with the commutative diagram (4.2). Assume that both N_1 and N_2 satisfy the unstable m_u-cone condition and the stable m_s-cone condition for $m_u, m_s \geq 1$, and that (SS1)–(SS4) are satisfied. Independently, consider two h-sets $\widetilde{D}_1^{\mathrm{u}}$ and $\widetilde{D}_2^{\mathrm{s}}$ given by*

$$(\widetilde{D}_1^{\mathrm{u}})_{\mathrm{c}} := \overline{B_u(0, r_a)} \times \overline{B_s\left(0, d_b r_b + \frac{r_a}{m_u}\right)},$$

$$(\widetilde{D}_2^{\mathrm{s}})_{\mathrm{c}} := \overline{B_u\left(0, d_a r_a + \frac{r_b}{m_s}\right)} \times \overline{B_s(0, r_b)}$$

via c_{N_1} and c_{N_2}, respectively, where r_a, r_b, d_a, d_b denote positive numbers given in (SS4)–(SS5). Write the nonsingular matrix $P \equiv P_2^{-1} P_1$ in (4.2) in the block form

$$P = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix},$$

where the first row $(p_{11} \; p_{12})$ acts on the u-dimensional vector a_1 and the second row $(p_{21} \; p_{22})$ acts on the s-dimensional vector b_1. Finally, assume that $\widetilde{D}_1^u \overset{T_{x,12}}{\Longrightarrow} \widetilde{D}_2^s$, and that

$$(4.17) \qquad m_s^{-2} < \frac{m_u^2 \cdot \min(1, \underline{\mathfrak{p}_2}^2 - \sigma_{21}\overline{\mathfrak{p}_2} - 2\sigma_{12}^2/m_u^2)}{2\overline{\mathfrak{p}_1}^2 + m_u^2 \sigma_{21}\overline{\mathfrak{p}_2}},$$

where $\overline{\mathfrak{p}_i}$ and $\underline{\mathfrak{p}_i}$ are the maximal and the minimal eigenvalues of the matrix p_{ii}, respectively, and σ_{ij} denotes the maximal singular value of p_{ij}. Then (4.4) holds.

PROOF. First we set

$$(\widetilde{D}_1^u)_c^- = \partial B_u(0, r_a) \times \overline{B_s\left(0, d_b r_b + \frac{r_a}{m_u}\right)},$$

$$(\widetilde{D}_1^u)_c^+ = \overline{B_u(0, r_a)} \times \partial B_s\left(0, d_b r_b + \frac{r_a}{m_u}\right)$$

$$(\widetilde{D}_2^s)_c^- := \partial B_u\left(0, d_a r_a + \frac{r_b}{m_s}\right) \times \overline{B_s(0, r_b)},$$

$$(\widetilde{D}_2^s)_c^+ := \overline{B_u\left(0, d_a r_a + \frac{r_b}{m_s}\right)} \times \partial B_s(0, r_b)$$

corresponding to the h-set structure.

We embed $T_{x,12}(\widetilde{D}_1^u)$ and $\widetilde{D}_2^s$ into $(N_1)_{\overline{y}}$ and $(N_2)_{\overline{y}}$, respectively, so that the origin of both $T_{x,12}(\widetilde{D}_1^u)$ and $\widetilde{D}_2^s$ is the point $q \in S_\varepsilon \cap (N_1 \cap N_2)_{\overline{y}}$. We write the embedded sets as $T_{x,12}(\widetilde{D}_1^u)$ and $\widetilde{D}_2^s$ again, respectively. Without loss of generality, we may assume that the representation of q in the (a_2, b_2, y_2)-coordinates is given by $q = (0, 0, \overline{y})$. Note that, in this case, $D_1^u \cap \widetilde{D}_1^u$ is a family of horizontal disks in $\widetilde{D}_1^u$ and $D_2^s \cap \widetilde{D}_2^s$ is a family of vertical disks in $\widetilde{D}_2^s$.

Define two sets $\widehat{D}_1^u$ and $\widehat{D}_2^s$ as follows (see also Figure 5):

$$(\widehat{D}_1^u)_c := (\widetilde{D}_1^u)_c \cup \bigcup_{z \in (\widetilde{D}_1^u)_c^-} (C_{m_u}^u(z) \cap \{z = (a_1, b_1, y_1) \mid |a_1| \geq r_a\}),$$

$$(\widehat{D}_2^s)_c := (\widetilde{D}_2^s)_c \cup \bigcup_{z \in (\widetilde{D}_2^s)_c^+} (C_{m_s}^s(z) \cap \{z = (a_2, b_2, y_2) \mid |b_2| \geq r_b\}).$$

Unstable m_u-cone and stable m_s-cone conditions imply that $D_1^u \cap \Gamma \subset \widehat{D}_1^u$ and $D_2^s \cap \Gamma \subset \widehat{D}_2^s$, where D_1^u and D_2^s are given in (SS5) and $\Gamma = \{y = \overline{y}\}$. Indeed, D_1^u is given by a tube $B_s(W^s(S_\varepsilon)_c, d_b r_b)$ in $(N_1)_c$. The unstable manifold $W^u(q)$ is contained in the unstable m_u-cone $C_{m_u}^u(q)$. If $W^u(S_\varepsilon)_c$ runs on the range
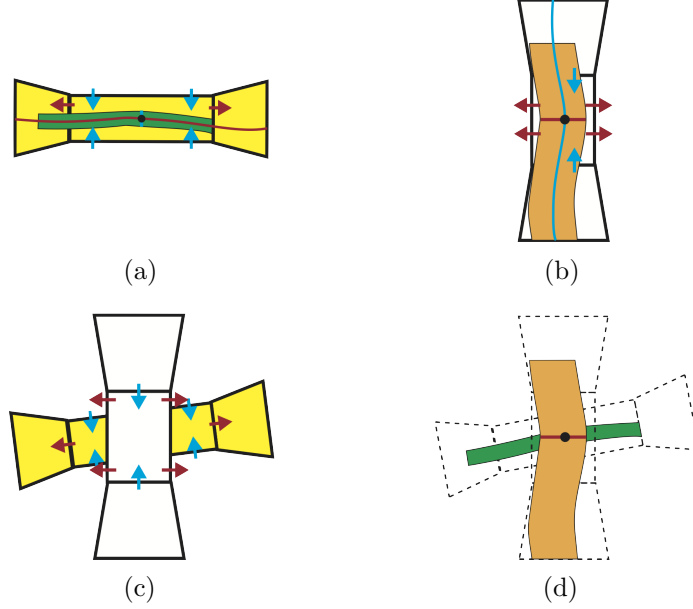
FIGURE 5. Illustration of Proposition 4.32. (a) The set $\widehat{D}_1^u \cap \Gamma$ with $u = s = 1$, where $\Gamma = \{y = \bar{y}\}$. The red curve represents the slice of the unstable manifold $W^s(S_\varepsilon)$ by Γ. The green tube represents the set $D_1^u \cap \Gamma$. The definition of $\widehat{D}_1^u$ and unstable m_u-cone conditions indicate the inclusion $D_1^u \cap \Gamma \subset \widehat{D}_1^u$. Note that $D_1^u \cap \widetilde{D}_1^u$ consists of a collection of horizontal disks. (b) The set $\widehat{D}_2^s \cap \Gamma$ with $u = s = 1$, where $\Gamma = \{y = \bar{y}\}$. The blue curve represents the slice of the stable manifold $W^s(S_\varepsilon)$ by Γ. The orange tube represents the set $D_2^s \cap \Gamma$. The definition of $\widehat{D}_2^s$ and stable m_s-cone conditions indicate the inclusion $D_2^s \cap \Gamma \subset \widehat{D}_2^s$. Note that $D_2^s \cap \widetilde{D}_2^s$ consists of a collection of vertical disks. (c) Covering relation $\widetilde{D}_1^u \overset{T_{x,12}}{\Longrightarrow} \widetilde{D}_2^s$ and (4.17). The inequality (4.17) indicates that two cones outside $\widetilde{D}_1^u$ and $\widetilde{D}_2^s$ are sufficiently sharp satisfying $(\pi_{a_2,b_2} \circ (T_{x,12} \times I_1))\overline{(\widehat{D}_1^u \setminus \widetilde{D}_1^u)} \cap \pi_{a_2,b_2}\overline{(\widehat{D}_2^s \setminus \widetilde{D}_2^s)} = \emptyset$.

$\{|a| \leq r_a\}$ in the a-direction, the enclosure of $W^u(S_\varepsilon)_c$ in the b-direction is bounded by $\{|b| \leq r_a/m_u\}$. Similar arguments yield that the enclosure of the tube $B_s(W^u(S_\varepsilon)_c, d_b r_b) \cap \{|a| \leq r_a\}$ in the b-direction is bounded by $\{|b| \leq d_b r_b + r_a/m_u\}$. Relations between unstable manifolds through $q \in S_\varepsilon$ and $C_{m_u}^u(q)$ thus yield that $D_1^u \cap \Gamma \subset \widehat{D}_1^u$. The similar arguments for D_2^s yield the second. In particular, the covering relation $\pi_{a_1,b_1}\widehat{D}_1^u \overset{T_{x,12}}{\Longrightarrow} \pi_{a_2,b_2}\widehat{D}_2^s$ implies (SS5). Note that this argument is independent of $q \in S_\varepsilon \cap N_1 \cap N_2$, in particular, $\pi_y(q)$.

By the construction of $\widehat{D}_1^u$ and $\widehat{D}_2^s$, it is sufficient to prove the following two conditions to verify $\pi_{a_1,b_1}\widehat{D}_1^u \overset{T_{x,12}}{\Longrightarrow} \pi_{a_2,b_2}\widehat{D}_2^s$:

- $\widetilde{D}_1^u \overset{T_{x,12}}{\Longrightarrow} \widetilde{D}_2^s$,
- $(\pi_{a_2,b_2} \circ (T_{x,12} \times I_1))\overline{(\widehat{D}_1^u \setminus \widetilde{D}_1^u)} \cap \pi_{a_2,b_2}\overline{(\widehat{D}_2^s \setminus \widetilde{D}_2^s)} = \emptyset$.

The former is just one of our assumptions. The latter concerns intersections between subsets of cones. Since an m-cone $C_m^{\mathrm{u}}(z)$ is a collection of lines inside $C_m^{\mathrm{u}}(z)$ through z, it is sufficient to consider the location of lines through base sets.

Now we consider the image of the unstable m_u-cone $\mathcal{C}_{\mathrm{c}} \equiv \bigcup\limits_{z \in (\widetilde{D}_1^{\mathrm{u}})_{\mathrm{c}}^-} C_{m_u}^{\mathrm{u}}(z)$

under $\widetilde{T}_{\mathrm{c}} \equiv (T_{x,12})_{\mathrm{c}} \times I_1$. Since $\widetilde{T}_{\mathrm{c}}$ is the nonsingular affine map, any lines are mapped into lines via $\widetilde{T}_{\mathrm{c}}$. Choose a point $(a_{12}, b_{12}, y_2) \in (\widetilde{D}_1^{\mathrm{u}})_{\mathrm{c}}^-$. The boundary $\partial \mathcal{C}_{\mathrm{c}}$ is a subset of $\bigcup\limits_{(a_{12},b_{12},y_2) \in (\widetilde{D}_1^{\mathrm{u}})_{\mathrm{c}}^-} \{ |a_{11} - a_{12}|^2 = m_u^2(|b_{11} - b_{12}|^2 + |y_1 - y_2|^2) \}.$

The transformation $\widetilde{T}_{\mathrm{c}} \colon (a_{1i}, b_{1i}, y_i) \mapsto (a_{2i}, b_{2i}, y_i)$ yields

$$
\begin{aligned}
|a_{21} - a_{22}|^2 &= |p_{11}(a_{11} - a_{12}) + p_{12}(b_{11} - b_{12})|^2 \\
&= m_u^2(|p_{21}(a_{11} - a_{12}) + p_{22}(b_{11} - b_{12})|^2 + |y_1 - y_2|^2).
\end{aligned}
$$

The triangular inequality yields

$$
\begin{aligned}
&|p_{11}(a_{11} - a_{12}) + p_{12}(b_{11} - b_{12})|^2 \\
&\quad \le |p_{11}(a_{11} - a_{12})|^2 + 2|p_{11}(a_{11} - a_{12})||p_{12}(b_{11} - b_{12})| + |p_{12}(b_{11} - b_{12})|^2 \\
&\quad \le 2(|p_{11}(a_{11} - a_{12})|^2 + |p_{12}(b_{11} - b_{12})|^2) \\
&\quad \le 2(\overline{\mathfrak{p}_1}^2 |a_{11} - a_{12}|^2 + \sigma_{12}^2 |b_{11} - b_{12}|^2), \\
&|p_{21}(a_{11} - a_{12}) + p_{22}(b_{11} - b_{12})|^2 + |y_1 - y_2|^2 \\
&\quad \ge (|p_{21}(a_{11} - a_{12})| - |p_{22}(b_{11} - b_{12})|)^2 + |y_1 - y_2|^2 \\
&\quad = |p_{21}(a_{11} - a_{12})|^2 - 2|p_{21}(a_{11} - a_{12})||p_{22}(b_{11} - b_{12})| \\
&\qquad + |p_{22}(b_{11} - b_{12})|^2 + |y_1 - y_2|^2 \\
&\quad \ge |p_{21}(a_{11} - a_{12})|^2 - 2\sigma_{21}\overline{\mathfrak{p}_2}|a_{11} - a_{12}||b_{11} - b_{12}| + \underline{\mathfrak{p}_2}^2 |b_{11} - b_{12}|^2 + |y_1 - y_2|^2 \\
&\quad \ge -\sigma_{21}\overline{\mathfrak{p}_2}(|a_{11} - a_{12}|^2 + |b_{11} - b_{12}|^2) + \underline{\mathfrak{p}_2}^2 |b_{11} - b_{12}|^2 + |y_1 - y_2|^2.
\end{aligned}
$$

Thus

$$
\begin{aligned}
&2(\overline{\mathfrak{p}_1}^2 |a_{11} - a_{12}|^2 + \sigma_{12}^2 |b_{11} - b_{12}|^2) \\
&\quad \ge m_u^2 \left\{ -\sigma_{21}\overline{\mathfrak{p}_2}(|a_{11} - a_{12}|^2 + |b_{11} - b_{12}|^2) + \underline{\mathfrak{p}_2}^2 |b_{11} - b_{12}|^2 + |y_1 - y_2|^2 \right\}.
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
&(2\overline{\mathfrak{p}_1}^2 + m_u^2 \sigma_{21}\overline{\mathfrak{p}_2})|a_{11} - a_{12}|^2 \\
&\quad \ge m_u^2 \left\{ \left(\underline{\mathfrak{p}_2}^2 - \sigma_{21}\overline{\mathfrak{p}_2} - \frac{2\sigma_{12}^2}{m_u^2} \right) |b_{11} - b_{12}|^2 + |y_1 - y_2|^2 \right\} \\
&\quad \ge m_u^2 \cdot \min\left(1, \underline{\mathfrak{p}_2}^2 - \sigma_{21}\overline{\mathfrak{p}_2} - \frac{2\sigma_{12}^2}{m_u^2} \right) \{ |b_{11} - b_{12}|^2 + |y_1 - y_2|^2 \},
\end{aligned}
$$

equivalently,

$$|a_{11} - a_{12}|^2 \geq \frac{m_u^2 \cdot \min(1, \underline{\mathfrak{p}_2}^2 - \sigma_{21}\overline{\mathfrak{p}_2} - 2\sigma_{12}^2/m_u^2)}{2\overline{\mathfrak{p}_1}^2 + m_u^2\sigma_{21}\overline{\mathfrak{p}_2}} \left\{ |b_{11} - b_{12}|^2 + |y_1 - y_2|^2 \right\}.$$

Let

$$\overline{m}^2 := \frac{m_u^2 \cdot \min(1, \underline{\mathfrak{p}_2}^2 - \sigma_{21}\overline{\mathfrak{p}_2} - 2\sigma_{12}^2/m_u^2)}{2\overline{\mathfrak{p}_1}^2 + m_u^2\sigma_{21}\overline{\mathfrak{p}_2}}.$$

The above inequality indicates that any line in $\widehat{D}_1^{\mathrm{u}}$ is mapped into another one included in $C_{\overline{m}}^{\mathrm{u}}(z)$ for some z. By our construction, we know that any (half) lines in $\widetilde{T}\widehat{D}_1^{\mathrm{u}} \setminus \widetilde{D}_1^{\mathrm{u}}$ lie on ones with vertices in $T_{x,12}\widetilde{D}_1^{\mathrm{u}} \cap (\widetilde{D}_2^{\mathrm{s}} \setminus (\widetilde{D}_2^{\mathrm{s}})^+)$. Such vertices can be also chosen as points on $T_{x,12}(\widetilde{D}_1^{\mathrm{u}})^-$, which are disjoint from $\widetilde{D}_2^{\mathrm{s}}$. Remark that $T_{x,12}\widetilde{D}_1^{\mathrm{u}} \cap (\widetilde{D}_2^{\mathrm{s}})^+ = \emptyset$ follows from the covering relation $\widetilde{D}_1^{\mathrm{u}} \overset{T_{x,12}}{\Longrightarrow} \widetilde{D}_2^{\mathrm{s}}$.

For any point $z \in T_{x,12}\widetilde{D}_1^{\mathrm{u}} \cap \widetilde{D}_2^{\mathrm{s}}$, we easily know that $C_{\overline{m}}^{\mathrm{u}}(z) \cap C_{m_s}^{\mathrm{s}}(z) = \{z\}$ if $(\overline{m}m_s)^2 > 1$. This implies that any lines in $\widetilde{T}\widehat{D}_1^{\mathrm{u}} \setminus \widetilde{D}_1^{\mathrm{u}}$ are disjoint from $C_{m_s}^{\mathrm{s}}(z)$ with $z \in T_{x,12}\widetilde{D}_1^{\mathrm{u}} \cap \widetilde{D}_2^{\mathrm{s}}$. This fact holds for any $z \in T_{x,12}\widetilde{D}_1^{\mathrm{u}} \cap \widetilde{D}_2^{\mathrm{s}}$.

On the other hand, for any point $w \in (\widetilde{D}_2^{\mathrm{s}})^+$, there is a point $z \in T_{x,12}\widetilde{D}_1^{\mathrm{u}} \cap \widetilde{D}_2^{\mathrm{s}}$ such that $C_{m_s}^{\mathrm{s}}(w) \subset C_{m_s}^{\mathrm{s}}(z)$, which follows from the property of cones and the structure of $T_{x,12}\widetilde{D}_1^{\mathrm{u}} \cap \widetilde{D}_2^{\mathrm{s}}$ from the covering relation $\widetilde{D}_1^{\mathrm{u}} \overset{T_{x,12}}{\Longrightarrow} \widetilde{D}_2^{\mathrm{s}}$. Consequently, we know that the set $\widehat{\widetilde{D}_2^{\mathrm{s}} \setminus \widetilde{D}_2^{\mathrm{s}}}$ is contained in the union of $C_{m_s}^{\mathrm{s}}(z)$ with $z \in T_{x,12}\widetilde{D}_1^{\mathrm{u}} \cap \widetilde{D}_2^{\mathrm{s}}$.

Combining these observations, we obtain $\widetilde{T}(\widehat{\widetilde{D}_1^{\mathrm{u}} \setminus \widetilde{D}_1^{\mathrm{u}}}) \cap \widehat{\widetilde{D}_2^{\mathrm{s}} \setminus \widetilde{D}_2^{\mathrm{s}}} = \emptyset$. $\square$

4.5. Invariant sets on slow manifolds. We move to discussions of invariant sets on slow manifolds. When $\varepsilon > 0$, the dynamics on slow manifolds can generally exhibit nontrivial dynamics. In other words, slow manifolds can have nontrivial invariant sets, such as equilibria, periodic orbits and so on, for slow dynamics. Such sets play a key role in describing nontrivial global solutions such as singularly perturbed homoclinic or heteroclinic orbits.

Consider the slow manifold S_ε in a fast-saddle-type block M satisfying stable and unstable cone conditions. Theorem 3.5 implies that S_ε is represented by the graph of a function $x = h^\varepsilon(y)$ continuously depending on y and ε including $\varepsilon = 0$. Substituting $h^\varepsilon(y)$ into (1.1), one sees that the y-equation will decouple from the x-equation. Since S_ε is parameterized by y, the resulting decoupled equation

$$y' = \varepsilon g(h^\varepsilon(y), y, \varepsilon)$$

describes the dynamics on S_ε. After time rescaling $\tau = t/\varepsilon$ we obtain

$$(4.18) \qquad \qquad \dot{y} = g(h^\varepsilon(y), y, \varepsilon), \qquad \cdot = \frac{d}{d\tau},$$

to show that the dynamics on slow manifolds is reduced to the *regular perturbation problem*. Taking the limit $\varepsilon \to 0$ in (4.18) we obtain the dynamics on the

critical manifold $S_0 \subset \{f(x, y, 0) = 0\}$:

$$\dot{y} = g(h^0(y), y, 0). \tag{4.19}$$

When we want to study dynamics on slow manifolds, such as the existence of equilibria and their stability, (4.18) as well as (4.19) are in the center of our considerations.

In general, what we know about S_ε is just the fact that it is the graph of a function $x = h^\varepsilon(y)$, and its concrete description is quite difficult to obtain. As for the normally hyperbolic critical manifold $S_0 = \{x = h^0(y)\}$, we know that it is a subset of the nullcline $\{f(x, y, 0) = 0\}$. Thanks to this property, we can study the details of S_0 such as the differential of h^0 as a y-function via the implicit function differential equation $f_x(h^0(y), y, 0)(\partial h^0/\partial y) + f_y(h^0(y), y, 0) = 0$. On the contrary, the function $h^\varepsilon(y)$ does not possess such simple and useful properties. For example, $f(h^\varepsilon(y), y, \varepsilon) \equiv 0$ does not necessarily hold. In general, $h^\varepsilon(y)$ is the solution of the following nonlinear partial differential equation:

$$\varepsilon \frac{\partial h^\varepsilon}{\partial y}(y) \, g(h^\varepsilon(y), y, \varepsilon) = f(h^\varepsilon(y), y, \varepsilon). \tag{4.20}$$

In order to calculate the differential $\partial h^\varepsilon/\partial y$ for studying the stability of equilibria on slow manifolds, for example, we have to solve equation (4.20) rigorously. Nevertheless, from the viewpoint of rigorous numerics, it will be more reasonable to regard h^ε as h^0 with small errors than to solve (4.20) directly. With this in mind, we consider (4.18) as the differential inclusion

$$\dot{y} \in \big\{ g(h^\varepsilon(y), y, \varepsilon) \mid \varepsilon \in [0, \varepsilon_0], \, (h^\varepsilon(y), y) \in M \big\},$$

where M is a fast-saddle-type block containing S_ε. The right-hand side possesses $h^0(y)$ as the representative of the enclosure.

As an example of dynamics on S_ε via rigorous numerics, we consider the validation of equilibria for (4.18) on S_ε as well as their stability for $\varepsilon \in (0, \varepsilon_0]$. Thanks to regular perturbation relationship between (4.18) and (4.19), it is natural to assume that equilibria of (4.18) on S_ε will be close to those of (4.19) on S_0. We then rewrite (4.18) as

$$\dot{y} = g(h^0(y), y, 0) + \{g(h^\varepsilon(y), y, \varepsilon) - g(h^0(y), y, 0)\}.$$

Using the Mean Value Theorem, the error term has the following expression:

$$g(h^\varepsilon(y), y, \varepsilon) - g(h^0(y), y, 0) = g(h^\varepsilon(y), y, \varepsilon) - g(h^0(y), y, \varepsilon)$$
$$+ g(h^0(y), y, \varepsilon) - g(h^0(y), y, 0),$$

$$g(h^\varepsilon(y), y, \varepsilon) - g(h^0(y), y, \varepsilon) = \frac{\partial g}{\partial x}(\widetilde{x}, y, \varepsilon)(h^\varepsilon(y) - h^0(y)) \quad \text{for some } \widetilde{x} \in M,$$

$$g(h^0(y), y, \varepsilon) - g(h^0(y), y, 0) = \frac{\partial g}{\partial \varepsilon}(h^0(y), y, \widetilde{\varepsilon})\varepsilon \qquad \qquad \text{for some } \widetilde{\varepsilon} \in [0, \varepsilon_0].$$

The difference between $h^\varepsilon(y)$ and $h^0(y)$ can be estimated as $|(h^\varepsilon(y) - h^0(y))_{x_i}| \le \beta_{x_i}$. Here β_{x_i} corresponds to the size of the i-th component in the x-coordinate of a fast-saddle-type block M obtained by rigorous numerics. It is precisely computable. Such an estimate can be realized because both $h^\varepsilon(y)$ and $h^0(y)$ belong to an identical h-set. Finally we obtain the following estimate:

$$(4.21) \quad \{g(h^\varepsilon(y), y, \varepsilon) - g(h^0(y), y, 0)\}$$
$$\in \frac{\partial g}{\partial x}(\widetilde{x}, y, \varepsilon)\beta_{x_i}[-1, 1] + \frac{\partial g}{\partial \varepsilon}(h^0(y), y, \widehat{\varepsilon})[0, \varepsilon_0].$$

Let y_0 be the numerical zero of $g(h^0(y), y, 0) = 0$. The principal part $g(h^0(y), y, 0)$ can be then expanded as

$$g(h^0(y), y, 0) = g(h^0(y_0), y_0, 0) + \left(g_x h_y^0 + g_y\right)_{y=y_0}(y - y_0) + \text{h.o.t.}$$

The Jacobian matrix $J_0 \equiv g_x h_y^0 + g_y$ at $\varepsilon = 0$ can be explicitly calculated via the Implicit Function Theorem to obtain

$$g_x h_y^0 + g_y = -g_x (f_x)^{-1} f_y + g_y.$$

One can explicitly calculate eigenvalues of J_0, which will be close to those of the Jacobian matrix of $J_\varepsilon \equiv \partial g / \partial y$ for $\varepsilon > 0$. On the other hand, the chain rule yields

$$J_\varepsilon = \frac{\partial g}{\partial x}(h^\varepsilon(y), y, \varepsilon)\frac{\partial h^\varepsilon}{\partial y}(y) + \frac{\partial g}{\partial y}(h^\varepsilon(y), y, \varepsilon)$$

and it contains $\partial h^\varepsilon / \partial y$ from (4.20). We additionally need to write the Fréchet differential of error terms in the form $g(h^\varepsilon(y), y, \varepsilon) - (\partial h^\varepsilon / \partial y) \cdot (y - y_0)$, when we try to validate the unique existence or hyperbolicity of equilibria via cone conditions [40] or Lyapunov conditions [24], for example. Instead, we apply a topological tool with J_0 to validate the invariant sets on slow manifolds.

Here we construct isolating blocks on slow manifolds. Using enclosures like (4.21) and eigenvalues of J_0, we can compute the *rigorous* enclosure of vector fields on crossing sections on slow manifolds.

Consider the case $l = 1$, which is our original setting. Assume that we have an ε-parameter family of slow manifolds $\{S_\varepsilon\}_{\varepsilon \in (0, \varepsilon_0]}$ as well as the critical manifold S_0, which can be validated by the method in Section 3. Then one would have obtained a fast-saddle-type block M surrounding $\{S_\varepsilon\}_{\varepsilon \in [0, \varepsilon_0]}$. The schematic illustration of such a procedure is shown in Figure 6. Let $y_0 \in S_0$ be a numerical zero of $g(x, y, 0)$. One expects that there exists an equilibrium y_ε on S_ε for sufficiently small $\varepsilon > 0$. We calculate the enclosure of $g(h^\varepsilon(y), y, \varepsilon)$ on

(a)

(b)

(c)

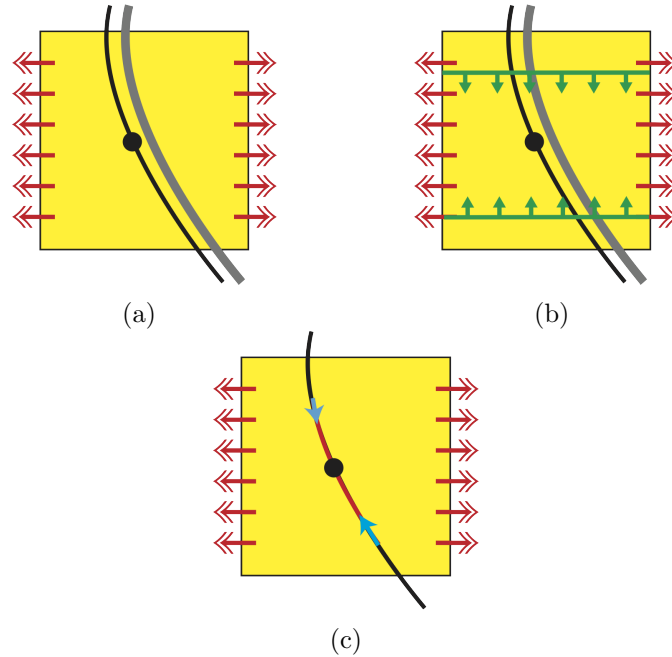Figure 6. Invariant sets on slow manifolds. (a) The critical manifold S_0 ($\varepsilon = 0$, colored ash), and the perturbed slow manifold S_ε ($\varepsilon \in (0, \varepsilon_0]$, colored black) via Theorem 3.5. The yellow rectangle with red arrows describes a fast-saddle-type block with the fast-exit. The critical manifold S_0 is a subset of the nullcline $f(x, y, 0) = 0$. In general, S_ε exhibits a nontrivial structure such as equilibria (drawn by a black ball). (b) Validation of vector fields on S_ε ($\varepsilon \in [0, \varepsilon_0]$). One cannot detect the position of the slow manifold S_ε in advance. In order to detect the vector field on such invariant manifolds, one puts a section S (colored green) having an intersection with S_ε and validates the vector field on S with rigorous numerics. This contains information of the vector field on $S \cap S_\varepsilon$. (c) An illustration of the isolating block containing an equilibrium on S_ε. Analysis in (b) contains the information of the rigorous vector field on S_ε, which gives an isolating neighbourhood colored red. General topological arguments such as the mapping degree or the Conley index guarantee the existence of nontrivial invariant sets. In the case of illustration described here, there exists an equilibrium on S_ε, namely, an equilibrium of the full system (1.1).

$\{y = y_0 \pm \delta\}$ for small δ in such a situation. It can be done by

$$g(h^\varepsilon(y), y, \varepsilon)|_{y=y_0 \pm \delta} = g(h^0(y), y, 0) + \{g(h^\varepsilon(y), y, \varepsilon) - g(h^0(y), y, 0)\}$$

$$= g(h^0(y_0), y_0, 0) + (g_x h_y^0 + g_y)_{y=y_0}(y - y_0) + \text{h.o.t.}$$

$$\in \{g(h^0(y_0), y_0, 0) \pm J_0 \delta + \text{h.o.t.}\} + \frac{\partial g}{\partial x}(\widetilde{x}, y, \varepsilon)\beta_x[-1, 1] + \frac{\partial g}{\partial \varepsilon}(h^0(y), y, \widetilde{\varepsilon})\varepsilon,$$

where $\beta_x > 0$ can be determined by the size of M. Remark that the enclosure of all terms on the right-hand side is rigorously computable and that they make sense for all $\varepsilon \in [0, \varepsilon_0]$. The principal term is $\pm J_0 \delta$. $g(h^0(y_0), y_0, 0)$ is very close

to 0 since y_0 is assumed to be a numerical zero of $g(h^0(y_0), y_0, 0)$. If

$$(4.22) \qquad g(h^\varepsilon(y), y, \varepsilon)|_{y=y_0-\delta} > 0 \quad \text{and} \quad g(h^\varepsilon(y), y, \varepsilon)|_{y=y_0+\delta} < 0$$

are validated on $M \times [0, \varepsilon_0]$, then the set $B_\varepsilon := \{(h^\varepsilon(y), y, \varepsilon) \in M \times [0, \varepsilon_0] \mid y \in [y_0 - \delta, y_0 + \delta]\}$ is an isolating block on S_ε for $\dot{y} = g(h^\varepsilon(y), y, \varepsilon)$ with the attracting boundary. Similarly, if

$$(4.23) \qquad g(h^\varepsilon(y), y, \varepsilon)|_{y=y_0-\delta} < 0 \quad \text{and} \quad g(h^\varepsilon(y), y, \varepsilon)|_{y=y_0+\delta} > 0$$

are validated on $M \times [0, \varepsilon_0]$, then the set $B_\varepsilon = \{(h^\varepsilon(y), y, \varepsilon) \in M \times [0, \varepsilon_0] \mid y \in [y_0 - \delta, y_0 + \delta]\}$ is an isolating block on S_ε for $\dot{y} = g(h^\varepsilon(y), y, \varepsilon)$ with the repelling boundary. In both cases, the general Conley index theory [7] yields that $\mathrm{Inv}(B_\varepsilon) \neq \emptyset$ for all $\varepsilon \in (0, \varepsilon_0]$. $\mathrm{Inv}(B_\varepsilon)$ is actually the ε-parameter family of invariant sets on slow manifolds. The direction of vector fields corresponds to the stability of $\mathrm{Inv}(B_\varepsilon)$. Moreover, Proposition 2.17 shows the existence of equilibria in $\mathrm{Inv}(B_\varepsilon)$ for (4.18), which leads to an equilibrium in the full system (1.1). Note that isolating blocks can be constructed by the same implementations as in Section 2.3.

REMARK 4.33. In terms of the Conley index theory, sets $\{B_\varepsilon\}_{\varepsilon \in [0, \varepsilon_0]}$ are *singular isolating neighbourhoods*. Indeed, for $\varepsilon = 0$, B_0 consists of the curve of hyperbolic equilibria and hence it is not an isolating neighbourhood. On the other hand, for $\varepsilon \in (0, \varepsilon_0]$ it is an isolating neighbourhood for $\dot{y} = g(h^\varepsilon(y), y, \varepsilon)$.

The general theory of the Conley index can be referred e.g. to [7], [26], [32]. In particular, [26] gives an explanation of singular perturbation version of the Conley index. More advanced topics for singular perturbation problems with the Conley index are shown in [12], [15], [14], for example.

REMARK 4.34. All validations of vector fields such as (4.22) and (4.23) make sense *only on slow manifolds* $\{S_\varepsilon\}_{\varepsilon \in [0, \varepsilon_0]}$, since the decoupled vector field (4.18) makes sense only on slow manifolds. In other words, estimates of vector fields discussed here do not always give reasonable information on vector fields off slow manifolds. Calculations of vector fields discussed here and Section 4.1 should be thus considered independently.

A simpler way to study the dynamics on slow manifolds will be to focus on the dynamics (4.19), namely, neglect the error term $g(h^\varepsilon(y), y, \varepsilon) - g(h^0(y), y, 0)$. We then study the dynamics (4.18) on slow manifolds via the regular perturbation theory. The essence of this idea is the same as ours, but such results make sense only for sufficiently small $\varepsilon > 0$. One of our main purposes here is to study the slow dynamics for *all* $\varepsilon \in (0, \varepsilon_0]$ and hence such simpler idea is not sufficient to our study.

4.6. Unstable manifolds of invariant sets on slow manifolds. We discuss the unstable manifold of invariant sets in the full system. Combining the covering-exchange property with graph representations of locally invariant manifolds, the standard consequence of covering relation (Proposition 2.14) yields the existence of connecting orbits between slow manifolds for $(1.1)_\varepsilon$. However, it does not always mean the existence of connecting orbits between invariant sets for $(1.1)_\varepsilon$. This difference comes from the fact that the role of slow variable y changes between $\varepsilon = 0$ and $\varepsilon > 0$. When we apply the covering relations to connecting orbits between equilibria for $(1.1)_0$: namely (1.3), a natural approach is to prove the existence of orbits connecting points on the stable and unstable manifolds of equilibria at a certain $\overline{y}$. This immediately yields validation of a connecting orbit at $y = \overline{y}$.

On the other hand, if $\varepsilon > 0$, a parameter y becomes a time dependent variable and slow manifolds do not always consist of a family of equilibria or general invariant sets. In fact, if unstable manifolds of normally hyperbolic invariant manifolds are perturbed for $\varepsilon > 0$, then the negatively invariant manifolds (e.g. unstable manifolds) of *invariant sets* are drastically collapsed in general. In particular, dimensions of h-sets corresponding to unstable manifolds of invariant sets change. The standard argument of covering relations is not thus sufficient to validating connecting orbits in $(1.1)_\varepsilon$.

To overcome this difficulty, we apply further structure of slow manifolds, called the Fenichel fibering, to validate (un)stable manifolds of invariant sets on slow manifolds. A direct consequence of discussions in the previous subsection is stated as follows. This result can be generalized to arbitrary dimensions, but we omit the details because we do not make use of such generalized arguments in this paper.

LEMMA 4.35. *Let N be a fast-saddle-type block for $(1.1)_\varepsilon$ satisfying stable and unstable cone conditions, and $I_\varepsilon \subset \operatorname{int} N$ be a subset of 1-dimensional slow manifold in N. Assume that I_ε is an isolating block on the slow manifold whose boundaries are repelling (cf. (4.23)). Then a trivial collection $\{I_\varepsilon\}$ is admissible with respect to $\varphi_\varepsilon(t, \cdot)$ for all sufficiently small $t > 0$ (cf. Definition 2.12). Similarly, assume that I_ε is an isolating block on the slow manifold whose boundaries are attracting (cf. (4.22)). Then a trivial collection $\{I_\varepsilon\}$ is back-admissible with respect to $\varphi_\varepsilon(t, \cdot)$ for all sufficiently small $t > 0$. Here we regard the flow φ_ε in the full system as that restricted to I_ε in both cases. This restriction makes sense since slow manifolds are locally invariant.*

Since I_ε is 1-dimensional, if I_ε is repelling, the covering relation $I_\varepsilon \overset{\varphi_\varepsilon(t, \cdot)}{\Longrightarrow} I_\varepsilon$ is nothing but $I_\varepsilon \subset \varphi_\varepsilon(t, I_\varepsilon)$. Similarly, if I_ε is attracting, the back-covering relation $I_\varepsilon \overset{\varphi_\varepsilon(t, \cdot)}{\Longleftarrow} I_\varepsilon$ is nothing but $I_\varepsilon \subset \varphi_\varepsilon(-t, I_\varepsilon)$. For simplicity, we write k-covering

relations $N \xRightarrow{f} N$ for an h-set N and a continuous map f by $\left\{N \xRightarrow{f} N\right\}_k$. Let $\left\{N \xLeftarrow{f} N\right\}_k$ be defined in a similar manner.

Lemma 4.35 gives the other proof of existence of an invariant set in I_ε. Note that this invariant set is actually invariant for the full system $(1.1)_\varepsilon$.

A key essence of the Fenichel fibering are cone conditions on validated locally invariant manifolds. In the case of fast-slow systems, the property that the unstable fiber $\mathrm{Im}(h^z_{f,\mathrm{u}}) \cap N$ in $W^{\mathrm{u}}(S_\varepsilon)$ for $z \in S_\varepsilon$ becomes a horizontal disk in N is nontrivial, which is due to dynamics in the slow direction.

From now on we relate the fiber bundle structure of slow manifolds to covering relations. The fiber bundle structure of $W^{\mathrm{u}}(I_\varepsilon)$ for $I_\varepsilon \subset S_\varepsilon$ leads to the construction of a deformation retract r_λ of unstable manifolds so that the fiber bundle $W^{\mathrm{u}}(I_\varepsilon)$ r_λ-covers a fast-exit face. Namely, the following lemma holds.

LEMMA 4.36. *Consider* (3.1). *Let* $N \subset \mathbb{R}^{n+1}$ *be a fast-saddle-type block with* $u = 1$ *satisfying the stable cone and the unstable fiber-cone conditions (Appendix B) such that the coordinate representation* N_{c} *is actually given by* $N_{\mathrm{c}} = \overline{B_1} \times \overline{B_s} \times [0,1]$. *Here* $*_{\mathrm{c}}$ *denotes the image of objects under the homeomorphism* c_N. *Also, let* N^{exit} *be a fast-exit face of* N *with*

$$\pi_a N_{\mathrm{c}}^{\mathrm{exit}} = \{1\} \subset \partial B_1, \qquad (N_{\mathrm{c}}^{\mathrm{exit}})^- = \{y_\pm\} \subset [0,1] \quad \text{with } 0 < y_- < y_+ < 1,$$

and $I_\varepsilon \subset \mathrm{int}\, N$ *be a connected subset of* 1-*dimensional slow manifold in* N. *Assume that, for any* $z \in I_\varepsilon$, *the Lipschitz function* $(b,y) = b^z_{\mathrm{u},\varepsilon}(a)$ *is a horizontal disk in* N *representing* $W^{\mathrm{u}}(z)$. *We also assume that, at* $z_\pm \in \partial I_\varepsilon$ *with* $z_+ > z_-$, $\pi_y((b^{z_+}_{\mathrm{u},\varepsilon})_c(1)) > y_+$ *and* $\pi_y((b^{z_-}_{\mathrm{u},\varepsilon})_c(1)) < y_-$ *hold. Then* $I_\varepsilon \xRightarrow{r_1 \circ b^*_{\mathrm{u},\varepsilon}} N^{\mathrm{exit}}$, *where* $r_\lambda \circ b^z_{\mathrm{u},\varepsilon}$ *is the deformation retract given by*

$$(4.24) \quad (r_\lambda \circ b^z_{\mathrm{u},\varepsilon})_c(a) := (\lambda + (1-\lambda)a, b^z_{\mathrm{u},\varepsilon}(\lambda + (1-\lambda)a)), \quad \lambda \in [0,1],\ z \in I_\varepsilon.$$

If $\pi_a N^{\mathrm{exit}} = \{-1\}$, *then the same statement holds by replacing* r_λ *in* (4.24) *by*

$$(4.25) \quad (r_\lambda \circ b^z_{\mathrm{u},\varepsilon})_c(a) := (-\lambda + (1-\lambda)a, b^z_{\mathrm{u},\varepsilon}(-\lambda + (1-\lambda)a)), \quad \lambda \in [0,1],\ z \in I_\varepsilon.$$

PROOF. Our assumptions imply that $r_\lambda \circ b^{z_\pm}_{\mathrm{u},\varepsilon}(\overline{B_1}) \cap N^{\mathrm{exit}} = \emptyset$. Note that graphs of $b^{z_\pm}_{\mathrm{u},\varepsilon}$ lie on the unstable manifold $W^{\mathrm{u}}(I_\varepsilon)$, which implies $r_\lambda \circ b^z_{\mathrm{u},\varepsilon}(\overline{B_1}) \cap (N^{\mathrm{exit}})^+ = \emptyset$ for all $z \in I_\varepsilon$. The rest of assumptions in Propositions 2.8 or 2.9 is easy to check. $\qquad \square$

If we additionally assume that I_ε is a singular isolating neighbourhood, then we can describe unstable manifolds of invariant sets.

PROPOSITION 4.37. *In addition to assumptions in Lemma 4.36, we assume that* I_ε *is an isolating block on the slow manifold which is either repelling or attracting. Then there exists a point* $z \in \mathrm{Inv}(I_\varepsilon)$ *such that* $W^{\mathrm{u}}(z) \cap N^{\mathrm{exit}} \neq \emptyset$.

Proof. Since I_ε is isolated, Lemma 4.35 yields the covering relation

$$\left\{ I_\varepsilon \overset{\varphi_\varepsilon(t, \,\cdot\,)}{\Longrightarrow} I_\varepsilon \right\}_k \quad \text{for arbitrary } k \geq 1 \text{ and all sufficiently small } t > 0 \text{ if } I_\varepsilon$$

is repelling. Similarly, if I_ε is attracting, the covering relation $\left\{ I_\varepsilon \overset{\varphi_\varepsilon(t, \,\cdot\,)}{\Longleftarrow} I_\varepsilon \right\}_k$ holds for arbitrary $k \geq 1$.

We only deal with the attracting case. By Proposition 2.11 there exists a point $z_k \in I_\varepsilon$ such that

$$(4.26) \qquad \varphi_\varepsilon(-k't, z_k) \in I_\varepsilon \quad \text{for all } k' = 0, \ldots, k, \quad W^{\mathrm{u}}(z_k) \cap N^{\mathrm{exit}} \neq \emptyset.$$

Since I_ε is compact, then the sequence $\{z_k\}_{k \geq 1}$ contains a convergent subsequence. Let z_∞ be the limit. We can choose a sufficiently small $t_\varepsilon > 0$ such that $\varphi_\varepsilon([-t_\varepsilon, 0), z) \cap I_\varepsilon = \emptyset$ holds for all $z \in \partial I_\varepsilon$, since I_ε is an isolating block. Note that ∂I_ε means the boundary with respect to the slow manifold in N. Obviously, we can choose a limit point z_∞ from (4.26) with $t = t_\varepsilon$. The properties of t_ε, z_∞ and (4.26) thus imply $\varphi_\varepsilon([-kt_\varepsilon, 0], z_\infty) \subset I_\varepsilon$ for all $k \in \mathbb{N}$, which yields $\varphi_\varepsilon((-\infty, 0], z_\infty) \subset I_\varepsilon$. The inclusion $\varphi_\varepsilon([0, \infty), z_\infty) \subset I_\varepsilon$ is obvious since I_ε is attracting. As a consequence, $z_\infty \in \mathrm{Inv}(I_\varepsilon)$. By continuity of the graph $b^z_{\mathrm{u}, \varepsilon}$ with respect to z, then $W^{\mathrm{u}}(z_\infty)$ is well-defined and $W^{\mathrm{u}}(z_\infty) \cap N^{\mathrm{exit}} \neq \emptyset$. $\qquad \square$

The validation of I_ε being an isolating block on slow manifolds can be done by arguments in Section 4.5. On the other hand, if we apply Lemma 4.36, we have to know the location of fibers $W^{\mathrm{u}}(z)$ for all $z \in I_\varepsilon$. The unstable fiber-cone condition indicates that, for $z \in I_\varepsilon$, the fiber $W^{\mathrm{u}}(z)$ is included in the unstable cone $C^{\mathrm{u}}_{(\mathrm{sl}_{\mathrm{u}})^{-1}}(z)$ where sl_{u} is the slope of unsatble fibers whose detail is shown in Appendix B. We can apply this cone to validating assumptions in Lemma 4.36. Namely, the following lemma holds, which can be validated by rigorous numerics.

Lemma 4.38. *Consider* (3.1) *with fixed* $\varepsilon \in [0, \varepsilon_0]$. *Let* $N \subset \mathbb{R}^{n+1}$ *be a fast-saddle-type block with* $u = 1$ *satisfying the stable cone and the unstable fiber-cone conditions such that the coordinate representation* N_{c} *is actually given by* $N_{\mathrm{c}} = \overline{B_1} \times \overline{B_s} \times [0, 1]$. *Also, let* N^{exit} *be a fast-exit face of* N *with*

$$\pi_a N_c^{\mathrm{exit}} = \{1\}, \quad (N_c^{\mathrm{exit}})^- = \{y^\pm\} \quad \text{with } 0 < y_- < y_+ < 1$$

and $I_\varepsilon \subset \mathrm{int}\, N$ *be a connected subset of* 1*-dimensional slow manifold in* N. *Assume that the unstable fiber-cone condition (Appendix B) is satisfied in* N *with the slope of fiber* sl_{u} *and that, at* $z_\pm \in \partial I_\varepsilon$, *the unstable cones* $C^{\mathrm{u}}_{\mathrm{sl}_{\mathrm{u}}^{-1}}(z_\pm)$ *satisfy the following properties*:

$$(4.27) \qquad\qquad (C^{\mathrm{u}}_{\mathrm{sl}_{\mathrm{u}}^{-1}}(z_\pm)) \cap \partial N \subset N^{f,-},$$

$$(4.28) \qquad\qquad \inf(\pi_y(C^{\mathrm{u}}_{\mathrm{sl}_{\mathrm{u}}^{-1}}(z_+)) \cap \{a = 1\}) > y_+,$$

$$(4.29) \qquad\qquad \sup(\pi_y(C^{\mathrm{u}}_{\mathrm{sl}_{\mathrm{u}}^{-1}}(z_-)) \cap \{a = 1\}) < y_-.$$

Then all assumptions in Lemma 4.36 *hold.*

Proof. The unstable cone condition indicates, as mentioned, that the graph of the Lipschitz function $b_{u,\varepsilon}^z$ is contained in $C_{(\mathrm{sl_u})^{-1}}^u(z)$ for all $z \in I_\varepsilon$. Since cones have the identical shape for $z \in I_\varepsilon$ and I_ε is 1-dimensional, then the condition $\{(a, (b_{u,\varepsilon}^z)_c(a)) \mid a \in \overline{B_1}\} \subset N$ for all $z \in I_\varepsilon$ obviously follows from (4.27). Inequalities $\pi_y((b_{u,\varepsilon}^{z+})_c(1)) > y_+$ and $\pi_y((b_{u,\varepsilon}^{z-})_c(1)) < y_-$ are direct consequences of (4.28) and (4.29), respectively. $\square$

5. Topological existence theorems of singularly perturbed trajectories

Now we are ready to state our verification theorems. In this section we derive existence theorems for periodic and heteroclinic orbits which can be applied to (1.1).

5.1. Existence of periodic orbits. First we state the existence theorems for periodic orbits near singular orbits.

Theorem 5.1 (Existence of periodic orbits). *Consider* (1.1). *Assume that there exist $\rho \in \mathbb{N}$, $\varepsilon_0 > 0$ such that* (1.1) *admits the following ε ($\in [0, \varepsilon_0]$)-parameter family of sets in $\mathbb{R}^{n+1}$ (see also Figure 7 (a)):*

- $\mathcal{S}_\varepsilon^j$, $j = 0, \ldots, \rho$: *a fast-saddle-type block with $u = 1$ satisfying the covering-exchange property with respect to $\mathcal{F}_\varepsilon^{j-1}$ defined below and for $T^{j-1} > 0$.*
- $\mathcal{F}_\varepsilon^j$, $j = 0, \ldots, \rho$: *a fast-exit face of $\mathcal{S}_\varepsilon^j$. Identify $\mathcal{F}_\varepsilon^{-1}$ with $\mathcal{F}_\varepsilon^\rho$.*

Then the following statements hold:

(a) *When $\varepsilon \in (0, \varepsilon_0]$, (1.1) admits a periodic orbit $\{\Gamma_\varepsilon(t) \mid t \in \mathbb{R}\}$ with a sequence of positive numbers $0 < t_f^0 < t_s^1 < t_f^1 < \ldots < t_f^\rho < t_s^0 \equiv T_\varepsilon < \infty$ such that*

$$\Gamma_\varepsilon(T_\varepsilon) = \Gamma_\varepsilon(0) \in \mathcal{F}_\varepsilon^0,$$
$$\Gamma_\varepsilon([t_f^j, t_s^{j+1}]) \subset \mathcal{S}_\varepsilon^{j+1}, \quad \Gamma_\varepsilon(t_s^{j+1}) \in \mathcal{F}_\varepsilon^{j+1} \quad (j = 0, \ldots, \rho - 1),,$$
$$\Gamma_\varepsilon([t_f^\rho, T_\varepsilon]) \subset \mathcal{S}_\varepsilon^0.$$

(b) *When $\varepsilon = 0$, (1.1) admits the collection of heteroclinic orbits $\{\Gamma^j = \{x_{y_j}(t)\}_{t \in \mathbb{R}}\}_{j=0}^\rho$ with*

$$\lim_{t \to -\infty} x_{y_j}(t) = p_j \in \mathcal{S}_0^j \qquad \text{with } f(p_j, y_j, 0) = 0,$$
$$\lim_{t \to +\infty} x_{y_j}(t) = q_j \in \mathcal{S}_0^{j+1} \quad \text{with } f(q_j, y_j, 0) = 0 \quad (j = 0, \ldots, \rho - 1),$$
$$\lim_{t \to -\infty} x_{y_\rho}(t) = p_\rho \in \mathcal{S}_0^\rho \qquad \text{with } f(p_\rho, y_\rho, 0) = 0,$$
$$\lim_{t \to +\infty} x_{y_\rho}(t) = q_\rho \in \mathcal{S}_0^0 \qquad \text{with } f(q_\rho, y_\rho, 0) = 0,$$
$$\Gamma^j \cap \mathcal{F}_\varepsilon^j \neq \emptyset \qquad \qquad (j = 0, \ldots, \rho)$$

bridging 1-*dimensional normally hyperbolic invariant manifolds* $S_0^j \subset \mathcal{S}_0^j$ *and* $S_0^{j+1} \subset \mathcal{S}_0^{j+1}$ *for* $j = 0, \dots, \rho - 1$. *In case that* $j = \rho$ *such a heteroclinic orbit connects* $S_0^\rho \subset \mathcal{S}_0^\rho$ *and* $S_0^0 \subset \mathcal{S}_0^0$.
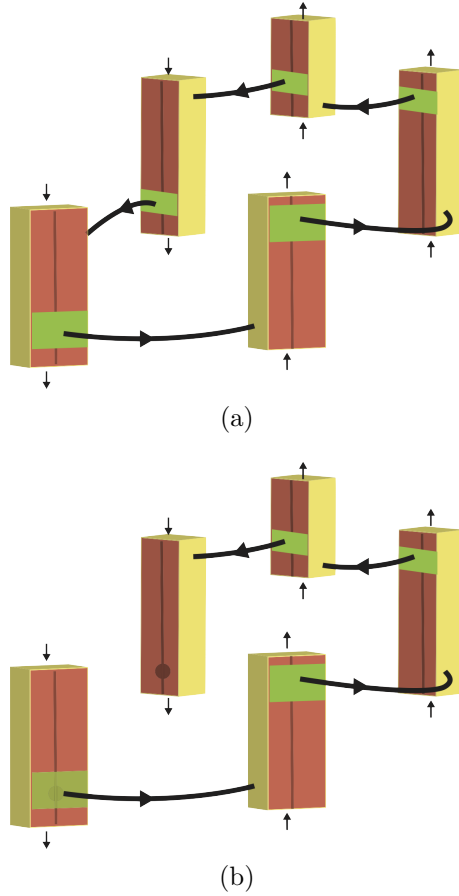

(a)

(b)

FIGURE 7. Illustration of $\{\mathcal{S}_\varepsilon^j, \mathcal{F}_\varepsilon^j\}_{j=0}^\rho$ in Theorems 5.1 and 5.6, $\rho = 4$. Rectangular parallelepipeds (colored red and yellow) are fast-saddle-type blocks with fast-exit faces (colored green) satisfying the covering-exchange property. Vertical black arrows show the slow vector field. Bold black curves outside blocks describe heteroclinic orbits $\{\Gamma^j\}_{j=0}^\rho$. Black lines inside blocks correspond to limit critical manifolds. Their union generates a singular limit orbit Γ_0. Theorem 5.1 claims that there exists a family of periodic orbits $\{\Gamma_\varepsilon\}_{\varepsilon \in (0, \varepsilon_0]}$ near Γ_0.

PROOF. First consider the case $\varepsilon \in (0, \varepsilon_0]$. By Proposition 4.5 and our assumption, there exist h-sets $\widetilde{\mathcal{S}}_\varepsilon^j \subset \mathcal{S}_\varepsilon^j$ such that we get the sequence of covering relations

$$\mathcal{F}_\varepsilon^\rho \overset{\varphi_\varepsilon(T_\rho, \cdot)}{\Longrightarrow} \widetilde{\mathcal{S}}_\varepsilon^0 \overset{P_\varepsilon^0}{\Longrightarrow} \mathcal{F}_\varepsilon^0 \overset{\varphi_\varepsilon(T_0, \cdot)}{\Longrightarrow} \widetilde{\mathcal{S}}_\varepsilon^1 \overset{P_\varepsilon^1}{\Longrightarrow} \dots \overset{\varphi_\varepsilon(T_{\rho-1}, \cdot)}{\Longrightarrow} \widetilde{\mathcal{S}}_\varepsilon^\rho \overset{P_\varepsilon^\rho}{\Longrightarrow} \mathcal{F}_\varepsilon^\rho,$$

where $P_\varepsilon^j : \mathcal{S}_\varepsilon^j \to \partial \mathcal{S}_\varepsilon^j$ is the Poincaré map of $\mathcal{S}_\varepsilon^j$. Then our statement is just a consequence of Proposition 2.11. The case $\varepsilon = 0$ is just a consequence of Proposition 2.14. All arguments with respect to horizontal and vertical disks are valid due to Corollary 3.7. $\square$

REMARK 5.2. The standard Exchange Lemma says that the period when trajectories stay near slow manifolds in $O(\varepsilon)$-distance is $O(1/\varepsilon)$ (actually, discussed in [20], $O(\exp(-c/\varepsilon))$ for some $c > 0$). The period T_ε is thus expected to be $O(1/\varepsilon)$ for $\varepsilon \in (0, \varepsilon_0]$.

Theorem 5.1 can be generalized as stated below, replacing covering-exchange pairs by covering-exchange sequences.

COROLLARY 5.3 (Validation of periodic orbits with slow shadowing). *Consider* (1.1). *Assume that there exist* $\rho \in \mathbb{N}$, $\varepsilon_0 > 0$ *such that* (1.1) *admits the following* ε ($\in [0, \varepsilon_0]$)*-parameter family of sets in* $\mathbb{R}^{n+1}$ (*see also Figure 8*):

- $\{N_\varepsilon^{j,i}\}_{i=0,\ldots,m_j}^{j=0,\ldots,\rho}$: *a sequence of h-sets with* $u = 1$ *which forms a covering-exchange sequence with* $\mathcal{F}_\varepsilon^{j-1}$ *defined below.*
- $\mathcal{F}_\varepsilon^j$, $j = 0, \ldots, \rho$: *a fast-exit face of* N_ε^{j,m_j}. *Identify* $\mathcal{F}_\varepsilon^{-1}$ *with* $\mathcal{F}_\varepsilon^\rho$.

Then all statements in Theorem 5.1 *hold with* $\mathcal{S}_\varepsilon^j = \bigcup_{i=0}^{m_j} N_\varepsilon^{j,i}$.
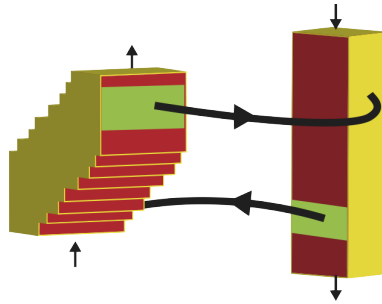

FIGURE 8. Illustration of $\{\mathcal{S}_\varepsilon^j, \mathcal{F}_\varepsilon^j\}_{j=0}^\rho$ in Corollary 5.3, $\rho = 2$. Rectangular parallelepipeds (colored red and yellow) are fast-saddle-type blocks with fast-exit faces (colored green) satisfying the covering-exchange property and slow shadowing. Vertical black arrows show the slow vector field. Bold black curves outside blocks describe heteroclinic orbits $\{\Gamma^j\}_{j=0}^\rho$. The union of heteroclinic orbits Γ^j and critical manifolds inside blocks generates a singular limit orbit Γ_0. Theorem 5.1 claims that there exists a family of periodic orbits $\{\Gamma_\varepsilon\}_{\varepsilon \in (0, \varepsilon_0]}$ near Γ_0.

PROOF. We only consider the case $\varepsilon \in (0, \varepsilon_0]$. By Propositions 4.12, 4.14, 4.15 and our assumptions, there is a sequence of h-sets $\{M_\varepsilon^{j,i}\}_{i=0,\ldots,m_j}^{j=0,\ldots,\rho}$ with

$M^{j,i}_\varepsilon \subset N^{j,i}_\varepsilon$ which admits a sequence of covering relations

$$\mathcal{F}^\rho_\varepsilon \overset{\varphi_\varepsilon(T_\rho, \cdot)}{\Longrightarrow} M^{0,0}_\varepsilon \overset{P^{0,0}_\varepsilon}{\Longrightarrow} M^{0,1}_\varepsilon \overset{P^{0,1}_\varepsilon}{\Longrightarrow} \cdots \overset{P^{0,m_0-1}_\varepsilon}{\Longrightarrow} M^{0,m_0}_\varepsilon$$

$$\overset{P^{0,m_0}_\varepsilon}{\Longrightarrow} \mathcal{F}^0_\varepsilon \overset{\varphi_\varepsilon(T_0, \cdot)}{\Longrightarrow} M^{1,0}_\varepsilon \overset{P^{1,0}_\varepsilon}{\Longrightarrow} \cdots \overset{P^{\rho,m_\rho-1}_\varepsilon}{\Longrightarrow} M^{\rho,m_\rho}_\varepsilon \overset{P^{\rho,m_\rho}_\varepsilon}{\Longrightarrow} \mathcal{F}^\rho_\varepsilon,$$

where $P^{j,i}_\varepsilon \colon (N^{j,i}_\varepsilon)_{\leq \overline{y}_{j,i} + \overline{h}_{j,i}} \to \partial(N^{j,i}_\varepsilon)_{\leq \overline{y}_{j,i} + \overline{h}_{j,i}}$ is the Poincaré map in the family $(N^{j,i}_\varepsilon)_{\leq \overline{y}_{j,i} + \overline{h}_{j,i}}$ with appropriate choices of $\overline{y} = \overline{y}_{j,i}$ and $\overline{h} = \overline{h}_{j,i}$ following Propositions 4.12, 4.14 and 4.15. Then our statement is just a consequence of Proposition 2.11. $\qquad\square$

Note that points on the slow manifold S^j_ε are contained in all h-sets $\{M^{i,j}_\varepsilon\}$ in the proof of Corollary 5.3. This observation implies that true orbits in the full system (1.1) *shadow* trajectories on S^j_ε via covering relations (cf. [27]). This fact gives us a suggestion to describe true trajectories for fast-slow systems with multi-dimensional slow variables from the viewpoint of shadowing.

Corollary 5.3 can be further generalized as stated below, at least in the case $u = s = 1$, replacing covering-exchange sequences by covering-exchange sequences with extended cones.

COROLLARY 5.4 (Validation of periodic orbits with slow shadowing and m-cones). *Consider* (1.1). *Assume that there exist $\rho \in \mathbb{N}$, $\varepsilon_0 > 0$ such that* (1.1) *admits the following ε $(\in [0, \varepsilon_0])$-parameter family of sets in $\mathbb{R}^3$:*

- $\{N^{j,i}_\varepsilon\}^{j=0,\dots,\rho}_{i=0,\dots,m_j}$: *a sequence of fast-saddle-type blocks with $u = 1$ which admits a sequence of faxt-exit faces $\{N^{j,\mathrm{exit}}_\varepsilon\}^{j=0,\dots,\rho}$ with $N^{j,\mathrm{exit}}_\varepsilon \subset N^{j,m_j}_\varepsilon$.*

The collection $(\mathcal{F}^{j-1}_\varepsilon, \{N^{j,i}_\varepsilon\}_{i=0,\dots,m_j}, N^{j,\mathrm{exit}}_\varepsilon, C^{\mathrm{u}}_{m^u_j}, C^{\mathrm{s}}_{m^s_j})$ forms a covering-exchange sequence with extended cones $C^{\mathrm{u}}_{m^u_j}, C^{\mathrm{s}}_{m^s_j}$. Associating sets $C^{\mathrm{u}}_{m^u_j}, C^{\mathrm{s}}_{m^s_j}$ and $\mathcal{F}^j_\varepsilon$ are defined below.

- $C^{\mathrm{u}}_{m^u_j}$ $(j = 0, \dots, \rho)$: *the unstable m^u_j-cones of N^{j,m_j}_ε. Identify $C^{\mathrm{u}}_{m^u_{-1}}$ with $C^{\mathrm{u}}_{m^u_\rho}$.*
- $C^{\mathrm{s}}_{m^s_j}$ $(j = 0, \dots, \rho)$: *the stable m^s_j-cone of $N^{j,0}_\varepsilon$.*
- $\mathcal{F}^j_\varepsilon$ $(j = 0, \dots, \rho)$: *a fast-exit face $(C^{\mathrm{u}}_{m^u_j})^{\mathrm{exit}}$ of $C^{\mathrm{u}}_{m^u_j}$. Identify $\mathcal{F}^{-1}_\varepsilon$ with $\mathcal{F}^\rho_\varepsilon$.*

Then all statements in Theorem 5.1 hold with $S^j_\varepsilon = C^{\mathrm{s}}_{m^s_j} \cup \bigcup_{i=0}^{m_j} N^{j,i}_\varepsilon \cup C^{\mathrm{u}}_{m^u_j}$.

PROOF. Replace $\mathcal{F}^j_\varepsilon$ and $M^{j,0}_\varepsilon$ in the proof of Corollary 5.3 by $N^{j,\mathrm{exit}}_\varepsilon \overset{P^{m^u_j}_\varepsilon}{\Longrightarrow} \mathcal{F}^j_\varepsilon$ and $\widetilde{M}^{j,0}_\varepsilon$, respectively. Here $P^{m^u_j}_\varepsilon \colon C^{\mathrm{u}}_{m^u_j} \to \partial C^{\mathrm{u}}_{m^u_j}$ is the Poincaré map and $\widetilde{M}^{j,0}_\varepsilon \subset (N^{j,0}_\varepsilon \cup C^{\mathrm{s}}_{m^s_j})$ is the h-set corresponding to $\widetilde{M}_1$ in Proposition 4.30. $\qquad\square$

REMARK 5.5. The key essence of our validation near slow manifolds consists of the following three pieces:

(a) Construction of fast-saddle-type blocks.
(b) Stable and unstable cone conditions.
(c) Slow shadowing condition (if necessary).

As long as validations of the above procedures pass, we can extend slow manifolds *in an arbitrary range* keeping the existence of points near slow manifolds which exit their neighbourhoods after time $T = O(1/\varepsilon)$, as stated in the Exchange Lemma.

5.2. Existence of connecting orbits. Similar settings and arguments to the previous subsection yield the existence of heteroclinic orbits near singular orbits.

THEOREM 5.6 (Existence of heteroclinic orbits). *Consider* (1.1). *Assume that there exist $\rho \in \mathbb{N}$ and $\varepsilon_0 > 0$ such that (1.1) admits the following ε ($\in [0, \varepsilon_0]$)-parameter family of sets in $\mathbb{R}^{n+1}$ (see also Figure 7 (b)):*

- $\mathcal{S}_{\varepsilon}^j$, $j = 0$: *a fast-saddle-type block satisfying stable and unstable cone conditions. Moreover, it contains the nonempty maximal invariant sets $S_{\varepsilon,u}$.*
  $j = \rho$: *a fast-saddle-type block satisfying the covering-exchange property with respect to $\mathcal{F}_{\varepsilon}^{\rho-1}$ (defined below) except the third condition. Moreover, it contains the nonempty maximal invariant set $S_{\varepsilon,s}$. The set $S_{\varepsilon,s}$ is contained in an attracting isolating block on the slow manifold in $\mathcal{S}_{\varepsilon}^{\rho}$.*
  $j = 1, \ldots, \rho - 1$: *a fast-saddle-type block satisfying the covering-exchange property with respect to $\mathcal{F}_{\varepsilon}^{j-1}$ defined below and for $T^{j-1} > 0$. All $\mathcal{S}_{\varepsilon}^j$ are assumed that $u(\mathcal{S}_{\varepsilon}^j) \equiv u = 1$*
- $\mathcal{F}_{\varepsilon}^j$, $j = 0, \ldots, \rho - 1$: *a fast-exit face of $\mathcal{S}_{\varepsilon}^j$.*
  If $j = 0$, the invariant set $S_{\varepsilon,u}$ admits an isolating block $B(S_{\varepsilon,u})$ on the slow manifold in $\mathcal{S}_{\varepsilon}^0$ such that all assumptions in Lemma 4.38 as well as fiber-cone condition (Appendix B) in $\mathcal{S}_{\varepsilon}^0$ hold.

Then the following statements hold:

(a) *When $\varepsilon \in (0, \varepsilon_0]$, (1.1) admits a heteroclinic orbit $\{\Gamma_{\varepsilon}(t) \mid t \in \mathbb{R}\}$ with a sequence of positive numbers $0 < t_f^0 < t_s^1 < t_f^1 < \ldots < t_s^{\rho-1} < t_f^{\rho-1} < \infty$ such that*

$$\mathrm{dist}(\Gamma_{\varepsilon}(t), S_{\varepsilon.u}) \to 0 \quad as\ t \to -\infty,$$
$$\Gamma_{\varepsilon}((-\infty, 0]) \subset W^{\mathrm{u}}(S_{\varepsilon,u}) \subset \mathcal{S}_{\varepsilon}^0, \quad \Gamma_{\varepsilon}(0) \in \mathcal{F}_{\varepsilon}^0,$$
$$\Gamma_{\varepsilon}([t_f^{j-1}, t_s^j]) \subset \mathcal{S}_{\varepsilon}^j, \quad \Gamma_{\varepsilon}(t_s^j) \in \mathcal{F}_{\varepsilon}^j \quad (j = 1, \ldots, \rho - 1),$$
$$\Gamma_{\varepsilon}([t_f^{\rho-1}, \infty)) \subset W^{\mathrm{s}}(S_{\varepsilon,s}) \subset \mathcal{S}_{\varepsilon}^{\rho},$$
$$\mathrm{dist}(\Gamma_{\varepsilon}(t), S_{\varepsilon.s}) \to 0 \quad as\ t \to +\infty.$$

(b) *When $\varepsilon = 0$, (1.1) admits the collection of heteroclinic orbits $\{\Gamma^j = \{x_{y_j}(t)\}_{t \in \mathbb{R}}\}_{j=0}^{\rho}$ with*

$$\lim_{t \to -\infty} x_{y_j}(t) = p_j \in \mathcal{S}_0^j \qquad \text{with } f(p_j, y_j, 0) = 0,$$

$$\lim_{t \to +\infty} x_{y_j}(t) = q_j \in \mathcal{S}_0^{j+1} \qquad \text{with } f(q_j, y_j, 0) = 0,$$

$$\Gamma^j \cap \mathcal{F}_\varepsilon^j \neq \emptyset \qquad\qquad (j = 0, \ldots, \rho - 1),$$

bridging 1-dimensional normally hyperbolic invariant manifolds $S_0^j \subset \mathcal{S}_0^j$ and $S_0^{j+1} \subset \mathcal{S}_0^{j+1}$ for $j = 0, \ldots, \rho - 1$.

PROOF. Consider the case $\varepsilon \in (0, \varepsilon_0]$. We deal with the case that $B(S_{\varepsilon,u})$ is attracting. By Propositions 4.5, 4.37 and our assumptions, we get the sequence of covering relations

$$\left\{ B(S_{\varepsilon,u}) \overset{\varphi_{\varepsilon(t,\,\cdot\,)}}{\Longleftarrow} B(S_{\varepsilon,u}) \right\}_k \overset{r_1 \circ b_{u,\varepsilon}^*}{\Longrightarrow} \mathcal{F}_\varepsilon^0 \overset{\varphi_\varepsilon(T_0,\,\cdot\,)}{\Longrightarrow} \widetilde{\mathcal{S}}_\varepsilon^1$$

$$\overset{P_\varepsilon^1}{\Longrightarrow} \mathcal{F}_\varepsilon^1 \overset{\varphi_\varepsilon(T_1,\,\cdot\,)}{\Longrightarrow} \widetilde{\mathcal{S}}_\varepsilon^2 \overset{P_\varepsilon^2}{\Longrightarrow} \cdots \overset{\varphi_\varepsilon(T_{\rho-1},\,\cdot\,)}{\Longrightarrow} \mathcal{S}_\varepsilon^\rho,$$

for all sufficiently small $t > 0$ and arbitrary $k \in \mathbb{N}$. Here r_λ is the deformation retract given by (4.24) or (4.25), $b_{u,\varepsilon}^z$ is the horizontal disk at $z \in B(S_{\varepsilon,u})$ whose graph is the unstable manifold $W^u(z)$ and $P_\varepsilon^j : \mathcal{S}_\varepsilon^j \to \partial \mathcal{S}_\varepsilon^j$ is the Poincaré map of $\mathcal{S}_\varepsilon^j$. The stable manifold $W^s(S_{\varepsilon,s})$ is given by a vertical disk b_ε^u in $\mathcal{S}_\varepsilon^\rho$ with (b, y)-coordinates as the $s(\mathcal{S}_\varepsilon^\rho)$-dimensional direction of the h-set $\mathcal{S}_\varepsilon^\rho$. Our statement is just a consequence of Propositions 4.37 and 2.14. See Definition 2.13 about horizontal and vertical disks. All arguments with respect to horizontal and vertical disks are valid due to Corollary 3.7. In the case that $B(S_{\varepsilon,u})$ is repelling, the same arguments are valid when replacing the covering relation $\left\{ B(S_{\varepsilon,u}) \overset{\varphi_{\varepsilon(t,\,\cdot\,)}}{\Longleftarrow} B(S_{\varepsilon,u}) \right\}_k$ by $\left\{ B(S_{\varepsilon,u}) \overset{\varphi_{\varepsilon(t,\,\cdot\,)}}{\Longrightarrow} B(S_{\varepsilon,u}) \right\}_k$.

The case $\varepsilon = 0$ is the same as that of Theorem 5.1. Remark that we do not need the sequence of covering relations

$$\left\{ B(S_{\varepsilon,u}) \overset{\varphi_{\varepsilon(t,\,\cdot\,)}}{\Longleftarrow} B(S_{\varepsilon,u}) \right\}_k \overset{r_1 \circ b_{u,\varepsilon}^*}{\Longrightarrow} \mathcal{F}_\varepsilon^0$$

to prove our statements in this case. $\qquad\qquad\qquad\qquad\qquad \square$

Theorem 5.6 can be generalized as stated below, replacing covering-exchange pairs by covering-exchange sequences.

COROLLARY 5.7 (Validation of heteroclinic orbits with slow shadowing). *Consider (1.1). Assume that there exist $\rho \in \mathbb{N}$ and $\varepsilon_0 > 0$ such that (1.1) admits the following ε $(\in [0, \varepsilon_0])$-parameter family of sets in $\mathbb{R}^{n+1}$:*

- *$\{N_\varepsilon^{j,i}\}_{i=0,\ldots,m_j}^{j=0,\ldots,\rho}$ with $m_0 = 0$. $j = 1, \ldots, \rho - 1$: a sequence of h-sets which forms a covering-exchange sequence with $\mathcal{F}_\varepsilon^{j-1}$ defined below.*

$j = \rho$: *a sequence of h-sets which forms a covering-exchange sequence with $\mathcal{F}_\varepsilon^{\rho-1}$ defined below except the last assumption in Definition 4.16. Blocks $N_\varepsilon^{0,0}$ and $N_\varepsilon^{\rho,m_\rho}$ contain nonempty maximal invariant sets $S_{\varepsilon,u}$ and $S_{\varepsilon,s}$, respectively. The invariant set $S_{\varepsilon,s}$ is contained in an attracting isolating block on the slow manifold in $N_\varepsilon^{\rho,m_\rho}$. All $N_\varepsilon^{j,i}$ are assumed that $u(N_\varepsilon^{j,i}) \equiv u = 1$.*

- $\mathcal{F}_\varepsilon^j$, $j = 0, \ldots, \rho - 1$: *a fast-exit face of N_ε^{j,m_j}.*
  If $j = 0$, the invariant set $S_{\varepsilon,u}$ admits an isolating block $B(S_{\varepsilon,u})$ on the slow manifold in $N_\varepsilon^{0,0}$ such that all assumptions in Lemma 4.38 as well as fiber-cone condition (Appendix B) in $N_\varepsilon^{0,0}$ hold.

Then all statements in Theorem 5.6 hold with $\mathcal{S}_\varepsilon^j = \bigcup\limits_{i=0}^{m_j} N_\varepsilon^{j,i}$.

PROOF. We only consider the case $\varepsilon \in (0, \varepsilon_0]$ and $B(S_{\varepsilon,u})$ is attracting. By Propositions 4.37, 4.12, 4.14, 4.15 and our assumptions, there is a sequence of h-sets $\{M_\varepsilon^{j,i}\}_{i=0,\ldots,m_j}^{j=0,\ldots,\rho}$ with $M_\varepsilon^{j,i} \subset N_\varepsilon^{j,i}$ which admits a sequence of covering relations

$$\left\{ B(S_{\varepsilon,u}) \overset{\varphi_{\varepsilon(t,\,\cdot\,)}}{\Longleftarrow} B(S_{\varepsilon,u}) \right\}_k \overset{r_1 \circ b_{\mathrm{u},\varepsilon}^*}{\Longrightarrow} \mathcal{F}_\varepsilon^0 \overset{\varphi_\varepsilon(T_0,\,\cdot\,)}{\Longrightarrow} M_\varepsilon^{1,0} \overset{P_\varepsilon^{1,0}}{\Longrightarrow} M_\varepsilon^{1,1} \overset{P_\varepsilon^{1,1}}{\Longrightarrow} \cdots$$

$$\cdots \overset{P_\varepsilon^{1,m_1-1}}{\Longrightarrow} M_\varepsilon^{1,m_1} \overset{P_\varepsilon^{1,m_1}}{\Longrightarrow} \mathcal{F}_\varepsilon^1 \overset{\varphi_\varepsilon(T_1,\,\cdot\,)}{\Longrightarrow} M_\varepsilon^{2,0} \overset{P_\varepsilon^{2,0}}{\Longrightarrow} \cdots \overset{P_\varepsilon^{\rho,m_\rho-1}}{\Longrightarrow} M_\varepsilon^{\rho,m_\rho}$$

for arbitrary $k \in \mathbb{N}$ and sufficiently small $t > 0$. Here r_λ is the deformation retract given by (4.24) or (4.25), $b_{\mathrm{u},\varepsilon}^z$ is the horizontal disk at $z \in B(S_{\varepsilon,u})$ whose graph is the unstable manifold $W^{\mathrm{u}}(z)$. Also, $P_\varepsilon^{j,i} \colon (N_\varepsilon^{j,i})_{\leq \overline{y}_{j,i} + \overline{h}_{j,i}} \to \partial(N_\varepsilon^{j,i})_{\leq \overline{y}_{j,i} + \overline{h}_{j,i}}$ is the Poincaré map in $(N_\varepsilon^{j,i})_{\leq \overline{y}_{j,i} + \overline{h}_{j,i}}$ with appropriate choices of $\overline{y} = \overline{y}_{j,i}$ and $\overline{h} = \overline{h}_{j,i}$ following Propositions 4.12, 4.14 and 4.15. The stable manifold $W^{\mathrm{s}}(S_{\varepsilon,s})$ is given by a vertical disk $b_\varepsilon^{\mathrm{s}}$ in $N_\varepsilon^{\rho,m_\rho}$ with (b, y)-coordinates as the $s(N_\varepsilon^{\rho,m_\rho})$-dimensional direction of the h-set $N_\varepsilon^{\rho,m_\rho}$. Our statement is just a consequence of Propositions 4.37 and 2.14. All arguments with respect to horizontal and vertical disks are valid due to Corollary 3.7 and Proposition 4.37. $\qquad\square$

Corollary 5.7 can be further generalized as stated below, at least in the case $u = s = 1$, replacing covering-exchange sequences by covering-exchange sequences with extended cones.

COROLLARY 5.8 (Validation of heteroclinic orbits with slow shadowing and m-cones). *Consider (1.1). Assume that there exist $\rho \in \mathbb{N}$ and $\varepsilon_0 > 0$ such that (1.1) admits the following ε ($\in [0, \varepsilon_0]$)-parameter family of sets in $\mathbb{R}^3$:*

- $\{N_\varepsilon^{j,i}\}_{i=0,\ldots,m_j}^{j=0,\ldots,\rho}$, *with $m_0 = 0$. $j = 1, \ldots, \rho-1$: a sequence of fast-saddle-type blocks which admits a sequence of fast-exit faces $\{N_\varepsilon^{j,\mathrm{exit}}\}^{j=0,\ldots,\rho}$ with $N_\varepsilon^{j,\mathrm{exit}} \subset N_\varepsilon^{j,m_j}$. Moreover, the collection*

$$(\mathcal{F}_\varepsilon^{j-1}, \{N_\varepsilon^{j,i}\}_{i=0,\ldots,m_j}, N_\varepsilon^{j,\mathrm{exit}}, C_{m_j^u}^{\mathrm{u}}, C_{m_j^s}^{\mathrm{s}})$$

forms a covering-exchange sequence with extended cones $C^{\mathrm{u}}_{m^u_j}, C^{\mathrm{s}}_{m^s_j}$. *Associating sets* $C^{\mathrm{u}}_{m^u_j}, C^{\mathrm{s}}_{m^s_j}$ *and* $\mathcal{F}^j_\varepsilon$ *are defined below.*

$j = \rho$: *a sequence of h-sets which forms a covering-exchange sequence with cones with* $\mathcal{F}^{\rho-1}_\varepsilon$ *and* $C^{\mathrm{s}}_{m^s_\rho}$ *defined below, except the last assumption in Definition* 4.16.

Blocks $N^{0,0}_\varepsilon$ *and* $N^{\rho,m_\rho}_\varepsilon$ *contain nonempty maximal invariant sets* $S_{\varepsilon,u}$ *and* $S_{\varepsilon,s}$, *respectively. The invariant set* $S_{\varepsilon,s}$ *is contained in an attracting isolating block on the slow manifold in* $N^{\rho,m_\rho}_\varepsilon$. *All* $N^{j,i}_\varepsilon$ *are assumed that* $u(N^{j,i}_\varepsilon) \equiv u = 1$.

- $C^{\mathrm{u}}_{m^u_j}$ $(j = 1, \dots, \rho - 1)$: *the unstable m^u_j-cones of* N^{j,m_j}_ε.
- $C^{\mathrm{s}}_{m^s_j}$ $(j = 1, \dots, \rho)$: *the stable m^s_j-cones of* $N^{j,0}_\varepsilon$.
- $\mathcal{F}^j_\varepsilon$ $(j = 0, \dots, \rho - 1)$: *a fast-exit face* $(C^{\mathrm{u}}_{m^u_j})^{\mathrm{exit}}$ *of* $C^{\mathrm{u}}_{m^u_j}$.
  If $j = 0$, the invariant set $S_{\varepsilon,u}$ *admits an isolating block* $B(S_{\varepsilon,u})$ *on the slow manifold in* $N^{0,0}_\varepsilon$ *such that all assumptions in Lemma* 4.38 *hold with the fast-exit face* $N^{0,\mathrm{exit}}_\varepsilon$ *of* $N^{0,0}_\varepsilon$.

Then all statements in Theorem 5.6 *hold with*

$$
\mathcal{S}^j_\varepsilon = \begin{cases} \displaystyle\bigcup_{i=0}^{m_j} N^{j,i}_\varepsilon \cup C^{\mathrm{u}}_{m^u_j} = N^{0,0}_\varepsilon \cup C^{\mathrm{u}}_{m^u_0} & \text{if } j = 0, \\[2em] C^{\mathrm{s}}_{m^s_j} \cup \displaystyle\bigcup_{i=0}^{m_j} N^{j,i}_\varepsilon & \text{if } j = \rho, \\[2em] C^{\mathrm{s}}_{m^s_j} \cup \displaystyle\bigcup_{i=0}^{m_j} N^{j,i}_\varepsilon \cup C^{\mathrm{u}}_{m^u_j} & \text{otherwise.} \end{cases}
$$

PROOF. Replace $\mathcal{F}^j_\varepsilon$ and $M^{j,0}_\varepsilon$ in the proof of Theorem 5.7 by $N^{j,\mathrm{exit}}_\varepsilon \overset{P^{m^u_j}_\varepsilon}{\Longrightarrow} \mathcal{F}^j_\varepsilon$ and $\widetilde{M}^{j,0}_\varepsilon$, respectively. Here $P^{m^u_j}_\varepsilon : C^{\mathrm{u}}_{m^u_j} \to \partial C^{\mathrm{u}}_{m^u_j}$ is the Poincaré map and $\widetilde{M}^{j,0}_\varepsilon \subset (N^{j,0}_\varepsilon \cup C^{\mathrm{s}}_{m^s_j})$ is the h-set corresponding to $\widetilde{M}_1$ in Proposition 4.30. $\qquad\square$

REMARK 5.9. Changing the choice of covering-exchange sequences and fast-saddle-type blocks containing nontrivial invariant sets, we can obtain various types of singularly perturbed global orbits near singular orbits. For example, in Theorem 5.6, further assuming $\mathcal{S}^0_\varepsilon = \mathcal{S}^\rho_\varepsilon$ and $S_{\varepsilon,u} = S_{\varepsilon,s} = \{p_\varepsilon\}$, an equilibrium, there exists an ε-family of homoclinic orbits $\{H_\varepsilon\}_{\varepsilon \in (0,\varepsilon_0]}$ of p_ε.

We can replace the fourth condition in the covering-exchange property,

$$
\mathcal{F}^j_\varepsilon \overset{\varphi_\varepsilon(T^j,\,\cdot\,)}{\Longrightarrow} \widetilde{\mathcal{S}}^{j+1}_\varepsilon,
$$

by a sequence of covering relations

$$
\mathcal{F}^j_\varepsilon \overset{\varphi_\varepsilon(T^j,\,\cdot\,)}{\Longrightarrow} M^j_1 \overset{\varphi_\varepsilon(T^j_1,\,\cdot\,)}{\Longrightarrow} M^j_2 \overset{\varphi_\varepsilon(T^j_2,\,\cdot\,)}{\Longrightarrow} \cdots \overset{\varphi_\varepsilon(T^j_{k-1},\,\cdot\,)}{\Longrightarrow} M^j_k \overset{\varphi_\varepsilon(T^j_k,\,\cdot\,)}{\Longrightarrow} \widetilde{\mathcal{S}}^{j+1}_\varepsilon
$$

for h-sets $\{M_i^j\}_{i=1}^k$ and positive numbers $\{T_i^j\}_{i=1}^k$ to prove the same statements as Theorems 5.1 and 5.6. These are applications to Proposition 2.11 or 2.14 and such extensions are useful for validating trajectories with complex behavior.

6. Sample validation results for the FitzHugh–Nagumo system

In this section we provide several examples of singularly perturbed orbits with computer assistance. Our sample system is the FitzHugh–Nagumo system given by

$$(6.1) \qquad \begin{cases} u' = v, \\ v' = \delta^{-1}(cv - f(u) + w), \\ w' = \varepsilon c^{-1}(u - \gamma w), \end{cases}$$

where $a \in (0, 1/2)$, c, γ and δ are positive parameters and $f(u) = u(u-a)(1-u)$. System (6.1) is well known as the system of traveling wave solutions $(U, W) = (\psi_U(x - ct), \psi_W(x - ct))$ of the following PDE:

$$(6.2) \qquad \begin{cases} U_t = U_{xx} + f(U) - W, \\ W_t = \varepsilon(U - \gamma W). \end{cases}$$

Throughout this section, the map π_α, $\alpha = u, v, w$, denotes the projection onto the α-component.

6.1. Strategy and parameters. Following arguments of the previous sections, we validate global orbits. The following implementation is basically common in our computations. In particular, we concentrate on the construction of covering-exchange sequences associated with slow shadowing sequences.

STEP 1. Fix $\varepsilon_0 > 0$ and several parameters.

STEP 2. For constructing the j-th slow shadowing sequence $\{N_\varepsilon^{j,i}\}_{i=0}^{m_j}$ ($j = 0, \ldots, \rho > 0$) with $\pi_w(N_\varepsilon^{j,i}) = [y_{j,i}^-, y_{j,i}^+]$, we set identical positive numbers a_0, b_0 and $\overline{h}$ in (SS4) in advance. Before constructing each $N_\varepsilon^{j,i}$, we compute approximate equilibria $\{(u_{j,i}, 0, w_{j,i})\}_{i=0}^{m_j}$ for $(6.1)_0$ (i.e. $\varepsilon = 0$) along a (normally hyperbolic) branch of the nullcline. For simplicity, compute them with $|w_{j,i+1} - w_{j,i}| \equiv \overline{h}$ for all i so that $\{w_{j,i}\}_{i=0}^{m_j}$ is monotonously increasing (resp. decreasing) in the case $q = +1$ (resp. $q = -1$).

Also, let $|y_{j,i}^+ - w_{j,i}| = |w_{j,i} - y_{j,i}^-| \equiv H/2$ for some $H > 0$ for further simplicity. Set $\pi_w(N_\varepsilon^{j,\text{exit}}) = [y_{j,m_j}^- + \overline{h}, y_{j,m_j}^+ - \delta]$ in the case of $q = +1$. Similarly, set $\pi_w(N_\varepsilon^{\text{exit}}) = [y_{j,m_j}^- + \delta, y_{j,m_j}^+ - \overline{h}]$ in the case of $q = -1$. Here $\delta > 0$ denotes an arbitrarily small number. For validation of (SS5), we apply Proposition 4.32. Under these settings, verify the slow shadowing condition (4.5). We revisit this verification later in Lemma 6.1.

In the case of validations of heteroclinic orbits, we additionally need to validate $W^{\mathrm{u}}(I_\varepsilon)$ for an isolating block I_ε on slow manifolds. To this end, verify all assumptions in Lemma 4.38.

STEP 3. If necessary, verify the unstable m-cone condition for an appropriate $m > 1$ in a block of the form (4.11) (Section 4.4). Also, construct the fast-exit face $(C^{\mathrm{u}}_m)^{\mathrm{exit}}$ of unstable m-cones following (4.15). Similarly, verify the stable m-cone condition for an appropriate $m > 1$ in a block of the form (4.12), if necessary. When we apply the predictor-corrector method for constructing fast-saddle-type block (Section 2.3.3), vertices of m-cones are slid in general. In this case, we cut the cone so that the fast-exit face is parallel to the y-axis. As a consequence, the length ℓ_{u} or ℓ_{s} of extended cones shortens at most $|f_x(\overline{x}, \overline{y})^{-1} f_y(\overline{x}, \overline{y}) H|$, where H is the height of the block in the y-direction. See Figure 9.
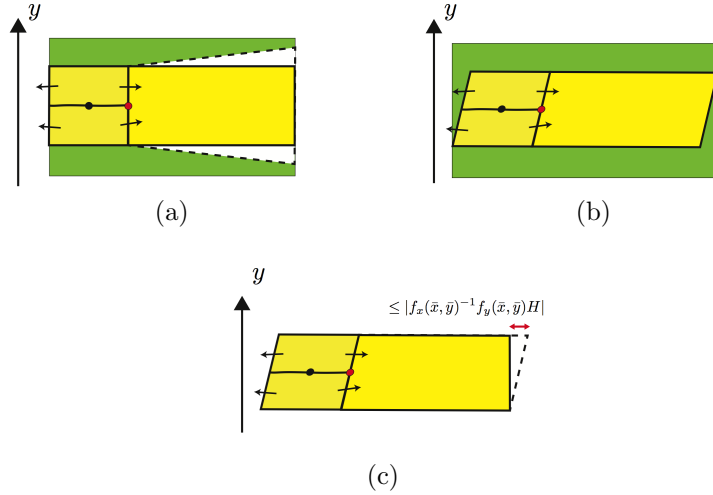

FIGURE 9. Cutting m-cones. (a) A fast-saddle-type block N with the basic form (Section 2.3.2) and associated unstable m-cone. The union of yellow and white regions represents $N \cup C^{\mathrm{u}}_m$. The yellow region represents $(N \cup C^{\mathrm{u}}_m) \cap \{y \in \pi_w(N)\}$. The x-components of block N are identical for y. (b) A fast-saddle-type block with the predictor-corrector form (Section 2.3.3) and associated unstable m-cone. The x-components of the block N as well as the cone C^{u}_m are slid, since the x-component of the center point also depends on y; $(x, y) = (\overline{x} + (dx/dy)(\overline{y}) \cdot (y - \overline{y}), y)$. The fast-exit face thus has a nontrivial angle to the y-axis. (c) Cut the edge of unstable m-cones. The resulting fast-exit face is parallel to the y-axis. The length ℓ_{u} of the extended cone shortens at most $|f_x(\overline{x}, \overline{y})^{-1} f_y(\overline{x}, \overline{y}) H|$, where H is the height of the block in the y-direction.

STEP 4. Solve initial value problems of ODEs for setting a fast-exit face of each block as an initial data and verify (CE4) in Definition 4.2 or *Drop* $N^{j-1,\mathrm{exit}}_\varepsilon \overset{\varphi_\varepsilon(T^j_k, \cdot)}{\Longrightarrow} (N^{j,0}_\varepsilon)_{\leq y^+_{j,0} - \overline{h}}$. This operation consists of direct applications of

interval arithmetics and ODE solver libraries such as CAPD [3]. Although our computations here are operated in full systems, slow dynamics can be regarded as the small error since our interest in this step is mainly the behavior of fast dynamics.

Validation of assumptions in Lemma 4.38 in Step 2 can be done as follows.

LEMMA 6.1 (Validation of assumptions in Lemma 4.38). *Let N be a fast-saddle-type block with $\pi_w(N) = [y_N^-, y_N^+]$, and I be an isolating block containing an equilibrium in the validated slow manifold S_ε with $0 < \mathrm{dist}(S_\varepsilon, N^{f,-}) < \mathrm{diam}(\pi_a(N)) - r_a$, where $r_a > 0$, and $\mathrm{dist}(S_\varepsilon, N^{f,+}) \geq r_b > 0$. Assume that N satisfies the stable m-cone condition and the unstable fiber-L-cone condition with*

$$(6.3) \qquad\qquad \mathrm{sl_u}\{\mathrm{diam}(\pi_a(N)) - r_a\} < r_b,$$

where $\mathrm{sl_u} = m(1 + L^{-2})^{1/2}$ is the slope of instable fibers stated in Appendix B. Also assume that I can be chosen in

$$(6.4) \quad N \cap \{y_N^- + \mathrm{sl_u}\{\mathrm{diam}(\pi_a(N)) - r_a\} \leq y \leq \{y_N^+ - \mathrm{sl_u}\{\mathrm{diam}(\pi_a(N)) - r_a\}\}$$

and that any subset $\widetilde{I} \subset S_\varepsilon$ containing I is also an isolating block with the same isolating information as I. Then we can choose an isolating block $\overline{I}_\varepsilon \subset S_\varepsilon$ and the fast-exit face N^{exit} with $\pi_w(N^{\mathrm{exit}}) = [y^-, y^+]$ and

$$(6.5) \quad y_N^- + 2\,\mathrm{sl_u}\{\mathrm{diam}(\pi_a(N)) - r_a\} \leq y^- < y^+ \leq y_N^+ - 2\,\mathrm{sl_u}\{\mathrm{diam}(\pi_a(N)) - r_a\}$$

so that assumptions in Lemma 4.38 hold.

PROOF. For each $p \in I$, any point $q \in W^u(p)$ satisfies $|\pi_b(q) - \pi_b(p)| < \mathrm{sl_u}\{\mathrm{diam}(\pi_a(N)) - r_a\}$, which is a consequence of properties of unstable cones. The same property holds for y-components. Therefore assumptions concerning with (6.3) and (6.4) imply (4.27). Similarly, it immediately holds that we can choose $\overline{I}_\varepsilon$ and N^{exit} with (6.5) so that (4.28)–(4.29) are satisfied. $\square$

In practical computations, the most difficult part is Step 4. In general, the larger both the fast-exit face $N_\varepsilon^{j,\mathrm{exit}}$ and the target h-set in *Drop*, i.e. $(N_\varepsilon^{j,0})_{\leq y_{j,0}^+ - \overline{h}}$, are, the easier validations of covering relations are. However, if we validate covering relations described in Step 4, the resulting h-set $M_\varepsilon^{j,i}$ is very close to the slow exit, i.e. $(N_\varepsilon^{j,i})_{y_{j,i}^+}$ (resp. $(N_\varepsilon^{j,i})_{y_{j,i}^-}$) in the case $q = +1$ (resp. $q = -1$). In particular, the next fast-exit face $N_\varepsilon^{j,\mathrm{exit}}$ becomes too thin to validate the next *Drop*. Procedures in Step 2 as well as Step 4 thus look incompatible with each other. Nevertheless, an appropriate choice of the slow shadowing ratio χ avoids this inconsistency.

LEMMA 6.2 (Validity of Step 2). *For each $j = 1, \dots, \rho$, consider the j-th slow shadowing sequence $\{N_\varepsilon^{j,i}\}_{i=0}^{m_j}$ in Step 2. Assume that a covering relation*

$N_\varepsilon^{j-1,\mathrm{exit}} \overset{\varphi_\varepsilon(T_j, \cdot)}{\Longrightarrow} (N_\varepsilon^{j,0})_{\leq y_{j,0}^+ - \overline{h}}$ *holds. Let* χ_j *be the slow shadowing ratio satisfying*

$$(6.6) \qquad \chi_j \leq 1 - \frac{H}{\overline{h}}\left(\frac{|w_{j,m_j} - w_{j,0}| - H}{\overline{h}}\right)^{-1} = 1 - \frac{H}{\overline{h}}\left(\frac{m_j\overline{h} - H}{\overline{h}}\right)^{-1}$$

with $m_j\overline{h} > H$. *Finally assume that* $\{N_\varepsilon^{j,i}\}_{i=0}^{m_j}$ *is the slow shadowing sequence with the identical ratio* χ_j. *Then we can choose a fast-exit face* $N_\varepsilon^{j,\mathrm{exit}}$ *so that* $\pi_w(N_\varepsilon^{j,\mathrm{exit}}) = [y_{j,m_j}^- + \overline{h}, y_{j,m_j}^+ - \delta]$ *(resp.* $\pi_w(N_\varepsilon^{\mathrm{exit}}) = [y_{j,m_j}^- + \delta, y_{j,m_j}^+ - \overline{h}])$ *in the case of* $q = +1$ *(resp.* $q = -1$*), where* $\delta > 0$ *is an arbitrarily small number. As a consequence, Steps 2 and 4 are valid simultaneously.*

Proof. We only prove the case $q = +1$. The case $q = -1$ is similar. By Proposition 4.14 with χ_j, we can construct a covering relation $M_\varepsilon^{j,0} \overset{P_\varepsilon^{j,0}}{\Longrightarrow} M_\varepsilon^{j,1}$, where

$$M_\varepsilon^{j,0} \subset (N_\varepsilon^{j,0})_{y_{j,0}^+ - \overline{h}} \quad \text{and} \quad M_\varepsilon^{j,1} \subset (N_\varepsilon^{j,1})_{y_{j,0}^+ - (1-\chi_j)\overline{h}} = (N_\varepsilon^{j,1})_{y_{j,1}^+ - (2-\chi_j)\overline{h}}.$$

The last equality follows from the choice of $y_{j,i}^\pm$ and $w_{j,i}$ in Step 2. Repeating this argument, two h-sets describing the i-th covering relation $M_\varepsilon^{j,i-1} \overset{P_\varepsilon^{j,i-1}}{\Longrightarrow} M_\varepsilon^{j,i}$ are located on $(N_\varepsilon^{j,i-1})_{I_{i-1}}$ and $(N_\varepsilon^{j,i})_{I_i}$, respectively, where

$$I_i = y_{j,i}^+ - \{(i+1) - i\chi_j\}\overline{h}.$$

We thus obtain $I_{m_j} = y_{j,m_j}^+ - \{(m_j+1) - m_j\chi_j\}\overline{h}$. If the ratio χ_j can be chosen satisfying (6.6), we obtain

$$\{(m_j+1) - m_j\chi_j\}\overline{h} \geq \left\{(m_j+1) - m_j\left\{1 - \frac{H}{\overline{h}}\left(\frac{|w_{j,m_j} - w_{j,0}| - H}{\overline{h}}\right)^{-1}\right\}\right\}\overline{h}$$

$$= \left\{1 + \frac{m_jH}{\overline{h}}\left(\frac{m_j\overline{h} - H}{\overline{h}}\right)^{-1}\right\}\overline{h}.$$

Consequently,

$$y_{j,m_j}^+ - \{(m_j+1) - m_j\chi_j\}\overline{h} \leq y_{j,m_j}^+ - \overline{h} - \frac{m_jH}{\overline{h}}\left(\frac{m_j\overline{h} - H}{\overline{h}}\right)^{-1}\overline{h}$$

$$= y_{j,m_j}^+ - \overline{h} - \left(\frac{m_j\overline{h}H}{m_j\overline{h} - H}\right) \leq y_{j,m_j}^+ - \overline{h} - H = y_{j,m_j}^- - \overline{h}.$$

Obviously, the slow shadowing pair with the ratio χ_j satisfies the slow shadowing condition with the ratio χ' for all $\chi' \in [\chi_j, 1]$. Therefore, arranging several $\chi_{j,i}$'s in the slow shadowing pair $\{N_\varepsilon^{j,i-1}, N_\varepsilon^{j,i}\}$, we can take the h-set M_ε^{j,m_j} on $(N_\varepsilon^{j,m_j})_{y_{j,i}^-}$. Statements of the lemma follow from the same arguments as in the proof of Proposition 4.15. Schematic pictures of the proof are shown in Figure 10. $\square$
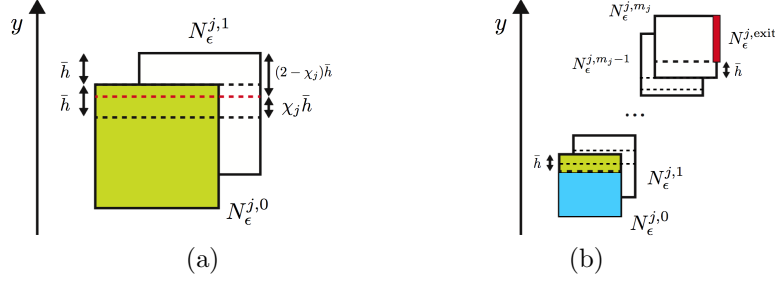
FIGURE 10. Schematic pictures of Lemma 6.2 ($q = +1$). Dotted lines denote sections $(N_\varepsilon^{j,i})_y$ for each i. (a) If the sequence of fast-saddle-type blocks $\{N_\varepsilon^{j,i}\}_{i=0}^{m_j}$ is a slow shadowing sequence with the ratio χ_j, the section $(N_\varepsilon^{j,i+1})_y$ where the covering relation $M_\varepsilon^{j,i} \overset{P_\varepsilon^{j,i}}{\Longrightarrow} M_\varepsilon^{j,i+1} \subset (N_\varepsilon^{j,i+1})_y$ gets lower than the usual version ($\chi_j = 1$). (b) Repeating the procedure in (a) sufficiently many times, we can take the section $(N_\varepsilon^{j,m_j})_y$ before $Jump$ at the bottom of N_ε^{j,m_j}; namely, $(N_\varepsilon^{j,m_j})_{y_{j,m_j}^-}$. As a consequence, we can take the fast-exit face $N_\varepsilon^{j,\text{exit}}$ large keeping the target h-set $(N_\varepsilon^{j,0})_{\leq y}$ in $Drop$ large.

Thanks to Lemma 6.2, we replace Step 2 by the following, which enables us to verify assumptions of results in Section 5 with large fast-exit faces and large target h-sets in $Drop$:

STEP 2'. Replace the statement "verify the slow shadowing condition" in Step 2 by "verify the slow shadowing condition with the slow shadowing ratio χ_j", where $\chi_j < 1$ is a given number satisfying (6.6).

If Steps 1, 2', 3 and 4 pass, then all assumptions of either Theorem 5.1, Corollaries 5.3 or 5.4 are satisfied in the case of periodic orbits for all $\varepsilon \in (0, \varepsilon_0]$. Similarly, all assumptions of either Theorem 5.6, Corollaries 5.7 or 5.8 are satisfied in the case of heteroclinic or homoclinic orbits.

In Step 2, the predictor-corrector approach discussed in Section 2.3.3 is used for choosing local coordinates around (normally hyperbolic) invariant manifolds. A concrete form for validations is shown in Appendix A.

Note that all our examples below are cases with $u = 1$. In such cases, one can directly verify the covering relation $\mathcal{F}_\varepsilon^{j-1} \overset{\varphi_\varepsilon(T_{j,} \cdot)}{\Longrightarrow} M_\varepsilon^{j,0}$ by checking all assumptions in Proposition 2.9. In Step 4, we apply this proposition to validating covering relations.

Below we list parameters we deal with in computations in our settings except the ones which arise in (6.1), before moving to practical computational examples. These parameters are set for each branch of slow manifolds which we try to find. Let $j \in \{0, 1, \ldots, \rho\}$ be the number of branches.

- $\overline{h}$: The height in the w-direction for slow shadowing condition (4.5). A sequence of equilibria $\{(u_i, v_i, w_i)\}$ for (1.3) is set so that $|w_{i+1} - w_i| \equiv \overline{h}$ for all i.

- H: The height of fast-saddle-type blocks in the w-direction. Each w-interval of length H corresponds to the set K in Section 2.3.2.

- $\overline{w}_0$: A given number such that $|\overline{w}_0 - w_0| < \overline{h}$, where w_0 is the w-component of an equilibrium (u_0, v_0, w_0). The equilibrium (u_0, v_0, w_0) is computed numerically and becomes the center of the fast-saddle-type block $N_\varepsilon^{j,0}$; the target h-set of *Drop*.

- $\overline{w}_{m_j}$: A given number such that $|\overline{w}_{m_j} - w_{m_j}| < \overline{h}$, where w_{m_j} is the w-component of an equilibrium $(u_{m_j}, v_{m_j}, w_{m_j})$. The equilibrium $(u_{m_j}, v_{m_j}, w_{m_j})$ is computed numerically and becomes the center of the fast-saddle-type block N_ε^{j,m_j}; the last h-set of the j-th slow shadowing sequence containing a fast-exit face.

- χ: The slow shadowing ratio given by

$$\chi = 1 - \frac{H}{\overline{h}} \left(\frac{|\overline{w}_{m_j} - \overline{w}_0| - H - 2\overline{h}}{\overline{h}} \right)^{-1}.$$

One can easily check that χ satisfies (6.6).

- r_a, r_b: The lengths of spaces in fast-saddle-type blocks introduced in (4.3). For simplicity, these numbers are identical for all blocks. Moreover, they are assumed to be identical to each other.

- d_a, d_b: Positive numbers less than 1 introduced in (SS5).

- m_u, m_s: Positive numbers determining the sharpness of unstable and stable m-cones, respectively.

- ℓ_u, ℓ_s: Positive numbers determining the length ℓ of unstable and stable m-cones in (4.11) and (4.12), respectively. By using these numbers, we compute bounds of the departure time T_{dep} in (4.13) to construct the fast-exit face $(C_m^{\text{u}})^{\text{exit}}$ of the unstable m-cone in (4.15), and the arrival time T_{arr} in (4.14). The arrival time T_{arr} is used to validate *Drop* to the target h-set $(N_\varepsilon^{j,0} \cup C_m^{\text{s}})_{y_0^+ - \overline{h}}$ in the w-direction corresponding to (4.16), as stated in Proposition 4.30.

  If we apply the predictor-corrector form (Section 2.3.3) to constructing fast-saddle-type blocks, the practical length ℓ_u is set as $\ell_u - |f_u(\overline{u})^{-1}|H$, following Step 3 and Figure 9. The factor $f_u(\overline{u})^{-1}$ is the differential of u by w at the center point $(\overline{u}, \overline{v}, \overline{w})$ via the implicit function differential for (6.1). Details are shown in Appendix A.

All computations are done by MacBook Air 2011 model (1.6 GHz, Intel Core i5 Processor, 4GB Memory), GCC version 4.2.1 with `-O2` option and CAPD library [3] version 3.0.

6.2. Demonstration 1: slow shadowing sequences with the ratio χ.
Validations of not only slow manifolds near critical manifolds consisting of equilibria for (1.3) but also the existence of trajectories which shadow slow manifolds are our starting points of the whole considerations. Slow manifolds for (6.1) are now expected to be near the nullcline $\{v = 0, f(u) = w\}$. The aim of this subsection is to test how large slow manifolds can be validated in terms of slow shadowing sequence. Following Step 2 at the beginning of this section, we validate slow shadowing sequences. Note that validations in this section also verify *Jump* in Proposition 4.15.

As a demonstration, we fix $a = 0.3$, $\gamma = 10.0$, $\delta = 9.0$ and $c \in [0.799, 0.801]$. These parameters are also used in Sections 6.7 and 6.8.

Computer Assisted Result 6.3. Consider (6.1) with $a = 0.3$, $\gamma = 10.0$ and $\delta = 9.0$. Then for all $c \in [0.799, 0.801]$ and $\varepsilon \in [0, 5.0 \times 10^{-5}]$, the branch of slow manifolds near the nullcline $\{v = 0, f(u) = w\}$ in $\{-1.765629966434 \times 10^{-1} \le u \le 2.017612584956 \times 10^{-3}, -6.0 \times 10^{-4} \le w \le 0.099\}$ is validated. In particular, the slow shadowing condition with $q = -1$ between blocks around this slow manifold is validated with parameters listed in "First branch" in Table 1.

Similarly, the branch of slow manifolds near the nullcline $\{v = 0, f(u) = w\}$ in $\{0.8504842978868 \le u \le 1.021440903396, -1.58 \times 10^{-2} \le w \le 0.07\}$ is validated. In particular, the slow shadowing condition with $q = +1$ between blocks around this slow manifold is validated with parameters listed in "Third branch" in Table 1.

Parameters	First branch	Third branch
χ	0.8807339449541285	0.8786764705882354
$\overline{h}$	0.003	0.003
H	0.0065	0.0066
d_a	0.75	0.75
d_b	0.7	0.75
r_a	0.008	0.008
r_b	0.0085	0.0078
m_u	100	100
m_s	100	100
computation time	0.566 sec.	0.467 sec.

Table 1. Validation parameters of slow shadowing in Computer Assisted Result 6.3.

This validation result implies that we have already validated trajectories with appropriately chosen initial data, say h-sets, which shadow slow manifolds with

an arbitrary length *for all* $\varepsilon \in (0, 5.0 \times 10^{-5}]$, as long as slow shadowing is validated. Moreover, *Jump* has been also validated for any fast-exit face with an appropriate height from the bottom for $q = +1$, and an appropriate width from the top for $q = -1$. Remark that the range of our validating slow manifolds is not the limit of our verifications. Validations of slow shadowing sequences are just iterations of Step 2 and can be validated very fast, if we have fast solver of linear algebra. Notice that slow shadowing sequences for $\varepsilon \in (0, \varepsilon_0]$ validate trajectories which shadow slow manifolds *without solving any differential equations for all* $\varepsilon \in (0, \varepsilon_0]$.

On the other hand, there is a trade-off for validating slow shadowing sequences. For example, if we increase the value of ε, say 6.0×10^{-5}, the slow shadowing condition (4.5) violates, since the slow speed becomes faster than expansion and contraction of h-sets in hyperbolic directions around slow manifolds. Factors determining (4.5) are $\overline{h}, H, d_a, d_b, r_a, r_b$ as well as eigenvalues and size of fast-saddle-type blocks. One expects that, the larger parameters, say $\overline{h}, r_a, r_b$ are, the easier the validation of (4.5) will be. However, in such a case, the covering relation in Proposition 4.32 is often failed. In particular, assumption (SS5) is violated. This is mainly because the distance between two centers (u_i, v_i, w_i) and $(u_{i+1}, v_{i+1}, w_{i+1})$ becomes larger and hence the affine map $T_{x,12}$ moves h-sets larger, if we increase $\overline{h}, r_a, r_b$. This is also the case if we increase parameters d_a and d_b.

It is also natural to think whether we can validate the whole branch of slow manifolds by *one* block in order to avoid such a trade-off. We then verify how large ε and a block N can be chosen so that the slow shadowing condition (4.5) holds. The following result is a sample validated one.

COMPUTER ASSISTED RESULT 6.4. Consider (6.1) with $a = 0.3$, $\gamma = 10.0$ and $\delta = 9.0$. Then for all $c \in [0.799, 0.801]$ and $\varepsilon \in [0, 0.002]$, the branch of slow manifolds near the nullcline $\{v = 0, f(u) = w\}$ with $\{0.005 \leq w \leq 0.055\}$ around $u = -0.07454149842$ is validated by a fast-saddle-type block N satisfying cone conditions with

$$\overline{h} = H = 0.05, \quad r_a = r_b = 0.0085.$$

Moreover, $c^{-1}(u - \gamma w) < 0$ and the slow shadowing condition (4.5) are satisfied in N.

This result indicates that the normal hyperbolicity around the first branch of nullcline is sufficiently strong comparing with slow dynamics with $\varepsilon = 0.002$. Computer assisted results in Sections 6.7 and 6.8 imply, however, that one fast-saddle-type block is not sufficient to validate the whole branch of slow manifolds near homoclinic orbits, while the validation of fast-saddle-type blocks is violated

with the present methodology if we set $\overline{h} = H$ larger than 0.05. These observations indicate that the slow shadowing is necessary to validate slow manifolds with a wide range of w at the cost of the narrower range of ε.

6.3. Demonstration 2: m-cones. Next, we present a demonstration for m-cones. When we want to construct a covering-exchange sequence, we need to verify the covering relation $\mathcal{F}_\varepsilon^1 \overset{\varphi_\varepsilon(T_0, \cdot)}{\Longrightarrow} \mathcal{S}_\varepsilon$, where $\mathcal{F}_\varepsilon^1$ is a fast-exit face of a fast-saddle-type block and $\mathcal{S}_\varepsilon$ is another block. In order to verify this covering relation, we solve a differential equation with the initial data $\mathcal{F}_\varepsilon^1$. On the other hand, we can replace $\mathcal{F}_\varepsilon^1$ by a fast-exit face $\mathcal{F}_\varepsilon^2$ of an extended cone, thanks to discussions in Section 4.4. Here we solve (6.1) with two initial data, $\mathcal{F}_\varepsilon^1$ and $\mathcal{F}_\varepsilon^2$, to see the following two points:

(1) accuracy of solution orbits, and

(2) verification of covering relations.

As an example, we set $a = 0.01$, $\gamma = 0.0$, $\delta = 5.0$, $c \in [0.495, 0.505]$ and $\varepsilon \in [0.0, 5.0 \times 10^{-6}]$. These are parameter values used in Section 6.4. In demonstrating computations, we used the ODE solver in CAPD library based on Lohner's method discussed in [39]. The order of Taylor expansion is set $p = 6$ and time step size is set $\Delta t = 0.0001$. Computational result is shown in Figure 11.

First we compare the case when we use stable m-cones with the case when we do not use stable m-cones. Figures 11 (a) and (b) show the same computational results. The only difference is whether or not a stable m-cone is validated. In this example, we validate the stable 29-cone with the length $\ell_s = 0.108232$ of a fast-saddle-type block around $(0.956575, 0, 0.0392) \in \mathbb{R}^3$ (see Figure 11 (a)). We solved ODE with initial data

$$\mathcal{F}_\varepsilon^2 \cap \{w \in [0.039249948844, 0.03926994850296]\}$$

after dividing it into uniform 30 small pieces. Here $\mathcal{F}_\varepsilon^2$ denotes the fast-exit face of unstable 21-cone with the length $\ell_u = 0.0247787$ of a fast-saddle-type block around the origin in $\mathbb{R}^3$. In this example, we solved ODE in 174345 steps.

In general, validated fast-saddle-type blocks corresponding to $\mathcal{S}_\varepsilon$ are very small, as shown in Figure 11 (b) (colored pink). On the other hand, validated trajectories are quite bigger than blocks. In our example, validated trajectories are already bigger than the block, which implies that we can *never* validate $\mathcal{F}_\varepsilon^1 \overset{\varphi_\varepsilon(T_0, \cdot)}{\Longrightarrow} \mathcal{S}_\varepsilon$ in this setting. A direct settlement of this problem would be a refinement of the initial data, which leads to huge computational costs in many cases and is not realistic. Instead, we consider the problem with the help of stable m-cones, which is shown in Figure 11 (a). In this case, the target block corresponding to $\mathcal{S}_\varepsilon$ becomes big enough to validate covering relations. Thanks to Section 4.4, we can discuss validation of trajectories with extended cones, which is much easier than verifications without cones.
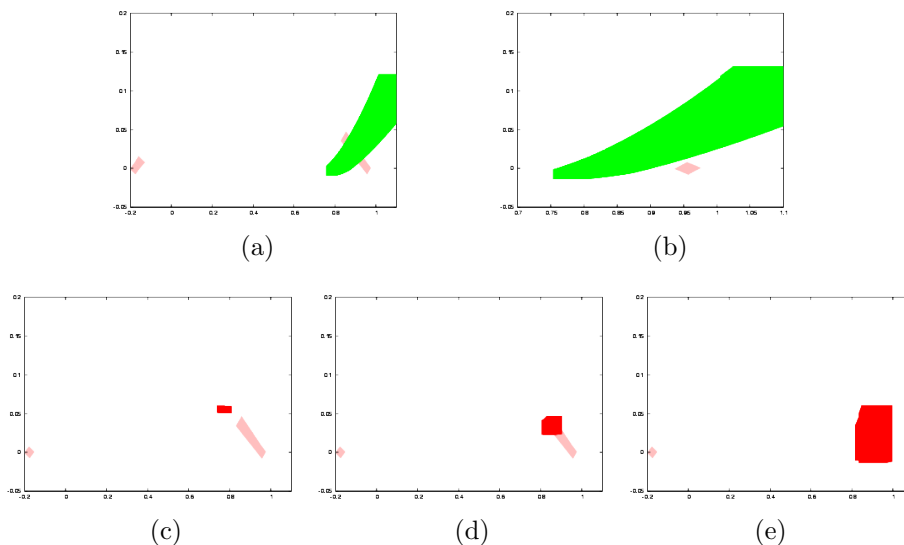
FIGURE 11. Comparison of solution enclosures with and without m-cones. Horizontal axis: u. Vertical axis: v. Each figure represents the projection of trajectories and cones on the (u, v)-plane. (a) Validation of solution orbits (green) with initial data $\mathcal{F}_\varepsilon^2$. Pink regions are the union of fast-saddle-type blocks and extended cones. Computational step of ODEs is 174345. (b) The same computational result as (a). In this case the stable m-cone is not validated. The tiny pink region around $(u, v) = (0.956721, 0)$ is the validated fast-saddle-type block. Readers see that enclosures of trajectories are much bigger than the fast-saddle-type block. (c) Validation of solution orbits (red) with initial data $\mathcal{F}_\varepsilon^1$ (i.e. without unstable m-cones) and the same time steps as (a). Validated trajectories do not arrive at the target region $\mathcal{S}_\varepsilon$ yet. (d) Validated trajectories with additional time step computations to (c). More precisely, computation step of ODEs is 192510. Enclosures become bigger and bigger. (e) Validated trajectories with additional time step computations to (d). More precisely, computation step of ODEs is 211575. Enclosures are already bigger than stable cones, which implies that validation of covering relations cannot be realized.

Next, we compare the case when we use unstable m-cones with the case when we do not use unstable m-cones. Computational result with unstable m-cone is Figure 11 (a). Figures 11 (c)–(e) show enclosure of trajectories with the initial data $\mathcal{F}_\varepsilon^1$, namely, a fast-exit face of a small fast-saddle-type block. If we do not use unstable m-cones, we need to solve ODEs for longer time steps than in the case when we use unstable m-cones (Figure 11 (c)). Such extra computations cause larger enclosure of solutions and there is little hope to validate covering relations, as indicated in Figures 11 (d), (e).

6.4. Periodic orbits. We go to validations of global orbits for (6.1). Our first example is validation of periodic orbits. As a demonstration we set $a = 0.01$,

$\gamma = 0.0$ and $\delta = 5.0$ and $c \approx 0.5$. All validations of covering-exchange sequences with extended cones yield the following computer assisted result.

COMPUTER ASSISTED RESULT 6.5. Consider (6.1) with $a = 0.01$, $\gamma = 0.0$ and $\delta = 5.0$. Then for all $c \in [0.495, 0.505]$ the following trajectories are validated.

(a) At $\varepsilon = 0$, there is a singular heteroclinic chain H_0 consisting of
  - heteroclinic orbits from p_0 to q_0, and from p_1 to q_1,
  - branches M^0, M^1 of nullcline $\{v = 0, f(u) = w\}$. M^0 contains p_0 and q_0. Similarly, M^1 contains p_1 and q_1.

Equilibria p_0, q_0, p_1 and q_1 are validated by

$$|\pi_{u,v}(p_0) - (-0.177098234, 2.18166218 \times 10^{-6})| < 2.49103628 \times 10^{-2},$$
$$|\pi_w(p_0) - 0.0395| < 3.25 \times 10^{-3},$$
$$|\pi_{u,v}(q_1) - (0.956125336, -5.98704406 \times 10^{-6})| < 2.10571434 \times 10^{-2},$$
$$|\pi_w(q_1) - 0.03932| < 3.3 \times 10^{-3},$$
$$|\pi_{u,v}(p_1) - (0.850811351, -9.47162333 \times 10^{-6})| < 2.44899980 \times 10^{-2},$$
$$|\pi_w(p_1) - 0.10602| < 3.3 \times 10^{-3},$$
$$|\pi_{u,v}(q_0) - (-0.282970990, 1.85607735 \times 10^{-6})| < 2.12552591 \times 10^{-2},$$
$$|\pi_w(q_0) - 0.10675| < 3.25 \times 10^{-3}.$$

(b) For all $\varepsilon \in (0, 5.0 \times 10^{-6}]$, there exists a periodic orbit H_ε near H_0. Parameters for validations are listed in Table 2.

We omit computation times for slow shadowing and *Jump*, since they take only a few seconds as stated in Section 6.2.

6.5. Heteroclinic cycles. The second example is a family of heteroclinic cycles. Consider the cubic curve $w = f(u)$. One can see that it is symmetric with respect to the inflection point

$$(u_{\mathrm{inf}}, w_{\mathrm{inf}}) := \left(\frac{1+a}{3}, \frac{(1+a)(1-2a)(2-a)}{27} \right)$$

and we can choose (u_3, w_3) from the curve $w = f(u)$ to be the point that is symmetric to the origin $(u, w) = (0, 0)$ with respect to the inflection point. Deng [9] shows that $\gamma_0 := 9/(2-a)(1-2a)$ and $c_0 := (1-2a)/\sqrt{2}$ with $\delta = 1.0$ admit a heteroclinic loop of $(0,0)$ and (u_3, w_3) for sufficiently small ε. Symmetry of the cubic curve $w = f(u)$ with respect to $(u_{\mathrm{inf}}, w_{\mathrm{inf}})$ implies that the vector field (6.1) with $(\gamma, c) = (\gamma_0, c_0)$ is symmetric under the following transformation:

$$\widetilde{u} = \frac{2(1+a)}{3} - u, \qquad \widetilde{v} = -v, \qquad \widetilde{w} = \frac{2(1+a)(1-2a)(2-a)}{27} - w.$$

Parameters	On M^0 $(q = -1)$	On M^1 $(q = +1)$
χ	0.8922056384742952	0.8895212587880817
$\overline{h}$	0.0025	0.0023
H	0.0065	0.0066
d_a	0.8	0.8
d_b	0.7	0.8
r_a	0.008	0.008
r_b	0.0085	0.008
m_u	21 (around p_0)	55 (around p_1)
m_s	28 (around q_0)	29 (around q_1)
ℓ_u	0.0247787 (around p_0)	0.0186724 (around p_1)
ℓ_s	0.113375 (around q_0)	0.108232 (around q_1)
	$N_\varepsilon^{0,\text{exit}} \overset{\varphi_\varepsilon(T_0,\,\cdot\,)}{\Longrightarrow} N_\varepsilon^{1,0}$	$N_\varepsilon^{1,\text{exit}} \overset{\varphi_\varepsilon(T_1,\,\cdot\,)}{\Longrightarrow} N_\varepsilon^{0,0}$
T_i	0.059	0.0266
	(with $\Delta t = 1.0 \times 10^{-4}$)	(with $\Delta t = 2.0 \times 10^{-5}$)
computation time	70 min. 6 sec.	78 min. 53 sec.

TABLE 2. Validation parameters of slow shadowing in Computer Assisted Result 6.5.

It is thus sufficient to validate a heteroclinic orbit from the origin to $(u_3, 0, v_3)$ if we want to validate heteroclinic cycles. Here we validate heteroclinic cycles for concrete ε and specific a and δ. All validations of covering-exchange sequence with extended cones yield the following computer assisted result.

COMPUTER ASSISTED RESULT 6.6. Consider (6.1) with $a = 0.35$, $\gamma = \gamma_0$, $c = c_0$. Then, for $\delta = 1.0$, the following trajectories are validated.

(a) At $\varepsilon = 0$, there is a singular heteroclinic chain H_0 consisting of
- heteroclinic orbits from p_0 to q_1, and from p_1 to q_0,
- branches M^0, M^1 of nullcline $\{v = 0, f(u) = w\}$. M^0 contains p_0 and q_0. Similarly, M^1 contains p_1 and q_1.

Equilibria p_0 and q_1 are validated by

$$|\pi_{u,v}(p_0) - (2.51043396 \times 10^{-4}, -1.69061417 \times 10^{-8})| < 1.27332504 \times 10^{-2},$$

$$|\pi_w(p_0)| < 1.5 \times 10^{-3},$$

$$|\pi_{u,v}(q_1) - (0.999695758, -1.92238884 \times 10^{-5})| < 1.93244763 \times 10^{-2},$$

$$|\pi_w(q_1) - 1.01177747 \times 10^{-15}| < 2.5 \times 10^{-3}.$$

The equilibrium p_0 admits an attracting isolating block on S_ε contained in $\{(u, v, w) \in S_\varepsilon \cap \{u \le 8.10058152 \times 10^{-3}\} \mid w \in [-2.507 \times 10^{-4}, 2.732 \times 10^{-4}]\}$. Note that p_1 is symmetric to p_0 with respect to the inflection

Parameters	On M^0 ($q = -1$)	On M^1 ($q = +1$)
χ	0.9322799097065463	0.8866213151927438
$\overline{h}$	0.0001	0.0001
H	0.003	0.005
d_a	0.8	0.8
d_b	0.7	0.8
r_a	0.003	0.006
r_b	0.0045	0.008
m_u	14 (around p_0)	–
m_s	–	6 (around q_1)
$2\,\mathrm{sl}_u\{\mathrm{diam}(\pi_a(N)) - r_a\}$	$7.731100200015 \times 10^{-4}$	–
ℓ_u	0.0153705 (around p_0)	–
ℓ_s	–	0.0649544 (around q_1)
	$N_\varepsilon^{0,\mathrm{exit}} \overset{\varphi_\varepsilon(T_0,\cdot)}{\Longrightarrow} N_\varepsilon^{1,0}$	$N_\varepsilon^{1,\mathrm{exit}} \overset{\varphi_\varepsilon(T_1,\cdot)}{\Longrightarrow} N_\varepsilon^{0,0}$
T_i	0.045	–
	(with $\Delta t = 1.0 \times 10^{-4}$)	
computation time	36 min. 56 sec.	–

Table 3. Validation parameters of slow shadowing and isolating blocks in Computer Assisted Result 6.6.

point $(u_{\mathrm{inf}}, 0, w_{\mathrm{inf}})$. Similarly, q_0 is symmetric to q_1 with respect to the inflection point $(u_{\mathrm{inf}}, 0, w_{\mathrm{inf}})$.

(b) For all $\varepsilon \in (0, 5.0 \times 10^{-6}]$, there exists a heteroclinic cycle H_ε near H_0. The cycle H_ε consists of heteroclinic orbits from p_0 to q_1 and from p_1 to q_0.

Parameters for validations are listed in Table 3. Finally, assumptions of Lemma 6.1 are validated with the attracting isolating block $\{w \in [-3.0 \times 10^{-6}, 3.0 \times 10^{-6}]\}$ on the slow manifold containing p_0 and the stable 0.051-cone and the unstable fiber 5.0×10^{-5}-cone conditions.

6.6. Heteroclinic orbits. The third example is a family of heteroclinic orbits different from heteroclinic cycles.

In this example we apply the method in Section 4.5 to isolating blocks on slow manifolds, which validates equilibria on slow manifolds.

Let $(\overline{u}, 0, \overline{w}) \not\approx (0, 0, 0)$ be a point in $\mathbb{R}^3$ such that $f(\overline{u}) = \overline{w}$ and that $\overline{u} = \gamma \overline{w}$. Assume that $f_u(\overline{u}) \neq 0$. Then there is a function $u = h(w)$ which is unique in a small neighbourhood of $(\overline{u}, 0, \overline{w})$ such that $\overline{u} = h(\overline{w})$, $f(h(\overline{w})) = \overline{w}$ and $f(h(w)) = w$ hold in such a neighbourhood. This is due to the Implicit Function Theorem. Assuming that $f_u(u) \neq 0$ holds for all u in a given neighbourhood of $\overline{u}$,

(a1)   (a2)   (a3)
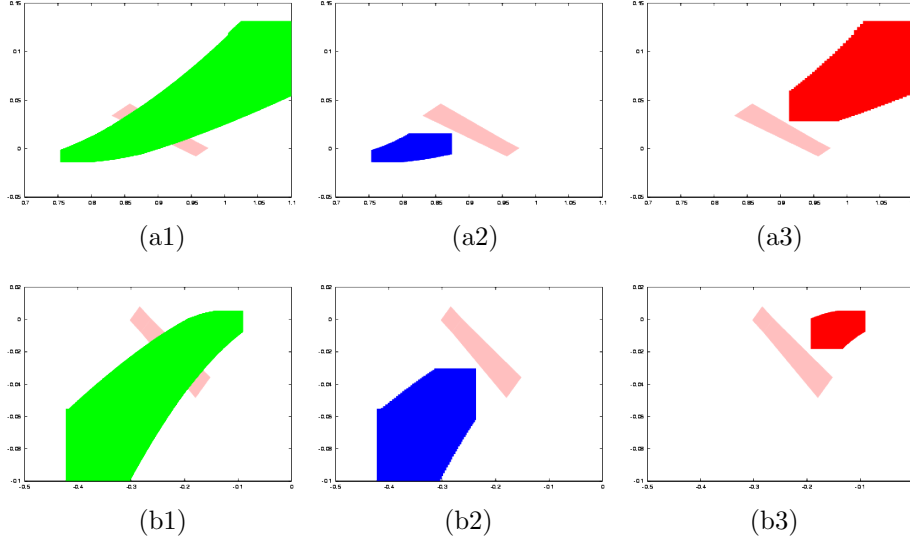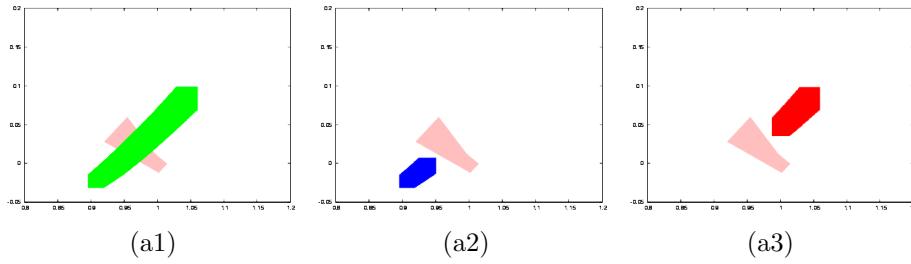
(b1)   (b2)   (b3)

FIGURE 12. Validation of covering-exchange sequences in Computer Assisted Result 6.5. Horizontal axis: u, vertical axis: v. Each figure represents the projection of trajectories and cones on (u, v)-plane. Validations of:
(a1) $\varphi_\varepsilon(T_0, (C^{\mathrm{u}}_{m_u})^{\mathrm{exit}}) \cap \{w \in [0.0365499948149, 0.0422098969893467]\}$.
(a2) $\varphi_\varepsilon(T_0, (C^{\mathrm{u}}_{m_u})^{\mathrm{exit}}) \cap \{w \in [0.0365499948149, 0.0365699944692267]\}$.
(a3) $\varphi_\varepsilon(T_0, (C^{\mathrm{u}}_{m_u})^{\mathrm{exit}}) \cap \{w \in [0.04218989733502, 0.0422098969893467]\}$.
(b1) $\varphi_\varepsilon(T_1, (C^{\mathrm{u}}_{m_u})^{\mathrm{exit}}) \cap \{w \in [0.1037942789937, 0.10913580929979]\}$.
(b2) $\varphi_\varepsilon(T_1, (C^{\mathrm{u}}_{m_u})^{\mathrm{exit}}) \cap \{w \in [0.1037942789937, 0.103814744627057]\}$.
(b3) $\varphi_\varepsilon(T_1, (C^{\mathrm{u}}_{m_u})^{\mathrm{exit}}) \cap \{w \in [0.109115343666433, 0.10913580929979]\}$.

(a1)   (a2)   (a3)

FIGURE 13. Validation of covering-exchange sequences in Computer Assisted Result 6.6. Horizontal axis: u, vertical axis: v. Each figure represents the projection of trajectories and cones on the (u, v)-plane. Validation of:
(a1) $\varphi_\varepsilon(T_0, (C^{\mathrm{u}}_{m_u})^{\mathrm{exit}}) \cap \{w \in [-5.3303301324 \times 10^{-4}, 4.89431324599 \times 10^{-4}]\}$;
(a2) $\varphi_\varepsilon(T_0, (C^{\mathrm{u}}_{m_u})^{\mathrm{exit}}) \cap \{w \in [-5.3303301324 \times 10^{-4}, -5.28187210691 \times 10^{-4}]\}$;
(a3) $\varphi_\varepsilon(T_0, (C^{\mathrm{u}}_{m_u})^{\mathrm{exit}}) \cap \{w \in [4.8458552205 \times 10^{-4}, 4.89431324599 \times 10^{-4}]\}$.

the above implicit function representation holds in the given neighbourhood. By using the implicit function differential, the vector field $g(u, v, w) = c^{-1}(u - \gamma w)$ is rewritten as

$$g(u, v, w) = \frac{1}{c}\left(\frac{1}{f_u(\overline{u})} - \gamma\right)(w - \overline{w}) + \frac{1}{c}\left((u - \overline{u}) - \frac{w - \overline{w}}{f_u(\overline{u})}\right)$$

near $(h(\overline{w}), 0, \overline{w}) \in \mathbb{R}^3$. Remark that $\overline{u} = \gamma\overline{w}$. This expression leads to an effective estimate of vector fields on slow manifolds, as stated in Section 4.5.

All validations of slow shadowing sequences with extended cones and isolating blocks on slow manifolds yield the following computer assisted result.
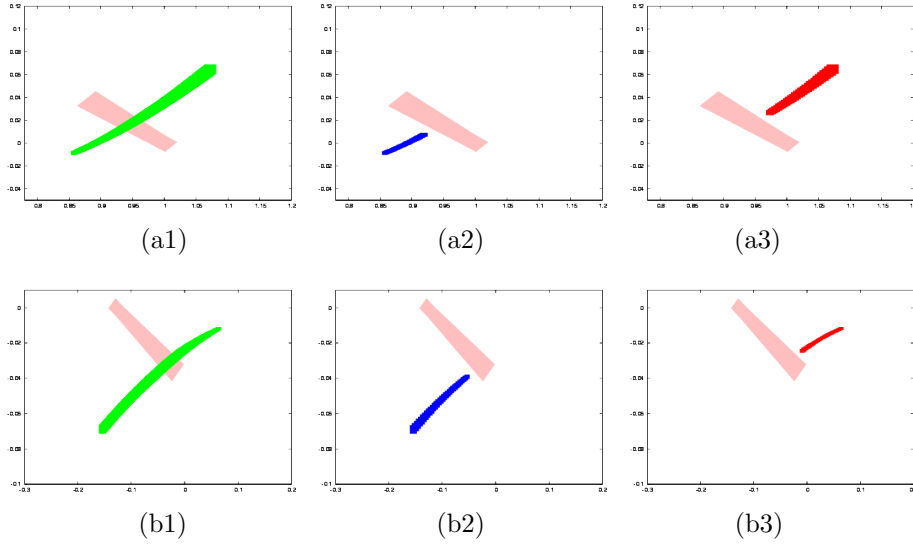

FIGURE 14. Validation of covering-exchange sequences in Computer Assisted Result 6.7. Horizontal axis: u, vertical axis: v. Each figure represents the projection of trajectories and cones on the (u, v)-plane. Validation of:

(a1) $\varphi_\varepsilon(T_0, (C^{\mathrm{u}}_{m_u})^{\mathrm{exit}}) \cap \{w \in [-7.34476926213333 \times 10^{-4},$
      $0.00117805243070567]\}$.

(a2) $\varphi_\varepsilon(T_0, (C^{\mathrm{u}}_{m_u})^{\mathrm{exit}}) \cap \{w \in [-7.34476926213333 \times 10^{-4}, -7.20912888221$
      $\times 10^{-4}]\}$.

(a3) $\varphi_\varepsilon(T_0, (C^{\mathrm{u}}_{m_u})^{\mathrm{exit}}) \cap \{w \in [0.00116448839213337, 0.00117805243070567]\}$.

(b1) $\varphi_\varepsilon(T_1, (C^{\mathrm{u}}_{m_u})^{\mathrm{exit}}) \cap \{w \in [0.059360930386376, 0.06423749049711]\}$.

(b2) $\varphi_\varepsilon(T_1, (C^{\mathrm{u}}_{m_u})^{\mathrm{exit}}) \cap \{w \in [0.059360930386376, 0.05938299626923]\}$.

(b3) $\varphi_\varepsilon(T_1, (C^{\mathrm{u}}_{m_u})^{\mathrm{exit}}) \cap \{w \in [0.064215424614256, 0.06423749049711]\}$.

COMPUTER ASSISTED RESULT 6.7. Consider (6.1) with $a = 0.3$, $\gamma = 15.0$, $\delta = 5.0$.

(a) For each $c \in [0.628, 0.629]$, the following two kinds of trajectories are validated:

- At $\varepsilon = 0$, there is a singular heteroclinic chain H^0 consisting of a heteroclinic orbit from $p_0 = (0,0,0)$ to q_1 and a branch M^1 of the nullcline $\{v = 0, f(u) = w\}$ connecting q_1 and $p_1 = (u^1_{m_1}, 0, w^1_{m_1})$. Equilibria have the following estimates:

$$|\pi_{u,v}(p_0) - (8.00733430 \times 10^{-4}, 2.18164458 \times 10^{-7})| < 2.77714227 \times 10^{-2},$$
$$|\pi_w(p_0) - 3.3 \times 10^{-4}| < 2.75 \times 10^{-3},$$
$$|\pi_{u,v}(q_1) - (0.998891342, -1.20618902 \times 10^{-5})| < 2.11449578 \times 10^{-2},$$
$$|\pi_w(q_1) + 4.2 \times 10^{-4}| < 3.0 \times 10^{-3},$$
$$|\pi_{u,v}(p_1) - (0.883967690, 3.06376214 \times 10^{-7})| < 2.63725073 \times 10^{-2},$$
$$|\pi_w(p_1) - 0.05938| < 3.0 \times 10^{-3}.$$

- For all $\varepsilon \in (0, 1.0 \times 10^{-5}]$, there exists a heteroclinic orbit H^1_ε from p_0 to p_1 near H^1_0. The equilibrium p_1 admits an attracting isolating block on S_ε contained in $\{(u,v,w) \in S_\varepsilon \cap \{u \geq 0.861927234\} \mid w \in [5.8494 \times 10^{-2}, 6.0056 \times 10^{-2}]\}$.

Parameters for validations are listed in Table 4.

Parameters	On M^0 ($q = -1$)	On M^1 ($q = +1$)
χ	0.8949178448605274	0.8877665544332212
$\overline{h}$	0.00023	0.00022
H	0.0055	0.006
d_a	0.8	0.75
d_b	0.8	0.8
r_a	0.008	0.008
r_b	0.0085	0.008
m_u	50 (around p_0)	—
m_s	—	25.5 (around q_1)
$2\,\mathrm{sl_u}\{\mathrm{diam}(\pi_a(N)) - r_a\}$	$6.003396000012 \times 10^{-4}$	—
ℓ_u	0.0136582 (around p_0)	—
ℓ_s	—	0.117888 (around q_1)
	$N^{0,\mathrm{exit}}_\varepsilon \overset{\varphi_\varepsilon(T_0,\,\cdot\,)}{\Longrightarrow} N^{1,0}_\varepsilon$	$N^{1,\mathrm{exit}}_\varepsilon \overset{\varphi_\varepsilon(T_1,\,\cdot\,)}{\Longrightarrow} N^{0,0}_\varepsilon$
T_i	0.064	—
	(with $\Delta t = 1.0 \times 10^{-4}$)	
computation time	25 min. 25 sec.	—

TABLE 4. Validation parameters of slow shadowing and isolating blocks in Computer Assisted Result 6.7 with $c \in [0.628, 0.629]$.

Finally, assumptions of Lemma 6.1 are validated with the attracting isolating block $\{w \in [-2.119830199999 \times 10^{-3}, 2.779830199999 \times 10^{-3}]\}$ on the slow manifold containing p_0 and the stable 0.018-cone and the unstable fiber-5.0×10^5-cone conditions.

(b) For each $c \in [0.551, 0.553]$, there exist the following two kinds of trajectories:

- At $\varepsilon = 0$, there is a singular heteroclinic chain H_0^2 consisting of a heteroclinic orbit from p_1 to q_0 and a branch M^0 of the nullcline $\{v = 0, f(u) = w\}$ connecting q_0 and p_0. Equilibria have the following estimates:

$$|\pi_{u,v}(p_1) - (0.881572201, 1.13212050 \times 10^{-6})| < 2.63868472 \times 10^{-2},$$
$$|\pi_w(p_1) - 0.0602| < 4.0 \times 10^{-3},$$
$$|\pi_{u,v}(q_0) - (-0.125383244, 1.48465533 \times 10^{-5})| < 1.53394803 \times 10^{-2},$$
$$|\pi_w(q_0) - 0.0602| < 4.0 \times 10^{-3},$$
$$|\pi_{u,v}(p_0) - (-1.36886862 \times 10^{-3}, -8.28111689 \times 10^{-8})| < 2.38267941 \times 10^{-2},$$
$$|\pi_w(p_0)| < 4.0 \times 10^{-3}.$$

Parameters	On M^0 ($q = -1$)	On M^1 ($q = +1$)
χ	0.8484848484848486	0.8540145985401461
$\overline{h}$	0.0001	0.0002
H	0.008	0.008
d_a	0.8	0.75
d_b	0.8	0.8
r_a	0.0042	0.006
r_b	0.0045	0.0068
m_u	–	20 (around p_1)
m_s	26.5 (around q_0)	–
$2\,\mathrm{sl}_u\{\mathrm{diam}(\pi_a(N)) - r_a)\}$	–	$5.900088000012 \times 10^{-4}$
ℓ_u	–	0.0539878 (around p_1)
ℓ_s	0.113106 (around q_0)	–
	$N_\varepsilon^{0,\mathrm{exit}} \stackrel{\varphi_\varepsilon(T_0, \cdot)}{\Longrightarrow} N_\varepsilon^{1,0}$	$N_\varepsilon^{1,\mathrm{exit}} \stackrel{\varphi_\varepsilon(T_1, \cdot)}{\Longrightarrow} N_\varepsilon^{0,0}$
T_i	–	0.055
		(with $\Delta t = 1.0 \times 10^{-4}$)
computation time	–	26 min. 23 sec.

Table 5. Validation parameters of slow shadowing and isolating blocks in Computer Assisted Result 6.7 with $c \in [0.551, 0.553]$.

- For all $\varepsilon \in (0, 1.0 \times 10^{-5}]$, there exists a heteroclinic orbit H_ε^2 from p_1 to p_0 near H_0^2. The equilibrium p_0 admits an attracting isolating block on S_ε contained in $\{(u, v, w) \in S_\varepsilon \cap \{u \le 1.06035231 \times 10^{-2}\} \mid w \in [-1.040 \times 10^{-4}, 1.264 \times 10^{-3}]\}$.

Parameters for validations are listed in Table 5. Finally, assumptions of Lemma 6.1 are validated with the attracting isolating block $\{w \in [5.7895004400 \times 10^{-2}, 6.5304995600 \times 10^{-2}]\}$ on the slow manifold containing p_1 and the stable 0.17-cone and the unstable fiber-5.0×10^5-cone conditions.

In this case we have to care about the existence of an equilibrium on the slow manifold in S_ε^1. Easy calculations yield that S_ε^1 possesses at most one equilibrium.

6.7. Homoclinic orbits. The final example is a family of homoclinic orbits. Set $a = 0.3$, $\gamma = 10.0$, $\delta = 9.0$ and $c \approx 0.8$. All validations of covering-exchange sequences with extended cones yield the following computer assisted result.

COMPUTER ASSISTED RESULT 6.8. Consider (6.1) with $a = 0.3$, $\gamma = 10.0$ and $\delta = 9.0$. Then for all $c \in [0.799, 0.801]$ the following trajectories are validated.

(a) At $\varepsilon = 0$, there is a singular heteroclinic chain H_0 consisting of
   - heteroclinic orbits from p_0 to q_1, and from p_1 to q_0,
   - branches M^0, M^1 of the nullcline $\{v = 0, f(u) = w\}$. M^0 contains p_0 and q_0. Similarly, M^1 contains q_1 and p_1.

   Equilibria p_0, q_1, p_1 and q_0 are validated by

$$|\pi_{u,v}(p_0) - (-4.22605959 \times 10^{-3}, 4.67157371 \times 10^{-7})| < 2.90338686 \times 10^{-2},$$
$$|\pi_w(p_0) - 1.94 \times 10^{-3}| < 3.25 \times 10^{-3},$$
$$|\pi_{u,v}(q_1) - (0.996532931, -4.66925590 \times 10^{-6})| < 2.13488628 \times 10^{-2},$$
$$|\pi_w(q_1) - 2.02 \times 10^{-3}| < 3.3 \times 10^{-3},$$
$$|\pi_{u,v}(p_1) - (0.870742129, -2.92737408 \times 10^{-6})| < 2.84335647 \times 10^{-2},$$
$$|\pi_w(p_1) - 0.06362| < 3.3 \times 10^{-3},$$
$$|\pi_{u,v}(q_0) - (-0.129665170, 3.70642116 \times 10^{-6})| < 2.20129903 \times 10^{-2},$$
$$|\pi_w(q_0) - 0.06335| < 3.25 \times 10^{-3}.$$

(b) For all $\varepsilon \in (0, 5.0 \times 10^{-6}]$, there exists a homoclinic orbit H_ε of p_0 near H_0. The equilibrium p_0 admits an attracting isolating block on S_ε contained in

$$\{(u, v, w) \in S_\varepsilon \cap \{u \le 2.10010357 \times 10^{-2}\} \mid w \in [-8.15 \times 10^{-5}, 9.40 \times 10^{-5}]\}.$$

Parameters for validations are listed in Table 6.

Parameters	On M^0 $(q = -1)$	On M^1 $(q = +1)$
χ	0.8812568505663135	0.8792535675082328
$\overline{h}$	0.00023	0.00022
H	0.0065	0.0066
d_a	0.8	0.8
d_b	0.75	0.8
r_a	0.008	0.008
r_b	0.0085	0.0078
m_u	50 (around p_0)	52 (around p_1)
m_s	45 (around q_0)	55 (around q_1)
$2\,\mathrm{sl}_\mathrm{u}\{\mathrm{diam}(\pi_a(N)) - r_a\}$	$3.019812000006 \times 10^{-4}$	–
ℓ_u	0.0441793 (around p_0)	0.0370798 (around p_1)
ℓ_s	0.12217 (around q_0)	0.0984568 (around q_1)
	$N_\varepsilon^{0,\mathrm{exit}} \overset{\varphi_\varepsilon(T_0,\,\cdot\,)}{\Longrightarrow} N_\varepsilon^{1,0}$	$N_\varepsilon^{1,\mathrm{exit}} \overset{\varphi_\varepsilon(T_1,\,\cdot\,)}{\Longrightarrow} N_\varepsilon^{0,0}$
T_i	0.067	0.0288
	(with $\Delta t = 1.0 \times 10^{-4}$)	(with $\Delta t = 2.0 \times 10^{-5}$)
computation time	5 min. 57 sec.	44 min. 6 sec.

TABLE 6. Validation parameters of slow shadowing and isolating blocks in Computer Assisted Result 6.8.

Finally, assumptions of Lemma 6.1 are validated with the attracting isolating block $\{w \in [-1.15901 \times 10^{-3}, 5.03901 \times 10^{-3}]\}$ on the slow manifold and the stable 0.0085-cone and the unstable fiber-5.0×10^5-cone conditions.

REMARK 6.9. In this case we easily know that the only equilibrium of (6.1) for $\varepsilon > 0$ is $(u, v, w) = (0, 0, 0)$ and the dynamics around $(0, 0, 0)$ can be easily understood to show that $(0, 0, 0)$ is attracting on the slow manifold in $\mathcal{S}_\varepsilon^0$.

Note that appropriate arrangements of computational schemes can improve the accuracy of validated trajectories and the validation range of parameters including ε.

In many articles, the uniqueness of global orbits is also discussed. From the geometrical viewpoint, typical arguments for the uniqueness require the transversality of locally invariant manifolds. In our case, however, transversality in the sense of differential manifolds is not mentioned. In other words, we cannot prove the uniqueness of validated orbits in the current setting.

6.8. Continuation of homoclinic orbits. Finally, we verify the ε-continuation of homoclinic orbits validated in the previous subsection. It is very hard to validate trajectories of $(6.1)_\varepsilon$ for all $\varepsilon \in (0, \varepsilon_0]$ with large ε_0 at a time. In most cases, covering relations in the fast dynamics fail. To overcome this difficulty,
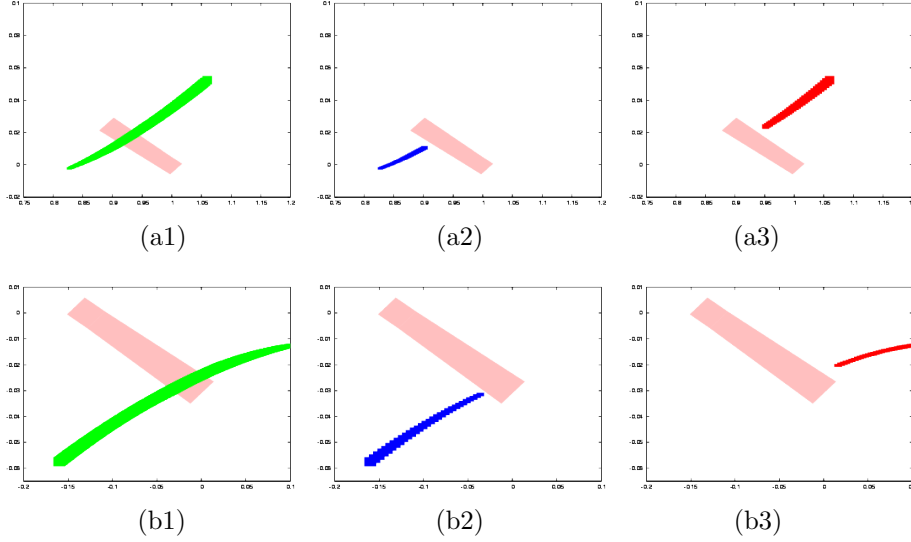
FIGURE 15. Validation of covering-exchange sequences in Computer Assisted Result 6.8. Horizontal axis: u, vertical axis: v. Each figure represents the projection of trajectories and cones on the (u, v)-plane. Validation of:
(a1) $\varphi_\varepsilon(T_0, (C_{m_u}^{\mathrm{u}})^{\mathrm{exit}}) \cap \{w \in [-4.98044338856 \times 10^{-4},$
    $0.00295819387292]\}$.
(a2) $\varphi_\varepsilon(T_0, (C_{m_u}^{\mathrm{u}})^{\mathrm{exit}}) \cap \{w \in [-4.98044338856 \times 10^{-4},$
    $-4.4138469604 \times 10^{-4}]\}$.
(a3) $\varphi_\varepsilon(T_0, (C_{m_u}^{\mathrm{u}})^{\mathrm{exit}}) \cap \{w \in [0.002901534230104, 0.00295819387292]\}$.
(b1) $\varphi_\varepsilon(T_1, (C_{m_u}^{\mathrm{u}})^{\mathrm{exit}}) \cap \{w \in [0.06094493968776, 0.0668173357334667]\}$.
(b2) $\varphi_\varepsilon(T_1, (C_{m_u}^{\mathrm{u}})^{\mathrm{exit}}) \cap \{w \in [0.06094493968776, 0.0609654725410667]\}$.
(b3) $\varphi_\varepsilon(T_1, (C_{m_u}^{\mathrm{u}})^{\mathrm{exit}}) \cap \{w \in [0.06679680288016, 0.0668173357334667]\}$.

we divide a closed interval $[0, \varepsilon_0]$ into sub-intervals

$$[0, \varepsilon_0^1] \cup [\varepsilon_0^1, \varepsilon_0^2] \cup \ldots \cup [\varepsilon_0^{m-1}, \varepsilon_0^m], \quad \varepsilon_0^m = \varepsilon_0$$

and validate trajectories within each sub-interval.

In our demonstrating example, we divide the interval $[0, 5.0 \times 10^{-5}]$ into

$$[0, 5.0 \times 10^{-6}] \cup [5.0 \times 10^{-6}, 1.2 \times 10^{-5}] \cup [1.2 \times 10^{-5}, 2.2 \times 10^{-5}]$$
$$\cup [2.2 \times 10^{-5}, 3.2 \times 10^{-5}] \cup [3.2 \times 10^{-5}, 4.2 \times 10^{-5}] \cup [4.2 \times 10^{-5}, 5.0 \times 10^{-5}].$$

Validating homoclinic orbits for each sub-interval and summarizing them, we obtain the following. Let p_0, q_0, p_1 and q_1 be points corresponding to those stated in Computer Assisted Result 6.8.

COMPUTER ASSISTED RESULT 6.10. Consider (6.1) with $a = 0.3$, $\gamma = 10.0$ and $\delta = 9.0$. Then for all $c \in [0.799, 0.801]$ and for all $\varepsilon \in (0, 5.0 \times 10^{-5}]$, there exists a homoclinic orbit H_ε of p_0 near H_0, where H_0 is the singular homoclinic

orbit for $(6.1)_0$ obtained in Computer Assisted Result 6.8. Several numerical data for validations of H_ε are listed in Tables 7–10.

ε	$\overline{w}^0_{m_0}$	$\overline{h}$	H	χ	ℓ_u
$[0.0, 5.0 \times 10^{-6}]$	0.00175	0.00023	0.0065	0.8812568505663135	0.04417933
$[5.0 \times 10^{-6}, 1.2 \times 10^{-5}]$	0.00175	0.00023	0.0065	0.8812568505663135	0.0441793
$[1.2 \times 10^{-5}, 2.2 \times 10^{-5}]$	0.00175	0.00023	0.0065	0.8813652126300421	0.0441725
$[2.2 \times 10^{-5}, 3.2 \times 10^{-5}]$	0.00175	0.00023	0.0065	0.8813652126300421	0.0441528
$[3.2 \times 10^{-5}, 4.2 \times 10^{-5}]$	0.00175	0.00023	0.0065	0.8813652126300421	0.0441429
$[4.2 \times 10^{-5}, 5.0 \times 10^{-5}]$	0.0019	0.00025	0.0065	0.8809523809523810	0.0441474

TABLE 7. Main information about blocks around $p_0 = (u^0_{m_0}, v^0_{m_0}, w^0_{m_0})$ with $|w^0_{m_0} - \overline{w}^0_{m_0}| < \overline{h}$. The sharpness m_u of unstable cones is an identical value: $m_u = 50$. The ratios d_a, d_b in (4.5) are set as identical values: $d_a = 0.8$, $d_b = 0.75$. The space length parameters r_a, r_b of fast-saddle-type blocks are set as identical values: $r_a = 0.008$, $r_b = 0.0085$.

ε	$\overline{w}^1_0$	$\overline{h}$	H	χ	ℓ_s
$[0.0, 5.0 \times 10^{-6}]$	0.0633	0.00023	0.0065	0.8812568505663135	0.12217
$[5.0 \times 10^{-6}, 1.2 \times 10^{-5}]$	0.0633	0.00023	0.0065	0.8812568505663135	0.12217
$[1.2 \times 10^{-5}, 2.2 \times 10^{-5}]$	0.0635	0.00023	0.0065	0.8813652126300421	0.122164
$[2.2 \times 10^{-5}, 3.2 \times 10^{-5}]$	0.0635	0.00023	0.0065	0.8813652126300421	0.122145
$[3.2 \times 10^{-5}, 4.2 \times 10^{-5}]$	0.0635	0.00023	0.0065	0.8813652126300421	0.122141
$[4.2 \times 10^{-5}, 5.0 \times 10^{-5}]$	0.0635	0.00025	0.0065	0.8809523809523810	0.122144

TABLE 8. Main information about blocks around $q_0 = (u^1_0, v^1_0, w^1_0)$ with $|w^1_0 - \overline{w}^1_0| < \overline{h}$. The sharpness m_s of stable cones is an identical value: $m_s = 45$. The ratios d_a, d_b in (4.5) are set as identical values: $d_a = 0.8$, $d_b = 0.75$. The space length parameters r_a, r_b of fast-saddle-type blocks are set as identical values: $r_a = 0.008$, $r_b = 0.0085$.

7. Conclusion

We have proposed a topological methodology for validating singularly perturbed periodic, homoclinic and heteroclinic orbits for fast-slow systems with its applicability to concrete systems with computer assistance. The main features of our proposing methodology are the following.

- Our central strategy consists of applications of well-known topological tools, covering relations, cones and isolating blocks, with taking account of the singular perturbation structure of normally hyperbolic invariant manifolds. This is one of the points where our approach differs from preceding works such as [1].

ε	$\overline{w}^1_{m_1}$	$\overline{h}$	H	χ	ℓ_u
$[0.0, 5.0 \times 10^{-6}]$	0.0636	0.00022	0.0066	0.8792535675082328	0.0370798
$[5.0 \times 10^{-6}, 1.2 \times 10^{-5}]$	0.0636	0.00022	0.0066	0.8792535675082328	0.0370798
$[1.2 \times 10^{-5}, 2.2 \times 10^{-5}]$	0.0636	0.00022	0.0066	0.8792535675082328	0.0370719
$[2.2 \times 10^{-5}, 3.2 \times 10^{-5}]$	0.0636	0.00022	0.0066	0.8792535675082328	0.0370534
$[3.2 \times 10^{-5}, 4.2 \times 10^{-5}]$	0.0636	0.00022	0.0066	0.8792535675082328	0.0370464
$[4.2 \times 10^{-5}, 5.0 \times 10^{-5}]$	0.0634	0.0003	0.0065	0.8786764705882354	0.0370888

TABLE 9. Main information about blocks around $p_1 = (u^1_{m_1}, v^1_{m_1}, w^1_{m_1})$ with $|w^1_{m_1} - \overline{w}^1_{m_1}| < \overline{h}$. The sharpness m_u of unstable cones is an identical value: $m_u = 52$. The ratios d_a, d_b in (4.5) are set as identical values: $d_a = 0.8$, $d_b = 0.8$ for $\varepsilon \in (0.0, 4.2 \times 10^{-5}]$, $d_a = 0.75$, $d_b = 0.75$ for $\varepsilon \in [4.2 \times 10^{-5}, 5.0 \times 10^{-5}]$. The space length parameters r_a, r_b of fast-saddle-type blocks are set as identical values: $r_a = 0.008$, $r_b = 0.0078$.

ε	$\overline{w}^0_0$	$\overline{h}$	H	χ	ℓ_s
$[0.0, 5.0 \times 10^{-6}]$	0.0019	0.00022	0.0066	0.8792535675082328	0.0984568
$[5.0 \times 10^{-6}, 1.2 \times 10^{-5}]$	0.0019	0.00022	0.0066	0.8792535675082328	0.0984568
$[1.2 \times 10^{-5}, 2.2 \times 10^{-5}]$	0.0019	0.00022	0.0066	0.8792535675082328	0.0984452
$[2.2 \times 10^{-5}, 3.2 \times 10^{-5}]$	0.0019	0.00022	0.0066	0.8792535675082328	0.098424
$[3.2 \times 10^{-5}, 4.2 \times 10^{-5}]$	0.0019	0.00022	0.0066	0.8792535675082328	0.0984201
$[4.2 \times 10^{-5}, 5.0 \times 10^{-5}]$	0.0018	0.0003	0.0065	0.8786764705882354	0.0984281

TABLE 10. Main information about blocks around $q_1 = (u^0_0, v^0_0, w^0_0)$ with $|w^0_0 - \overline{w}^0_0| < \overline{h}$. The sharpness m_s of stable cones is an identical value: $m_s = 55$. The ratios d_a, d_b in (4.5) are set as identical values: $d_a = 0.8$, $d_b = 0.8$ for $\varepsilon \in (0.0, 4.2 \times 10^{-5}]$, $d_a = 0.75$, $d_b = 0.75$ for $\varepsilon \in [4.2 \times 10^{-5}, 5.0 \times 10^{-5}]$. The space length parameters r_a, r_b of fast-saddle-type blocks are set as identical values: $r_a = 0.008$, $r_b = 0.0078$.

- All our procedures can be validated via rigorous numerics (e.g. interval arithmetics). As a consequence, one can validate the continuation of singular limit orbits for all $\varepsilon \in (0, \varepsilon_0]$ with a given $\varepsilon_0 > 0$ with computer assistance.

A main concept for realizing the above points simultaneously is the covering-exchange; the singular perturbation's version of covering relations. We have proposed not only its primitive form but also its generalization; a collection of local behavior named "slow shadowing", "drop" and "jump". The notion of covering-exchange describes the behavior of trajectories in the full system which shadow normally hyperbolic invariant manifolds as well as their stable and unstable manifolds, even for sufficiently small ε. In particular, this notion enables us to validate such trajectories *without solving any differential equations*

ε	$N_\varepsilon^{0,\text{exit}} \overset{\varphi_\varepsilon(T_0,\,\cdot\,)}{\Longrightarrow} N_\varepsilon^{1,0}$	$N_\varepsilon^{1,\text{exit}} \overset{\varphi_\varepsilon(T_1,\,\cdot\,)}{\Longrightarrow} N_\varepsilon^{0,0}$
$[0.0, 5.0 \times 10^{-6}]$	5 min. 57 sec.	44 min. 6 sec.
$[5.0 \times 10^{-6}, 1.2 \times 10^{-5}]$	6 min. 0 sec.	44 min. 9 sec.
$[1.2 \times 10^{-5}, 2.2 \times 10^{-5}]$	6 min. 8 sec.	44 min. 39 sec.
$[2.2 \times 10^{-5}, 3.2 \times 10^{-5}]$	5 min. 59 sec.	44 min. 49 sec.
$[3.2 \times 10^{-5}, 4,2 \times 10^{-5}]$	5 min. 57 sec.	45 min. 23 sec.
$[4.2 \times 10^{-5}, 5.0 \times 10^{-5}]$	5 min. 58 sec.	46 min. 17 sec.
$[0.0, 5.0 \times 10^{-5}]$	(Total) 35 min. 59 sec.	(Total) 269 min. 23 sec.

TABLE 11. Computation times of *Drop* in Computer Assisted Result 6.10. It has taken about 13.4 hours for validating *Drop* in $\{\varepsilon \in [0, 5.0 \times 10^{-5}]\}$ in our computation environments. As for the slow shadowing and *Jump*, it totally takes less than 30 seconds to validate.

in practical computations. Moreover, as mentioned above, it exceeds the limit of the multiple timescale parameter range: "sufficiently small ε", to ε with an explicit range in practical applications with computer assistance.

Of course, further applications of topological tools such as covering relations and the Conley index enable us to prove the existence of trajectories with complicated behavior. We believe that all these ideas will overcome difficulties of a broad class of singular perturbation problems.

We end this paper proposing further directions of our arguments.

7.1. Bridging validation results to "direct" approach. Our methodology concentrates on validations of trajectories for $\varepsilon \in (0, \varepsilon_0]$ on the basis of geometric singular perturbation theory. One of challenges concerning with continuation of trajectories is the integration of singularly perturbed trajectories with those by a *direct approach*; namely, validation of trajectories without taking account of singular perturbation structure of systems for positive ε bounded away from 0 (e.g. [1]). In our validation examples, under a specific choice of parameter values, solution trajectories have been validated for $\varepsilon \in (0, 5.0 \times 10^{-5}]$ in the case of homoclinic orbits for the FitzHugh–Nagumo system (6.1). In the case of (6.1), validations of each branch of slow manifolds are completed in less than one second.

This parameter range is still far from validation ranges with direct approaches; $\varepsilon = 0.01$ in [1], for example. Validations of slow shadowing for further range of ε are not easy, which is mainly because there is a trade-off between the slow shadowing condition (4.5) and the covering relation in assumption (SS5), as discussed in Section 6.2. Both conditions are essential for describing the slow shadowing, and hence our task for overcoming this situation will be development of an improved topological and numerical method, like *multi-step methods*

in numerical initial value problems, keeping the essence of slow shadowing phenomena. Obviously, if one can validate each global branch of normally hyperbolic slow manifolds by *one* block, there is no problem in this direction as indicated in Computer Assisted Result 6.4, in which case we can apply the primitive form of covering-exchange in Section 4.1. We mention another work for validating global trajectories with an explicit range of ε in Section 7.5, which indicates that this direction makes sense for solving bridging problems. After improvements of validations of slow shadowing or blocks containing slow manifolds with computer assistance, there will be a possibility that validated trajectories can be further continued in ε and, we hope, be connected to validated ones with direct approach under appropriate choice of various parameters or solvers. Further works in this direction from the viewpoint of topological and numerical analysis are the ones of our future studies.

7.2. Uniqueness and stability of validated orbits. In our topological statements (Theorems 5.1, 5.6, Corollaries 5.3, 5.4, 5.7 and 5.8), only the existence of desired orbits is stated. As for the uniqueness of such orbits, we have no answers yet. More precisely, these theorems require only topological transversality in terms of covering relations, which is weaker than the transversality in terms of differentiable manifolds. If we can verify even transversality in the sense of differentiable manifolds, then the Exchange Lemma gives us the local uniqueness of singularly perturbed orbits. Transversality is also essential for discussion of stability of connecting orbits. Jones [18] discusses stability of homoclinic orbits for (6.1) as the traveling wave solutions of FitzHugh–Nagumo PDE (6.2) from geometric viewpoints. For this stability argument, some information about the nature of transversality between two manifolds plays a central role.

One of the standard approaches to transversality of manifolds is to use *Melnikov integrals* via solving variational equations. If we take rigorous numerics into account, the C^1-Lohner method [39] provides the variational information of trajectories as well as solutions themselves and will give an answer to solve transversality problems.

7.3. Covering-exchange for fast-slow systems with multi-dimensional slow variables. Our validation arguments can be applied only to fast-slow systems with one-dimensional slow variable at present. It is natural to ask whether our concept is applicable to fast-slow systems with multi-dimensional slow variables. The key problem concerning this extension is how we should track true trajectories near slow manifolds in terms of covering-exchange properties. Systems with multi-dimensional slow variables can admit rich behavior even on slow manifolds. Validation of slow manifolds by one fast-saddle-type block

as well as concrete vector fields is thus more difficult than systems with one-dimensional slow variables because of nonlinearity of manifolds. It is not thus realistic to extend Theorems 5.1 and 5.6 to systems with multi-dimensional slow variable directly from the viewpoint of practical computations.

By the way, existence theorems of trajectories with covering relations imply that *true trajectories shadow reference orbits*. In the case of singular perturbation problems, reference orbits correspond to singular orbits consisting of heteroclinic orbits and trajectories on slow manifolds. The slow shadowing discussed in Section 4.3 reflects the aspect of shadowing trajectories in fast-slow systems with one-dimensional slow variable. This expectation will be valid even for systems with multi-dimensional slow variables. Such an extended sequence will yield the existence of true trajectories which shadow those on slow manifolds. Higher dimensional extension of the slow shadowing condition or its analogue will give an explicit criterion for validations with rigorous numerics in reasonable processes.

7.4. Extension of procedures for slow manifolds with non-hyperbolic points. Our validation arguments are based on the assumption that limiting critical manifolds are normally hyperbolic everywhere in consideration. Our procedure in this paper can be never applied if there is a point which is non-hyperbolic, like fold points, inside limiting critical manifolds. Nevertheless, there are a lot of cases where interesting phenomena in singular perturbation problems are involved by such non-hyperbolic points.

One of the famous examples is *canard solution*, which was discovered and first analyzed by Benoit, Callot, Diener and Diener [2]. A canard solution is a solution of a singular perturbed system which is contained in the intersection of an attracting slow manifold and a repelling one. Canard solutions provide a rich phenomenon such as *canard explosion*, namely, a transition from a small limit cycle to a relaxation oscillation through a sequence of *canard cycles* [10]. Since canard cycles involve fold points as jump points, our current implementations can never validate canard cycles.

Krupa and Szmolyan [22] discuss the extension of geometric singular perturbation theory for slow manifolds including fold and canard points via the technique of blow-up. Simultaneously, Liu [23] discusses the Exchange Lemma in case that the critical manifold contains loss-of-stability turning points, the corresponding phenomenon is called the *delay of stability loss*. In [23], the Fenichel-type coordinate which overcomes non-normal hyperbolicity is constructed by the other type of invariant manifold theorem proved by Chow–Liu–Yi [6]. We also mention sequential works by Schecter and Szmolyan [28]–[31], which discuss the singularly perturbed Riemann–Dafermos solutions as the small perturbation of composite waves consisting of constant waves, shock waves and rarefaction

waves in systems of conservation laws. There, non-hyperbolic points called gain-of-stability turning points naturally arise in invariant manifolds. They derive the General Exchange Lemma for describing behavior of tracking manifolds near such non-hyperbolic manifolds with the help of blow-ups.

We have to take such mathematical techniques into account, if we deal with invariant manifolds which are not normally hyperbolic with computer assistance.

7.5. Remark on the work [8] and a direction to improvements of our works. Another work for validating global trajectories for the fast-slow system (1.1) with an explicit range of ε has appeared online just before the first version of this paper has been released: the work by Czechowski and Zgliczyński [8].

Authors of [8] focus on the FitzHugh–Nagumo equation (6.1) with specific parameters and provide a topological methodology for validating periodic orbits for $\varepsilon \in (0, \varepsilon_0]$ with an explicit $\varepsilon_0 > 0$. The central tools in [8] are *isolating segments*, which correspond to fast-saddle-type blocks in this paper, and covering relations. Unlike our methodology, they do *not* refer to the normally hyperbolic structure of slow manifolds, but only topological conditions around slow manifolds; namely, transversality of vector fields in normally hyperbolic directions and monotonicity in slow directions. The most remarkable result there will be that, in their example, they validate each branch of slow manifolds *by one isolating segment* (i.e. *fast-saddle-type block*). As a consequence, for specific a and γ, they validate a periodic orbit for $\varepsilon \in (0, 0.0015]$, which will be big enough to bridge a gap between singular perturbation approach and direct approach. Indeed, they also show that the classical Newton–Moore method can be applied to validate a locally unique periodic orbit at $\varepsilon = 0.0015$, which indicates that their methodology solves the bridging problem mentioned in Section 7.1.

These results indicate the following observation; if we can validate each branch of slow manifolds by one fast-saddle-type block, global trajectories will be validated for $\varepsilon \in (0, \varepsilon_0]$ with ε_0 large enough to bridge singular perturbation approach and direct approach. This expectation also follows from Computer Assisted Result 6.4.

In general, (normally hyperbolic) slow manifolds have nonlinear structure. There is thus little hope to construct a fast-saddle-type block around the global branch of slow manifolds for general systems, without taking account of nonlinearity in, at least, critical manifolds as well as their normal hyperbolicity. Our results as well as [8] send us to the study of a systematic procedure of *nonlinear*, *smooth* and *global* fast-saddle-type blocks (e.g., [12]) satisfying cone conditions for an explicit range $\varepsilon \in (0, \varepsilon_0]$.

Appendix A. Concrete terms for (6.1) according to Section 2.3.3

Here we show concrete forms of error terms and Jacobian matrices calculated from the vector field (6.1) for validating fast-saddle-type blocks, and cone conditions with the predictor-corrector approach (Section 2.3.3).

A.1. Fast-saddle-type blocks. We list various terms in Section 2.3.3 in the case of FitzHugh–Nagumo system for readers' accessibility. Let $(\overline{u}, 0, \overline{w})$ be a numerical zero of $-f(u) + w$, where $f(u) = u(u - a)(1 - u), a \in (0, 1/2)$.

Use the coordinates $(\widetilde{u}, \widetilde{v}, \widetilde{w})$ given by as follows:

(A.1) $$u = \widetilde{u} + (\overline{u} + f_u(\overline{u})^{-1}\widetilde{w}), \qquad v = \widetilde{v}, \qquad w = \widetilde{w} + \overline{w},$$

where $(\overline{u}, 0, \overline{w})$ is a (numerical) root of $\{v = 0, cv - f(u) + w = 0\}$ and $f_u(\overline{u})$ is assumed to be invertible. The transformed vector field as the extended system then is

$$\begin{aligned}
\widetilde{u}' &= u' - f_u(\overline{u})^{-1}\widetilde{w}' = v - f_u(\overline{u})^{-1}\varepsilon c^{-1}(u - \gamma w) \\
&= \widetilde{v} - \varepsilon c^{-1} f_u(\overline{u})^{-1}(\widetilde{u} + (f_u(\overline{u})^{-1} - \gamma)\widetilde{w} + \overline{u} - \gamma\overline{w}), \\
\widetilde{v}' &= \delta^{-1}(cv - f(u) + w) \\
&= \delta^{-1}\{c\widetilde{v} - f\left(\widetilde{u} + \overline{u} + f_u(\overline{u})^{-1}\widetilde{w}\right) + \widetilde{w} + \overline{w}\}, \\
\widetilde{w}' &= \varepsilon c^{-1}(\widetilde{u} + (f_u(\overline{u})^{-1} - \gamma)\widetilde{w} + \overline{u} - \gamma\overline{w}), \\
\eta' &= 0,
\end{aligned}$$

where we introduced an auxiliary variable η to be $\varepsilon = \varepsilon_0\eta$, $\eta \in [0, 1]$, with constant ε_0.

The fast component of the vector field is

$$\begin{pmatrix} \widetilde{u}' \\ \widetilde{v}' \end{pmatrix} = \begin{pmatrix} \widetilde{v} - \varepsilon c^{-1} f_u(\overline{u})^{-1}(\widetilde{u} + (f_u(\overline{u})^{-1} - \gamma)\widetilde{w} + \overline{u} - \gamma\overline{w}) \\ \delta^{-1}\left\{c\widetilde{v} - f\left(\widetilde{u} + \overline{u} + f_u(\overline{u})^{-1}\widetilde{w}\right) + \widetilde{w} + \overline{w}\right\} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 \\ -\delta^{-1}f_u(\overline{u}) & \delta^{-1}c \end{pmatrix}\begin{pmatrix} \widetilde{u} \\ \widetilde{v} \end{pmatrix} + \begin{pmatrix} -\varepsilon c^{-1} f_u(\overline{u})^{-1}(\widetilde{u} + (f_u(\overline{u})^{-1} - \gamma)\widetilde{w} + \overline{u} - \gamma\overline{w}) \\ \delta^{-1}\{-f(u) + f(\overline{u}) + f_u(\overline{u})\widetilde{u} + \widetilde{w}\} \end{pmatrix}.$$

Denote (6.1) by $X = F(X, Y, \varepsilon)$, $Y = \varepsilon G(X, Y, \varepsilon)$, where $X = (u, v)^{\mathrm{T}}$ and $Y = w$; namely,

$$F(X, Y, \varepsilon) = \begin{pmatrix} v \\ \delta^{-1}(cv - f(u) + w) \end{pmatrix}, \qquad G(X, Y, \varepsilon) = c^{-1}(u - \gamma w).$$

The Jacobian matrices at $(\overline{u}, 0, \overline{w})$ are

$$F_X|_{(\overline{u}, 0, \overline{w})} = \begin{pmatrix} 0 & 1 \\ -\delta^{-1}f_u(\overline{u}) & \delta^{-1}c \end{pmatrix}, \quad (F_X|_{(\overline{u}, 0, \overline{w})})^{-1} = \frac{\delta}{f_u(\overline{u})}\begin{pmatrix} \delta^{-1}c & -1 \\ \delta^{-1}f_u(\overline{u}) & 0 \end{pmatrix},$$

$$F_Y|_{(\overline{u}, 0, \overline{w})} = \begin{pmatrix} 0 \\ \delta^{-1} \end{pmatrix}, \qquad\qquad (F_X^{-1} \circ F_Y)|_{(\overline{u}, 0, \overline{w})} = \begin{pmatrix} -1/f_u(\overline{u}) \\ 0 \end{pmatrix}.$$

Around the numerical equilibrium $(\overline{X}, \overline{Y}) \equiv ((\overline{u}, 0), \overline{w}) = (\overline{u}, 0, \overline{w})$, we introduce the affine transformation $T\colon (Z, W) \mapsto (X, Y)$ of the predictor-corrector form by

$$(X, Y) = T(Z, W) := (PZ + \overline{X} - (F_X^{-1} \circ F_Y)|_{(\overline{u}, 0, \overline{w})} W,\ W + \overline{Y}),$$

where P is the nonsingular matrix such that $P^{-1} F_X|_{(\overline{u}, 0, \overline{w})} P = \Lambda = \mathrm{diag}(\lambda_1, \lambda_2)$. Then, denoting $Z = (z_1, z_2)^{\mathrm{T}}$,

$$\begin{pmatrix} z_1' \\ z_2' \end{pmatrix} = \begin{pmatrix} \lambda_1 z_1 \\ \lambda_2 z_2 \end{pmatrix} + P^{-1} \begin{pmatrix} 0 \\ -\delta^{-1} W \end{pmatrix} + \widehat{F}(X, Y, \varepsilon) + \varepsilon \begin{pmatrix} -G(X, Y, \varepsilon)/f_u(\overline{u}) \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} \lambda_1 z_1 \\ \lambda_2 z_2 \end{pmatrix} + P^{-1} \left(\begin{pmatrix} 0 \\ -\delta^{-1} W \end{pmatrix} + \delta^{-1} \begin{pmatrix} 0 \\ -(f(u) - f_u(\overline{u})(u - \overline{u})) + W \end{pmatrix} \right.$$
$$\left. + \varepsilon \begin{pmatrix} -G(X, Y, \varepsilon)/f_u(\overline{u}) \\ 0 \end{pmatrix} \right)$$

$$= \begin{pmatrix} \lambda_1 z_1 \\ \lambda_2 z_2 \end{pmatrix} + P^{-1} \begin{pmatrix} -\varepsilon G(X, Y, \varepsilon)/f_u(\overline{u}) \\ -\delta^{-1}(f(u) - f_u(\overline{u})(u - \overline{u})) \end{pmatrix}.$$

Note that the linear order term of W-variable in the error term is eliminated.

Let $N \subset \mathbb{R}^3$ be an h-set (in the (u, v, w)-coordinate) containing $(\overline{u}, 0, \overline{w})$. Then the Mean Value Theorem implies that the higher order term $f(u) - f_u(\overline{u})(u - \overline{u})$ is included in the enclosure

$$\left\{ \frac{f''(\widetilde{u})}{2} (u - \overline{u})^2 \ \middle|\ u, \widetilde{u} \text{ such that } (u, v, w), (\widetilde{u}, v, w) \in N \right\}.$$

Thanks to the term $(u - \overline{u})^2$, this set is in general very small if N is small. In our example, $f''(u) = -6u + 2(a + 1)$. Obviously, the first component $-\varepsilon G(X, Y, \varepsilon)/f_u(\overline{u})$ is very small if $\varepsilon > 0$ is sufficiently small. Note that the denominator $f_u(\overline{u})$ is bounded away from 0 since we focus on validations of normally hyperbolic invariant manifolds. Finally, for a given compact set N and $\varepsilon_0 > 0$, we obtain enclosures of the error term

$$P^{-1} \begin{pmatrix} [-\varepsilon c^{-1}(u - \gamma w)/f_u(\overline{u}) \mid (u, v, w) \in N,\ \varepsilon \in [0, \varepsilon_0]] \\ [\delta^{-1}\{3\widetilde{u} - (a + 1)\}(u - \overline{u})^2 \mid u, \widetilde{u} \text{ such that } (u, v, w), (\widetilde{u}, v, w) \in N] \end{pmatrix}$$
$$\subset \begin{pmatrix} [\delta_1^-, \delta_1^+] \\ [\delta_2^-, \delta_2^+] \end{pmatrix},$$

which can be computed by interval arithmetics and, in principle, returns very small enclosures for small N and ε_0.

A.2. Jacobian matrices for cone conditions. The Jacobian matrix in these coordinates at $(\widetilde{u}, \widetilde{v}, \widetilde{w}, \eta)$ ignoring the differential of η'-term is

(A.2)
$$\begin{pmatrix} -\varepsilon(cf_u(\overline{u}))^{-1} & 1 & \varepsilon(cf_u(\overline{u}))^{-1}(-f_u(\overline{u})^{-1} + \gamma) & -\varepsilon_0(cf_u(\overline{u}))^{-1}(\widetilde{u} + (f_u(\overline{u})^{-1} - \gamma)\widetilde{w} + \overline{u} - \gamma\overline{w}) \\ -\delta^{-1}f_u(u) & \delta^{-1}c & (\delta f_u(\overline{u}))^{-1}\{-f_u(u) + f_u(\overline{u})\} & 0 \\ \varepsilon c^{-1} & 0 & \varepsilon c^{-1}(f_u(\overline{u})^{-1} - \gamma) & \varepsilon_0 c^{-1}(\widetilde{u} + (f_u(\overline{u})^{-1} - \gamma)\widetilde{w} + \overline{u} - \gamma\overline{w}) \end{pmatrix},$$

where the variable u in the second row corresponds to (A.1). We additionally transform $(\widetilde{u}, \widetilde{v})$ linearly to (z_1, z_2) by $(\widetilde{u}, \widetilde{v}) = P(z_1, z_2)$, where P is the nonsingular matrix such that

$$P^{-1} \begin{pmatrix} 0 & 1 \\ -\delta^{-1} f_u(\overline{u}) & \delta^{-1} c \end{pmatrix} P = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}.$$

The Jacobian matrix (A.2) in the new coordinates $(z_1, z_2, \widetilde{w})$ is then written as

$$\text{(A.3)} \quad \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$+ \begin{pmatrix} F_{11} & F_{12} & \varepsilon(cf_u(\overline{u}))^{-1}(-f_u(\overline{u})^{-1}+\gamma) & -\varepsilon_0(cf_u(\overline{u}))^{-1}(\widetilde{u}+(f_u(\overline{u})^{-1}-\gamma)\widetilde{w}+\overline{u}-\gamma\overline{w}) \\ F_{21} & F_{22} & (\delta f_u(\overline{u}))^{-1}\{-f_u(u)+f_u(\overline{u})\} & 0 \\ \varepsilon c^{-1} & 0 & \varepsilon c^{-1}(f_u(\overline{u})^{-1}-\gamma) & \varepsilon_0 c^{-1}(\widetilde{u}+(f_u(\overline{u})^{-1}-\gamma)\widetilde{w}+\overline{u}-\gamma\overline{w}) \end{pmatrix},$$

$$\begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix} = P^{-1} \begin{pmatrix} -\varepsilon(cf_u(\overline{u}))^{-1} & 0 \\ \delta^{-1}(f_u(\overline{u})-f_u(u)) & 0 \end{pmatrix} P.$$

In order to verify cone conditions in a given h-set N containing $(\overline{u}, 0, \overline{w})$, we compute corresponding maximal singular values in N. All such calculations are done by interval arithmetics. As for calculations of the enclosure of $f_u(\overline{u}) - f_u(u)$ in N, it is useful to use the mean value form

$$f_u(\overline{u}) - f_u(u) \in \big\{ -f_{uu}(\widehat{u})(u - \overline{u}) \mid u, \widehat{u} \text{ such that } (u, v, w), (\widehat{u}, v, w) \in N \big\}$$
$$= \big\{ (6\widehat{u}^2 - 2(a+1))(u - \overline{u}) \mid u, \widehat{u} \text{ such that } (u, v, w), (\widehat{u}, v, w) \in N \big\}.$$

We apply the matrix (A.3) as the Fréchet differential of the vector field in (3.1) to validating cone conditions discussed in Section 3. The m-cone conditions can be treated in the similar manner.

Appendix B. Invariant foliation validation for $W^u(S_\varepsilon)$

In this section, we provide a sufficient condition such that the invariant foliation of $W^u(S_\varepsilon)$ in a given fast-saddle-type block N is derived with computer assistance. Let N be a fast-saddle-type block satisfying the stable m-cone condition for some $m > 0$ in the coordinate (a, b, y, ε) given in Section 3, and $W^u(S_\varepsilon) = \{b = h_u(a, y, \varepsilon)\}$ be the validated unstable manifold of a slow manifold branch S_ε in N for $\varepsilon \in (0, \varepsilon_0]$ (Theorem 3.5). Let $\widehat{K} \subset \mathbb{R}^l$ be a compact set as that in the proof of Theorem 3.5, and $\widehat{N}$ be the compact set such that $\pi_y \widehat{N} = \widehat{K}$ and that $\widehat{N} \cap \{y \in K\} = N$. We use the same notations in Section 3 in the following arguments.

The whole considerations are reduced to vector fields on $W^u(S_\varepsilon)$. Vector field on $W^u(S_\varepsilon)$ is

(B.1)
$$\begin{cases} a' = Aa + F_1(a, h_u(a, y, \eta), y, \eta), \\ y' = \sigma \eta g(a, h_u(a, y, \eta), y, \eta) + \delta \rho(y) n_y, \\ \eta' = 0, \end{cases}$$

where n_y is the unit *outward* normal vector for $\partial \widehat{K}$. Let $z_i = (a_i, b_i, y_i, \eta_i) \in N$, $i = 1, 2$, and $\Delta a = a_1 - a_2$ and so on. Then we have

$$\Delta a' = A\Delta a + F_1(a, h_u(a_1, y_1, \eta_1), y_1, \eta_1) - F_1(a, h_u(a_2, y_2, \eta_2), y_2, \eta_2)$$
$$= A\Delta a + \frac{\partial F_1}{\partial a}(\widetilde{z})\Delta a + \frac{\partial F_1}{\partial b}(\widetilde{z})(h_u(a_1, y_1, \eta_1)$$
$$- h_u(a_2, y_2, \eta_2)) + \frac{\partial F_1}{\partial y}(\widetilde{z})\Delta y + \frac{\partial F_1}{\partial \eta}(\widetilde{z})\Delta \eta,$$
$$\Delta y' = \sigma\{\eta_1 g(a_1, h_u(a_1, y_1, \eta_1), y_1, \eta_1) - \eta_2 g(a_2, h_u(a_2, y_2, \eta_2), y_2, \eta_2)\}$$
$$= \Big\{ \sigma g(a, h_u(a_1, y_1, \eta_1), y_1, \eta_1)\Delta \eta$$
$$+ \sigma \eta_2 \Big(\frac{\partial g}{\partial a}(\widetilde{z})\Delta a + \frac{\partial g}{\partial b}(\widetilde{z})(h_u(a_1, y_1, \eta_1) - h_u(a_2, y_2, \eta_2))$$
$$+ \frac{\partial g}{\partial y}(\widetilde{z})\Delta y + \frac{\partial g}{\partial \eta}(\widetilde{z})\Delta \eta \Big) \Big\},$$
$$\Delta \eta' = 0,$$

for some $\widetilde{z} \in N$. The following definition plays a central role for constructing invariant foliations.

DEFINITION B.1. Let N be a fast-saddle-type block satisfying the stable m-cone condition for some $m > 0$. Let $\sigma_{1,a}$, $\sigma_{1,b}$, $\sigma_{1,y,\eta}$, $\sigma_{\text{slow},a}$, $\sigma_{\text{slow},b}$ and $\sigma_{\text{slow},y,\eta}$ be the upper bounds of maximal singular values of the following matrices in $N \times [0, 1]$:

$$\sigma_{1,a} : \left(\frac{\partial F_1}{\partial a}(z) \right), \qquad \sigma_{1,b} : \left(\frac{\partial F_1}{\partial b}(z) \right), \qquad \sigma_{1,y,\eta} : \left(\frac{\partial F_1}{\partial y}(z) \quad \frac{\partial F_1}{\partial \eta}(z) \right),$$
$$\sigma_{\text{slow},a} : \left(\frac{\partial g}{\partial a}(z) \right), \quad \sigma_{\text{slow},b} : \left(\frac{\partial g}{\partial a}(z) \right), \quad \sigma_{\text{slow},y,\eta} : \left(\eta\frac{\partial g}{\partial y}(z) \quad g(z) + \eta\frac{\partial g}{\partial \eta}(z) \right),$$

with $z = (a, b, y, \eta) \in N \times [0, 1]$. Fix $L > 0$. We shall call that N satisfies *the unstable fiber-L-cone condition*, or simply *the unstable fiber-cone condition*, if

(B.2) $\quad \lambda_A > \beta \equiv \sigma_{1,a} + m(1 + L^{-2})^{1/2}\sigma_{1,b} + L^{-1}\sigma_{1,y,\eta}$
$$+ \sigma(L\sigma_{\text{slow},a} + m(L^2 + 1)^{1/2}\sigma_{\text{slow},b} + \sigma_{\text{slow},y,\eta}).$$

Now we have the following theorem.

Theorem B.2. *Let N be a fast-saddle-type block satisfying the cone condition, the stable m-cone condition with assumptions in Theorem 3.5 (a) and the unstable fiber-L-cone condition. Taking $\widehat{K}$ smaller, if necessary, assume that the same conditions hold in $\widehat{N}$. Then, for each $z \in S_\varepsilon (\subset N)$, there exists a Lipschitzian function $y = h^z_{f,u}(a)$ such that*

$$W^u(z) = \{(a, h_u(a, h^z_{f,u}(a), \varepsilon), h^z_{f,u}(a), \varepsilon)\} \subset \widehat{N}$$

is defined locally. The Lipschitz constant of $h^z_{f,u}$ has an upper bound $\mathrm{sl_u} := m(1+L^{-2})^{1/2}$ for any z.

Since y is not a parameter for (B.1) with $\eta > 0$, then the unstable fiber $W^u(z)$ is not always defined globally in N. A sufficient condition which $W^u(z)$ is defined *globally* in a given block is discussed in Sections 4.6 and 6.1.

Proof. The existence of unstable fibers are proved by the same arguments as that of $W^u(S_\varepsilon)$ in $\widehat{N}$, replacing $\widehat{N}$ by the cone

$$C^{f,u}_L(z) \equiv \{z\} + \{(a, \zeta) \mid |a - \pi_a(z)| \geq L|\zeta - \pi_\zeta(z)|\}, \quad \zeta = (y, \eta)$$

for $z \in S_\varepsilon$. Let

$$M^{f,u}_L(t) := -\langle \Delta a(t), \Delta a(t) \rangle + L^2 \left(\langle \Delta y(t), \Delta y(t) \rangle + \langle \Delta \eta(t), \Delta \eta(t) \rangle \right).$$

Consider the time differential of $M^{f,u}_L$ in the *reverse time direction* $\tau = -t$. Then we have

$$\frac{1}{2}\frac{d}{d\tau}M^{f,u}_L = -\frac{1}{2}\frac{d}{dt}M^{f,u}_L = -L^2(\langle \Delta y, \Delta y' \rangle + \langle \Delta \eta, \Delta \eta' \rangle) + \langle \Delta a, \Delta a' \rangle$$

$$= \langle \Delta a, A\Delta a \rangle + \left\langle \Delta a, \frac{\partial F_1}{\partial a}(\widetilde{z})\Delta a + \frac{\partial F_1}{\partial b}(\widetilde{z})(h_1 - h_2) \right.$$

$$\left. + \frac{\partial F_1}{\partial y}(\widetilde{z})\Delta y + \frac{\partial F_2}{\partial \eta}(\widetilde{z})\Delta \eta \right\rangle$$

$$- \sigma L^2 \left\langle \Delta y, \eta \frac{\partial g}{\partial a}(\widetilde{z})\Delta a + \eta \frac{\partial g}{\partial b}(\widetilde{z})(h_1 - h_2) \right.$$

$$\left. + \eta \frac{\partial g}{\partial y}(\widetilde{z})\Delta y + \left(g_1 + \eta \frac{\partial g}{\partial \eta}(\widetilde{z}) \right)\Delta \eta \right\rangle,$$

where $g_1 = g(a_1, h_u(a_1, y_1, \eta_1), y_1, \eta_1)$ and $h_i = h_u(a_i, y_i, \eta_i)$, $i = 1, 2$. We further have

$$\frac{1}{2}\frac{d}{d\tau}M^{f,u}_L \geq (\lambda_A - \sigma_{1,a})|\Delta a|^2 - \sigma_{1,b}|\Delta a||h_1 - h_2| - \sigma_{1,y,\eta}|\Delta a||\Delta y\ \Delta \eta|$$

$$- \sigma L^2 \{\eta \sigma_{\mathrm{slow},a}|\Delta a| + \eta \sigma_{\mathrm{slow},b}|h_1 - h_2| + \sigma_{\mathrm{slow},y,\eta}|\Delta y\ \Delta \eta|\}|\Delta y|,$$

where $|\Delta y\ \Delta \eta| = (|\Delta y|^2 + |\Delta \eta|^2)^{1/2}$, etc. Now let $z_1, z_2 \in W^u(S_\varepsilon)$. Then the Lipschitzian function h_u has the following estimate, which follows from the stable m-cone condition:

$$|h_1 - h_2| \leq m|\Delta a\ \Delta y\ \Delta \eta|.$$

Moreover, if $z_2 \in W^u(S_\varepsilon) \cap C_m^s(z_1)^c \cap \{M_L^{f,u} = 0\}$, we further have

$$|h_1 - h_2| \leq m|\Delta a \, \Delta y \, \Delta \eta| \leq m(L^2 + 1)^{1/2}|\Delta y \, \Delta \eta|.$$

Hence we have

(B.3)
$$\frac{1}{2}\frac{d}{d\tau}M_L^{f,u} \geq L^2 \{\lambda_A - \beta\} |\Delta y \, \Delta \eta|^2,$$

where β is given in (B.2). The differential is thus positive if $\lambda_A > \beta$, unless $\Delta y = \Delta \eta = 0$. Summarizing the above argument we have

LEMMA B.3. *Let N be a fast-saddle-type block satisfying the stable m-cone conditions such that the slow manifold S_ε as well as $W^u(S_\varepsilon) = \{b = h_u(a, y, \varepsilon)\}$ is validated within N for $\varepsilon \in (0, \varepsilon_0]$. Let $M_L^{f,u}(t) := -|a_1(t) - a_2(t)|^2 + L^2|\zeta_1(t) - \zeta_2(t)|^2$, where $(a_i, \zeta_i) \equiv (a_i, y_i, \eta_i) \in W^u(S_\varepsilon)$, $i = 1, 2$, with some $L > 0$. Assume that the unstable fiber-L-cone condition is satisfied in N. Then $dM_L^{f,u}/d\tau > 0$ on $\{M_L^{f,u} = 0\}$ unless $(y_1, \varepsilon_1) = (y_2, \varepsilon_2)$, where $\tau = -t$.*

Go back to the proof of the theorem. Lemma B.3 indicates that the conic part $\{|a| = L|\zeta - \widehat{\zeta}|\}$ of the boundary $\partial C_L^{f,u}(z_1)$ with $z_1 \in S_\varepsilon$ is the entrance in the t time direction. Then the existence argument in the proof of Theorem 3.5 with time-reversal flow yields the existence of a point $z_2 \in S_{\widehat{a}} \equiv \{(a, y, \eta) \in C_L^{f,u}(z_1) \mid a = \widehat{a}\} \cap W^u(S_\varepsilon)$ such that $\varphi_\varepsilon(t, z_2) \in C_L^{f,u}(\varphi_\varepsilon(t, z_1))$ for all $t \leq 0$, as long as $\pi_y S_{\widehat{a}} \subset \text{int } \widehat{K}$ holds. Since such a point can be chosen for any $\widehat{a}$, this argument gives a component $\zeta = (y, \eta)$ for each $\widehat{a}$. The y-component of ζ gives a function $y = h_{f,u}^z(a)$.

The rest is the well-definedness and Lipschitz continuity of $h_{f,u}^z$. The inequality (B.3) also yields

(B.4)
$$\frac{1}{2}\frac{d}{d\tau}|\Delta y \, \Delta \eta|^2 \geq \{\lambda_A - \beta\} |\Delta y \, \Delta \eta|^2.$$

If, for fixed ε, both points (a, y_1) and (a, y_2) are on the graph $\{y = h_{f,u}^z(a)\}$, these points satisfy $M_L^{f,u}(\tau) > 0$ and stay in N for all $\tau \in [0, \infty)$, equivalently, $t \in (-\infty, 0]$. The inequality (B.4) with the unstable fiber-L-cone condition in N indicates that the difference $|\Delta y \, \Delta \eta|$ diverges as $\tau \to \infty$, equivalently $t \to -\infty$. However, it contradicts the fact that N is compact, which shows the well-definedness of $y = h_{f,u}^z(a)$ and its Lipschitz continuity.

For fixed ε, the slope of unstable fiber, namely the Lipschitz constant of $h_{f,u}^z$ is estimated as follows. Let $(a_i, b_i, y_i, \varepsilon)$, $i = 1, 2$, be points on $W^u(z)$. Note that we have to pay attention to the variance of b-component. First we have

$$|b_1 - b_2| = |h_u(a_1, y_1, \varepsilon) - h_u(a_2, y_2, \varepsilon)| \leq m(|a_1 - a_2|^2 + |y_1 - y_2|^2)^{1/2}.$$

Second we have

$$|y_1 - y_2| = |h_{f,u}(a_1) - h_{f,u}(a_2)| \leq L^{-1}|a_1 - a_2|.$$

Summarizing these estimates, we have

$$(|b_1 - b_2|^2 + |y_1 - y_2|^2)^{1/2} \leq m(1 + L^{-2})^{1/2}|a_1 - a_2|,$$

namely, the slope of unstable fiber in $W^u(S_\varepsilon)$ has the upper bound $m(1 + L^{-2})^{1/2} \equiv \mathrm{sl_u}$. □

REMARK B.4. If we formally set $m = 0$ in (B.2), the unstable fiber-cone condition turns out to be independent of differentials of vector fields in the b-component. In Jones's lecture note [19], whole arguments concerning with invariant foliations are developed *after straightening $W^u(S_\varepsilon)$ and $W^s(S_\varepsilon)$*; namely $\{b = 0\}$ and $\{a = 0\}$, respectively. In our arguments, on the other hand, the locally invariant manifold $W^u(S_\varepsilon)$ is kept bended. Instead, we estimate the strength of bend by m, which enables us to validate invariant foliations without any helps of nonlinear straightening transformations.

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