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MULTIPLICITY OF SOLUTIONS FOR p-LAPLACIAN TYPE ELLIPTIC PROBLEMS WITH ELECTROMAGNETIC FIELDS AND CRITICAL NONLINEARITY

Zhongyi Zhang

Abstract. We consider a class of p-Laplacian type elliptic problems with electromagnetic fields and critical nonlinearity in bounded domains. New results about the existence and multiplicity of solutions to these problems are obtained by using the concentration-compactness principle and variational method.

1. Introduction

In this paper we deal with the existence and multiplicity of solutions to the following p-Laplacian type elliptic problems with electromagnetic fields and critical nonlinearity:

$$\begin{cases} \Big[g\Big(\displaystyle\int_\Omega |\nabla_A u|^p\,dx\Big)\Big]\Delta_{p,A}u = \lambda h(x,|u|^p)|u|^{p-2}u + |u|^{p^*-2}u, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where $\Delta_{p,A}u(x) := \mathrm{div}(|\nabla u + iA(x)u|^{p-2}(\nabla u + iA(x)u)$, here i is the imaginary unit, $\Omega \subset \mathbb{R}^N$ is an open bounded domain with smooth boundary and λ is a positive parameter, $p^* = Np/(N-p)$ is the critical exponent according to

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the Sobolev embedding. Functions $h\colon \overline{\Omega}\times\mathbb{R}\to\mathbb{R}$, $g\colon \mathbb{R}^+\to\mathbb{R}^+$ are continuous functions that satisfy the following conditions:

(G1) There exists $\alpha_0>0$ such that $g(t)\geq\alpha_0$ for all $t\geq 0$.
(G2) There exists σ such that $1<p/\sigma<p^*$ and $G(t)\geq\sigma g(t)t$ for all $t\geq 0$, where $G(t)=\int_0^t g(s)\,ds$.
(H1) $h(x,s)\in C(\Omega\times\mathbb{R},\mathbb{R})$, $h(x,-s)=-h(x,s)$ for all $s\in\mathbb{R}$.
(H2) $\lim\limits_{|s|\to\infty} h(x,s)/s^{(p^*-p)/p}=0$ uniformly for $x\in\Omega$.
(H3) $\lim\limits_{|s|\to 0^+} h(x,s)/s^{1/\sigma-1}=\infty$ uniformly for $x\in\Omega$.

There is a vast literature concerning the existence and multiplicity of solutions for (1.1) with no magnetic field, namely $A(x)\equiv 0$, $g(t)\equiv 1$ and $p=2$, starting from the celebrated paper by Brezis and Nirenberg [2]. For example, Li and Zou [28] obtained infinitely many solutions with odd nonlinearity. Chen and Li [7] established the existence of infinitely many solutions by using the minimax procedure. For more related results, we refer the interested readers to [3], [5], [14], [15], [17], [22], [34] and references therein.

On the one hand, for the special case of problem (1.1) with $A(x)\equiv 0$, $g(t)=at+b$ and $p=2$, equation (1.1) reduces to the following Dirichlet problem of Kirchhoff type:

$$\begin{cases} -\left(a+b\displaystyle\int_\Omega |\nabla u|^2\,dx\right)\Delta u=f(x,u), & x\in\Omega,\\ u|_{\partial\Omega}=0, \end{cases}\tag{1.2}$$

where $\Omega\subset\mathbb{R}^N$, problem (1.2) is a generalization of a model introduced by Kirchhoff [23]. More precisely, Kirchhoff proposed a model given by the equation

$$\rho\frac{\partial^2 u}{\partial t^2}-\left(\frac{\rho_0}{h}+\frac{E}{2L}\int_0^L\left|\frac{\partial u}{\partial x}\right|^2 dx\right)\frac{\partial^2 u}{\partial x^2}=0,\tag{1.3}$$

where ρ,ρ_0,h,E,L are constants, which extends the classical D'Alembert's wave equation, by considering the effects of changes in the length of strings during vibrations. Equation (1.2) is related to the stationary analogue of problem (1.3). Problem (1.2) received much attention only after Lions [26] proposed an abstract framework to the problem. Some important and interesting results can be found in, for example, [11], [10], [18], [20], [21], [24], [30], [32], [39]. We note that the results dealing with problem (1.2) with critical nonlinearity are relatively scarce. For the case $p\neq 2$, by means of a direct variational method, the authors proved the existence and multiplicity of solutions to a class of p-Kirchhoff-type problem with Dirichlet boundary data [12]. In [29], the author showed the existence of infinite solutions to the p-Kirchhoff type quasilinear elliptic equation. But they did not give any further information on the sequence of solutions.

For $A(x) \not\equiv 0$, there are also many works dealing with the magnetic case. The first one seems to be [13] where the existence of standing waves was obtained for $\hbar > 0$ fixed and for special classes of magnetic fields. If A and W are periodic functions, the existence of various types of solutions for fixed $\hbar > 0$ was proved in [1] by applying minimax arguments. Concerning semiclassical bound states, it is proved in [25] that for $\hbar > 0$ sufficiently small there exists a least energy solution which concentrates near the global minimum of W. A multiplicity result for solutions was obtained in [8] by using a topological argument. There it was also proved that the magnetic potential A contributes only to the phase factor of solitary solutions for $\hbar > 0$ sufficiently small. In [9] single-bump bound states were considered by using perturbation methods, these concentrate near a non-degenerate critical point of W as $\hbar \to 0$. For the critical growth case, Wang [37] studied the electromagnetic Schrödinger equations

$$-(\nabla + iA(x))^2 u(x) + \lambda V(x)u(x) = K(x)|u|^{2^*-2}u \quad \text{for } x \in \mathbb{R}^N. \tag{1.4}$$

By applying the linking theorem twice to the corresponding functional, they established the existence results. Chabrowski and Szulkin [4] considered problems (1.2) under the assumption that V changes sign, by using a minmax type argument based on a topological linking, they obtained a solution in the Sobolev space which is defined in the paper. For $K(x) \equiv 1$, Han [19] studied problem (1.4) and established the existence of nontrivial solutions in the critical case by means of variational method. For more results, we refer the reader to [35]–[37] and the references therein.

Motivated by the above, we aim to show the existence of infinitely many solutions of problem (1.1) and a sequence of infinitely many arbitrarily small solutions converging to zero, by using a new version of the symmetric mountain-pass lemma due to Kajikiya [31].

To the best of our knowledge, the existence and multiplicity of solutions to problem (1.1) has not been studied so far by variational methods. As we shall see, problem (1.1) can be viewed as an elliptic equation coupled with a non-local term. The competing effect of the non-local term with the critical nonlinearity and the lack of compactness of the embedding of $W_0^{1,p}(\Omega, \mathbb{C})$ into the space $L^{p^*}(\Omega, \mathbb{C})$ prevents us from using variational methods in a standard way. Some new estimates for such Kirchhoff equation involving Palais–Smale sequences, which are key points to apply this kind of theory, are needed to be re-established. We mainly follow the idea of [16], [31]. Let us point out that although the idea was used before for other problems, the adaptation of the procedure to our problem is not trivial since due to the appearance of non-local term we must consider our problem in a suitable space and so we need more delicate estimates.

Our main result of this paper is the following.

THEOREM 1.1. *Suppose that* (G1)–(G2) *and* (H1)–(H3) *hold. Then there exists* $\lambda^* > 0$ *such that for any* $\lambda \in (0, \lambda^*)$ *problem* (1.1) *has a sequence of nontrivial solutions* $\{u_n\}$ *and* $u_n \to 0$ *as* $n \to \infty$.

2. Preliminary lemmas

We shall denote by $W_0^{1,p}(\Omega, \mathbb{C})$ the closure of $C_0^\infty(\Omega, \mathbb{C})$ under the norm induced by

$$\|u\|^p = \int_\Omega |\nabla_A u|^p \, dx,$$

where $\nabla_A u := \nabla u + iAu$. Similar to the diamagnetic inequality [13], we have the following inequality:

$$|\nabla_A u(x)| \geq |\nabla |u(x)||, \quad \text{for } u \in W_0^{1,p}(\Omega, \mathbb{C}).$$

Indeed, since A is real-valued

$$|\nabla |u|(x)| = \left| \mathrm{Re}\left(\nabla u \, \frac{\overline{u}}{|u|} \right) \right| = \left| \mathrm{Re}(\nabla u + iAu) \frac{\overline{u}}{|u|} \right| \leq |\nabla u + iAu|$$

(the bar denotes complex conjugation). Thus, if $u \in W_0^{1,p}(\Omega, \mathbb{C})$, $|u|$ belongs to the usual Sobolev space $W_0^{1,p}(\Omega, \mathbb{R})$. Moreover, the embedding $W_0^{1,p}(\Omega, \mathbb{C}) \hookrightarrow L^q(\Omega, \mathbb{C})$ is continuous for each $1 \leq q \leq p^*$ and it is compact for $1 \leq q < p^*$.

Consider the energy functional $J\colon W_0^{1,p}(\Omega, \mathbb{C}) \to \mathbb{R}$ defined by

$$J(u) = \frac{1}{p} G(\|u\|^p) - \lambda \int_\Omega H(x, |u|^p) \, dx - \frac{1}{p^*} \int_\Omega |u|^{p^*} \, dx.$$

It is well-known that a critical point of J is a weak solution of problem (1.1) (see [38]). Denote by J' the derivative operator of J in the weak sense. Then

$$\begin{aligned} \langle J'(u), v \rangle = \mathrm{Re} \bigg\{ & g(\|u\|^p) \int_\Omega (|\nabla_A u|^{p-2} \nabla_A u \cdot \overline{\nabla_A v}) \, dx \\ & - \int_\Omega |u|^{p^*-2} u \overline{v} \, dx - \lambda \int_\Omega h(x, |u|^p) |u|^{p-2} u \overline{v} \, dx \bigg\}, \end{aligned}$$

for all $u, v \in W_0^{1,p}(\Omega, \mathbb{C})$.

Hereafter, we denote by $\lambda_1 > 0$ the best constant of the compact embedding $W_0^{1,p}(\Omega, \mathbb{C}) \hookrightarrow L^p(\Omega, \mathbb{C})$ which is given by

$$\lambda_1 = \inf_{u \in W_0^{1,p}(\Omega, \mathbb{C}) \setminus \{0\}} \frac{\displaystyle\int_\Omega |\nabla_A u|^p \, dx}{\left(\displaystyle\int_\Omega |u|^p \, dx \right)^{1/p}}.$$

Denote by S the best Sobolev constant of the embedding $W_0^{1,p}(\Omega,\mathbb{R})\hookrightarrow L^{p^*}(\Omega,\mathbb{R})$ which is given by

$$S = \inf_{u\in W_0^{1,p}(\Omega,\mathbb{R})\setminus\{0\}} \frac{\displaystyle\int_\Omega |\nabla u|^p\,dx}{\bigg(\displaystyle\int_\Omega |u|^{p^*}\,dx\bigg)^{p/p^*}}.$$

It is well-known that S is independent of Ω and it is never achieved, except when $\Omega=\mathbb{R}^N$.

To use variational methods, we give some results related to the Palais–Smale compactness condition. Recall that a sequence (u_n) is called a Palais–Smale sequence of J at the level c if $J(u_n)\to c$ and $J'(u_n)\to 0$.

DEFINITION 2.1 (see [40]). Let X be a reflexive Banach space and X^* its topological dual. The mapping $A\colon X\to X^*$ is said to have type (S_+) if any sequence u_n in X satisfying $u_n\rightharpoonup u_0$ in X and

$$\limsup_{n\to+\infty}\langle A(u_n),u_n-u_0\rangle\le 0$$

contains a convergent subsequence.

For each $u\in W_0^{1,p}(\Omega,\mathbb{C})$, we define $A\colon W_0^{1,p}(\Omega,\mathbb{C})\to W_0^{-1,p'}(\Omega,\mathbb{C})$ by

$$\langle A(u),\varphi\rangle=\int_\Omega |\nabla u|^{p-2}\nabla u\nabla\varphi\,dx,\quad\text{for all }\varphi\in W_0^{1,p}(\Omega,\mathbb{C}).$$

REMARK 2.2. It is easy to prove that the operator A is of type (S_+).

We recall the second concentration-compactness principle of Lions [27].

LEMMA 2.3 (see [27]). *Let $\{u_n\}$ be a weakly convergent sequence to u in $W_0^{1,p}(\Omega,\mathbb{R})$ such that $|u_n|^{p^*}\rightharpoonup\nu$ and $|\nabla u_n|\rightharpoonup\mu$ in the sense of measures. Then, for some at most countable index set I,*

(a) $\nu=|u|^{p^*}+\sum_{j\in I}\delta_{x_j}\nu_j$, $\nu_j>0$,
(b) $\mu\ge|\nabla u|^p+\sum_{j\in I}\delta_{x_j}\mu_j$, $\mu_j>0$,
(c) $\mu_j\ge S\nu_j^{p/p^*}$,

where S is the best Sobolev constant, $x_j\in\mathbb{R}^N$, δ_{x_j} are Dirac measures at x_j and μ_j,ν_j are constants.

Under assumptions (H_1) and (H_2), we have

$$h(x,|s|^p)|s|^p=o\big(|s|^{p^*}\big),\qquad H(x,|s|^p)=o\big(|s|^{p^*}\big),$$

which means that, for all $\varepsilon>0$, there exist $a(\varepsilon),b(\varepsilon)>0$ such that

$$|h(x,|s|^p)|s|^p|\le a(\varepsilon)+\varepsilon|s|^{p^*},\tag{2.1}$$

$$|H(x,|s|^p)|\le b(\varepsilon)+\varepsilon|s|^{p^*}.\tag{2.2}$$

Hence, for some $c(\varepsilon) > 0$,

$$H(x, |s|^p) - \frac{\sigma}{p} h(x, |s|^p)|s|^p \le c(\varepsilon) + \varepsilon |s|^{p^*}. \tag{2.3}$$

LEMMA 2.4. *Suppose that* (G1)–(G2) *and* (H1)–(H3) *hold. Then, for any* $\lambda > 0$, *the functional* J *satisfies the local* $(\mathrm{PS})_c$ *condition in*

$$c \in \bigg(-\infty, \frac{p^*\sigma - p}{2pp^*} (\alpha_0 S)^{N/p} - \lambda c\bigg(\frac{p^*\sigma - p}{2pp^*\lambda} \bigg) |\Omega| \bigg)$$

in the following sense: if

$$J(u_n) \to c < \frac{p^*\sigma - p}{2pp^*} (\alpha_0 S)^{N/p} - \lambda c\bigg(\frac{p^*\sigma - p}{2pp^*\lambda} \bigg) |\Omega|$$

and $J'(u_n) \to 0$ *for some sequence in* $W_0^{1,p}(\Omega, \mathbb{C})$, *then* $\{u_n\}$ *contains a subsequence converging strongly in* $W_0^{1,p}(\Omega)$.

PROOF. Let $\{u_n\}$ be a sequence in $W_0^{1,p}(\Omega, \mathbb{C})$ such that

$$J(u_n) = \frac{1}{p} G(\|u_n\|^p) - \lambda \int_\Omega H(x, |u_n|^p)\, dx - \frac{1}{p^*} \int_\Omega |u_n|^{p^*} dx = c + o(1), \tag{2.4}$$

$$\begin{aligned} \langle J'(u_n), v\rangle = \mathrm{Re} \bigg\{ & g\big(\|u_n\|^p\big) \int_\Omega \big(|\nabla_A u_n|^{p-2} \nabla_A u_n \cdot \overline{\nabla_A v}\big)\, dx \\ & - \int_\Omega |u_n|^{p^*-2} u_n \overline{v}\, dx - \lambda \int_\Omega h(x, |u_n|^p) |u_n|^{p-2} u_n \overline{v}\, dx \bigg\} = o(1)\|u_n\|. \end{aligned} \tag{2.5}$$

From (G2) we see that

$$G(\|u_n\|^p) - \sigma g(\|u_n\|^p)\|u_n\|^p \ge 0 \quad \text{for all } n.$$

By (2.4) and (2.5), we have

$$\begin{aligned} c + o(1)\|u_n\| &= J(u_n) - \frac{\sigma}{p} \langle J'(u_n), u_n\rangle \\ &= \frac{1}{p} G(\|u_n\|^p) - \frac{\sigma}{p} g\big(\|u_n\|^p\big)\|u_n\|^p + \bigg(\frac{\sigma}{p} - \frac{1}{p^*} \bigg) \int_\Omega |u_n|^{p^*} dx \\ &\quad - \lambda \int_\Omega H(x, |u_n|^p)\, dx + \frac{\sigma}{p} \lambda \int_\Omega h(x, |u_n|^p)|u_n|^p\, dx \\ &\ge \frac{p^*\sigma - p}{pp^*} \int_\Omega |u_n|^{p^*} dx - \lambda \int_\Omega H(x, |u_n|^p)\, dx + \frac{\sigma}{p} \lambda \int_\Omega h(x, |u_n|^p)|u_n|^p\, dx, \end{aligned}$$

i.e.

$$\frac{p^*\sigma - p}{pp^*} \int_\Omega |u_n|^{p^*} dx \le \lambda \int_\Omega \bigg(H(x, |u_n|^p) - \frac{\sigma}{p} h(x, |u_n|^p)|u_n|^p \bigg) dx + c + o(1)\|u_n\|.$$

Then, by inequality (2.3), we have

$$\bigg(\frac{p^*\sigma - p}{pp^*} - \lambda\varepsilon \bigg) \int_\Omega |u_n|^{p^*} dx \le \lambda c(\varepsilon)|\Omega| + c + o(1)\|u_n\|.$$

Setting $\varepsilon = (p^*\sigma - p)/(2pp^*\lambda)$, we get

$$\int_\Omega |u_n|^{p^*}\,dx \le M + o(1)\|v_n\|, \tag{2.6}$$

where $o(1) \to 0$ and M is some positive number. On the other hand, by (2.2) and (2.6), we have

$$\begin{aligned} c + o(1)\|u_n\| &= J(u_n) \\ &= \frac{1}{p}G(\|u_n\|^p) - \lambda \int_\Omega H(x, |u_n|^p)\,dx - \frac{1}{p^*}\int_\Omega |u_n|^{p^*}\,dx \\ &\ge \frac{\alpha_0\sigma}{p}\|u_n\|^p - \lambda b(\varepsilon)|\Omega| - \left[\frac{1}{p^*} + \lambda\varepsilon\right]\int_\Omega |u_n|^{p^*}\,dx. \end{aligned} \tag{2.7}$$

Therefore, (2.6) and (2.7) imply that $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega, \mathbb{C})$. Hence, by the diamagnetic inequality, $\{|u_n|\}$ is bounded in $W_0^{1,p}(\Omega, \mathbb{R})$. Then, for some subsequence, there is $u \in W_0^{1,p}(\Omega, \mathbb{C})$ such that $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega, \mathbb{C})$. We claim that

$$\int_\Omega |u_n|^{p^*}\,dx \to \int_\Omega |u|^{p^*}\,dx. \tag{2.8}$$

In order to prove this claim, we suppose that

$$|\nabla|u_n||^p \rightharpoonup |\nabla|u||^p + \mu \quad \text{and} \quad |u_n|^{p^*} \rightharpoonup |u|^{p^*} + \nu \quad (\text{weak}^* \text{ sense of measures}).$$

Using the concentration compactness-principle, we obtain a countable index set I, sequences $\{x_j\} \subset \mathbb{R}^N$, and $\{\mu_j\}, \{\nu_j\} \subset (0, \infty)$ such that

$$\nu = \sum_{j\in I} \delta_{x_j}\nu_j, \quad \mu \ge \sum_{j\in I} \delta_{x_j}\mu_j \quad \text{and} \quad \mu_j \ge S\nu_j^{p/p^*} \tag{2.9}$$

for all $j \in I$, where δ_{x_j} is the Dirac measure mass at $x_j \in \overline{\Omega}$. Let $\psi \in C_0^\infty(\mathbb{R}^N, [0,1])$ such that $0 \le \psi \le 1$,

$$\psi(x) = \begin{cases} 1 & \text{if } |x| < 1, \\ 0 & \text{if } |x| \ge 2, \end{cases} \tag{2.10}$$

and $|\nabla\psi|_\infty \le 2$.

For $\varepsilon > 0$ and $j \in I$, denote $\psi_\varepsilon^j(x) = \psi((x - x_j)/\varepsilon)$. Since $\{u_n\psi_\varepsilon^j\}$ is bounded in $W_0^{1,p}(\Omega, \mathbb{C})$ and ψ_ε^j takes values in $\mathbb{R}$, a direct calculation shows that

$$\langle J'(u_n), \psi_\varepsilon^j u_n\rangle \to 0 \quad \text{and} \quad \overline{\nabla_A(u_n\psi_\varepsilon^j)} = i\overline{u_n}\nabla\psi_\varepsilon^j + \psi_\varepsilon^j\overline{\nabla_A u_n},$$

that is,

$$g(\|u_n\|^p)\int_\Omega |\nabla_A u_n|^p\psi_\varepsilon^j\,dx = -g(\|u_n\|^p)\mathrm{Re}\left(\int_{\mathbb{R}^N} i|\nabla_A u_n|^{p-2}\overline{u_n}\nabla_A u_n\overline{\nabla_A\psi_\varepsilon^j}\,dx\right) + \lambda\int_\Omega h(x,|u_n|^p)|u_n|^p\psi_\varepsilon^j\,dx + \int_\Omega |u_n|^{p^*}\psi_\varepsilon^j\,dx + o_n(1). \tag{2.11}$$

Hence, by Hölder's inequality, we obtain

$$\begin{aligned}\limsup_{n\to\infty}&\left|\mathrm{Re}\int_{\mathbb{R}^N} i|\nabla_A u_n|^{p-2}\overline{u_n}\nabla_A u_n\overline{\nabla\psi_\varepsilon^j}\,dx\right| \\ &\le \limsup_{n\to\infty}\left(\int_{\mathbb{R}^N}|\nabla_A u_n|^p\,dx\right)^{(p-1)/p}\left(\int_{\mathbb{R}^N}|\overline{u_n\nabla_A\psi_\varepsilon^j}|^p\,dx\right)^{1/p} \\ &\le C_1\left(\int_{B(x_j,2\varepsilon)}|u|^p|\nabla_A\psi_\varepsilon^j|^p\,dx\right)^{1/p} \\ &\le C_1\left(\int_{B(x_j,2\varepsilon)}|\nabla_A\psi_\varepsilon^j|^N\,dx\right)^{1/N}\left(\int_{B(x_j,2\varepsilon)}|u|^{p^*}\,dx\right)^{1/p^*} \\ &\le C_2\left(\int_{B(x_j,2\varepsilon)}|u|^{p^*}\,dx\right)^{1/p^*}\to 0 \quad \text{as } \varepsilon\to 0.\end{aligned} \tag{2.12}$$

Since ψ_ε^j has compact support, letting $n\to\infty$ in (2.11), we deduce from (2.11) and (2.12) that

$$\alpha_0\int_\Omega\psi_\varepsilon^j\,d\mu \le C_2\left(\int_{B(x_j,2\varepsilon)}|u|^{p^*}dx\right)^{1/p^*}+\lambda\int_{B(x_j,2\varepsilon)}h(x,|u|^p)|u|^p\,dx+\int_\Omega\psi_\varepsilon^j\,d\nu.$$

Letting $\varepsilon\to 0$, we obtain $\alpha_0\mu_j\le\nu_j$. Therefore,

$$(\alpha_0 S)^{N/p}\le\nu_j. \tag{2.13}$$

We will prove that this inequality is not possible. Let us assume that $(\alpha_0 S)^{N/p}\le\nu_{j_0}$ for some $j_0\in I$. Since

$$c = J(u_n)-\frac{\sigma}{p}\langle J'(u_n),u_n\rangle + o_n(1),$$

it follows that

$$\begin{aligned}c &= \lim_{n\to\infty}\left(J(u_n)-\frac{\sigma}{p}\langle J'(u_n),u_n\rangle\right) \\ &\ge\left(\frac{\sigma}{p}-\frac{1}{p^*}\right)\int_\Omega|u_n|^{p^*}dx-\lambda\int_\Omega\left[H(x,|u_n|^p)-\frac{\sigma}{p}h(x,|u_n|^p)|u_n|^p\right]dx \\ &\ge\left(\frac{p^*\sigma-p}{pp^*}-\lambda\varepsilon\right)\int_\Omega\psi_\varepsilon^{j_0}|u_n|^{p^*}dx-\lambda c(\varepsilon)|\Omega|.\end{aligned}$$

Letting $\varepsilon = (p^*\sigma - p)/(2pp^*\lambda)$ and $n \to \infty$, we obtain

$$\begin{aligned} c &\geq \frac{p^*\sigma - p}{2pp^*} \sum_{j\in J} \psi_\varepsilon^{j_0}(x_j)\nu_j - \lambda c\bigg(\frac{p^*\sigma - p}{2pp^*\lambda}\bigg)|\Omega| \\ &\geq \frac{p^*\sigma - p}{2pp^*}(\alpha_0 S)^{N/p} - \lambda c\bigg(\frac{p^*\sigma - p}{2pp^*\lambda}\bigg)|\Omega|. \end{aligned}$$

This is impossible. Then $I = \emptyset$, and hence $u_n \to u$ in $L^{p^*}(\Omega, \mathbb{C})$.

Then, using (2.8) and the fact that $u_n \to u$ in $L^{p^*}(\Omega, \mathbb{C})$, we have

$$\lim_{n\to\infty} \operatorname{Re} \int_\Omega h(x, |u_n|^p)|u_n|^{p-2}\overline{(u_n - u)}\, dx = 0, \tag{2.14}$$

$$\lim_{n\to\infty} \operatorname{Re} \int_\Omega |u_n|^{p^*-2}u_n\overline{(u_n - u)}\, dx = 0. \tag{2.15}$$

From $\langle J'(u_n), u_n - u\rangle = o_n(1)$, we deduce that

$$\begin{aligned} \langle J'(u_n), u_n - u\rangle = \operatorname{Re}\bigg\{ & g(\|u_n\|^p) \int_\Omega |\nabla_A u_n|^{p-2}\nabla_A u_n \overline{\nabla_A(u_n - u)}\, dx \\ & - \lambda \int_\Omega h(x, |u_n|^p)|u_n|^{p-2}\overline{(u_n - u)}\, dx - \int_\Omega |u_n|^{p^*-2}u_n\overline{(u_n - u)}\, dx\bigg\} = o_n(1). \end{aligned}$$

This, (2.14) and (2.15) imply

$$\lim_{n\to\infty} g(\|u_n\|^p) \operatorname{Re} \int_\Omega |\nabla_A u_n|^{p-2}\nabla u_n \overline{\nabla_A(u_n - u)}\, dx = 0.$$

Since u_n is bounded and g is continuous, up to subsequence, there is $t_0 \geq 0$ such that

$$g(\|u_n\|^p) \to g(t_0^p) \geq \alpha_0, \quad \text{as } n \to \infty,$$

and so

$$\lim_{n\to\infty} \operatorname{Re} \int_\Omega |\nabla_A u_n|^{p-2}\nabla_A u_n \overline{\nabla_A(u_n - u)}\, dx = 0.$$

Thus, by the (S_+) property, $u_n \to u$ strongly in $W_0^{1,p}(\Omega, \mathbb{C})$. □

3. Existence of a sequence of arbitrarily small solutions

In this section, we prove the existence of infinitely many solutions of (1.1) which tend to zero. Let X be a Banach space and denote

$$\Sigma = \{A \subset X \setminus \{0\} : A \text{ is closed in } X \text{ and symmetric with respect to the orgin}\}.$$

For $A \in \Sigma$, we define genus $\gamma(A)$ as

$$\gamma(A) = \inf\{m \in N : \exists\, \varphi \in C(A, \mathbb{R}^m \setminus \{0\},\ -\varphi(x) = \varphi(-x))\}.$$

If there is no mapping φ as above for any $m \in N$, then $\gamma(A) = +\infty$. Let Σ_k denote the family of closed symmetric subsets A of X such that $0 \notin A$ and $\gamma(A) \geq k$. We list some properties of the genus (see [31], [33]).

PROPOSITION 3.1. *Let A and B be closed symmetric subsets of X which do not contain the origin. Then the following hold*:

(a) *If there exists an odd continuous mapping from A to B, then $\gamma(A) \leq \gamma(B)$.*
(b) *If there is an odd homeomorphism from A to B, then $\gamma(A) = \gamma(B)$.*
(c) *If $\gamma(B) < \infty$, then $\gamma(\overline{A \setminus B}) \geq \gamma(A) - \gamma(B)$.*
(d) *The n-dimensional sphere S^n has a genus of $n+1$ by the Borsuk–Ulam theorem.*
(e) *If A is compact, then $\gamma(A) < +\infty$ and there exists $\delta > 0$ such that $U_\delta(A) \in \Sigma$ and $\gamma(U_\delta(A)) = \gamma(A)$, where $U_\delta(A) = \{x \in X : \|x - A\| \leq \delta\}$.*

The following version of the symmetric mountain-pass lemma is due to Kajikiya [31].

LEMMA 3.2. *Let E be an infinite-dimensional space, $J \in C^1(E, \mathbb{R})$ and suppose the following conditions hold*:

(C_1) *J is even, bounded from below, $J(0) = 0$ and J satisfies the local Palais–Smale condition, i.e. for some $\overline{c} > 0$, every sequence $\{u_k\}$ in E such that $\lim\limits_{k\to\infty} J(u_k) = c < \overline{c}$ and $\lim\limits_{k\to\infty} \|J'(u_k)\|_{E^*} = 0$ has a convergent subsequence.*
(C_2) *For each $k \in N$, there exists $A_k \in \Sigma_k$ such that $\sup\limits_{u \in A_k} J(u) < 0$.*

Then either (R_1) *or* (R_2) *below holds.*

(R_1) *There exists a sequence $\{u_k\}$ such that $J'(u_k) = 0$, $J'(u_k) < 0$ and $\{u_k\}$ converges to zero.*
(R_2) *There exist two sequences $\{u_k\}$ and $\{v_k\}$ such that $J'(u_k) = 0$, $J(u_k) < 0$, $u_k \neq 0$, $\lim\limits_{k\to\infty} u_k = 0$, $J'(v_k) = 0$, $J(v_k) < 0$, $\lim\limits_{k\to\infty} v_k = 0$, and $\{v_k\}$ converges to a non-zero limit.*

REMARK 3.3. From Lemma 3.2 we have a sequence $\{u_k\}$ of critical points such that $J(u_k) \leq 0$, $u_k \neq 0$ and $\lim\limits_{k\to\infty} u_k = 0$.

In order to get infinitely many solutions we need some lemmas. Let $\varepsilon = 1/p^*\lambda$, from (2.2) we have

$$
\begin{aligned}
J(u) &:= \frac{1}{p} G(\|u\|^p) - \lambda \int_\Omega H(x, |u|^p)\,dx - \frac{1}{p^*} \int_\Omega |u|^{p^*}\,dx \\
&\geq \frac{\alpha_0 \sigma}{p} \int_\Omega |\nabla_A u|^p\,dx - \left(\frac{1}{p^*} + \varepsilon\lambda\right) \int_\Omega |u|^{p^*}\,dx - \lambda b(\varepsilon)|\Omega| \\
&= \frac{\alpha_0 \sigma}{p} \int_\Omega |\nabla_A u|^p\,dx - \frac{2}{p^*} \int_\Omega |u|^{p^*}\,dx - \lambda b\left(\frac{1}{p^*\lambda}\right)|\Omega| \\
&\geq L_1 \|u\|^p - L_2 \|u\|^{p^*} - L_3 \lambda,
\end{aligned}
$$

where L_1, L_2, L_3 are some positive constants.

Let $Q(t) = L_1 t^p - L_2 t^{p^*} - L_3\lambda$. Then $J(u) \geq Q(\|u\|)$. Furthermore, there exists

$$\lambda_* := \frac{pL_1}{NL_3}\left(\frac{pL_1}{p^*L_2}\right)^{(N-p)/p}$$

such that for $\lambda \in (0, \lambda_*)$, Q attains its positive maximum, that is, there exists

$$R_1 = \left(\frac{pL_1}{p^*L_2}\right)^{(N-p)/p^2}$$

such that $e_1 = Q(R_1) = \max_{t\geq 0} Q(t) > 0$. Therefore, for $e_0 \in (0, e_1)$, we may find $R_0 < R_1$ such that $Q(R_0) = e_0$. Now we define

$$\chi(t) = \begin{cases} 1, & 0 \leq t \leq R_0, \\ \dfrac{L_1 t^p - \lambda L_3 - e_1}{L_2 t^{p^*}}, & t \geq R_1, \\ C^\infty, \quad \chi(t) \in [0,1], & R_0 \leq t \leq R_1. \end{cases}$$

Then it is easy to see $\chi \in [0, 1]$ and χ is C^∞. Since the functional J is not bounded from below, we could not use the theory directly. So we follow [16] to consider a truncated functional of J. Let $\varphi(u) = \chi(\|u\|)$ and consider the perturbation of J:

$$T(u) := \frac{1}{p}G(\|u\|^p) - \frac{1}{p^*}\varphi(u)\int_\Omega |u|^{p^*}\,dx - \lambda\varphi(u)\int_\Omega H(x, |u|^p)\,dx. \tag{3.1}$$

Then

$$T(u) \geq L_1\|u\|^p - L_2\varphi(v)\|u\|^{p^*} - L_3\lambda = \overline{Q}(\|u\|),$$

where $\overline{Q}(t) = L_1 t^p - L_2\chi(t)t^{p^*} - L_3\lambda$ and

$$\overline{Q}(t) = \begin{cases} Q(t), & 0 \leq t \leq R_0, \\ e_1, & t \geq R_1. \end{cases}$$

From the above arguments, we have the following:

LEMMA 3.4. *Let T be defined as in* (3.1). *Then*

(a) *$T \in C^1(W_0^{1,p}(\Omega, \mathbb{C}), \mathbb{R})$ and T is even and bounded from below.*
(b) *If $T(u) < e_0$, then $\overline{Q}(\|u\|) < e_0$, consequently, $\|u\| < R_0$ and $J(u) = T(u)$.*
(c) *There exists λ^* such that, for $\lambda \in (0, \lambda^*)$, T satisfies a local* $(\mathrm{PS})_c$ *condition for*

$$c < e_0 \in \left(0, \min\left\{e_1, \frac{p^*\sigma - p}{2pp^*}(\alpha_0 S)^{N/p} - \lambda c\left(\frac{p^*\sigma - p}{2pp^*\lambda}\right)|\Omega|\right\}\right).$$

LEMMA 3.5. *Suppose that* (G1)–(G2) *and* (H3) *hold. Then for any $k \in \mathbb{N}$, there exists $\delta = \delta(k) > 0$ such that $\gamma(\{u \in W_0^{1,p}(\Omega, \mathbb{C}) : T(u) \leq -\delta(k)\}\setminus\{0\}) \geq k$.*

PROOF. Firstly, by (H_3) of Theorem 1.1, for any fixed $u \in W_0^{1,p}(\Omega, \mathbb{C})$, $u \neq 0$, we have

$$H(x, |\rho u|^p) \geq M(\rho)|\rho u|^{p/\sigma} \quad \text{with} \quad M(\rho) \to \infty \quad \text{as} \quad \rho \to 0.$$

On the other hand, by integrating (G2), we obtain

$$G(t) \leq \frac{G(t_0)}{t_0^{1/\sigma}} t^{1/\sigma} = C_0 t^{1/\sigma} \quad \text{for all} \quad t \geq t_0 > 0. \tag{3.2}$$

Secondly, given any $k \in N$, let E_k be a k-dimensional subspace of $W_0^{1,p}(\Omega, \mathbb{C})$. There then exists a positive constant δ such that

$$\|u\| \leq \delta |u|_{p/\sigma} \quad \text{for all } u \in E_k.$$

Therefore for any $u \in E_k$ with $\|u\| = 1$ and ρ small enough, by (3.2) and (H3), we have

$$\begin{aligned} T(\rho u) &= \frac{1}{p} G(\|\rho u\|^p) - \frac{1}{p^*} \varphi(u) \int_\Omega |\rho u|^{p^*} \, dx - \lambda \varphi(u) \int_\Omega H(x, |\rho u|^p) \, dx \\ &\leq \frac{C_0}{p} \rho^{p/\sigma} - \frac{\lambda M(\rho)}{\delta^{p/\sigma}} \rho^{p/\sigma} \leq \left(\frac{C_0}{p} - \frac{\lambda M(\rho)}{\delta^{p/\sigma}} \right) \rho^{p/\sigma} = -\delta(k) < 0, \end{aligned}$$

since $\lim_{|\rho| \to 0} M(\rho) = +\infty$. That is,

$$\{u \in E_k : \ \|u\| = \rho\} \subset \{u \in W_0^{1,p}(\Omega, \mathbb{C}) : T(u) \leq -\delta(k)\} \setminus \{0\}.$$

This completes the proof. $\square$

PROOF OF THEOREM 1.1. Recall that

$$\Sigma_k = \{A \in W_0^{1,p}(\Omega, \mathbb{C}) \setminus \{0\} : A \text{ is closed and } \ A = -A, \ \gamma(A) \geq k\}$$

and define $c_k = \inf\limits_{A \in \Sigma_k} \sup\limits_{u \in A} T(u)$. By Lemmas 3.4 (a) and 3.5, we know that $-\infty < c_k < 0$. Therefore, assumptions (C_1) and (C_2) of Lemma 3.2 are satisfied. This means that T has a sequence of solutions $\{u_n\}$ converging to zero. Hence, Theorem 1.1 follows by Lemma 3.4 (b). $\square$

4. A special case of problem (1.1)

We consider the following the special case of problem (1.1):

$$\begin{aligned} -\left(\alpha + \beta \int_\Omega |\nabla u|^p \, dx\right) \Delta_p u &= \lambda f(x, u) + |u|^{p^*-2} u && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{4.1}$$

where Ω is a bounded smooth domain of $\mathbb{R}^N$, $1 < p < N < 2p$, α and β are positive constants.

Set $g(t) = \alpha + \beta t$. Then, $g(t) \geq \alpha$ and

$$G(t) = \int_0^1 g(s) \, ds = \alpha t + \frac{1}{2} \beta t^2 \geq \frac{1}{2} (\alpha + \beta t) t = \sigma g(t) t,$$

where $\sigma = 1/2$. Hence conditions (G1) and (G2) are satisfied.

For this case, a typical example of a function satisfying conditions (F1)–(F3) is given by

$$f(x,t) = \sum_{i=1}^{k} a_i(x)|t|^{q_i-2}t,$$

where $k \geq 1$, $1 < q_i < p/\sigma$ and $a_i \in C(\overline{\Omega})$. In view of Theorem 1.1, we have the following corollary.

Corollary 4.1. *Suppose that* (F1)–(F3) *hold. There then exists* $\lambda^* > 0$ *such that for any* $\lambda \in (0, \lambda^*)$*, problem* (4.1) *has a sequence of nontrivial solutions* $\{u_n\}$ *and* $u_n \to 0$ *as* $n \to \infty$.

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Zhongyi Zhang
College of Mathematics
Changchun Normal University
Changchun 130032, Jilin, P.R. CHINA
and
College of Mathematics
Jilin University
Changchun 130011, Jilin, P.R. CHINA
E-mail address: zhyzhang66@163.com