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FRACTIONAL ORDER SEMILINEAR VOLTERRA INTEGRODIFFERENTIAL EQUATIONS IN BANACH SPACES

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Abstract. Sufficient conditions are established for the existence results of fractional order semilinear Volterra integrodifferential equations in Banach spaces. Results are obtained by using the theory of fractional cosine families and fractional powers of operators.

1. Introduction

The integrodifferential equations in Banach spaces have attracted much interest. Prüss [20] considered the solvability behavior on the real line of linear integrodifferential equations in a general Banach space and gave several applications to integral partial differential equations. Grimmer [5] established general conditions to ensure the existence of a resolvent operator for an integrodifferential equation in Banach spaces. Fitzgibbon [4] studied the existence, continuation, and behavior of solutions to an abstract semilinear Volterra integrodifferential equation. Keyantuo and Lizama [8] characterized existence and uniqueness of solutions for a linear integrodifferential equation in Hölder spaces. Londen [12] proved an existence result on a nonlinear Volterra integrodifferential equation in real reflexive Banach spaces by using the theory of maximal monotone operators. Prüss [22] studied linear Volterra integrodifferential equations in Banach

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spaces in case the main part of the equation generates an analytic C_0-semigroup. Travis and Webb [23] studied the existence of solutions to semilinear second order Volterra integrodifferential equations in Banach spaces by using the theory of strongly continuous cosine families. Mainini and Mola [14] considered in an abstract setting, an instance of the Coleman–Gurtin model for heat conduction with memory. Engler [3] constructed global weak solution of scalar second-order quasilinear hyperbolic integrodifferential equations with singular kernels. Prüss [21] studied the existence, positivity, regularity, compactness and integrability of the resolvent for a class of Volterra equations of scalar type. Hernández [6] studied the existence of strict and classical solutions for a class of abstract nonautonomous Volterra integrodifferential equations in Banach spaces. Lang and Chang [10] investigated the local existence and uniqueness of solutions to integrodifferential equations with infinite delay. Jawahdou [7] studied the existence of mild solutions for initial value problems for semilinear Volterra integrodifferential equations in Banach spaces.

In recent years, fractional differential equations have received increasing attention due to its applications in physics, chemistry, materials, engineering, biology, finance, we refer to [19], [13], [15]. Fractional order derivatives have the memory property and can describe many phenomena that integer order derivatives cannot characterize.

Consider the following fractional semilinear differential equation:

$$\begin{cases} {}^C D_t^\alpha u(t) = Au(t) & \text{for } t > 0, \\ u(0) = x, u^{(k)}(0) = 0 & \text{for } k = 1, \dots, m-1, \end{cases} \tag{1.1}$$

where $\alpha > 0$, m is the smallest integer greater than or equal to α, ${}^C D_t^\alpha$ is the α-order Caputo fractional derivative operator, $A\colon D(A) \subset X \to X$ is a closed densely defined linear operator on a Banach space X.

Bazhlekova [1] introduced the notion of solution operator for (1.1) as follows.

Definition 1.1. A family $\{C_\alpha(t)\}_{t\geq 0} \subset \mathcal{B}(X)$ is called a *solution operator* for (1.1) if the following conditions are satisfied:

(a) $C_\alpha(t)$ is strongly continuous for $t \geq 0$ and $C_\alpha(0) = I$ (the identity operator on X);
(b) $C_\alpha(t)D(A) \subset D(A)$ and $AC_\alpha(t)\xi = C_\alpha(t)A\xi$ for all $\xi \in D(A)$, $t \geq 0$;
(c) $C_\alpha(t)\xi$ is a solution of $x(t) = \xi + \int_0^t g_\alpha(t-s)Ax(s)\,ds$ for all $\xi \in D(A)$, $t \geq 0$, we refer to equality (2.3) concerning the definition of $g_\alpha(t)$.

A is called the *infinitesimal generator* of $C_\alpha(t)$. Note that in some literature the solution operator also is called the fractional resolvent family or fractional resolvent operator function, see [2], [11]. As a matter of fact, the solution operator $C_2(t)$ is a cosine family, in this paper, for $\alpha \in (1,2]$, the solution operator

$C_\alpha(t)$ is called *strongly continuous α-order fractional cosine family*, or α-order cosine family, for short.

Chen and Li [2] developed a purely algebraic notion, called the α-resolvent operator function: A family $\{S_\alpha(t)\}_{t\geq 0}$ of bounded linear operators of a Banach space X is called an *α-resolvent operator function* if the following conditions are satisfied:

(a) $S_\alpha(t)$ is strongly continuous for $t \geq 0$ and $S_\alpha(0) = I$ (the identity operator),
(b) $S_\alpha(t)S_\alpha(s) = S_\alpha(s)S_\alpha(t)$ for all $t, s \geq 0$, and
(c) there holds for all $t, s \geq 0$ that

$$S_\alpha(s)J_t^\alpha S_\alpha(t) - J_s^\alpha S_\alpha(s)S_\alpha(t) = J_t^\alpha S_\alpha(t) - J_s^\alpha S_\alpha(s),$$

where J_t^α is the α-order Riemann–Liouville fractional integral operator.

It has been proved in [2] that a family $\{S_\alpha(t)\}_{t\geq 0}$ is an α-resolvent operator function if and only if it is a solution operator (or an α-times resolvent family, see [11]) for problem (1.1).

Peng and Li [18] developed a novel operator theory for problem (1.1) with the order $\alpha \in (0,1)$.

DEFINITION 1.2 ([18]). Let $0 < \alpha < 1$. A one-parameter family $\{T_\alpha(t)\}_{t\geq 0}$ of bounded linear operators of X is called a *strongly continuous fractional semigroup* of order α (or α-order fractional semigroup, for short) if it possesses the following two properties:

(a) for every $x \in X$, the mapping $t \mapsto T(t)x$ is continuous over $[0,\infty)$;
(b) $T_\alpha(0) = I$, and for all $t, s \geq 0$,

$$\begin{aligned} &\int_0^{t+s} \frac{T_\alpha(\tau)\,d\tau}{(t+s-\tau)^\alpha} - \int_0^{t} \frac{T_\alpha(\tau)\,d\tau}{(t+s-\tau)^\alpha} - \int_0^{s} \frac{T_\alpha(\tau)\,d\tau}{(t+s-\tau)^\alpha} \\ &\qquad = \alpha \int_0^t \int_0^s \frac{T_\alpha(\tau_1)T_\alpha(\tau_2)}{(t+s-\tau_1-\tau_2)^{1+\alpha}}\,d\tau_1 d\,\tau_2, \end{aligned} \tag{1.2}$$

where the integrals are understood in the sense of strong operator topology.

For $\alpha \in (0,1)$, it is proved that a family of bounded linear operators is a solution operator for (1.1) if and only if it is a fractional semigroup. Moreover, it is shown that problem (1.1) is well-posed if and only if its coefficient operator generates an α-order semigroup.

Keyantuo [9] investigated a general framework for connections between ordinary non-homogeneous equations in Banach spaces and fractional Cauchy problems. When the underlying operator generates a strongly continuous semigroup, using a subordination argument, the fractional evolution equation is well-posed.

In this paper we are concerned with the fractional order semilinear Volterra integrodifferential equation

$$(1.3)\qquad \begin{cases} {}^{\mathrm{C}}D_t^{\alpha}u(t) = Au(t) + \displaystyle\int_0^t h(t,s,u(s))\,ds + f(t) & \text{for } t \in \mathbb{R}_+, \\ u(0) = x, \quad u'(0) = y, \end{cases}$$

where $\mathbb{R}_+ = [0,\infty)$, $\alpha \in (1,2]$, ${}^{\mathrm{C}}D_t^{\alpha}$ is the α-order Caputo fractional derivative operator, A is the infinitesimal generator of a strongly continuous fractional cosine family $\{C_\alpha(t)\}_{t\geq 0}$ on a Banach space X, h is a nonlinear unbounded operator from $\mathbb{R}_+ \times \mathbb{R}_+ \times X$ to X, f is a function from $\mathbb{R}_+$ to X and $x, y \in X$.

The paper is organized as follows. In Section 2, we give the basic notations and preliminary facts. In Section 3, we give the sufficient conditions for the existence of equation (1.3). At last, an example is presented to illustrate the main results.

2. Preliminaries

Let X be a Banach space with norm $\|\cdot\|$. By $\mathcal{B}(X)$ we denote the space of all bounded linear operators on X. Let $1 \leq p < \infty$. By $L^p([0,T];X)$ we denote the space of X-valued Bochner integrable functions $f\colon [0,T] \to X$ with the norm

$$(2.1)\qquad \|f\|_{L^p([0,T];X)} = \bigg(\int_0^T \|f(t)\|^p\,dt\bigg)^{1/p}.$$

By $C([0,T];X)$, resp. $C^1([0,T];X)$, we denote the spaces of functions $f\colon [0,T] \to X$, which are continuous, resp. continuously differentiable. $C([0,T];X)$ and $C^1([0,T];X)$ are Banach spaces endowed with the norms

$$(2.2)\qquad \|f\|_C = \sup_{t\in[0,T]} \|f(t)\|_X, \qquad \|f\|_{C^1} = \sup_{t\in[0,T]} \sum_{k=0}^{1} \|f^{(k)}(t)\|_X.$$

Let I be the identity operator on X. If A is a linear operator on X, then $R(\lambda, A) = (\lambda I - A)^{-1}$ denotes the resolvent operator of A. For the sake of simplicity, we use the following notation for $\alpha > 0$:

$$(2.3)\qquad g_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}, \quad t > 0,$$

where $\Gamma(\alpha)$ is the gamma function. If $\alpha = 0$, we set $g_0(t) = \delta(t)$, the delta distribution.

Definition 2.1. The *Riemann–Liouville fractional integral of order $\alpha \geq 0$* as defined

$$(2.4)\qquad J_t^{\alpha}u(t) = \int_0^t g_\alpha(t-s)u(s)\,ds,$$

where $u(t) \in L^1([0,T];X)$.

The set of the Riemann–Liouville fractional integral operators $\{J_t^\alpha\}_{\alpha\geq 0}$ is a semigroup, i.e. $J_t^\alpha J_t^\beta = J_t^{\alpha+\beta}$ for all $\alpha, \beta \geq 0$.

Definition 2.2. The *Riemann–Liouville fractional derivative of order* $\alpha \in (1,2]$ as defined

$$D_t^\alpha u(t) = \frac{d^2}{dt^2} J_t^{2-\alpha} u(t), \tag{2.5}$$

where $u(t) \in L^1([0,T];X)$, $D_t^\alpha u(t) \in L^1([0,T];X)$.

Definition 2.3. The *Caputo fractional derivative of order* $\alpha \in (1,2]$ as defined

$$^{\mathrm{C}}D_t^\alpha u(t) = D_t^\alpha (u(t) - u(0) - u'(0)t), \tag{2.6}$$

where $u(t) \in L^1([0,T];X) \cap C^1([0,T];X)$, $D_t^\alpha u(t) \in L^1([0,T];X)$.

The Laplace transform for the Riemann–Liouville fractional integral is given by

$$L\{J_t^\alpha u(t)\} = \frac{1}{\lambda^\alpha}\widehat{u}(\lambda), \tag{2.7}$$

where $\widehat{u}(\lambda)$ is the Laplace of u given by

$$\widehat{u}(\lambda) = \int_0^\infty e^{-\lambda t} u(t)\, dt, \quad \operatorname{Re}\lambda > \omega. \tag{2.8}$$

The Laplace transform for Caputo derivative is given by

$$L\{^{\mathrm{C}}D_t^\alpha u(t)\} = \lambda^\alpha \widehat{u}(\lambda) - u(0)\lambda^{\alpha-1} - u'(0)\lambda^{\alpha-2}. \tag{2.9}$$

Definition 2.4. The *fractional sine family* $S_\alpha\colon \mathbb{R}_+ \to \mathcal{B}(X)$ associated with C_α is defined by

$$S_\alpha(t) = \int_0^t C_\alpha(s)\, ds. \tag{2.10}$$

Remark 2.5. For $x \in X$, define

$$S'(0)x = \frac{dS_\alpha(t)x}{dt}\bigg|_{t=0}.$$

From Definitions 2.4 and 1.1, it is clear that $S'(0) = I$ (the identity operator on X).

Definition 2.6. The *fractional Riemann–Liouville family* $P_\alpha\colon \mathbb{R}_+ \to \mathcal{B}(X)$ associated with C_α is defined by

$$P_\alpha(t) = J_t^{\alpha-1} C_\alpha(t). \tag{2.11}$$

Definition 2.7. The α-order cosine family $C_\alpha(t)$ is called *exponentially bounded* if there are constants $M \geq 1$ and $\omega \geq 0$ such that

$$\|C_\alpha(t)\| \leq Me^{\omega t}, \quad t \geq 0. \tag{2.12}$$

An operator A is said to belong to $\mathcal{C}^\alpha(M,\omega)$ if problem (1.1) has an α-order cosine family $C_\alpha(t)$ satisfying (2.12).

3. Existence of solutions

For $\alpha \in (1,2)$, we assume $A \in \mathcal{C}^\alpha(M,\omega)$ and let $C_\alpha(t)$ be the corresponding α-order cosine family. We have (see [1], (2.5) and (2.6))

$$\{\lambda^\alpha : \operatorname{Re}\lambda > \omega\} \subset \rho(A), \tag{3.1}$$

and

$$\lambda^{\alpha-1}R(\lambda^\alpha, A)\xi = \int_0^\infty e^{-\lambda t}C_\alpha(t)\xi\, dt, \quad \operatorname{Re}\lambda > \omega,\ \xi \in X. \tag{3.2}$$

By (2.11) and (3.2), we have

$$R(\lambda^\alpha, A)\xi = \int_0^\infty e^{-\lambda t}P_\alpha(t)\xi\, dt, \quad \operatorname{Re}\lambda > \omega,\ \xi \in X. \tag{3.3}$$

For a fractional cosine family $C_\alpha(t)$, we define $E = \{x \in X : C_\alpha(t)x$ is continuously differentiable on $\mathbb{R}_+\}$. By the identity $\lambda^\alpha R(\lambda^\alpha, A) - I = AR(\lambda^\alpha, A)$, (3.2) and (3.3), we have that $P_\alpha(t)E \subset D(A)$, $t \in \mathbb{R}_+$, and

$$\frac{d}{dt}C_\alpha(t)x = AP_\alpha(t)x, \quad x \in E,\ t \in \mathbb{R}_+. \tag{3.4}$$

By (2.10) and (3.2), we have

$$\lambda^{\alpha-2}R(\lambda^\alpha, A)\xi = \int_0^\infty e^{-\lambda t}S_\alpha(t)\xi\, dt, \quad \operatorname{Re}\lambda > \omega,\ \xi \in X. \tag{3.5}$$

Lemma 3.1. *Let A be the infinitesimal generator of an α-order cosine family $C_\alpha(t)$ and $S_\alpha(t)$ be the corresponding α-order sine family. Then*:

(a) *For all $x \in D(A)$ and $t \geq 0$,*

$$S_\alpha(t)x \in D(A) \quad \text{and} \quad AS_\alpha(t)x = S_\alpha(t)Ax.$$

(b) *For all $x \in D(A)$ and $t \geq 0$,*

$$S_\alpha(t)x = tx + J_t^\alpha S_\alpha(t)Ax.$$

Proof. (a) Fix some $\mu^\alpha \in \rho(A)$, for $\lambda > \max\{\omega, 0\}$ and $x \in X$,

$$\int_0^\infty e^{-\lambda t}S_\alpha(t)\mu^{\alpha-2}R(\mu^\alpha, A)x\, dt = \lambda^{\alpha-2}R(\lambda^\alpha, A)\mu^{\alpha-2}R(\mu^\alpha, A)x$$
$$= \mu^{\alpha-2}R(\mu^\alpha, A)\lambda^{\alpha-2}R(\lambda^\alpha, A)x = \int_0^\infty e^{-\lambda t}\mu^{\alpha-2}R(\mu^\alpha, A)S_\alpha(t)x\, dt.$$

From the uniqueness theorem of the Laplace transform, it follows that

$$R(\mu^\alpha, A)S_\alpha(t) = S_\alpha(t)R(\mu^\alpha, A).$$

This implies (a).

(b) For $x \in D(A)$, $\lambda > \omega \geq 0$,

$$\begin{aligned}\int_0^\infty \lambda^2 e^{-\lambda t} tx\, dt &= x = \lambda^\alpha R(\lambda^\alpha, A)x - R(\lambda^\alpha, A)Ax \\ &= \int_0^\infty \lambda^2 e^{-\lambda t} S_\alpha(t)x\, dt - \int_0^\infty \lambda^2 e^{-\lambda t} J_t^\alpha S_\alpha(t)Ax\, dt.\end{aligned}$$

Hence, (b) follows from the uniqueness theorem of Laplace transform. □

Since $A \in \mathcal{C}^\alpha(M, \omega)$ for $\alpha \in (1, 2)$, then from Theorem 3.3 in [1], it follows that A generates an analytic semigroup $T(t)$ of angle $(\alpha - 1)\pi/2$. We suppose that $0 \in \rho(A)$, then for $\beta \in (0, 1)$, we can define the fractional powers operator $(-A)^{-\beta}$ as follows:

$$(-A)^{-\beta} = \frac{\sin \pi\beta}{\pi} \int_0^\infty \tau^{-\beta}(\tau I - A)^{-1}\, d\tau.$$

Definition 3.2. Let A be the infinitesimal generator of an analytic semigroup $T(t)$. For every $\beta > 0$ we define $(-A)^\beta = ((-A)^{-\beta})^{-1}$. For $\beta = 0$, $(-A)^\beta = I$.

We collect some basic properties of fractional powers $(-A)^\beta$ in the following lemma.

Lemma 3.3 ([17]). *Assume $(-A)^\beta$ is defined by Definition 3.1, then:*

(a) *$(-A)^\beta$ is a closed operator with domain $D((-A)^\beta) = R((-A)^{-\beta})$ (the range of $(-A)^{-\beta}$).*
(b) *For $\beta \geq \gamma > 0$, $D((-A)^\beta) \subset D((-A)^\gamma)$.*
(c) *$D((-A)^\beta)$ is dense in X for every $\beta \geq 0$.*
(d) *If β, γ are real, then $(-A)^{\beta+\gamma}x = (-A)^\beta(-A)^\gamma x$ for every $x \in D((-A)^\eta)$ where $\eta = \max(\beta, \gamma, \beta + \gamma)$.*

By (c), (d) of Lemma 3.3, we see that for $\beta \in (0, 1)$,

$$(-A)^\beta = (-A)^{\beta-1}(-A). \tag{3.6}$$

We note that $D((-A)^\beta)$ is a Banach space equipped with the norm $\|x\|_\beta = \|(-A)^\beta x\|$, $x \in D((-A)^\beta)$. By X_β we denote this Banach space.

Lemma 3.4. *Let A be the infinitesimal generator of an α-order cosine family $C_\alpha(t)$ on X. By $P_\alpha(t)$ we denote the corresponding Riemann–Liouville family. If $k \colon \mathbb{R}_+ \to X$ is continuously differentiable and $v(t) = \int_0^t P_\alpha(t - s)k(s)\, ds$, then $v(t) \in D(A)$ for $t \geq 0$, and*

$$Av(t) = \int_0^t C_\alpha(t - S)k'(\tau)\, ds + C_\alpha(t)k(0) - k(t). \tag{3.7}$$

Proof. Since $k\colon \mathbb{R}_+ \to X$ is continuously differentiable, we have

$$v(t) = \int_0^t P_\alpha(t-s)k(s)\,ds = \int_0^t P_\alpha(t-s)\bigg(\int_0^s k'(\tau)\,d\tau + k(0)\bigg)\,ds \\ = \int_0^t \int_0^{t-\tau} P_\alpha(s)k'(\tau)\,ds\,d\tau + \int_0^t P_\alpha(t-s)k(0)\,ds. \tag{3.8}$$

From (2.11), (b) of Proposition 3.3 in [2], it follows that for all $x \in X$, $t \geq 0$,

$$\int_0^t P_\alpha(s)x\,ds \in D(A) \quad \text{and} \quad A\int_0^t P_\alpha(s)x\,ds = C_\alpha(t)x - x.$$

Then $v(t) \in D(A)$,

$$Av(t) = \int_0^t (C_\alpha(t-\tau)k'(\tau) - k'(\tau))\,d\tau + C_\alpha(t)k(0) - k(0) \\ = \int_0^t C_\alpha(t-s)k'(s)\,ds + C_\alpha(t)k(0) - k(t). \tag{3.9}$$

□

Lemma 3.5. *Let A be the infinitesimal generator of an α-order cosine family $C_\alpha(t)$ on X. Let $f\colon \mathbb{R}_+ \to X$ be continuously differentiable, $x, y \in D(A)$, and let*

$$\varphi(t) = C_\alpha(t)x + S_\alpha(t)y + \int_0^t P_\alpha(t-s)f(s)\,ds, \quad t \in [0,T],$$

then $\varphi(t) \in D(A)$ and φ satisfies

$$\begin{cases} {}^{\mathrm{C}}D_t^\alpha \varphi(t) = A\varphi(t) + f(t) \quad \text{for } t \in \mathbb{R}_+, \\ \varphi(0) = x, \quad \varphi'(0) = y. \end{cases}$$

Proof. From (3.4) and Lemma 3.3, it follows that $\varphi(t) \in D(A)$. It is clear that $\varphi(0) = x$. Since $f\colon \mathbb{R}_+ \to X$ is continuously differentiable, it is easy to show that $\varphi'(0) = y$. By (2.6), Remark 2.5 and Lemma 3.1, we have

$$\begin{aligned} {}^{\mathrm{C}}D_t^\alpha \varphi(t) &= {}^{\mathrm{C}}D_t^\alpha C_\alpha(t)x + {}^{\mathrm{C}}D_t^\alpha S_\alpha(t)y + {}^{\mathrm{C}}D_t^\alpha\bigg(\int_0^t P_\alpha(t-s)f(s)\,ds\bigg) \\ &= AC_\alpha(t)x + D_t^\alpha(S_\alpha(t)y - S_\alpha(0)y - tS'_\alpha(0)y) \\ &\quad + D_t^\alpha\bigg(\int_0^t P_\alpha(t-s)f(s)\,ds\bigg) \\ &\quad = AC_\alpha(t)x + D_t^\alpha(S_\alpha(t)y - ty) + \frac{d^2}{dt^2}J_t^{2-\alpha}(P_\alpha(t) * f(t)) \\ &= AC_\alpha(t)x + D_t^\alpha J_t^\alpha S_\alpha(t)Ay + \frac{d^2}{dt^2}\big(g_{2-\alpha}(t) * g_{\alpha-1}(t) * C_\alpha(t) * f(t)\big) \\ &= AC_\alpha(t)x + S_\alpha(t)Ay + \frac{d^2}{dt^2}(1 * C_\alpha(t) * f(t)) \\ &= AC_\alpha(t)x + AS_\alpha(t)y + \frac{d}{dt}(C_\alpha(t) * f(t)). \end{aligned}$$

By Lemma 3.4, we have

$$\frac{d}{dt}(C_\alpha(t) * f(t)) = A\int_0^t P_\alpha(t-s)f(s)\,ds + f(t).$$

Therefore, the proof is complete. □

We make the following assumptions on the functions h and f:

(A_1) $h\colon \mathbb{R}_+ \times \mathbb{R}_+ \times D \to X$ is continuous, where D is an open subset of X_β, $\beta \in [0,1)$.

(A_2) $h_1\colon \mathbb{R}_+ \times \mathbb{R}_+ \times D \to X$ is continuous, where h_1 denotes the derivative of h with respect to its first variable.

(A_3) $f\colon \mathbb{R}_+ \to X$ is continuously differentiable.

THEOREM 3.6. *Let $\alpha \in (1,2)$. Assume that $A \in \mathcal{C}^\alpha(M,\omega)$ and let $C_\alpha(t)$, $S_\alpha(t)$ and $P_\alpha(t)$ denote the corresponding α-order cosine family, α-order sine family and α-order Riemann–Liouville family, respectively. Assume that A^{-1} is compact. Let $x \in D$, $\beta \in (0,1)$ and let $(-A)^{\beta-1}y \in E$. If* (A_1)–(A_3) *are satisfied, then there exists $T > 0$ and a continuous function $u\colon [0,T] \to X_\beta$ such that, for $t \in [0,T]$,*

$$u(t) = C_\alpha(t)x + S_\alpha(t)y + \int_0^t P_\alpha(t-s)\int_0^s h(s,r,u(r))\,dr\,ds + \int_0^t P_\alpha(t-s)f(s)\,ds. \tag{3.10}$$

If, in addition, $x \in D(A)$ and $y \in E$, then the Caputo derivative ${}^{\mathrm{C}}D_t^\alpha u$ of the solution u of (3.10) *is continuous, $u \in D(A)$, and u satisfies*

$$\begin{cases} {}^{\mathrm{C}}D_t^\alpha u(t) = Au(t) + \displaystyle\int_0^t h(t,s,u(s))\,ds + f(t) & \text{for } t \in [0,T], \\ u(0) = x, \quad u'(0) = y. \end{cases} \tag{3.11}$$

PROOF. For $\delta > 0$, let $N_\delta(x) = \{x_1 \in X_\beta : \|x - x_1\|_\beta < \delta\}$. Let

$$\varphi(t) = C_\alpha(t)x + S_\alpha(t)y + \int_0^t P_\alpha(t-s)f(s)\,ds.$$

We can choose $\delta > 0$ and $T > 0$ such that $N_\delta(x) \subset D$ and for $r,s \in [0,T]$ and $x_1 \in N_\delta(x)$,

$$\|h(r,s,x_1)\| \le 1 + M(x,T), \ \ \|h_1(r,s,x_1)\| \le 1 + N(x,T); \tag{3.12}$$

for $t \in [0,T]$,

$$\|\varphi(t) - x\|_\beta < \delta/2; \tag{3.13}$$

for $t \in [0,T]$ and $x_1, x_2, x_3 \in N_\delta(x)$,

$$\left\|(-A)^{\beta-1}\left(-\int_0^t C_\alpha(t-s)\left(h(s,s,x_1)+\int_0^s h_1(s,r,x_2)\,dr\right)ds + \int_0^t h(t,s,x_3)\,ds\right)\right\| < \frac{\delta}{2}, \tag{3.14}$$

where $M(x,T) = \sup_{r,s\in[0,T]} \|h(r,s,x)\|$ and $N(x,T) = \sup_{r,s\in[0,T]} \|h_1(r,s,x)\|$. In fact, since $h\colon \mathbb{R}_+\times\mathbb{R}_+\times D \to X$ is continuous, given $\varepsilon > 0$, there exists $\delta > 0$ such that for $x_1 \in N_\delta(x)$ and $r,s \in [0,T]$, we have $\|h(s,r,x_1)-h(s,r,x)\| < \varepsilon$. Letting $\varepsilon \in (0,1)$, we obtain $\|h(r,s,x_1)\| \le 1 + M(x,T)$. Similarly, $\|h_1(r,s,x_1)\| \le 1 + N(x,T)$.

It is easy to show that $(-A)^\beta C_\alpha(t)x = C_\alpha(t)(-A)^\beta x$ for $x \in X_\beta$. Note that $t \in [0,T]$ and $C_\alpha(t)$ is strongly continuous for $t \ge 0$, then

$$\begin{aligned}\|(-A)^\beta(C_\alpha(t)x - x)\| &= \|(C_\alpha(t)-I)(-A)^\beta x\| \\ &\le \|C_\alpha(t)-I\|\|x\|_\beta = C(\alpha,T)\|x\|_\beta,\end{aligned} \tag{3.15}$$

where $C(\alpha,T) = \sup_{t\in[0,T]} \|C_\alpha(t)-I\|$. Since $\beta \in (0,1)$, then there exists a positive constant $M_0 > 0$ such that $\|(-A)^{\beta-1}\| \le M_0$ (see Lemma 6.3 in [17]).

Since $(-A)^{\beta-1}y \in E$, we have

$$\begin{aligned}\|(-A)^\beta S_\alpha(t)y\| = \|(-A)^{\beta-1}(-A)S_\alpha(t)y\| &= \left\|(-A)^{\beta-1}\frac{d}{dt}C_\alpha(t)y\right\| \\ &\le \|(-A)^{\beta-1}\|\left\|\frac{d}{dt}C_\alpha(t)y\right\| \le M_0M'(\alpha,T,y),\end{aligned} \tag{3.16}$$

where

$$M'(\alpha,T,y) = \sup_{t\in[0,T]} \frac{d}{dt}C_\alpha(t)y.$$

By Lemma 3.4, we have

$$\begin{aligned}\left\|(-A)^\beta\int_0^t P_\alpha(t-s)f(s)\,ds\right\| &= \left\|(-A)^{\beta-1}(-A)\int_0^t P_\alpha(t-s)f(s)\,ds\right\| \\ &\le \|(-A)^{\beta-1}\|\left\|(-A)\int_0^t P_\alpha(t-s)f(s)\,ds\right\| \\ &= \|(-A)^{\beta-1}\|\left\|\int_0^t C_\alpha(t-s)f'(s)ds + C_\alpha(t)f(0) - f(t)\right\| \\ &= \|(-A)^{\beta-1}\|(M_T'Me^{\omega T}T + M_T) \\ &\le M_0(M_T'Me^{\omega T}T + M_T),\end{aligned} \tag{3.17}$$

where $M'_T = \sup_{s\in[0,T]} \|f'(s)\|$, $M_T = \sup_{s\in[0,T]} \|C_\alpha(t)f(0) - f(t)\|$. Since

$$\begin{aligned}(3.18)\qquad \|\varphi(t) - x\|_\beta &= \|(-A)^\beta(\varphi(t) - x)\| \\ &\le \|(-A)^\beta(C_\alpha(t)x - x)\| + \|(-A)^\beta S_\alpha(t)y\| \\ &\quad + \left\|(-A)^\beta \int_0^t P_\alpha(t-s)f(s)\,ds\right\|.\end{aligned}$$

Put (3.15)–(3.17) into (3.18) to get

$$\begin{aligned}(3.19)\qquad \|\varphi(t) - x\|_\beta &\le C(\alpha,T)\|x\|_\beta + M_0M'(\alpha,T,y) \\ &\quad + M_0(M'_T M e^{\omega T}T + Me^{\omega T}M_T + M_T).\end{aligned}$$

For $t \in [0,T]$ and $x_1, x_2, x_3 \in N_\delta(x)$, we have

$$\begin{aligned}(3.20)\qquad &\left\|(-A)^{\beta-1}\left(-\int_0^t C_\alpha(t-s)\left(h(s,s,x_1) + \int_0^s h_1(s,r,x_2)\,dr\right)ds\right.\right. \\ &\qquad\qquad \left.\left. + \int_0^t h(t,s,x_3)\,ds\right)\right\| \\ &\le M_0\big\{TMe^{\omega T}(1 + M(x,T) + T(1 + N(x,T)) + T(1 + M(x,T))\big\}.\end{aligned}$$

From (3.19) and (3.20), it can be seen that we can choose $\delta > 0$ and $T > 0$ such that (3.13) and (3.14) hold, provided that δ and T satisfy the following inequalities:

$$\begin{aligned}(3.21)\qquad &C(\alpha,T)\|x\|_\beta + M_0M'(\alpha,T,y) \\ &\qquad + M_0(M'_T M e^{\omega T}T + Me^{\omega T}M_T + M_T) < \frac{\delta}{2},\end{aligned}$$

and

$$(3.22)\qquad M_0\big\{TMe^{\omega T}(1 + M(x,T) + T(1 + N(x,T)) + T(1 + M(x,T))\big\} < \frac{\delta}{2}.$$

Let $C := C([0,T];X_\beta)$ equipped with the norm $\|\phi\|_C = \sup_{t\in[0,T]} \|\phi(t)\|_\beta$. Let F be the closed convex bounded subset of $C([0,T];X_\beta)$, defined by

$$F = \left\{\phi \in C : \|\phi - \varphi\|_C \le \frac{\delta}{2}\right\}.$$

From $\|\phi(t) - x\|_\beta \le \|\phi - \varphi\|_C + \|\varphi(t) - x\|_\beta \le \delta$, it follows that $\phi(t) \in D$ for $\phi(t) \in F$, $t \in [0,T]$. Set the mapping Q on F by

$$(Q\phi)(t) = \varphi(t) + \int_0^t P_\alpha(t-s)\int_0^s h(s,r,\phi(r))\,dr\,ds, \quad t \in [0,T].$$

STEP 1. We show that Q maps F into F. Since

$$\frac{d}{ds}\int_0^s h(s,r,\phi(r))\,dr = \int_0^s h_1(s,r,\phi(r))\,dr + h(s,s,\phi(s)),$$

by (3.4), (3.6), (3.9), (3.14), we have

$$\begin{aligned}\|(Q\phi)(t) &- \varphi(t)\|_\beta = \|(-A)^\beta((Q\phi)(t)-\varphi(t))\| \\ &= \left\|(-A)^{\beta-1}\left(-\int_0^t \left(AP_\alpha(t-s)\int_0^s h(s,r,\phi(r))\,dr\,ds\right)\right)\right\| \\ &= \left\|(-A)^{\beta-1}\left[-\int_0^t C_\alpha(t-s)\left(\int_0^s h_1(s,r,\phi(r))\,dr + h(s,s,\phi(s))\right)ds\right.\right. \\ &\quad \left.\left. + \int_0^t h(t,s,\phi(s))\,ds\right]\right\| < \frac{\delta}{2}.\end{aligned}$$

It is easy to show that $Q\phi\colon [0,T]\to X_\beta$ is continuous in t on $[0,T]$. We see that Q maps F into F.

Step 2. We show that Q is continuous. By (A_1), (A_2), for every $\varepsilon>0$, there exists some $\delta>0$ such that for $\phi_1,\phi_2\in F$, $\|\phi_1-\phi_2\|_C<\delta$, $s\in[0,T]$,

$$\sup_{r\in[0,T]}\|h(s,r,\phi_1(r))-h(s,r,\phi_2(r))\| < \varepsilon,$$
$$\sup_{r\in[0,T]}\|h_1(s,r,\phi_1(r))-h_1(s,r,\phi_2(r))\| < \varepsilon.$$

Then

$$\begin{aligned}\|(Q\phi_1)(t) &- (Q\phi_2)(t)\|_\beta \\ &= \left\|(-A)^{\beta-1}\left[(-A)\int_0^t P_\alpha(t-s)\int_0^s h(s,r,\phi_1(r))\,dr\,ds\right\|\right. \\ &\quad \left. -(-A)\int_0^t P_\alpha(t-s)\int_0^s h(s,r,\phi_2(r))\,dr\,ds\right] \\ &= \left\|(-A)^{\beta-1}\left[-\int_0^t C_\alpha(t-s)\left(\int_0^s h_1(s,r,\phi_1(r))\,dr\right.\right.\right. \\ &\quad \left. -\int_0^s h_1(s,r,\phi_2(r))\,dr + h(s,s,\phi_1(s)) - h(s,s,\phi_2(s))\right)ds \\ &\quad \left.\left. + \int_0^t (h(t,s,\phi_1(s)) - \int_0^t (h(t,s,\phi_2(s)))\,ds\right]\right\| \\ &\le \|(-A)^{\beta-1}\|\left[\int_0^t Me^{\omega(t-s)}\left(\int_0^s \varepsilon\,dr+\varepsilon\right)ds + \int_0^t \varepsilon\,ds\right].\end{aligned}$$

This implies that Q is continuous.

Step 3. We show that the set $\{Q\phi : \phi\in F\}$ is equicontinuous. For $\phi\in F$, $0\le t\le t'\le T$, we have

$$\begin{aligned}\|(Q\phi)(t) &- (Q\phi)(t')\|_\beta \\ &\le \|(C_\alpha(t)-C_\alpha(t'))(-A)^\beta x\| + \|A(P_\alpha(t)-P_\alpha(t'))(-A)^{\beta-1}y\| \\ &\quad + \left\|(-A)^{\beta-1}\left[\int_0^t \left(C_\alpha(t-s)\left(h(s,s,\phi(s)) + \int_0^s h_1(s,r,\phi(r))\,dr\right)ds\right.\right.\right.\end{aligned}$$

$$- \int_0^{t'} C_\alpha(t'-s)\Big(h(s,s,\phi(s)) + \int_0^s h_1(s,r,\phi(r)\Big)\, dr\Big)\, ds\Big]\Big\|$$
$$+ \Big\|(-A)^{\beta-1}\Big(\int_0^t h(t,s,\phi(s))\, ds - \int_0^{t'} h(t',s,\phi(s))\, ds\Big)\Big\|$$
$$+ \Big\|(-A)^{\beta-1}\Big(\int_0^t C_\alpha(t-s)f'(s)\, ds - \int_0^{t'} C_\alpha(t'-s)f'(s)\, ds\Big)\Big\|$$
$$+ \|(-A)^{\beta-1}(C_\alpha(t)-C_\alpha(t'))f(0)\| + \|(-A)^{\beta-1}(f(t)-f(t'))\|.$$

Since $C_\alpha(t)$, $P_\alpha(t)$ are strongly continuous, it follows that

$$\|(C_\alpha(t)-C_\alpha(t'))(-A)^\beta x\| + \|A(P_\alpha(t)-P_\alpha(t'))(-A)^{\beta-1}y\| \to 0$$

as $|t-t'| \to 0$, and

$$\|(-A)^{\beta-1}(C_\alpha(t)-C_\alpha(t'))f(0)\| + \|(-A)^{\beta-1}(f(t)-f(t'))\| \to 0$$

as $|t-t'| \to 0$. By Lemma 2.1 in [23], since A^{-1} is compact, then for $0<\beta<1$, $(-A)^{\beta-1}$ is compact. The compactness of $(-A)^{\beta-1}$, the strong continuity of $C_\alpha(t), P_\alpha(t)$, together with (3.12) imply that

$$\Big\|(-A)^{\beta-1}\Big[\int_0^t \Big(C_\alpha(t-s)\Big(h(s,s,\phi(s)) + \int_0^s h_1(s,r,\phi(r))\, dr\Big)\, ds$$
$$- \int_0^{t'} C_\alpha(t'-s)(h(s,s,\phi(s)) + \int_0^s h_1(s,r,\phi(r))\, dr\Big) ds\Big]\Big\|$$
$$\le \Big\| \int_0^t (C_\alpha(t-s) - C_\alpha(t'-s))(-A)^{\beta-1}\Big(h(s,s,\phi(s)) + \int_0^s h_1(s,r,\phi(r))\, dr\Big) ds\Big\|$$
$$+ \|(-A)^{\beta-1}\| \Big\| \int_t^{t'} C_\alpha(t'-s)\Big(h(s,s,\phi(s)) + \int_0^s h_1(s,r,\phi(r))\, dr\Big) ds\Big\| \to 0$$

as $|t-t'| \to 0$. On the other hand, by (3.12),

$$\Big\|(-A)^{\beta-1}\Big(\int_0^t C_\alpha(t-s)f'(s)\, ds - \int_0^{t'} C_\alpha(t'-s)f'(s)\, ds\Big)\Big\|$$
$$\le \|(-A)^{\beta-1}\| \Big(\Big\|\int_0^t \int_t^{t'} h_1(r,s,\phi(s))\, dr\, ds\Big\| + \Big\|\int_t^{t'} h(t',s,\phi(s))\, ds\Big\|\Big) \to 0$$

as $|t-t'| \to 0$, and

$$\Big\|(-A)^{\beta-1}\Big(\int_0^t C_\alpha(t-s)f'(s)\, ds - \int_0^{t'} C_\alpha(t'-s)f'(s)\, ds\Big)\Big\|$$
$$\le \|(-A)^{\beta-1}\| \Big(\Big\|\int_0^t (C_\alpha(t-s) - C(t'-s))f'(s)\, ds\Big\|$$
$$+ \Big\|\int_t^{t'} C_\alpha(t'-s)f'(s)\, ds\Big\|\Big) \to 0$$

as $|t-t'| \to 0$. Therefore, $\{Q\phi : \phi \in F\}$ is equicontinuous.

STEP 4. We show that for any given $t \in [0,T]$, the set $\{Q\phi : \phi \in F\}$ is precompact in X_β. Since A^{-1} is compact, then for $\gamma \in (\beta, 1]$, $(-A)^{-\gamma} \colon X \to X_\beta$ is compact, we only need to prove that $\{(-A)^\gamma((Q\phi)(t) - \varphi(t)) : \phi \in F\}$ is bounded $\gamma \in (\alpha, 1]$. In fact, we have

$$\begin{aligned}
&\|(-A)^\gamma(Q\phi - \varphi)(t)\| \\
&\quad \leq \bigg\|(-A)^{\gamma-1}\int_0^t C_\alpha(t-s)\bigg(h(s,s,\phi(s)) + \int_0^s h_1(s,r,\varphi(r))\,dr\bigg)ds \\
&\qquad\qquad\qquad\qquad + (-A)^{\gamma-1}\int_0^t h(t,s,\phi(s))\,ds\bigg\|.
\end{aligned}$$

From (3.12), the boundedness is obtained. Therefore, it follows from the Arzela–Ascoli theorem, Q is compact. By the Schauder fixed point theorem, Q has a fixed point in F, which is a solution of (3.10). If $x \in D(A)$, $y \in E$, then by Lemma 3.4, the solution of (3.10) is a solution of (3.11). □

4. An example

Consider the fractional semilinear Volterra integrodifferential equation of order $\alpha \in (1, 2]$

$$(4.1)\qquad \begin{cases} {}^{\mathrm{C}}D_t^\alpha z(t,x) = \Delta z(t,x) + \displaystyle\int_0^t \rho(t,s,z(s,x))\,ds + \theta(t,x) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } t \in \mathbb{R}_+,\ x \in (0,\pi), \\ z(t,0) = z(t,\pi) \qquad\qquad\qquad\quad\ \text{for } t \in \mathbb{R}_+, \\ z(0,x) = \sigma(x), \quad z_t(0,x) = \mu(x) \quad \text{for } x \in (0,\pi), \end{cases}$$

where ${}^{\mathrm{C}}D_t^\alpha$ is the α-order Caputo fractional derivative operator. Let $X = L^2[0,\pi]$ and define $A \colon X \to X$ by $Aw = w''$ with the domain $D(A) = \{w \in X : w, w' \text{ are absolutely continuous},\ w'' \in X,\ w(0) = w(\pi) = 0\}$. Thus

$$Aw = -\sum_{n=1}^\infty n^2(w,w_n)w_n, \quad w \in D(A),$$

where $w_n(s) = \sqrt{\pi/2}\sin ns$, $n = 1, 2, \ldots$, is the orthonormal set of eigenvalues of A. It is easy to see that A is the infinitesimal generator of a strongly continuous cosine family $C(t)$, $t \in \mathbb{R}$, on X given by

$$C(t)w = \sum_{n=1}^\infty \cos nt(w,w_n)w_n, \quad w \in X.$$

From the subordinate principle (see Theorem 3.1 in [1]), it follows that A is the infinitesimal generator of α-order cosine family $C_\alpha(t)$ such that $C_\alpha(0) = I$, and

$$C_\alpha(t) = \int_0^\infty \varphi_{t,\alpha/2}(s)C(s)\,ds, \quad t > 0,$$

where $\varphi_{t,\alpha/2}(s) = t^{-\alpha/2}\phi_{\alpha/2}(st^{-\alpha/2})$, and

$$\phi_\gamma(z) = \sum_{n=0}^{\infty} \frac{(-z)^n}{n!\Gamma(-\gamma n + 1 - \gamma)}, \quad 0 < \gamma < 1.$$

If we take $\beta = 1/2$, then

$$(-A)^{1/2}w = \sum_{n=1}^{\infty} n(w, w_n)w_n, \quad w \in D((-A)^{1/2}).$$

The operator $(-A)^{-1/2}$ is given by

$$(-A)^{-1/2}w = \sum_{n=1}^{\infty} \frac{1}{n}(w, w_n)w_n, \quad w \in X.$$

It is easy to show that $(-A)^{-1/2}$ is compact. By Lemma 2.1 in [23], A^{-1} is compact. Let $\rho\colon \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ be continuous and continuously differentiable with respect to its first variable. Let $\theta\colon \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ be continuous and continuously differentiable with respect to its first variable. Let $h\colon \mathbb{R}_+ \times \mathbb{R}_+ \times X_{1/2} \to X$ be defined by $(h(t,s,w))(x) = \rho(t,s,w(x))$, $w \in X_{1/2}$, $x \in [0,\pi]$, and let $f\colon \mathbb{R}_+ \to X$ be defined by $(f(t))(x) = \theta(t,x)$, $x \in [0,\pi]$. Then we can rewrite (4.1) as (3.11). If $w \in D((-A)^{1/2})$, then w is absolutely continuous, $w' \in X$, $w(0) = w(\pi) = 0$, and $\|w\|_{1/2} = \|w'\|$ (see Chapter 6 in [16]). Let $t_1, s_1 \in [0,T]$, $w_1 \in X_{1/2}$. For every $\varepsilon > 0$, there exists a $\delta > 0$ such that if $t, s \in [0,T]$, $x \in [0,\pi]$, $v \in \mathbb{R}$, and $|t_1 - t| < \delta$, $|s_1 - s| < \delta$, $|w_1(x) - v| < \delta$, then $|\rho(t_1, s_1, w_1(x)) - \rho(t,s,v)| < \varepsilon$. Let $w \in X_{1/2}$, and $\|w_1 - w\|_{1/2} < \delta/\sqrt{\pi}$. Then

$$\begin{aligned}|w_1(x) - w(x)| &\le \left| \int_0^x (w_1'(r) - w'(r))\,dr \right| \\ &\le \int_0^x |w_1'(r) - w'(r)|\,dr \le \sqrt{\pi}\|w_1' - w'\| = \sqrt{\pi}\|w_1 - w\|_{1/2}.\end{aligned}$$

Hence, for $|t_1 - t| < \delta$, $|s_1 - s| < \delta$, $|w_1(x) - v| < \delta$, we have

$$\|h(t_1, s_1, w) - h(t,s,w)\| = \int_0^\pi |\rho(t_1, s_1, w_1(x)) - \rho(t,s,w(x))|^2\,dx \le \pi\varepsilon^2.$$

Therefore h is continuous. By a similar method, the conditions (A_2) and (A_3) are satisfied. By Theorem 3.6, the integrodifferential equation (4.1) has a local solution.

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