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MULTIPLE SOLUTIONS WITH PRESCRIBED MINIMAL PERIOD FOR SECOND ORDER ODD NEWTONIAN SYSTEMS WITH SYMMETRIES

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Abstract. For an orthogonal Γ-representation V (Γ is a finite group) and for an even Γ-invariant C^2-functional $f\colon V \to \mathbb{R}$ satisfying the condition $0 < \theta\nabla f(x) \bullet x \le \nabla^2 f(x)x \bullet x$ (for $\theta > 1$ and $x \in V \setminus \{0\}$), we consider the odd Newtonian system $\ddot{x}(t) = -\nabla f(x(t))$ and establish the existence of multiple periodic solutions with a minimal period p (for any given $p > 0$). As an example, we prove the existence of arbitrarily many periodic solutions with minimal period p for a specific D_n-symmetric Newtonian system.

1. Introduction

The purpose of this paper is to study the existence of multiple periodic solutions with a given minimal period $p > 0$ for the Newtonian systems of the

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type

$$\ddot{x} = -\nabla f(x), \quad x \in V := \mathbb{R}^n, \tag{1.1}$$

where V is an orthogonal Γ-representation (with Γ being a finite group), $f\colon V \to \mathbb{R}$ is a Γ-invariant even C^2-function satisfying the condition

$$0 < \theta \nabla f(x) \bullet x \le \nabla^2 f(x) x \bullet x \quad \text{for } x \in V \setminus \{0\}, \tag{1.2}$$

for some $\theta > 1$.

The problem of finding periodic solutions with minimal period for variational problems is not new and there is a large literature devoted to this topic. Beginning with the pioneering work of P. Rabinowitz (cf. [21]), the question of existence of such solutions in the first order Hamiltonian systems

$$\frac{dz}{dt} = J\nabla f(z), \qquad J = \begin{bmatrix} 0 & -\mathrm{Id} \\ \mathrm{Id} & 0 \end{bmatrix}, \tag{1.3}$$

was discussed in the works of Clark and Ekeland (cf. [6]), Ambrosetti and Mancini (cf. [2]), Girardi and Matzeu (cf. [13], [14]), Deng (cf. [7]), Ekeland and Hofer (cf. [9]), Zhang and Tang (cf. [23], 2013), Michalek and Tarantello (cf. [20]), Fei et al. (cf. [12], [10]), Liu and Wang (cf. [19]), and many others. These authors used various methods, such as the Mountain Pass lemma, finite-dimensional approximations, duality principle, index theory, and restrictions to Nehari manifold, to prove several interesting existence results for multiple periodic solutions with the prescribed minimal period.

On the other hand, although there are many existence results for multiple periodic solutions of (1.3) with prescribed minimal period, there are only few such results for system (1.1). The existence of such periodic solutions was discussed, for example, in papers by Long (cf. [16, 17]), Fei et al. (cf. [11]), Xiao (cf. [22]), and others. The main goal of this paper is to combine the Nehari manifold techniques with the H-fixed-point reduction method in order to show the existence of multiple periodic solutions with the prescribed minimal period for symmetric Newtonian systems satisfying condition (1.2). More precisely, by exploring Γ-symmetries of system (1.1), where Γ is a finite group, and by applying the H-fixed-point reduction (for a specific subgroup $H \subset \Gamma \times \mathbb{Z}_2 \times S^1$), we show that the second order odd Newtonian system (1.1), where V is a Γ-representation and f is a Γ-invariant function (Γ is a finite group), has multiple p-periodic solutions with the minimal period p. Depending on the size of the group Γ, this number of p-periodic solutions may be arbitrarily large. Our approach is based on the usage of the Nehari manifold techniques developed by Yu Ming Xiao in [22].

Our main result, Theorem 4.4, mainly states that for any Γ-symmetric Newtonian system (1.1), satisfying (1.2), and for any $p > 0$, there exist multiple

Γ-orbits of p-periodic solutions with the minimal period p. In fact, such an orbit exists for any "maximal" orbit type (which we call *twisted maximal type*) in the functional space $\mathscr{H}$ (see Subsection 3.1). In addition, we present an example of such symmetric systems (3.1), where Γ is a dihedral group D_n. Then, if $n = q_1^{\alpha_1} \cdot \ldots \cdot q_s^{\alpha_s}$, where $q_1 < \ldots < q_s$ are prime numbers, we define

$$\sigma(n) := \begin{cases} 2(q_1 + q_2 + \ldots + q_s) & \text{when } n \text{ is odd;} \\ 2 + 2(q_2 + \ldots + q_s) & \text{when } n \text{ is even.} \end{cases}$$

The obtained result shows, as an example, that the Newtonian system (3.1) with f given by (6.7), admits at least $\sigma(n)$ periodic solutions with the minimal period p. Notice that for any N, there is n such that $\sigma(n) \geq N$.

To the best of our knowledge, this is the first time the combined equivariant and Nehari techniques were applied to obtain the existence of multiple periodic solutions with a prescribed minimal period p for symmetric Newtonian systems.

This paper consists of five sections. In Section 2, various definitions concerning key concepts are listed. In the meantime, basic results pertaining to the equivariant topology, representation theory, and the Principle of Symmetric Criticality are elaborated (cf. Subsection 2.3). In Section 3, all the required assumptions are formulated for the Γ-symmetric system (3.1), which is reformulated as a symmetric variational problem (cf. Subsection 3.2). Properties of the Sobolev space $\widetilde{\mathscr{H}}$ of p-periodic functions and its H-fixed-point subspace $\mathscr{H}$ (cf. Subsection 3.1) are also discussed. Then this variational problem (3.8) is reduced to a $\Gamma \times \mathbb{Z}_2$-symmetric variational problem on $\mathscr{H}$ (cf. Subsection 3.1). In Section 4, properties of twisted subgroups are discussed and the notion of maximal twisted orbit type in $\mathscr{H}$ is introduced. The main result, Theorem 4.4, is stated in Subsection 4.2 and the proof of Theorem 4.4 is presented in Section 5. For completeness, all the auxiliary results needed in the proof are listed in Subsection 5.1. Next, the Nehari manifold for the variational functional J associated with (3.1) is defined and its properties are examined in Subsection 5.2. The Palais–Smale condition is established in Subsection 5.3 and the existence of the minimum of J on the Nehari manifold is proved in Subsection 5.4. Finally, we show that the function minimizing J on the H-fixed-point subspace of the Nehari manifold has the minimal period p (Subsection 5.5). In Section 6, an example of a system (3.1) is presented, which is symmetric with respect to an action of the dihedral group D_n and satisfying all the required properties of the main theorem. For this particular example, the existence of multiple (depending only on the number n) periodic solutions with an arbitrary period $p > 0$ is established (cf. Theorem 6.1).

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2. Preliminaries

Throughout this section, G is assumed to be a compact Lie group.

2.1. Equivariant jargon. For a subgroup $H \subset G$ (it is always assumed to be closed), we denote the normalizer of H in G by $N(H)$, the Weyl group of H in G by $W(H) := N(H)/H$ and the conjugacy class of H in G by (H). The set $\Phi(G)$ of all conjugacy classes in G admits a partial order which can be defined as follows: $(H) \le (K)$ if and only if $gHg^{-1} \subset K$ for some $g \in G$.

For a G-space X and $x \in X$, $G_x := \{g \in G : gx = x\}$ is called the *isotropy* of x and $G(x) := \{gx : g \in G\}$ is called the *orbit* of x. One can easily verify that $G(x) \simeq G/G_x$. Denote the orbit space of X by X/G, which is the set of all orbits in X under the action of G. Furthermore, we call the conjugacy class (G_x) the *orbit type* of x (or simply an *orbit type*) in X and put $\Phi(G;X) := \{(H) \in \Phi(G) : H = G_x \text{ for some } x \in X\}$.

Also, for a G-space X and a closed subgroup H of G, we adopt the following notations: $X_H := \{x \in X : G_x = H\}$, $X^H := \{x \in X : G_x \supset H\}$, $X_{(H)} := \{x \in X : (G_x) = (H)\}$, $X^{(H)} := \{x \in X : (G_x) \ge (H)\}$, among which X^H is called the H-fixed-point subspace of X.

Let X and Y be two G-spaces. A continuous map $f\colon X \to Y$ is said to be *equivariant* if $f(gx) = gf(x)$ for all $x \in X$ and $g \in G$. If the functional $f\colon X \to \mathbb{R}$ satisfies $f(gx) = f(x)$ for all $x \in X$ and $g \in G$, then f is called G-*invariant*. As is known (see, for instance, [8], [5]), for any subgroup $H \subset G$ and equivariant map $f\colon X \to Y$, the map $f^H\colon X^H \to Y^H$, with $f^H := f|_{X^H}$, is well-defined and $W(H)$-equivariant.

2.2. Isotypical decomposition of finite-dimensional representations. As is well-known, any compact group admits only countably many non-equivalent real (resp. complex) irreducible representations. Therefore, given a compact Lie group Γ, we always assume that a complete list of all real (resp. complex) irreducible Γ-representations, denoted by $\mathcal{V}_i$, $i = 0, 1, \ldots$ (resp. $\mathcal{U}_j$, $j = 0, 1, \ldots$), is available. Here we assume that $\mathcal{V}_0$ (resp. $\mathcal{U}_0$) stands for trivial real (resp. complex) one-dimensional G-representations. Let V (resp. U) be a finite-dimensional real (resp. complex) Γ-representation (without loss of generality, V (resp. U) can be assumed to be orthogonal (resp. unitary)). Then, it is possible to represent V (resp. U) as the direct sum

$$V = V_0 \oplus \ldots \oplus V_r, \tag{2.1}$$

$$(\text{resp. } U = U_0 \oplus \ldots \oplus U_s), \tag{2.2}$$

which is called the Γ-*isotypical decomposition of* V (resp. U), where the isotypical component V_i (resp. U_j) is *modeled* on the irreducible Γ-representation $\mathcal{V}_i$, $i = 0,\dots,r$ (resp. $\mathcal{U}_j$, $j = 0,\dots,s$), i.e. V_i (resp. U_j) contains all the irreducible subrepresentations of V (resp. U) which are equivalent to $\mathcal{V}_i$ (resp. $\mathcal{U}_j$).

Given an orthogonal Γ-representation V, denote the group of all Γ-equivariant linear invertible operators on V by $\mathrm{GL}^\Gamma(V)$. Then, the isotypical decomposition (2.1) induces a decomposition of $\mathrm{GL}^\Gamma(V)$:

$$\mathrm{GL}^\Gamma(V) = \bigoplus_{i=0}^{r} \mathrm{GL}^\Gamma(V_i). \tag{2.3}$$

For every isotypical component V_i from (2.1), one has $\mathrm{GL}^\Gamma(V_i) \simeq \mathrm{GL}(m_i,\mathbb{F})$, where $m_i = \dim V_i/\dim \mathcal{V}_i$ and $\mathbb{F}$ is a finite-dimensional division algebra, i.e. $\mathbb{F} = \mathbb{R},\mathbb{C}$ or $\mathbb{H}$, depending on the type of the irreducible representation $\mathcal{V}_i$.

2.3. Principle of symmetric criticality. Assume that $\mathfrak{H}$ is a Hilbert G-representation and $\varphi\colon \mathfrak{H} \to \mathbb{R}$ is a continuously differentiable G-invariant functional. Assume $(H) \in \Phi(G;\mathfrak{H})$ and let $\varphi^H : \mathfrak{H}^H \to \mathbb{R}$ be the restriction of φ to $\mathfrak{H}^H$. Then, since $\nabla\varphi$ is G-equivariant, we have $\nabla\varphi(\mathfrak{H}^H) \subset \mathfrak{H}^H$. Therefore

$$\nabla\varphi^H(x) = P_H\nabla\varphi(x) = \nabla\varphi(x), \quad x \in \mathfrak{H}^H,$$

where P_H is the orthogonal projection onto $\mathfrak{H}^H$. Consequently, if x is a critical point of φ^H then it is also a critical point of φ, i.e. H-symmetric critical points of φ are critical points of φ (Palais-Symmetric Criticality Principle).

3. Multiple periodic solutions for symmetric Newtonian systems

Assume that $p > 0$ is an arbitrary number. Let Γ be a finite group and $V = \mathbb{R}^n$ an orthogonal representation of Γ (Γ is acting on $\mathbb{R}^n$ by permuting the vector coordinates in $\mathbb{R}^n$). We are interested in the following second order Newtonian system

$$\begin{cases} \ddot{x}(t) = -\nabla f(x(t)), & t \in \mathbb{R},\ x(t) \in V, \\ x(t) = x(t+p), \quad \dot{x}(t) = \dot{x}(t+p), \end{cases} \tag{3.1}$$

where $f\colon V \to \mathbb{R}$ is a C^2-function satisfying the following assumptions:

(A1) f is even, i.e. $f(-x) = f(x)$, for all $x \in V$, $f(0) = 0$.

(A2) f is Γ-invariant, i.e. $f(\gamma x) = f(x)$ for all $\gamma \in \Gamma$ and $x \in V$.

(A3) There exists a constant $\theta > 1$ such that for all $x \in V \setminus \{0\}$

$$0 < \theta\nabla f(x) \bullet x \le \nabla^2 f(x)x \bullet x, \tag{3.2}$$

where $\bullet$ stands for the dot product in $\mathbb{R}^n$.

3.1. Sobolev spaces of p-periodic functions. Let $\widetilde{\mathscr{H}}$ denote the first Sobolev space of p-periodic functions from $\mathbb{R}$ to V, i.e.

$$\widetilde{\mathscr{H}} := H^1_p(\mathbb{R}, V) = \{x : \mathbb{R} \to V : x(0) = x(p),\ x|_{[0,p]} \in H^1([0,p]; V)\},$$

equipped with the inner product

$$\langle x, y\rangle := \int_0^p (\dot{x}(t) \bullet \dot{y}(t) + x(t) \bullet y(t))\, dt, \quad x, y \in H^1_p(\mathbb{R}, V).$$

Let $O(2)$ denote the group of 2×2-orthogonal matrices. Notice that $O(2) = SO(2) \cup SO(2)\kappa$, where $\kappa = \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$. It is convenient to identify a rotation $\left[\begin{smallmatrix} \cos\tau & -\sin\tau \\ \sin\tau & \cos\tau \end{smallmatrix}\right] \in SO(2)$ with $e^{i\tau} \in S^1 \subset \mathbb{C}$. Notice that $\kappa e^{i\tau} = e^{-i\tau}\kappa$.

Put $\widetilde{G} = \Gamma \times \mathbb{Z}_2 \times O(2)$, then the space $\widetilde{\mathscr{H}}$ is an orthogonal Hilbert representation of $\widetilde{G}$. Indeed, for $x \in \widetilde{\mathscr{H}}$ and $\gamma \in \Gamma$, $e^{i\tau} \in S^1$ we can define the group action as

$$(\gamma, \pm 1, e^{i\tau})x(t) = \pm\gamma x\bigg(t + \frac{p\tau}{2\pi}\bigg), \qquad (\gamma, \pm 1, e^{i\tau}\kappa)x(t) = \pm\gamma x\bigg(-t + \frac{p\tau}{2\pi}\bigg).$$

Γ is acting on $V = \mathbb{R}^n$ by permuting the vector coordinates in $\mathbb{R}^n$.

It is useful to identify a p-periodic function $x\colon \mathbb{R} \to V$ with a function $\widetilde{x}\colon S^1 \to V$ via the following commuting diagram:

$$\begin{tikzcd} \mathbb{R} \arrow[rr, "e"] \arrow[dr, "x"'] & & S^1 \arrow[dl, "\widetilde{x}"] \\ & V & \end{tikzcd} \qquad e^{\tau} = e^{i2\pi\tau/p}.$$

Using this identification, we will write $H^1(S^1_p, V)$ instead of $H^1_p(\mathbb{R}, V)$. In addition, notice that for $x \in \widetilde{\mathscr{H}}$, the isotropy group G'_x, where $G' := \Gamma \times S^1$, is twisted if and only if x is a non-constant periodic function.

Consider the $O(2)$-isotypical decomposition of $\widetilde{\mathscr{H}}$, which is

$$\widetilde{\mathscr{H}} = \overline{\bigoplus_{k=0}^{\infty} \mathbb{V}_k}, \quad \mathbb{V}_k := \{u_k \cos(2k\pi/p \cdot t) + v_k \sin(2k\pi/p \cdot t) : u_k, v_k \in V\}. \tag{3.3}$$

Each of the components $\mathbb{V}_k$ can be identified $\widetilde{G}$-equivariantly to the space $\mathbb{V}'_k$ of complex p-periodic functions defined by

$$\mathbb{V}'_k := \big\{e^{i2\pi kt/p}(x_k + iy_k) : x_k, y_k \in V\big\}, \quad \text{where } x_k = \frac{u_k + v_k}{2},\ y_k = \frac{u_k - v_k}{2}.$$

One can easily notice that the space $\mathbb{V}'_k$ is equivalent to the complexification $V^c := V \oplus V$ of V, where κ acts on vectors in V^c by conjugation and for $e^{i\tau} \in SO(2) \simeq S^1$, $e^{i\tau}z = e^{ik\tau} \cdot z$, where '$\cdot$' denotes complex multiplication and $z = x + iy \in V^c$. We will denote this $\widetilde{G}$-representation by $\mathfrak{v}_k(V^c)$. Let us define the space $\widetilde{h}^{2,p}(V^c)$ composed of all sequences $\mathfrak{z} := \{z_k\}_{k=0}^{\infty}$ such that $z_0 \in V$, $z_k \in \mathfrak{v}_k(V^c)$ for $k > 0$, and satisfying the condition

$$\|\mathfrak{z}\|^2 := \|z_0\|^2 + \sum_{k=1}^{\infty} \|z_k\|^2 \left(\frac{p}{2} + 4\pi^2 k^2\right) < \infty.$$

The space $\widetilde{h}^{2,p}(V^c)$ is clearly a Hilbert $\widetilde{G}$-representation, which is equivalent to $\widetilde{\mathscr{H}}$.

Denote by $H \subset \mathbb{Z}_2 \times O(2)$ the subgroup

$$H := \{(1,1), (-1,-1), (1,\kappa), (-1,-\kappa)\} =: \mathbf{D}_2^d.$$

We are interested in the H-fixed-point subspace $\mathscr{H} := \widetilde{\mathscr{H}}^H$. Notice that the normalizer $N(H) = \Gamma \times \mathbb{Z}_2 \times D_2$, therefore $W(H) = \Gamma \times \mathbb{Z}_2$, and $\mathscr{H} = \widetilde{\mathscr{H}}^H$ is a $W(H)$-orthogonal representation.

Clearly, each $\mathbb{V}_k$ in the decomposition (3.3) is $\Gamma \times \mathbb{Z}_2 \times O_2$-invariant and therefore

$$\mathscr{H} = \overline{\bigoplus_{k=0}^{\infty} \mathbb{V}_k^H}. \tag{3.4}$$

By the definition of fixed-point subspace, we can easily obtain that $x \in \mathscr{H}$ if and only if x is even and $-x(p/2+t) = -x(p/2-t) = x(t)$. It follows immediately that

$$\mathbb{V}_k^H := \begin{cases} \{0\} & \text{if } k \text{ is even,} \\ \{u_k \cos(2k\pi/p \cdot t) : u_k \in V\} & \text{if } k \text{ is odd.} \end{cases}$$

Then we have the following decomposition of $\mathscr{H}$:

$$\mathscr{H} = \overline{\bigoplus_{l=0}^{\infty} \mathbb{W}_{2l+1}}, \qquad \mathbb{W}_{2l+1} := \{u_{2l+1} \cos(2(2l+1)\pi/p \cdot t) : u_{2l+1} \in V\}, \tag{3.5}$$

where each of the components $\mathbb{W}_{2l+1}$ is Γ-equivariant to V.

For any function $x \in \mathscr{H} \subset \widetilde{\mathscr{H}}$, by using the isotypical decomposition (3.4) of $\mathscr{H}$, we have

$$\|x\|^2 = \int_0^p (\dot{x}(t) \bullet \dot{x}(t) + x(t) \bullet x(t))\, dt = \sum_{l=0}^{\infty} |u_{2l+1}|^2 \left(\frac{p}{2} + \frac{2\pi^2(2l+1)^2}{p}\right).$$

Notice that the norm $|||\cdot|||$ on $\mathscr{H}$, associated with the inner product

$$\langle x, y\rangle' := \int_0^p \dot{x}(t) \bullet \dot{y}(t)\, dt, \quad x, y \in \mathscr{H},$$

which is given by

$$|||x|||^2 := \sum_{l=0}^{\infty} |u_{2l+1}|^2 \left(\frac{2\pi^2(2l+1)^2}{p}\right), \tag{3.6}$$

is equivalent to the norm $\|\cdot\|$, therefore $(\mathscr{H}, \|\cdot\|)$ is equivalent to the Hilbert space $(\mathscr{H}, |||\cdot|||)$. Therefore, without loss of generality, we can assume in what follows that $\mathscr{H}$ is equipped with the norm (3.6). In order to simplify the notation, in what follows we will simply write $\|\cdot\|$ instead of $|||\cdot|||$ and we will denote the inner product $\langle\,\cdot\,,\,\cdot\,\rangle'$ by $\langle\,\cdot\,,\,\cdot\,\rangle$. In addition, notice that we also have

$$\|x\|_{L^2} \leq \|x\| \quad \text{for all } x \in \mathscr{H}. \tag{3.7}$$

3.2. Reformulation of (3.1) **as a symmetric variational problem.** Consider the space $\widetilde{\mathscr{H}}$ defined in Subsection 3.1 and define the following variational functional $\mathcal{J}\colon \widetilde{\mathscr{H}} \to \mathbb{R}$:

$$\mathcal{J}(x) := \int_0^p \left(\frac{1}{2}|\dot{x}(t)|^2 - f(x(t))\right) dt, \qquad |x(t)|^2 := x(t) \bullet x(t), \tag{3.8}$$

where $x \in \widetilde{\mathscr{H}}$. Notice that solutions to (3.1) are the critical points of $\mathcal{J}$, i.e. if $\nabla\mathcal{J}(x) = 0$ then x is a C^3-function satisfying (3.1) and vice versa. Therefore, finding solutions to (3.1) can be reduced to finding critical points of $\mathcal{J}$.

Let us make the following obvious remarks:

Remark 3.1. Under assumptions (A2), the functional $\mathcal{J}\colon \widetilde{\mathscr{H}} \to \mathbb{R}$ is $\Gamma \times O(2)$-invariant, thus $\nabla\mathcal{J}\colon \widetilde{\mathscr{H}} \to \widetilde{\mathscr{H}}$ is $\Gamma \times O(2)$-equivariant. Moreover, $\mathcal{J}$ is C^2-Fréchet differentiable and we have

$$\begin{aligned} \langle \nabla\mathcal{J}(x), y\rangle = D\mathcal{J}(x)y \ &= \int_0^p (\dot{x}(t) \bullet \dot{y}(t) - \nabla f(x) \bullet y(t))\, dt, \\ \langle \nabla^2\mathcal{J}(x)y, z\rangle = D^2\mathcal{J}(x)(y,z) \ &= \int_0^p (\dot{y}(t) \bullet \dot{z}(t) - \nabla^2 f(x(t))y(t) \bullet z(t),)\, dt, \end{aligned}$$

where $x, y, z \in \widetilde{\mathscr{H}}$.

Remark 3.2. Under assumption (A1), the gradient map $\nabla\mathcal{J}$ has additional symmetry $\mathbb{Z}_2$, i.e. it is $\Gamma \times \mathbb{Z}_2 \times O(2)$-equivariant. Let $\mathscr{H}$ be the H-fixed-point subspace of $\widetilde{\mathscr{H}}$ and J be the restriction of $\mathcal{J}$ to the H-fixed-point subspace, i.e. $\mathscr{H} := \widetilde{\mathscr{H}}^H$, $J := \mathcal{J}|_{\mathscr{H}}\colon \mathscr{H} \to \mathbb{R}$. Since $\nabla\mathcal{J}(\mathscr{H}) \subset \mathscr{H}$, it follows that $\nabla J = \nabla\mathcal{J}|_{\mathscr{H}}$ is a $\Gamma \times \mathbb{Z}_2$-equivariant completely continuous gradient field. Therefore, the critical points of J are also critical points of $\mathcal{J}$, and consequently they are the solutions to (3.1). In addition, notice that any non-zero critical point x of J is a non-constant p-periodic solution to (3.1), i.e. 0 is the unique stationary solution of (3.1) in $\mathscr{H}$.

Remark 3.3. Notice that the $\Gamma \times \mathbb{Z}_2$-equivariant operator $\nabla J(x)$ defined on the space $\mathscr{H}$ equipped with the norm (3.6) can be explicitly expressed by the formula

$$\nabla J(x) = x - j \circ L_o^{-1} \circ N_{\nabla f - \mathrm{Id}}(x), N_{\nabla f}(x)(t) := \nabla f(x(t)), \tag{3.9}$$

where $L_o x = \ddot{x}$, $L_o^{-1}\colon \mathscr{H} \to (H^3(S_p^1;V))^H$, $j\colon H^3(S_p^1;V) \to \widetilde{\mathscr{H}}$ is a compact operator.

4. Statement of the main result

Put $G = \Gamma \times \mathbb{Z}_2$.

4.1. Maximal orbit types in $\mathscr{H}$. Consider the Hilbert $\widetilde{G}$-representation introduced in above, where $\widetilde{G} := \Gamma \times \mathbb{Z}_2 \times O(2)$ and $G' := \Gamma \times S^1$. By identifying S^1 with $SO(2)$, we have $G' \subset \widetilde{G}$. Consider the following subgroups of G':

$$K^{\phi,l} := \{(\gamma, z) \in K \times S^1 : \phi(\gamma) = z^l\},$$

where $K \subset \Gamma$ is a subgroup, $\phi\colon K \to S^1$ is a homomorphism, and k is a fixed positive integer. In such a case, the group $K^{\phi,l}$ is called a *twisted* (by homomorphism ϕ) *l-folded* subgroup of G'. Following this practice, we denote the twisted one-folded group $K^{\phi,1}$ by K^ϕ and simply call it *twisted*. Notice that the normalizer $N(K^{\phi,l})$ of $K^{\phi,l}$ can be written as

$$N(K^{\phi,l}) = N_o \times S^1, \quad \text{where } K \subset N_o \subset N(K),$$
$$N_o := \{g \in N(K) : \phi(gkg^{-1}) = \phi(k) \text{ for all } k \in K\}.$$

Since $(K \times S^1)/K^{\phi,l} \simeq S^1$, we have the following exact sequence:

$$0 \longrightarrow S^1 \longrightarrow W(K^{\phi,l}) \longrightarrow N_o/K \longrightarrow 0,$$

thus there is a natural injective homomorphism from S^1 to $W(K^{\phi,l})$. Denote the set of conjugacy classes of all twisted isotropy groups in X by $\Phi^t(G';X) \subset \Phi(G';X)$, i.e. $\Phi^t(G';X)$ is the set of all *twisted orbit types* in X. Also, denote the subset of $\Phi^t(G';X)$ composed of all (H) such that H is a twisted l-folded subgroup by $\Phi_l^t(G',X)$. We have the following definition.

Definition 4.1. A one-twisted orbit type $(K^\phi) \in \Phi(G';X)$ is said to be a *twisted maximal orbit type* if (K^ϕ) is maximal in $\Phi_1^t(G';X)$ with respect to the natural order relation.

Definition 4.2. Let $x \in \mathscr{H} \setminus \{0\}$. Then the orbit type (G_x) is called to be *t-maximal*, if (G'_x) is twisted maximal orbit type in $\widetilde{\mathscr{H}}$.

Remark 4.3. Notice that if (G_x) is a t-maximal orbit type in $\mathscr{H} \setminus \{0\}$, then it is a maximal orbit type in $\mathscr{H} \setminus \{0\}$. Indeed, suppose that $H \subset G$ and (H) is the orbit type in $\mathscr{H} \setminus \{0\}$ for some x. Then there exists a subgroup $K \subset \Gamma$ and a homomorphism $\varphi\colon K \to \mathbb{Z}_2$ such that $H = K^\varphi$. If H is not maximal in $\mathscr{H} \setminus \{0\}$ then the homomorphism φ can be extended to $\varphi'\colon K' \to \mathbb{Z}_2$, where $K \subsetneq K'$ and $H' := K'^{\varphi'}$ is a G-isotropy for some $x' \in \mathscr{H} \setminus \{0\}$. Therefore, $\ker\varphi \subsetneq \ker\varphi'$. Plus, since x and x' are non-constant, then $G'_x, G'_{x'} \in \Phi^t(G';\mathscr{H})$. Let ϕ and ϕ' be the homomorphisms associated with G'_x and $G'_{x'}$, hence, $\ker\phi \subsetneq \ker\phi'$.

Therefore, we have $G'_x \subsetneq G'_{x'}$, and thus (G'_x) is not a maximal twisted orbit type in $\widetilde{\mathscr{H}}$ which is a contradiction with Definition 4.2.

4.2. Formulation of the main result.

Theorem 4.4. *Let $p > 0$ be an arbitrary number. Assume that the function $f\colon V \to \mathbb{R}$ is a C^2-functional satisfying conditions* (A1)–(A3)*. Then, for every maximal orbit type (K) in $\mathscr{H} \setminus \{0\}$, there exists at least one G-orbit of critical points of J in $\mathscr{H}$, corresponding to an orbit of p-periodic solutions to* (3.1) *in the space $\mathscr{H}$ with the minimal period exactly p. Moreover, if $\{(K_1), \dots, (K_m)\}$ is the collection of all t-maximal orbit types in $\mathscr{H}$, then system* (3.1) *has at least $\sum_{j=1}^{m} |\Gamma/K_j|$ periodic solutions with the minimal period exactly p.*

5. Proof of the main result

The proof follows the ideas from [22], which are based on the application of the concept of Nehari manifold. For the sake of completeness, we include all the related details. Full credit for the main idea of proof should be given to Yu Ming Xiao.

5.1. Auxiliary results.

Lemma 5.1. *Assume $f\colon V \to \mathbb{R}$ satisfies conditions* (A1)–(A3)*. Then*

$$0 < (1+\theta)f(x) \le \nabla f(x) \bullet x \quad \text{for all } x \in V \setminus \{0\}. \tag{5.1}$$

Proof. For any $x \in V \setminus \{0\}$, define the function $\phi\colon \mathbb{R} \to \mathbb{R}$ by $\phi(t) := f(tx)$ for all $t \in \mathbb{R}$. Clearly ϕ is of class C^1 and by direct computation, we have $\phi(0) = f(0) = 0$, $\phi'(t) = \nabla f(tx) \bullet x$ and $\phi''(t) = \nabla^2 f(tx)x \bullet x$. Therefore, by (A3), we obtain that for any $t > 0$, the following inequality holds:

$$0 < \theta \nabla f(tx) \bullet (tx) \le \nabla^2 f(tx)(tx) \bullet (tx),$$

which implies $0 < \theta\phi'(t) \le t\phi''(t)$. Thus,

$$\theta f(x) = \int_0^1 \theta\phi'(t)\,dt \le \int_0^1 t\phi''(t)\,dt = \nabla f(x) \bullet x - f(x).$$

Thus, it follows immediately that $0 < (1+\theta)f(x) \le \nabla f(x) \bullet x$. □

Lemma 5.2. *Assume $f\colon V \to \mathbb{R}$ satisfies conditions* (A1)–(A3)*. Put*

$$M := \max\{f(x) : |x| = 1\} \quad \text{and} \quad m := \min\{f(x) : |x| = 1\}.$$

Then, for all $x \in V$,

$$f(x) \le M|x|^{\theta+1} \quad \text{for } |x| \le 1 \quad \text{and} \quad f(x) \ge m|x|^{\theta+1} \quad \text{for } |x| \ge 1, \tag{5.2}$$

where $|x|^2 = x \bullet x$.

Proof. If $x = 0$, then $f(x) = 0$ and $|x| = 0$, so, obviously, the lemma is true. For any $x \in V \setminus \{0\}$, define the function $\phi\colon \mathbb{R} \to \mathbb{R}$ by $\phi(t) := f(tx)$ for all $t \in \mathbb{R}$. Clearly, ϕ is of class C^1 and $\phi'(t) = \nabla f(tx) \bullet x$. By the previous lemma, we have $0 < (1+\theta)f(tx) \leq \nabla f(tx) \bullet tx$, i.e. $0 < (1+\theta)\phi(t) \leq t\phi'(t)$. Assume $s \geq 1$, then

$$\int_1^s \frac{\phi'(t)}{\phi(t)}\,dt \geq \int_1^s \frac{1+\theta}{t}\,dt.$$

Consequently, $f(sx) = \phi(s) \geq s^{(1+\theta)}\phi(1) = s^{(1+\theta)}f(x)$. Therefore, we can conclude that if $|x| \geq 1$, then

$$f(x) = f\left(|x|\frac{x}{|x|}\right) \geq |x|^{(1+\theta)} f\left(\frac{x}{|x|}\right) \geq m|x|^{(1+\theta)},$$

if $|x| \leq 1$, then

$$f\left(\frac{x}{|x|}\right) \geq \frac{1}{|x|^{(1+\theta)}} f(x).$$

Therefore, it follows that

$$f(x) \leq |x|^{(1+\theta)} f\left(\frac{x}{|x|}\right) \leq M|x|^{(1+\theta)}. \qquad \square$$

5.2. Nehari manifold for J. We define the following set:

$$N := \{x \in \mathscr{H} \setminus \{0\} : \langle \nabla J(x), x\rangle = 0\}. \tag{5.3}$$

Proposition 5.3. *N is a complete sub-manifold of $\mathscr{H}$ of co-dimension one.*

Proof. Define the function $\Psi\colon \mathscr{H} \to \mathbb{R}$ by

$$\Psi(x) := \langle \nabla J(x), x\rangle \quad \text{for all } x \in \mathscr{H}.$$

Clearly, Ψ is of class C^1. We claim that zero is a regular value of Ψ on $\mathscr{H} \setminus \{0\}$. Indeed, notice that $N = \Psi^{-1}(0) \cap (\mathscr{H} \setminus \{0\})$, and for every $x \in N$ (remember $x \neq 0$) we have

$$0 = \langle \nabla J(x), x\rangle \Leftrightarrow \int_0^p |\dot{x}(t)|^2\,dt = \int_0^p \nabla f(x(t)) \bullet x(t)\,dt.$$

Since $D\Psi(x)v = \langle \nabla^2 J(x)v, x\rangle + \langle \nabla J(x), v\rangle$, thus by (A3)

$$\begin{aligned} D\Psi(x)x &= \langle \nabla^2 J(x)x, x\rangle + \langle \nabla J(x), x\rangle = \langle \nabla^2 J(x)x, x\rangle + 0 \\ &= \int_0^p (|\dot{x}(t)|^2 - \nabla^2 f(x(t))x(t) \bullet x(t))\,dt \\ &= \int_0^p \big(\nabla f(x(t)) \bullet x(t) - \nabla^2 f(x(t))x(t) \bullet x(t)\big)\,dt < 0. \end{aligned}$$

Consequently, $D\Psi(x)\colon \mathscr{H} \to \mathbb{R}$ is surjective for all $x \in N$, and therefore zero is a regular value of Ψ restricted to $\mathscr{H} \setminus \{0\}$. Thus N is a sub-manifold of co-dimension one. In order to show that N is complete, we notice that there exists $\varepsilon > 0$ such that $B_\varepsilon(0) \cap N = \emptyset$. Notice that, inequality (5.2) implies that

$\|\nabla^2 f(0)\| = 0$. For, by the Taylor formula $f(x) = \nabla^2 f(0)x \bullet x/2 + r(x)$, where $\lim_{x\to 0} r(x)/|x|^2 = 0$, thus by (5.2)

$$\frac{1}{2}\nabla^2 f(0)x \bullet x \leq M|x|^{\theta+1} + |r(x)|$$

and, consequently,

$$\frac{\nabla^2 f(0)x \bullet x}{|x|^2} \leq 2M|x|^{\theta-1} + \frac{2|r(x)|}{|x|^2},$$

which implies that the non-negative quadratic form $\nabla^2 f(0)x\bullet x$ is zero for any x. By continuity of $\nabla^2 f(x)$, there exists $\eta > 0$ such that for $|x| < \eta$ we have $\|\nabla^2 f(x)\| < 1$. The continuity of the inclusion $i\colon \mathscr{H} \to C(S^1_p;V)$ ($C(S^1_p;V)$ equipped with the norm $\|x\|_\infty := \max\{|x(t)| : t \in [0,p]\}$) implies that there exists $\varepsilon > 0$ such that if $\|x\| < \varepsilon$ then $\|x\|_\infty < \eta$. Then, by applying inequalities (A3), (5.2) and (3.7) we obtain

$$\begin{aligned}\Psi(x) &= \int_0^p \dot{x}(t) \bullet \dot{x}(t)\,dt - \int_o^p \nabla f(x(t)) \bullet x(t)\,dt \\ &\geq \|x\|^2 - \frac{1}{\theta}\int_0^p \nabla^2 f(x(t))x(t) \bullet x(t)\,dt \\ &\geq \|x\|^2 - \frac{1}{\theta}\|x\|^2_{L^2} \geq \|x\|^2 - \frac{1}{\theta}\|x\|^2 = \|x\|^2\left(1 - \frac{1}{\theta}\right) > 0.\end{aligned}$$

Consequently, if $x \in B_\varepsilon(0)$ then $x \notin N$. □

Proposition 5.4. *Assume $f\colon V \to \mathbb{R}$ satisfies conditions* (A1)–(A3). *Let $J\colon \mathscr{H} \to \mathbb{R}$ be defined as in Remark* 3.2 *and N be defined by* (5.3). *Then $x \in \mathscr{H} \setminus \{0\}$ is a critical point of J if and only if $x \in N$ and it is a critical point of $J|_N\colon N \to \mathbb{R}$.*

Proof. It is clear that if for $x \neq 0$ we have $\nabla J(x) = 0$ then $x \in N$ and x is also a critical point of $J|_N$. Suppose that $x \in N$ and $DJ(x)|_{T_x(N)} \equiv 0$. Since $T_x(N)$ is a subspace of co-dimension 1 such that $T_x(N) \oplus \operatorname{span}\{x\} = \mathscr{H}$, we have that for every $v \in \mathscr{H}$ there exists $u \in T_x(N)$ and $t \in \mathbb{R}$ so $v = u + tx$. Therefore,

$$DJ(x)v = DJ(x)u + tDJ(x)x = DJ_N(x)u + t\langle \nabla J(x), x\rangle = 0 + 0 = 0,$$

and consequently $\nabla J(x) \equiv 0$, i.e. x is a critical point of J. □

Lemma 5.5. *Assume $f\colon V \to \mathbb{R}$ satisfies conditions* (A1)–(A3). *Let $J\colon \mathscr{H} \to \mathbb{R}$ be defined as in Remark* 3.2 *and N be defined by* (5.3). *Put $S(\mathscr{H}) := \{w \in \mathscr{H} : \|w\| = 1\}$. Then there exists a differentiable function $\mathfrak{s}\colon S(\mathscr{H}) \to (0,\infty)$ such that*

(a) *for all $w \in S(\mathscr{H})$ we have $\langle \nabla J(\mathfrak{s}(w)w), \mathfrak{s}(w)w\rangle = 0$;*

(b) *if for some* $c > 0$ *and* $w \in S(\mathscr{H})$ *we have* $\langle \nabla J(\mathfrak{c}w), cw\rangle = 0$, *then* $c = \mathfrak{s}(w)$.

Proof. Define the function $\mathfrak{K}\colon S(\mathscr{H}) \times \mathbb{R} \to \mathbb{R}$ by the following formula:

$$\mathfrak{K}(w,s) := J(sw) = \frac{s^2}{2}\int_0^p |\dot w(t)|^2\,dt - \int_0^p f(sw(t))\,dt. \tag{5.4}$$

Then, for every $s > 0$, we have

$$\begin{aligned}\mathfrak{K}'_s(w,s) &= s\int_0^p |\dot w(t)|^2\,dt - \frac{1}{s}\int_0^p \nabla f(sw(t)) \bullet (sw(t))\,dt,\\ \mathfrak{K}''_s(w,s) &= \int_0^p |\dot w(t)|^2\,dt - \frac{1}{s^2}\int_0^p \nabla^2 f(sw(t))(sw(t)) \bullet (sw(t))\,dt,\end{aligned}$$

where $w \in S(\mathscr{H})$ and $s \in \mathbb{R}$. Let $w_o \in S(\mathscr{H})$ be fixed. Put

$$\eta_o := \frac{1}{2}\int_0^p |w_o(t)|^2\,dt > 0,$$

then there exists $\delta > 0$ such that for all $w \in \mathcal{M} := S(\mathscr{H}) \cap B_\delta(w_o)$ we have

$$\int_0^p |w(t)|^2\,dt > \eta_o.$$

Let $0 < \eta < \eta_o$. Put $A_w := \{t \in [0,p] : |w(t)|^2 \geq \eta/p\}$. We claim that $\inf\{\mu(A_w) : w \in \mathcal{M}\} = \alpha_o > 0$. Indeed, suppose for contradiction that there exists a sequence $w_n \in \mathcal{M}$ such that $\mu(A_{w_n}) \to 0$ as $n \to \infty$, then we have for all $n \in \mathbb{N}$

$$\begin{aligned}\eta < \eta_o < \int_0^p |w_n(t)|^2\,dt &= \int_{A_{w_n}} |w_n(t)|^2\,dt + \int_{A^c_{w_n}} |w_n(t)|^2\,dt\\ &\leq \int_{A_{w_n}} |w_n(t)|^2\,dt + \frac{\eta}{p}(p - \mu(A_{w_n})).\end{aligned}$$

Since $\int_{A_{w_n}} |w_n(t)|^2\,dt \to 0$ as $n \to \infty$, it follows that $\eta < \eta_o \leq \eta$ which is a contradiction. Therefore, for $s > 0$ being a sufficiently large number we have for all $w \in \mathcal{M}$

$$\begin{aligned}\mathfrak{K}(w,s) &= \frac{1}{2}s^2\|w\|^2 - \int_0^p f(sw(t))\,dt\\ &\leq \frac{1}{2}s^2 - \int_0^p f(sw(t))\,dt \leq \frac{1}{2}s^2 - \left(\frac{\eta}{p}\right)^{(\theta+1)/2} m\alpha_0 s^{\theta+1} < 0,\end{aligned} \tag{5.5}$$

so, for s sufficiently large, $\mathfrak{K}(w,s) < 0$. Similarly, for $s > 0$ being a sufficiently large number we also have, for all $w \in \mathcal{M}$,

$$\begin{aligned}\mathfrak{K}'_s(w,s) &= s\|w\|^2 - \int_0^p \nabla f(sw(t)) \bullet w(t)\,dt\\ &\leq s - \frac{\theta+1}{s}\int_0^p f(sw(t))\,dt \leq s - (1+\theta)\left(\frac{\eta}{p}\right)^{(\theta+1)/2} m\alpha_0 s^{\theta} < 0.\end{aligned} \tag{5.6}$$

Notice that for $s > 0$ being sufficiently small (say $0 < s < \alpha_o$), we have that for all $w \in S(\mathscr{H})$ and $t \in [0,p]$, $|w(st)| < 1$, therefore, by Lemma 5.2, $f(sw(t)) \leq M(s|w(t)|)^{\theta+1}$. Consequently, we have

$$\mathfrak{K}(w,s) = \frac{1}{2}s^2\|w\|^2 - \int_0^p f(sw(t))\,dt \geq \frac{1}{2}s^2 - Ms^{\theta+1}\int_0^p |w(t)|^{\theta+1}\,dt,$$

and therefore, for sufficiently small $s > 0$ we have $\mathfrak{K}(w,s) > 0$ for all $w \in S(\mathscr{H})$. Clearly, we also have that for sufficiently small $s > 0$, $\mathfrak{K}_s'(w,s) < 0$ for all $w \in S(\mathscr{H})$.

Put $c_1 = s$, where $s > 0$ is a number for which inequality (5.6) holds for all $w \in \mathcal{M}$, i.e. $\mathfrak{K}_s'(w,c_1) < 0$. On the other hand, one can choose $c_o = s$, where $s > 0$ is a sufficiently small number such that $\mathfrak{K}_s'(w,s) > 0$ for all $w \in \mathcal{M}$. Therefore, by the Intermediate Value theorem, for every $w \in \mathcal{M}$, there exists $s_o(w) \in (c_0, c_1)$ such that $\mathfrak{K}_s'(w, s_o(w)) = 0$. In order to prove the uniqueness, assume that for a given w in $\mathcal{M}$ there exist two $s_1, s_2 \in (c_o, c_1)$ such that $\mathfrak{K}_s'(w,s_i) = 0$, $i = 1,2$. Notice that, if $\mathfrak{K}_s'(w,s_o) = 0$, then

$$\int_0^p |\dot{w}(t)|^2\,dt = \frac{1}{s_o^2}\int_0^p \nabla f(s_o w(t)) \bullet (s_o w(t))\,dt,$$

and therefore (since $\theta > 1$)

$$\begin{aligned}\mathfrak{K}_s''(w,s_o) &= \int_0^p |\dot{w}(t)|^2\,dt - \frac{1}{s_o^2}\int_0^p \nabla^2 f(s_o w(t))(s_o w(t)) \bullet (s_o w(t))\,dt \\ &= \frac{1}{s_o^2}\int_0^p \big(\nabla f(s_o w(t)) \bullet (s_o w(t)) - \nabla^2 f(s_o w(t))(s_o w(t)) \bullet (s_o w(t))\big)\,dt < 0.\end{aligned}$$

This means we have

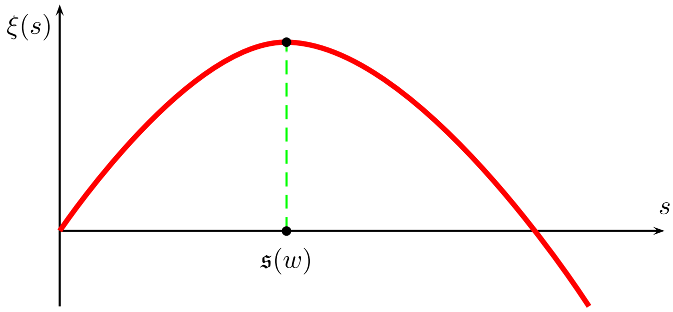
$$\mathfrak{K}_s'(w,s_o) = 0 \Rightarrow \mathfrak{K}_s''(w,s_o) < 0. \tag{5.7}$$

Therefore, since $\mathfrak{K}_s''(w,s_1) < 0$ and $\mathfrak{K}_s''(w,s_2) < 0$, there exists $s' \in (s_1,s_2)$ such that $\mathfrak{K}_s(w,s') = \min\{\mathfrak{K}_s(w,s) : s \in [s_1,s_2]\}$. Consequently, $\mathfrak{K}_s'(w,s') = 0$ and $\mathfrak{K}_s''(w,s') \geq 0$. However, on the other hand, by (5.7), we have $\mathfrak{K}_s''(w,s') < 0$ which is a contradiction. Differentiability of the function $\mathfrak{s}\colon S(\mathscr{H}) \to (0,\infty)$ follows from the Implicit Function theorem. $\square$

Remark 5.6. Assume $f\colon V \to \mathbb{R}$ satisfies conditions (A1)–(A3) and consider the function $\mathfrak{K}\colon S(\mathscr{H}) \times \mathbb{R} \to \mathbb{R}$ defined by (5.4) and the function $\mathfrak{s}\colon S(\mathscr{H}) \to (0,\infty)$ given in Lemma 5.5. The properties of the function $\mathfrak{K}$ can be summarized as follows:

(a) for every $w \in S(\mathscr{H})$ the function $\mathfrak{K}(w,\,\cdot\,)$ has a unique maximum at $\mathfrak{s}(w)$ and $\mathfrak{K}(w,\mathfrak{s}(w)) > 0$;
(b) for every $w \in S(\mathscr{H})$, $\lim_{s\to\infty} \mathfrak{K}(w,s) = -\infty$;
(c) for every $w \in S(\mathscr{H})$, $\mathfrak{K}_s'(w,\mathfrak{s}(w)) = 0$ and $\mathfrak{K}_s''(w,\mathfrak{s}(w)) < 0$.

For a given $w \in S(\mathscr{H})$ the function $\xi(s) := \mathfrak{K}(w, s)$ can be illustrated by the following graph:



Corollary 5.7. *Assume $f\colon V \to \mathbb{R}$ satisfies conditions* (A1)–(A3) *and consider the function $\mathfrak{K}\colon S(\mathscr{H}) \times \mathbb{R} \to \mathbb{R}$ defined by* (5.4) *and the function $\mathfrak{s}\colon S(\mathscr{H}) \to (0, \infty)$ given in Lemma* 5.5. *Let N be the Nehari manifold defined by* (5.3). *Define $\mathfrak{W}\colon S(\mathscr{H}) \to N$ by $\mathfrak{W}(w) := \mathfrak{s}(w)w$, for $w \in S(\mathscr{H})$. Then $\mathfrak{W}$ is a diffeomorphism.*

5.3. Palais–Smale condition.

Proposition 5.8. *Assume $f\colon V \to \mathbb{R}$ satisfies conditions* (A1)–(A3). *Let $J\colon \mathscr{H} \to \mathbb{R}$ be defined as in Remark* 3.2 *and let N be the Nehari manifold defined by* (5.3). *Then J satisfies the Palais–Smale condition* ((PS)*-condition for short*) *on N, i.e. if $x_n \in N$ is a sequence such that*

(a) *for some $a < b$, $a < J(x_n) < b$ for all $n \in \mathbb{N}$, and*
(b) $\lim_{n\to\infty} \nabla J(x_n) = 0$, *then $\{x_n\}$ contains a convergent to an element in N subsequence.*

Proof. Assume that a sequence $\{x_n\}$ satisfies conditions (a) and (b). Notice that, since $x_n \in N$, thus

$$\|x_n\|^2 = \int_0^p \nabla f(x_n(t)) \bullet x_n(t)\, dt. \tag{5.8}$$

By assumption, $J(x_n) < b$ for all $n \in \mathbb{N}$, thus by Lemma 5.1 and (5.8),

$$\begin{aligned} b > J(x_n) &= \frac{1}{2}\|x_n\|^2 - \int_0^p f(x_n(t))\, dt \\ &\geq \frac{1}{2}\|x_n\|^2 - \frac{1}{\theta+1}\int_0^p \nabla f(x_n(t)) \bullet x_n(t)\, dt \\ &= \frac{1}{2}\|x_n\|^2 - \frac{1}{1+\theta}\|x_n\|^2 = \|x_n\|^2\left(\frac{1}{2} - \frac{1}{\theta+1}\right). \end{aligned}$$

Since $\theta > 1$, thus we get, for all $n \in \mathbb{N}$,

$$\frac{b}{1/2 - 1/(\theta+1)} \geq \|x_n\|^2,$$

which implies that the sequence $\{x_n\}$ is bounded. By formula (3.9), we have that

$$\nabla J(x_n) = x_n - j \circ L^{-1} \circ N_{\nabla f}(x_n),$$

where j is a compact operator. Therefore, the sequence $y_n := j \circ L^{-1} \circ N_{\nabla f}(x_n)$, $n \in \mathbb{N}$ contains a convergent to x_* subsequence $\{y_{n_k}\}$. However, $\nabla J(x_{n_k}) \to 0$ as $k \to \infty$, thus $\lim_{k\to\infty} x_{n_k} = \lim_{k\to\infty} y_{n_k} = x_*$. Since N is complete, $x_* \in N$. □

5.4. Minimization of J on Nehari manifold.

LEMMA 5.9. *Assume $f\colon V \to \mathbb{R}$ satisfies conditions* (A1)–(A3). *Let $J\colon \mathscr{H} \to \mathbb{R}$ be defined as in Remark* 3.2 *and let N be the Nehari manifold defined by* (5.3). *Then for every $x \in N$, $J(x) > 0$.*

PROOF. By a similar argument as in the proof of Proposition 5.8, for every $x \in N$, we have

$$J(x) = \frac{1}{2}\|x\|^2 - \int_0^p f(x(t))\,dt \geq \|x\|^2\left(\frac{1}{2} - \frac{1}{\theta+1}\right) > 0. \qquad \square$$

LEMMA 5.10. *Assume $f\colon V \to \mathbb{R}$ satisfies conditions* (A1)–(A3). *Let $J\colon \mathscr{H} \to \mathbb{R}$ be defined as in Remark* 3.2 *and let N be the Nehari manifold defined by* (5.3). *Assume that H is a subgroup of $G := \Gamma \times \mathbb{Z}_2$ such that (H) is an orbit type in $\mathscr{H}$. Put $N^H := \{x \in N : gx = x$ for all $g \in H\}$. Then, there exists $x_o \in N^H$ such that $J(x_o) = \inf\{J(x) : x \in N^H\}$ and $\nabla J(x_0) = 0$. Moreover, $G_{x_o} \supset H$. In the case (H) is a maximal orbit type in $\mathscr{H}$, we have $G_{x_o} = H$.*

PROOF. Notice that N^H is a closed subset of N, thus it is a complete Hilbert manifold. On the other hand, since (H) is a twisted orbit type in $\mathscr{H}$, thus

$$(H) \in \Phi^t(G; \mathscr{H} \setminus \{0\}) = \Phi(G; S(\mathscr{H})).$$

Moreover, if $w \in S(\mathscr{H})$ then $\mathfrak{s}(w)w \in N$ and since $G_w = G_{\mathfrak{s}(w)w}$, we have

$$\Phi^t(G; S(\mathscr{H})) = \Phi^t(G; N),$$

thus $N^H \neq \emptyset$. Since J satisfies the (PS)-condition on N, it also satisfies the (PS)-condition on the submanifold N^H, thus by standard arguments, there exists $x_o \in N^H$ such that $J(x_o) = \inf\{J(x) : x \in N^H\}$. Denote by J_o^H the restriction of J to N^H. Then $\nabla J_o^H(x_o) = 0$ and therefore, by the Principle of Symmetric Criticality, $\nabla J|_N(x_o) = 0$. Then, by Proposition 5.4, $\nabla J(x_o) = 0$. □

5.5. Proof of minimality of period.

LEMMA 5.11. *Assume $f\colon V \to \mathbb{R}$ satisfies conditions* (A1)–(A3). *Let $J\colon \mathscr{H} \to \mathbb{R}$ be defined as in Remark* 3.2 *and let N be the Nehari manifold defined by* (5.3). *Assume that H is a subgroup of $G := \Gamma \times \mathbb{Z}_2$ such that (H) is a maximal orbit type in $\mathscr{H}$ and let $x_o \in N^H$ be such that $J(x_o) = \min\{J(x) : x \in N^H\}$. Then the minimal period of x_o is p.*

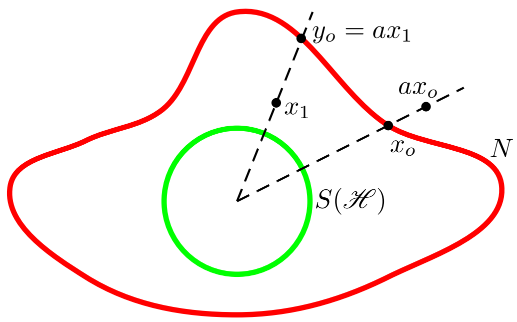


FIGURE 1. Minimization of J over Nehari manifold.

PROOF. Assume for contradiction that x_o has the minimal period p/k, where $k > 1$ (k has to be an odd number). Suppose that $H = K^{\varphi}$, thus we have $G_{x_o} = K^{\varphi}$. Define the function $x_1(t) := x_o(t/k)$, $t \in [0,p]$. Then $x_1 \in \mathscr{H}$ and $G_{x_1} = H$. Denote by $y_o \in N$ the element $y_o = \mathfrak{s}(x_1/\|x_1\|)x_1/\|x_1\| =: ax_1(t)$. Then we have $J(ax_o) \le J(x_o)$ and $J(y_o) \ge J(x_o)$.

$$\begin{aligned}
J(y_o) &= \frac{1}{2}\int_0^p |\dot{y}_o(t)|^2\,dt - \int_0^p f(y_o(t))\,dt \\
&= \frac{1}{2k^2}\int_0^p |a\dot{x}_o(t/k)|^2\,dt - \int_0^p f(ax_o(t/k))\,dt \\
&= \frac{1}{2k}\int_0^{p/k} |a\dot{x}_o(s)|^2\,ds - k\int_0^{p/k} f(ax_o(s))\,ds \\
&= \frac{1}{2k^2}\int_0^p |a\dot{x}_o(s)|^2\,ds - \int_0^p f(ax_o(s))\,ds \\
&< \frac{1}{2}\int_0^p |a\dot{x}_o(s)|^2\,ds - \int_0^p f(ax_o(s))\,ds = J(ax_o) \le J(x_o),
\end{aligned}$$

and we get a contradiction. See Figure 1. $\square$

6. Example of D_n-symmetric functional

Suppose Γ is the dihedral group $D_n \subset O(2)$, which is

$$D_n = \{1, \gamma, \dots, \gamma^{n-1}, \kappa, \gamma\kappa, \dots, \gamma^{n-1}\kappa\}, \quad \gamma = e^{2\pi i/n}.$$

Assume that $W := \mathbb{R}^n$ is the natural D_n-representation with the D_n-action defined on the generators as follows:

$$\gamma(x_1, \dots, x_n)^T := (x_2, \dots, x_n, x_1)^T, \tag{6.1}$$

$$\kappa(x_1, \dots, x_n)^T := (x_1, x_n, \dots, x_2), \tag{6.2}$$

where $(x_1,\dots,x_n)^T\in\mathbb{R}^n$. Clearly, the matrices of transformation equations (6.1) and (6.2) are

$$T_\gamma := \begin{bmatrix} 0 & 1 & 0 & \dots & 0\\ 0 & 0 & 1 & \dots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \dots & 1\\ 1 & 0 & 0 & \dots & 0 \end{bmatrix} \quad\text{and}\quad T_\kappa := \begin{bmatrix} 1 & 0 & 0 & \dots & 0\\ 0 & 0 & 0 & \dots & 1\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 1 & \dots & 0\\ 0 & 1 & 0 & \dots & 0 \end{bmatrix},$$

respectively. One can easily construct examples of a function $f\colon\mathbb{R}^n\to\mathbb{R}$ satisfying conditions (A1)–(A3) which is also D_n-symmetric.

For the complete list of irreducible D_n-representations and the corresponding basic degrees, we refer the reader, for instance, to [5, p. 174]. Here, we restrict ourselves with the data important for the present paper.

Put $m:=\lfloor n/2\rfloor$. Then W admits the isotypical decomposition

$$W := V_0\oplus\ldots\oplus V_m, \tag{6.3}$$

where $V_j\simeq\mathcal{V}_j$, $j=0,\dots,m$ (according to the convention introduced in [5]). Then, for any collection of real numbers, $\{\mu_j\}_{j=0}^m$, there exists a unique D_n-equivariant linear symmetric operator, $A\colon W\to W$, such that $\sigma(A):=\{\mu_j:0\le j\le m\}$ and the eigenspaces $E(\mu_j)=V_j$ (see [4]). If C is a matrix of such an operator, $A\colon W\to W$, then

(a) if n is odd, then

$$C=c_0\,\mathrm{Id}+\sum_{k=1}^m c_k\,\big[T_\gamma^k+T_\gamma^{-k}\big]\quad\text{and}\quad \mu_j=c_0+\sum_{k=1}^m 2c_k\cos\bigg(\frac{2\pi kj}{n}\bigg), \tag{6.4}$$

(b) if n is even, then

$$C=c_0\,\mathrm{Id}+\sum_{k=1}^{m-1} c_k[T_\gamma^k+T_\gamma^{-k}]+c_mT_\gamma^m \\ \text{and}\quad \mu_j=c_0+\sum_{k=1}^{m-1}2c_k\cos\bigg(\frac{2\pi kj}{n}\bigg)-c_m. \tag{6.5}$$

Put (see (6.3))

$$V := V_1\oplus\ldots\oplus V_m \tag{6.6}$$

and assume therefore that $\mu_j>0$, $j=1,\dots,m$ are given numbers. Let $A\colon V\to V$ be the corresponding linear operator, i.e. for n odd, the matrix C of A is

$$C=\sum_{k=1}^m c_k[T_\gamma^k+T_\gamma^{-k}]\quad\text{and}\quad \mu_j=\sum_{k=1}^m 2c_k\cos\bigg(\frac{2\pi kj}{n}\bigg)$$

and for n even, the matrix C of A is

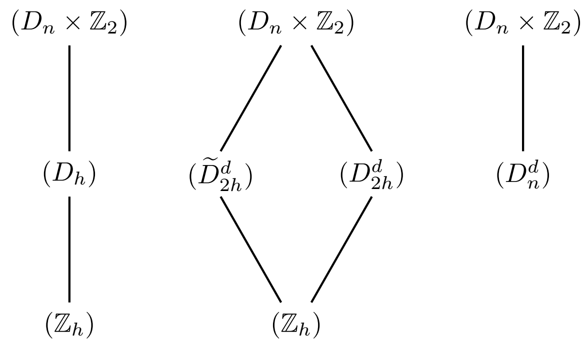
$$C = \sum_{k=1}^{m-1} c_k[T_\gamma^k + T_\gamma^{-k}] + c_m T_\gamma^m \quad \text{and} \quad \mu_j = \sum_{k=1}^{m-1} 2c_k \cos\left(\frac{2\pi kj}{n}\right) - c_m.$$

Since A is a positive operator, one can define $B := \sqrt{A}$. Let $f\colon V \to \mathbb{R}$ be defined by

$$f(x) := |Bx|^4 = (Bx \bullet Bx)^2 = (Ax \bullet x)^2, \quad x \in V. \tag{6.7}$$

One can easily verify that f is D_n-invariant and satisfies properties (A1)–(A3).

Assume that $n = q_1^{\alpha_1} \cdot \ldots \cdot q_s^{\alpha_s}$ is an integer, where $q_1 < \ldots < q_s$ are prime numbers. Notice that each of the subspaces $\mathbb{W}_{2l+1}$ in (3.5) is equivalent to the $D_n \times \mathbb{Z}_2$-representation V, consequently, the t-maximal orbit types in $\mathscr{H} \setminus \{0\}$ are the same as the maximal orbit types in the space $V \setminus \{0\}$. Consider the D_n-isotypical decomposition (6.6). Notice that it is also the isotypical decomposition of V with respect to the action of the group $G := D_n \times \mathbb{Z}_2$. An orbit type (H) in V is t-maximal if and only if it is maximal in one of the isotypical components $V_j \setminus \{0\}$. The lattices of the orbit types for the components V_j, $j = 0, \ldots, m$, are shown in Figure 2.



$0 < j < \frac{n}{2}$, l odd $\quad 0 < j < \frac{n}{2}$, l even $\quad n$ is even and $j = \frac{n}{2}$

Figure 2. Orbit types in the isotypical components V_j. Here $h = \gcd(n, j)$ and $l := \frac{n}{h}$.

We can recognize the maximal orbit types in $V \setminus \{0\}$ as follows.

If n is odd, then $n = q_1^{\alpha_1} \cdot \ldots \cdot q_s^{\alpha_s}$ with $q_1 > 2$. The maximal orbit types in irreducible representations are (D_{h_i}), where $h_i = q_1^{\alpha_1} \cdot \ldots \cdot q_i^{\alpha_i - 1} \cdot \ldots \cdot q_s^{\alpha_s}$, $i = 1, \ldots, s$. In such a case, the G-orbit of (D_{h_i})-type contains $|D_n \times \mathbb{Z}_2 / D_{h_i}| = 2q_i$ elements.

On the other hand, if n is even, then $q_1 = 2$, i.e. $n = 2^{\alpha_1} \cdot \ldots \cdot q_s^{\alpha_s}$. In this case, we have two types of the maximal orbit types in irreducible representations, which are (D_{h_i}) $i = 2, \ldots, s$, where $|D_n \times \mathbb{Z}_2/D_{h_i}| = 2q_i$ and (D_n^d), where $|D_n \times \mathbb{Z}_2/D_n^d| = 2$. Therefore, we have the following result.

Theorem 6.1. *Assume that $n = q_1^{\alpha_1} \cdot \ldots \cdot q_s^{\alpha_s}$ is an integer, where $q_1 < \ldots < q_s$ are prime numbers. Let $V \subset \mathbb{R}^n$ be the D_n-representation given by* (6.6) (*with the D_n action given by* (6.1) *and* (6.2)) *and let $f\colon V \to \mathbb{R}$ be given by* (6.7) (*where we assume that $\mu_j > 0$, $j = 1, \ldots, m$*). *Then system* (3.1) *has at least $\sigma(n)$ p-periodic solutions with the minimal period p, where $\sigma(n)$ is given by*:

(a) $\sigma(n) = 2(q_1 + \ldots + q_s)$, *when n is odd*;
(b) $\sigma(n) = 2 + 2(q_2 + \ldots + q_s)$, *when n is even.*

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