CLASSICAL MORSE THEORY REVISITED – I
BACKWARD λ-LEMMA AND HOMOTOPY TYPE

Joa Weber

―――――――――――

ABSTRACT. We introduce two tools, dynamical thickening and flow selectors, to overcome the infamous discontinuity of the gradient flow endpoint map near non-degenerate critical points. More precisely, we interpret the stable fibrations of certain Conley pairs (N, L), established in [2], as a *dynamical thickening of the stable manifold*. As a first application and to illustrate efficiency of the concept we reprove a fundamental theorem of classical Morse theory, Milnor's homotopical cell attachment theorem [1]. Dynamical thickening leads to a conceptually simple and short proof.

Consider a connected smooth manifold M of finite dimension n. Suppose $f \colon M \to \mathbb{R}$ is a smooth function and x is a non-degenerate critical point of f of Morse index k, that is $df_x = 0$ and in local coordinates the Hessian matrix $(\partial^2 f / \partial x^i \partial x^j)_{i,j}$ at x has precisely k negative eigenvalues, counting multiplicities, and zero is not an eigenvalue. Set $c := f(x)$ and assume for simplicity that the level set $\{f = c\}$ carries no critical point other than x.

Morse theory studies how the topology of sublevel sets $M^a = \{f \leq a\}$ changes when a runs through a critical value c. A fundamental tool is the concept of a flow, also called a 1-parameter group of diffeomorphisms of M. A common choice is the downward gradient flow $\{\varphi_s\}_{s \in \mathbb{R}}$, namely the one generated by the initial value problems $\frac{d}{ds}\varphi_s = -(\nabla f) \circ \varphi_s$ with $\varphi_0 = \mathrm{id}_M$. Existence is

―――――――

guaranteed, for instance, if the vector field is of compact support. Here ∇f
denotes the gradient vector field of f on M. It is uniquely determined by the
identity $df(\,\cdot\,) = g(\nabla f, \,\cdot\,)$ after fixing an auxiliary Riemannian metric g on M.
Key properties of the downward gradient flow are that f decays along flow lines
$s \mapsto \varphi_s p$, for $p \in M$, and that ∇f is orthogonal to level sets. Consequently
sublevel sets are forward flow invariant. As $df_x = 0$ if and only if $(\nabla f)_x = 0$,
any critical point x is a fixed point of the flow and non-degeneracy translates
into hyperbolicity.

By non-degeneracy of x its unstable manifold W^u and descending disk W_ε^u,

$$W^u = \Big\{ p \in M \ \Big| \ \lim_{s \to -\infty} \varphi_s p = x \Big\}, \qquad W_\varepsilon^u = W^u \cap \{ f \geq c - \varepsilon \},$$

are embedded open, respectively closed, disks in M of dimension $k = \mathrm{ind}(x)$;
actually an embedding $W_\varepsilon^u \hookrightarrow M$ as a closed k-disk exists only for every *suf-
ficiently small* $\varepsilon > 0$ (use the Morse lemma). The boundary $S_\varepsilon^u := \partial W_\varepsilon^u$ is
called a descending sphere. Consider instead the limit $s \to +\infty$ to get the stable
manifold W^s and ascending disk $W_\varepsilon^s = W^s \cap \{ f \leq c + \varepsilon \}$. They have analogous
properties except that they are of codimension k.

In [2], see [3, Theorem. 5.1] for details in the present finite dimensional case,
we implemented the structure of a disk bundle on the compact neighbourhood

$$N = N_x^{\varepsilon,\tau} := \{ p \in M \mid f(p) \leq c + \varepsilon, \ f(\varphi_\tau p) \geq c - \varepsilon \}_{\text{connected component of } x}$$

of x whenever $\varepsilon > 0$ is small and $\tau > 0$ is large. The fibers are codimension-k
disks with boundaries in the upper level set $\{ f = c + \varepsilon \}$ and parametrized by
their unique point of intersection, say q^T, with the unstable manifold. The fiber
over x is W_ε^s. Each point of a fiber $N(q^T)$ reaches the lower level set $\{ f = c - \varepsilon \}$
in time T under the downward gradient flow. Note that $\{ f = c - \varepsilon \}$ intersects
W^u in the descending $(k-1)$-sphere $S_\varepsilon^u = \partial W_\varepsilon^u$. Choose a tubular neighborhood
$\mathcal{D}$ of S_ε^u in $\{ f = c - \varepsilon \}$ to get a family of codimension-k disks $\mathcal{D}_q$, one for each
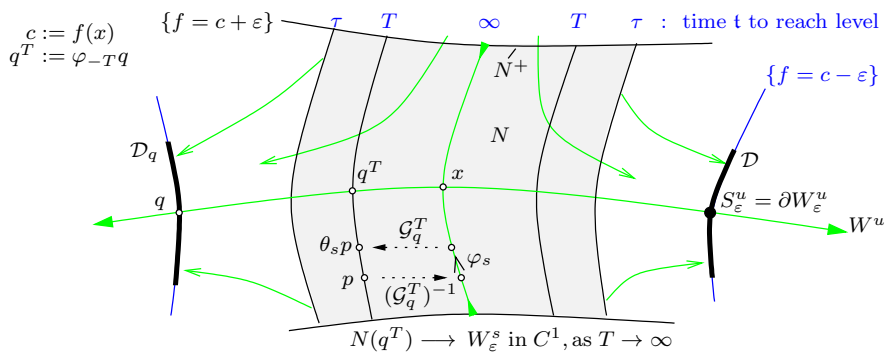$q \in S_\varepsilon^u$.

FIGURE 1. Dynamical thickening (N, θ) of the local stable manifold $(W_\varepsilon^s, \varphi|)$.

By [2], [3] we get a Lipschitz continuous ($C^{0,1}$) disk bundle

$$N = W_\varepsilon^s \overset{\cdot}{\bigcup_{T \geq \tau, q \in S_\varepsilon^u}} N(q^T), \qquad N(q^T) = \varphi_T^{-1}(\mathcal{D}_q) \cap \{f \leq c + \varepsilon\},$$

over $\varphi_{-\tau} W_\varepsilon^u$ which is $C^{1,1}$ away from the ascending disk W_ε^s. It is a key fact that the fibers are diffeomorphic to W_ε^s via C^1 maps $\mathcal{G}_q^T \colon W_\varepsilon^s \to N(q^T)$ which converge in C^1 to the identity on W_ε^s, as $T \to \infty$. Furthermore, the fibration is forward flow invariant in the sense that φ_s maps a fiber $N(q^T)$ into $N(\varphi_s q^T)$. Figure 1 illustrates the fibration and the qualitative behavior of the forward flow which is transverse to all fibers except the one over x which is invariant. Conjugation by the diffeomorphism $\mathcal{G}_q^T$ provides on each fiber $N(q^T)$ a copy θ_s of the forward flow φ_s on W_ε^s.

Now we reprove the cell attachment theorem.

THEOREM (Milnor [1, I Theorem.3.2]). *Let $f \colon M \to \mathbb{R}$ be a smooth function, and let x be a non-degenerate critical point with Morse index k. Setting $f(x) = c$, suppose that $f^{-1}[c-\varepsilon, c+\varepsilon]$ is compact and contains no critical point of f other than x, for some $\varepsilon > 0$. Then, for all sufficiently small ε, the set $M^{c+\varepsilon}$ has the homotopy type of $M^{c-\varepsilon}$ with a k-cell attached.*

PROOF. Fix a Riemannian metric on M. Without loss of generality assume that $-\nabla f$ is of compact support ([1]), so it generates a flow $\{\varphi_s\}_{s \in \mathbb{R}}$ on M. Pick constants $\varepsilon > 0$ small and $\tau > 0$ large in order to meet the assumptions in [3] of Theorem 5.1 (existence of the invariant fibration $N = N_x^{\varepsilon,\tau}$) and Definition 5.6 (induced fiberwise semi-flow θ). Figure 2 illustrates the proof: First deform $N \subset M^{c+\varepsilon}$ along θ towards the flow selector $\mathcal{S}^+$ and W_ε^u, then deform along φ.

Definition of flow selector (hypersurface transverse to two flows). View

$$\mathcal{S}^+ := \{\varphi_{-\mathfrak{s} \circ \mathfrak{t}^-(p)} p \mid p \in \mathcal{S}^-\} \subset N$$

as the graph of a function $\mathfrak{s} \circ \mathfrak{t}^-$ over an open subset $\mathcal{S}^- \subset f^{-1}(c - \varepsilon)$ where the coordinate lines are backward flow lines of φ starting at $\mathcal{S}^-$ with coordinate the backward time. By the flow box theorem this makes sense, as there is no singularity of ∇f on $\mathcal{S}^-$. By the graph property φ *will be transverse to* $\mathcal{S}^+$.

By [3, Theorem. 1.2] there is a C^0 time label function $\mathfrak{t} \colon N \to [\tau, \infty]$, of class C^1 as a function $N_\times := N \setminus W^s \to [\tau, \infty)$, which assigns to each point p the time it takes to reach the lower level set $f^{-1}(c - \varepsilon)$ under the gradient flow φ. The hypersurface $N^+ := \{p \in N \cap f^{-1}(c + \varepsilon) \mid \mathfrak{t}(p) > \tau\}$ is called the *entrance set* of N and $N_\times^+ := N^+ \setminus W^s$ its *regularization*; see Figure 1. As each point of $N_\times^+$ hits $f^{-1}(c - \varepsilon)$ under φ precisely once and transversely, the

---

([1]) Otherwise, substitute for $-\rho \nabla f$ where $\rho \colon M \to \mathbb{R}$ is a smooth compactly supported cut-off function with $\rho \equiv 1$ on the compact set $K := f^{-1}[c - \varepsilon, c + \varepsilon]$.

corresponding subset $\mathcal{S}_\times^- \subset f^{-1}(c - \varepsilon)$ is diffeomorphic to $N_\times^+$. The *time label function* $\mathfrak{t}^- \colon \mathcal{S}_\times^- \to (\tau, \infty)$ is defined by transferring the time labels of $N_\times^+$. It is of class C^1. Add the descending disk S_ε^u to define

$$\mathcal{S}^- := \mathcal{S}_\times^- \,\dot{\cup}\, S_\varepsilon^u = \{p \in f^{-1}(c - \varepsilon) \mid N^+ \cap \varphi_\mathbb{R} p \neq \emptyset\} \,\dot{\cup}\, S_\varepsilon^u, \qquad \mathcal{S}_\times^- \overset{\varphi}{\cong} N_\times^+,$$

as an open subset of $f^{-1}(c - \varepsilon)$; see Figure 2. Set $\mathfrak{t}^- = \infty$ on S_ε^u.

The function

$$\mathfrak{s} \colon (\tau, \infty) \to (\tau, 2\tau), \qquad \mathfrak{t} \mapsto 2\tau - \tau^2/\mathfrak{t},$$

is smooth and extends continuously to $[\tau, \infty]$ so that $\mathfrak{s}(\tau) = \tau$ with $\mathfrak{s}'(\tau) = 1$ and $\mathfrak{s}(\infty) = 2\tau$ with $\mathfrak{s}'(\infty) = 0$; see Figure 2 for the corresponding graph $\mathcal{S}^+$. Observe that critical points of $\mathfrak{s}$ correspond precisely to tangencies of θ to the hypersurface $\mathcal{S}_\times^+ := \mathcal{S}^+ \setminus W^u$. But $\mathfrak{s}$ admits no critical points on (τ, ∞), so θ *is transverse to* $\mathcal{S}_\times^+$. This proves that $\mathcal{S}_\times^+$ is a flow selector with respect to φ and θ.
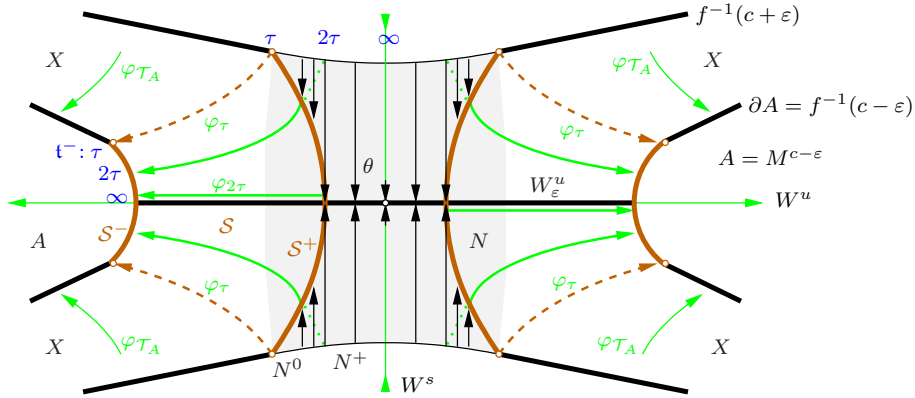

FIGURE 2. Flow selector $\mathcal{S}_\times^+ = \mathcal{S}^+ \setminus W^u$ with transverse flows θ and φ.

I. Strong deformation retraction. $r \colon M^{c+\varepsilon} \to M^{c-\varepsilon} \cup W_\varepsilon^u \cup X$ via θ: Let $\mathcal{S}$ be the region under the graph of $\mathcal{S}^+$, that is the region bounded by $\mathcal{S}^-$ and $\mathcal{S}^+$ and the hypersurfaces indicated by dashed arrows in Figure 2. The arrows are dashed to indicate that they do not belong to $\mathcal{S}$, but to the closure $\overline{\mathcal{S}}$. Consider the compact set $X := \big(f^{-1}[c - \varepsilon, c + \varepsilon] \setminus N\big) \cup \overline{\mathcal{S}}$ whose boundary is given by $f^{-1}(c - \varepsilon)$ and $\big(f^{-1}(c + \varepsilon) \setminus N^+\big) \cup \mathcal{S}^+$. Deforming $N \setminus \overline{\mathcal{S}}$ along the flow lines of θ until the flow line hits either the flow selector $\mathcal{S}^+$ or the descending disk W_ε^u, while not moving the other points of $M^{c+\varepsilon}$ at all, defines the required strong deformation retraction r. Continuity of r holds since θ is transverse to $\mathcal{S}_\times^+$ and $\mathcal{S}^+ \setminus \mathcal{S}_\times^+ = \mathcal{S}^+ \cap W_\varepsilon^u$ is reached under θ in infinite time just as is $W_\varepsilon^u \setminus \mathcal{S}$.

II. Homotopy equivalence. $M^{c-\varepsilon} \cup W^u_\varepsilon \cup X \sim M^{c-\varepsilon} \cup W^u_\varepsilon$ via φ: Given the pair of closed sets $A := M^{c-\varepsilon} \subset (X \cup A)$, consider the entrance time function $\mathcal{T}_A \colon X \cup A \to [0, \infty)$ which assigns to each point $p \in X \cup A$ the time it takes to reach A under φ. To see that $\mathcal{T}_A$ is well-defined note that A and $X \cup A$ are both forward flow invariant under φ. Indeed ∂A is a level set along which $-\nabla f$ is downward, hence inward, pointing. The (topological) boundary of $X \cup A$ is $\left(f^{-1}(c+\varepsilon) \setminus N^+ \right) \cup \mathcal{S}^+$ and $-\nabla f$ points inward along both pieces.

Given that φ is transverse to ∂X, the function $\mathcal{T}_A$ is lower and upper semi-continuous, hence continuous, because the subset A of $X \cup A$ is closed and forward flow invariant, respectively; cf. [2, Proof of Theorem B]. Since X is compact without critical points $\mathcal{T}_A$ is bounded. The map $h \colon [0,1] \times Z \to Z$ given by

$$h(\lambda, p) = \begin{cases} p & \text{if } p \in A = M^{c-\varepsilon}, \\ \varphi_{\lambda \mathcal{T}_A(p)} p & \text{if } p \in X, \\ \varphi_{\lambda 4\tau^2/\mathfrak{t}(p)} p & \text{if } p \in \overline{W^u_\varepsilon \setminus \mathcal{S}} = \varphi_{-2\tau} W^u_\varepsilon, \end{cases}$$

is continuous as it is defined by three continuous parts which agree on overlaps: $\mathcal{T}_A = 0$ on $A \cap X$ and $\mathcal{T}_A = 4\tau^2/\mathfrak{t} = 2\tau$ on $\varphi_{-2\tau} S^u_\varepsilon$. The inclusion $\iota \colon A \cup W^u_\varepsilon =: B \hookrightarrow Z := X \cup A \cup W^u_\varepsilon$ and $h_1 := h(1, \cdot) \colon Z \to B$ are reciprocal homotopy inverses. Indeed $\iota \circ h_1 = h_1 \sim h_0 = \mathrm{id}_Z$ and $h_1 \circ \iota = h_1|_B \sim h_0|_B = \mathrm{id}_B$. $\square$

Part two of h_1 unfortunately eliminates an outer piece of W^u_ε which we recover by $\varphi_{4\tau^2/\mathfrak{t}(\cdot)}(\cdot) \colon \varphi_{-2\tau} W^u_\varepsilon \to W^u_\varepsilon$. So h_1 does not restrict to the identity on W^u_ε, hence h is not a deformation retraction of $X \cup A \cup W^u_\varepsilon$ onto $A \cup W^u_\varepsilon$.

Perspectives. In the history of Morse theory discontinuity of the flow trajectory end point map φ_∞ obstructed to carry out, in a simple fashion, various constructions suggested by geometry, for instance, to extend continuously the inclusion map of an unstable manifold towards the closure. It will be a future research project to investigate the role of dynamical thickening and flow selectors in such cases. By [2] dynamical thickening can be defined in infinite dimensional contexts.

Added in proof. Flow selector was added to correct the discontinuity in the previous version. Flow selectors arose in cooperation with Pietro Majer (2015) in two flavors — via Conley blocks and via carving. Here we use a version of the Conley block technique.

References

[1] J. Milnor, *Morse theory*, based on lecture Notes by M. Spivak and R. Wells, Ann. Math. Stud. No. 51. Princeton University Press, Princeton, N.J., 1963.

[2] J. Weber, *Stable foliations and semi-flow Morse homology*, Ann. Scuola Norm. Sup. Pisa Cl. Sci., arXiv 1408.3842 (to appear).

[3] _____, *Contraction method and Lambda-Lemma*, São Paulo J. Math. Sci. **9** (2) (2015), 263–298.

Joa Weber
Instituto de Matemática, Estatística
e Computação Scientífica
Universidade Estadual de Campinas
Rua Sérgio Buarque de Holanda 651
SP 13083-859, Campinas, BRASIL

E-mail address: joa@ime.unicamp.br