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A VARIABLE EXPONENT DIFFUSION PROBLEM OF CONCAVE-CONVEX NATURE

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Abstract. We deal with the problem

$$\begin{cases} -\Delta u = \lambda u^{q(x)} & \text{if } x \in \Omega, \\ u = 0 & \text{if } x \in \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, $\lambda > 0$ is a parameter and the exponent $q(x)$ is a continuous positive function that takes values both greater than and less than one in $\overline{\Omega}$. It is therefore a kind of concave-convex problem where the presence of the interphase $q = 1$ in $\overline{\Omega}$ poses some new difficulties to be tackled. The results proved in this work are the existence of $\lambda^* > 0$ such that no positive solutions are possible for $\lambda > \lambda^*$, the existence and structural properties of a branch of minimal solutions, u_λ, $0 < \lambda < \lambda^*$, and, finally, the existence for all $\lambda \in (0, \lambda^*)$ of a second positive solution.

1. Introduction

This work is devoted to the analysis of positive solutions to the semilinear boundary value problem

$$\begin{cases} -\Delta u = \lambda u^{q(x)} & \text{if } x \in \Omega, \\ u = 0 & \text{if } x \in \partial\Omega, \end{cases} \tag{1.1}$$

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where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain and $\lambda > 0$ is a bifurcation parameter. The exponent q is assumed to be a positive function in $C^\gamma(\overline{\Omega})$, the main feature being the fact that

$$0 < \inf_{\Omega} q < 1 < \sup_{\Omega} q. \tag{1.2}$$

Thus, problem (1.1) can be regarded on its own right as a sort of "concave-convex" problem. To simplify the exposition, it will be also assumed that $N \geq 3$ and that (1.1) is subcritical, i.e.

$$q(x) \leq \frac{N+2}{N-2}, \quad x \in \Omega. \tag{1.3}$$

Reaction–Diffusion systems constitute an important and active area in the theory of nonlinear problems (see [13], [39], [33] and [32] for a global overview). One of their multiple branches is constituted by concave-convex problems, which have received some interest in the literature in recent times, including several kinds of boundary conditions and generalizations to other operators such as the p-Laplacian or fully nonlinear uniformly elliptic operators. The subject goes back to the pioneering works [28], [5], [15] and [16]. However, [3] is regarded as a first detailed analysis of the main properties of such type of problems, especially its bifurcation diagrams (see also [28], Section 1.1). Later extensions are [4], [14], that deal with Dirichlet conditions and the p–Laplacian operator; [9], dedicated to fully nonlinear uniformly elliptic operators with Dirichlet boundary conditions; [17], [36], dealing with flux–type nonlinear boundary conditions and source nonlinearities; [20], handling concave-convex terms of absorption nature, and [10], where a combination of concave absorption with a convex source leads to an interesting free boundary problem in $\mathbb{R}^N$. Of course, all these works are only a reduced sample of the previous research on the topic.

On the other hand, we would also like to mention the previous papers [12], [18], [19], [26], [29], [30] and [34], which have dealt with different types of elliptic problems involving variable exponents. However, at the best of our knowledge, no concave-convex problems such as (1.1) have been considered before in the literature.

It is expected that (1.1) exhibits similar features to other problems of concave-convex nature studied previously. However, a main difference that appears in our problem (1.1) when compared with previous ones is the presence of the interphase $q = 1$ in $\overline{\Omega}$. This fact raises subtle technical problems in several key points of the analysis. For example, let us mention the validity of the Palais–Smale condition, the existence of extremal solutions and the uniqueness of the small amplitude solutions for λ close to zero among others. Such issues are simpler to handle when the concave nonlinearity is well-separated from the convex one which is just the case for instance in [3].

To progress further, a more precise hypothesis on the exponent q should be introduced. It is implicit in (1.2) that the open sets $\Omega_- := \{x \in \Omega : q(x) < 1\}$, $\Omega_+ = \{x \in \Omega : q(x) > 1\}$ are nonempty. It will be assumed that $\Omega_\pm$ are the union of finitely many smooth domains. In addition, and to avoid unnecessary extra technicalities, we will also suppose that both $\Gamma_\pm := \partial\Omega_\pm \cap \Omega$ consist of closed manifolds, i.e. $\partial\Omega_\pm \cap \partial\Omega$ are whole components of $\partial\Omega$. Therefore, $\partial\Omega_\pm \cap \partial\Omega$ and $\Gamma_\pm$ are disjoint. All such conditions on q — including (1.3) — will be henceforth referred to as $(\mathrm{H})_q$.

We state now our main result.

THEOREM 1.1. *Let q be a positive function in $C^\gamma(\overline{\Omega})$ satisfying $(\mathrm{H})_q$. Then, problem* (1.1)

$$\begin{cases} -\Delta u = \lambda u^{q(x)} & \text{if } x \in \Omega, \\ u = 0 & \text{if } x \in \partial\Omega, \end{cases}$$

with $\lambda > 0$, has the following properties:

(a) *There exists $\lambda^* > 0$ such that no positive weak solutions exist if $\lambda > \lambda^*$.*

(b) *For every $0 < \lambda < \lambda^*$, there exists a minimal (classical) solution $u_\lambda \in C^{2,\alpha}(\overline{\Omega})$ for some $0 < \alpha < 1$. In addition,*

$$u_\lambda \to 0 \quad \text{as } \lambda \to 0+ \tag{1.4}$$

in $C^2(\overline{\Omega})$, meanwhile the mapping $\lambda \mapsto u_\lambda$ from $(0, \lambda^*)$ with values in $C^2(\overline{\Omega})$ is continuous from the left at every λ. Moreover, if v_λ is any family of positive solutions to (1.1) *such that $\|v_\lambda\|_{L^\infty(\Omega)} = O(1)$ as $\lambda \to 0$ then there exists a positive function $\varphi \in C^1(\Omega)$ which vanishes on $\partial\Omega$ such that*

$$v_\lambda(x) \le \lambda^{1/(1-q_-)}\varphi(x), \quad x \in \Omega,$$

for small λ, where $q_- = \inf_\Omega q < 1$.

(c) *The solution u_λ is linearly stable in the sense that, for all $\varphi \in H_0^1(\Omega)$,*

$$\int_\Omega |\nabla\varphi|^2 \ge \lambda \int_\Omega q(x) u_\lambda^{q(x)-1}\varphi^2. \tag{1.5}$$

(d) *Suppose that*

$$q(x) < \frac{N}{N-2} \quad \text{for } x \in \overline{\Omega}, \tag{1.6}$$

and that either $\Omega_+ \subset \overline{\Omega}_+ \subset \Omega$ or $q > 1$ on $\partial\Omega_+ \cap \partial\Omega$ provided such intersection is nonempty. Then, for every $0 < \lambda < \lambda^*$ there exists a second positive solution $v_\lambda \in C^{2,\alpha}(\overline{\Omega})$ for some $0 < \alpha < 1$. Moreover, a minimal solution u_{λ^*} exists at $\lambda = \lambda^*$ while the limit

$$u_{\lambda^*} = \lim_{\lambda \to \lambda^*-} u_\lambda,$$

holds in $C^2(\overline{\Omega})$.

Remark 1.2. (a) Observe that the existence of an extremal solution corresponding to $\lambda = \lambda^*$ is not provided in (a) of Theorem 1.1. Such existence is furnished in (d) under more restrictive conditions.

(b) Inequality (1.5) means that the first eigenvalue σ_1 of the linearization of (1.1) around $u = u_\lambda$ (see Lemma 2.3 and (2.7) below) is nonnegative.

(c) Regarding the existence and remaining properties of the minimal solution u_λ it should be remarked that no restriction as (1.3) on the size of q is required for them to hold (see Section 1 in [3]).

A more precise description on the global structure of the branch of minimal solutions u_λ is given in the next statement.

Theorem 1.3. *There exists* at most *a decreasing sequence of values λ_n in the interval $(0, \lambda^*)$ such that the following properties hold.*

(a) *The mapping $\lambda \mapsto u_\lambda$ is discontinuous at every λ_n.*

(b) *For every n, the mapping*

$$(\lambda_{n+1}, \lambda_n) \to C_0^1(\overline{\Omega}), \quad \lambda \mapsto u_\lambda,$$

is smooth, increasing and satisfies

$$\sigma_1(u_\lambda) > 0 \quad \text{for every } \lambda \in (\lambda_{n+1}, \lambda_n). \tag{1.7}$$

This fact holds also true for the initial interval (λ_1, λ^*).

Remark 1.4. (a) A possible quite likely option, which is not discarded in Theorem 1.3, is the nonexistence of such values λ_n. In that case, (b) is satisfied in the whole interval $(0, \lambda^*)$.

(b) Relation (1.7) asserts that u_λ is asymptotically stable with the possible exception of the values $\lambda = \lambda_n$.

The rest of the paper is organized as follows: Section 2 studies a variant (see (2.2) below) of our main problem (1.1), which is instrumental to get the existence of a minimal solution to (1.1). The smoothness of weak solutions to problem (1.1) and the eigenvalue problem associated to its linearization around a positive solution u are also analyzed in Section 2. The L^∞ estimates required to attain the existence of a second positive solution are obtained in Section 3. Section 4 is devoted to the proof of Theorem 1.1, while the analysis of the structure of the branch u_λ of minimal solutions is addressed in Section 5.

2. Auxiliary material

In this section, we will obtain some preliminary results, which will be instrumental in the proofs in Sections 3, 4 and 5. We begin by analyzing the regularity of solutions to a slight generalization of problem (1.1).

LEMMA 2.1. *Assume that q is a measurable function in Ω fulfilling the growth condition* (1.3). *Then, every weak solution $u \in H_0^1(\Omega)$ to the problem*

$$\begin{cases} -\Delta u = \lambda |u|^{q(x)-1}u & \text{if } x \in \Omega, \\ u = 0 & \text{if } x \in \partial\Omega, \end{cases} \tag{2.1}$$

verifies $u \in C^{1,\beta}(\overline{\Omega})$ for every $0 < \beta < 1$. If in addition $q \in C^{\gamma}(\overline{\Omega})$ then $u \in C^{2,\alpha}(\overline{\Omega})$ for some $0 < \alpha < 1$.

PROOF. The function $f(x,u) = |u|^{q(x)-1}u$ verifies a Carathéodory condition and

$$|f(x,u)| \le b(x)(1+|u|) \quad \text{for some } b \in L^{N/2}(\Omega).$$

It follows from [7] that $u \in L^p(\Omega)$ for all $p > 1$ and thus the same happens to $f(\,\cdot\,, u(\,\cdot\,))$. Then, it is a consequence of the L^p theory for elliptic equations and the embedding of Sobolev spaces that $u \in C^{1,\beta}(\overline{\Omega})$ for every $0 < \beta < 1$ (see [23]).

On the other hand, it can checked that the Nemytskiĭ operator N_f, $u(\,\cdot\,) \mapsto N_f(u)(\,\cdot\,)$, given by $N_f(u)(\,\cdot\,) = f(\,\cdot\,, u(\,\cdot\,))$, maps $C^1(\overline{\Omega})$ into $C^{\theta}(\overline{\Omega})$ with $\theta = \min\{\gamma, \inf_\Omega q\}$. Since $\partial\Omega$ is smooth, Schauder theory implies that $u \in C^{2,\alpha}(\overline{\Omega})$ for some $0 < \alpha < 1$ (we refer again to [23]). □

Next we deal with an auxiliary sublinear problem related to (1.1) and obtain some of its more relevant properties.

THEOREM 2.2. *Assume that $q \in C^{\gamma}(\overline{\Omega})$ satisfies $(\mathrm{H})_q$ and consider the problem*

$$\begin{cases} -\Delta u = \lambda \chi_{\Omega_-}(x) u^{q(x)} & \text{if } x \in \Omega, \\ u = 0 & \text{if } x \in \partial\Omega, \end{cases} \tag{2.2}$$

where χ_{Ω_-} stands for the characteristic function of Ω_-. Then the following assertions hold:

(a) *For every $\lambda > 0$ problem* (2.2) *admits a unique weak positive solution $\widetilde{u}_\lambda \in C^{1,\beta}(\overline{\Omega})$, $0 < \beta < 1$ arbitrary. Moreover, the mapping $\lambda \mapsto \widetilde{u}_\lambda$ is increasing.*

(b) *There exist $\lambda_0 > 0$ and a function $\varphi \in C^1(\overline{\Omega})$, verifying $\varphi(x) > 0$ for $x \in \Omega$ and $\varphi(x) = 0$ for $x \in \partial\Omega$, such that*

$$\widetilde{u}_\lambda(x) \le \lambda^{1/(1-q_-)}\varphi(x), \quad x \in \Omega,$$

for all $0 < \lambda \le \lambda_0$ where $q_- = \inf_\Omega q < 1$. In particular, $\widetilde{u}_\lambda \to 0$ as $\lambda \to 0+$ in $C^{1,\beta}(\overline{\Omega})$.

(c) *For all $q_1 \in \big(\inf_\Omega q, 1\big)$ there exist $\lambda_1 > 0$, and a function $\psi \in C(\overline{\Omega})$, verifying $\psi(x) > 0$ for $x \in \Omega$, $\psi(x) = 0$ for $x \in \partial\Omega$, such that*

$$\widetilde{u}_\lambda(x) \ge \lambda^{1/(1-q_1)}\psi(x), \quad x \in \Omega, \tag{2.3}$$

for $\lambda \geq \lambda_1$. In particular, $\widetilde{u}_\lambda \to \infty$ uniformly in compact sets of Ω as $\lambda \to \infty$.

Proof. The functional

$$J_0(u) = \int_\Omega \left(\frac{1}{2}|\nabla u|^2 - \frac{\lambda}{q+1}\chi_{\Omega_-}|u|^{q+1} \right)$$

is well-defined in $H_0^1(\Omega)$ since $\chi_{\Omega_-}|u|^{q+1} \leq u^2 + 1$, and it is coercive. In fact, if this is not the case there exists a sequence $\{u_n\}$ with $\|u_n\|_{H_0^1(\Omega)} \to +\infty$ and $J_0(u_n) \leq C$. Then

$$\int_\Omega \frac{\lambda}{q+1}\chi_{\Omega_-}|u_n|^{q+1} \to \infty.$$

This implies that $\|u_n\|_{L^2(\Omega)} \to \infty$. By setting $v_n = t_n^{-1}u_n$, $t_n = \|u_n\|_{L^2(\Omega)}$, it can be checked that

$$\int_\Omega \frac{\lambda}{q+1} t_n^{q-1}\chi_{\Omega_-}|v_n|^{q+1} \to 0.$$

Since $J_0(v_n) \leq o(1)$ it follows that $v_n \to 0$ in $H_0^1(\Omega)$ which is not possible since we have $\|v_n\|_{L^2(\Omega)} = 1$.

On the other hand, if we denote by λ_1 the first Dirichlet eigenvalue of the Laplacian in Ω and by ϕ_1 its corresponding associated positive eigenfunction, we have

$$\frac{J_0(t\phi_1)}{t^2} = \int_\Omega \left(\frac{\lambda_1}{2}\phi_1^2 - t^{q-1}\frac{\lambda}{q+1}\chi_{\Omega_-}\phi_1^{q+1} \right),$$

for $t > 0$. Hence the integral becomes negative if t is chosen small enough. Therefore, J_0 achieves the absolute minimum at some $u \in H_0^1(\Omega)$, $u \neq 0$, that can be assumed to be nonnegative. This function u is in particular a nonnegative weak solution to (2.2).

Observe next that an argument entirely similar to the one used in the proof of Lemma 2.1 shows that weak nonnegative solutions to (2.2) lie also in $C^{1,\beta}(\overline{\Omega})$ for every $0 < \beta < 1$. In particular they are positive in Ω with negative normal derivative

$$\frac{\partial u}{\partial \nu} < 0 \quad \text{on } \partial\Omega$$

(see [8]).

To prove uniqueness, assume now that u, v are weak positive solutions to (2.2). Then the following inequality holds ([8, p. 57]):

$$I := \left\langle \left(-\frac{\Delta u}{u} \right) - \left(-\frac{\Delta v}{v} \right), u^2 - v^2 \right\rangle \geq 0,$$

and equality is achieved only when $v = cu$ for some positive constant c. But

$$I = \lambda \int_{\Omega_-} (u^{q-1} - v^{q-1})(u^2 - v^2) \leq 0.$$

Thus, we get $v = cu$, and it follows that $c = 1$, which shows uniqueness.

To show the monotonicity of $\widetilde{u}_\lambda$ with respect to λ first observe that (2.2) possesses subsolutions whose amplitude can be as small as desired. In fact, for an arbitrary ball $B \subset \overline{B} \subset \Omega_-$ define $\underline{u}(x) = \varepsilon\phi_{1,B}(x)$ if $x \in \overline{B}$, $\underline{u}(x) = 0$ otherwise, where $\phi_{1,B}$ is a positive normalized first Dirichlet eigenfunction in B and $\varepsilon > 0$ a small parameter. On the other hand, $\widetilde{u}_{\lambda'}$ is a strict supersolution to (2.2) if $\lambda' > \lambda$. Since $\underline{u}$ can be found so that $\underline{u} \le \widetilde{u}_{\lambda'}$ we finally get $\widetilde{u}_\lambda < \widetilde{u}_{\lambda'}$ whenever $0 < \lambda < \lambda'$. Thus, $\widetilde{u}_\lambda$ increases with λ. This concludes the proof of part (a).

To prove (b), notice that $\{\widetilde{u}_\lambda\}$ is uniformly bounded in $L^\infty(\Omega)$ for $0 < \lambda < \lambda_0$. Thus, by using L^p estimates we can conclude that $\widetilde{u}_\lambda \to 0$ in $C^{1,\beta}(\overline{\Omega})$ (remark that the only nonnegative solution to (2.2) with $\lambda = 0$ is $u = 0$). On the other hand, by taking λ_0 small enough to have $\widetilde{u}_\lambda \le 1$ for $0 < \lambda \le \lambda_0$ we observe that $u = \widetilde{u}_\lambda$ satisfies,

$$-\Delta u \le \lambda u^{q_-}, \quad x \in \Omega,$$

with $q_- = \inf_\Omega q < 1$. Thus $\widetilde{u}_\lambda \le \lambda^{1/(1-q_-)}\varphi$ where φ is the solution to

$$\begin{cases} -\Delta\varphi = \varphi^{q_-} & \text{if } x \in \Omega, \\ \varphi = 0 & \text{if } x \in \partial\Omega. \end{cases} \tag{2.4}$$

This proves (b).

As for (c) we choose a smooth subdomain $D \subset \overline{D} \subset \Omega_-$ and set $u = \widehat{u}_\lambda$ the solution to the problem

$$\begin{cases} -\Delta u = \lambda u^{q(x)} & \text{if } x \in D, \\ u = 0 & \text{if } x \in \partial D. \end{cases} \tag{2.5}$$

The previous existence and uniqueness proof applies, of course, to (2.5). So, it is clear that $\widetilde{u}_\lambda \ge \widehat{u}_\lambda$. On the other hand, the function $\underline{u} = \lambda^{1/(1-\theta)}\widehat{u}_1$ defines a subsolution to (2.5) if $\theta = \inf_D q < 1$ and $\lambda \ge 1$ (observe that the value of θ can be modified through a different choice of D). Thus, $\widetilde{u}_\lambda \ge \lambda^{1/(1-\theta)}\widehat{u}_1$ for $\lambda \ge 1$. On the other hand, if ψ_0 is the solution to

$$\begin{cases} -\Delta\psi_0 = 0 & \text{if } x \in \Omega \setminus D, \\ \psi_0 = \widehat{u}_1 & \text{if } x \in \partial D, \\ \psi_0 = 0 & \text{if } x \in \partial\Omega, \end{cases}$$

then $\widetilde{u}_\lambda \ge \lambda^{1/(1-\theta)}\psi_0$ in $\Omega \setminus D$. Hence, (2.3) holds for ψ defined as $\widehat{u}_1$ in D and ψ_0 elsewhere. □

Our next purpose is to discuss the linearization of problem (1.1) around a possible positive solution u. We are therefore assuming that $u \in C^1(\overline{\Omega})$,

$u(x) > 0$ for $x \in \Omega$ together with $u(x) = 0$ and $\dfrac{\partial u}{\partial \nu}(x) < 0$ at every $x \in \partial\Omega$. This implies the existence of constants $k, K > 0$ such that

$$kd(x) \le u(x) \le Kd(x), \quad x \in \Omega. \tag{2.6}$$

From now on, the function $d(x)$ will stand for a smooth positive extension of the function $\operatorname{dist}(x, \partial\Omega)$ from a neighbourhood of $\partial\Omega$ to the whole of Ω.

Lemma 2.3. *Assume that $q \in C^\gamma(\overline{\Omega})$ satisfies $(\mathrm{H})_q$. Then, the eigenvalue problem*:

$$\begin{cases} -\Delta v = \lambda q u^{q(x)-1} v + \sigma v & \text{if } x \in \Omega, \\ v = 0 & \text{if } x \in \partial\Omega, \end{cases} \tag{2.7}$$

admits a first eigenvalue $\sigma = \sigma_1$ which exhibits the following properties.

(a) *Every associated weak eigenfunction $\phi \in H_0^1(\Omega)$ has constant sign in Ω. Moreover, $\phi \in C_0^1(\overline{\Omega}) \cap C^{2,\alpha}(\Omega)$, for some $0 < \alpha < 1$.*
(b) *σ_1 is the unique eigenvalue with a one-signed eigenfunction.*
(c) *σ_1 is simple.*

Proof. To prove the existence, consider the quadratic functional

$$J_1(v) = \frac{1}{2} \int_\Omega \left(|\nabla v|^2 - \lambda q u^{q-1} v^2 \right),$$

defined on the manifold $\mathcal{M} = \{v \in H_0^1(\Omega) : \int_\Omega v^2 = 1\}$. Then, J_1 is coercive on $\mathcal{M}$. In fact,

$$\int_\Omega \lambda q u^{q-1} v^2 = \int_\Omega \lambda q u^q \left(\frac{d}{u}\right) \left|\frac{v}{d}\right| |v| \le C \|u^q\|_{L^\infty(\Omega)} \|v\|_{H_0^1(\Omega)} \le C_\varepsilon + \varepsilon \|v\|^2_{H_0^1(\Omega)},$$

for every $v \in \mathcal{M}$, where $\varepsilon \in (0, 1/2)$ and $C_\varepsilon > 0$. Notice that here we used Hardy's inequality ([3], [6]). Thus,

$$J_1(v) \ge \frac{1}{2}(1 - \varepsilon) \|v\|^2_{H_0^1(\Omega)} - \frac{C_\varepsilon}{2}, \quad v \in \mathcal{M},$$

and J_1 is coercive on $\mathcal{M}$. Hence, $\sigma_1/2 := \inf_{\mathcal{M}} J_1$, is achieved at some $\phi \in \mathcal{M}$ that can be chosen nonnegative since ϕ may be replaced by $|\phi|$. Therefore

$$\int_\Omega \nabla\phi \nabla v - \lambda q u^{q-1} \phi v = \sigma_1 \int_\Omega \phi v, \tag{2.8}$$

for all $v \in H_0^1(\Omega)$. Since any other eigenpair $(\sigma, \psi) \in \mathbb{R} \times H_0^1(\Omega)$ must satisfy (2.8) with (σ_1, ϕ) replaced by (σ, ψ), it is clear that σ_1 is the first (i.e. the smallest) eigenvalue to (2.7). It also holds that σ_1 is the only eigenvalue with a one-signed associated eigenfunction: any eigenfunction ψ of a different eigenvalue σ must satisfy $\int_\Omega \phi\psi = 0$. Finally, if ϕ is any nonnegative eigenfunction associated to σ_1 then $\phi(x) > 0$ almost everywhere in Ω. In fact, $-\Delta\phi + M\phi \ge 0$ for some $M > 0$ and the strong maximum principle (see [23, Theorem 8.19]) yields the conclusion.

Let us next show that any other eigenfunction $\widetilde{\phi}$ associated to σ_1 is one-signed. In fact, if $\widetilde{\phi}_+ \neq 0$ (here $\widetilde{\phi}_\pm =$ stands for the positive/negative part of $\widetilde{\phi}$) then, by setting $v = \widetilde{\phi}_+$ in (2.8) we get,

$$\int_\Omega |\nabla \widetilde{\phi}_+|^2 - \lambda u^{q-1}\widetilde{\phi}_+^2 = \sigma_1 \int_\Omega \widetilde{\phi}_+^2.$$

This means that $\widetilde{\phi}_+$ is an eigenfunction and so $\widetilde{\phi}_+ > 0$. Hence $\widetilde{\phi}_- = 0$ as we wanted to prove.

Let us address now the more subtle regularity issue. If $\phi \in H_0^1(\Omega)$ is an eigenfunction then, Hardy's inequality implies that $f_1(x) := \lambda u^{q-1}\phi + \sigma_1\phi \in L^2(\Omega)$ and so $\phi \in H^2(\Omega) = W^{2,2}(\Omega)$. But applying again Hardy's inequality it follows that $\phi/d \in H^1(\Omega)$. Hence, $f_1 \in L^{q_1}(\Omega)$, $q_1 = 2^* = 2N/(N-2)$ and so $\phi \in W^{2,2^*}(\Omega)$, $\phi/d \in W^{1,2^*}(\Omega)$. We thus improve the regularity of the right hand side of (2.7) up to $f_1 \in L^{q_2}(\Omega)$, $q_2 = q_1^* = Nq_1/(N-q_1)$. In general, the right hand side f_1 improves up to $L^{q_n}(\Omega)$, $q_n = q_{n-1}^*$, provided it belongs to $L^{q_{n-1}}(\Omega)$. This bootstrap argument allows us to assert that ϕ is at least in $C_0^1(\overline{\Omega})$. The $C^{2,\alpha}(\Omega)$ assertion is then a consequence of classical interior regularity since $f_1 \in C^\alpha(\Omega)$.

The simplicity of σ_1 is proved by combining the ideas in [8] and [27]. In fact, let ϕ, ψ be two positive eigenfunctions associated to σ_1, fix $\varepsilon > 0$ small and set $\phi_\varepsilon = \phi + \varepsilon$, $\psi_\varepsilon = \psi + \varepsilon$. Then, integration by parts leads to the equality

$$I_\varepsilon := \int_\Omega \left(\nabla\phi_\varepsilon \nabla\widetilde{\phi}_\varepsilon - \nabla\psi_\varepsilon\nabla\widetilde{\psi}_\varepsilon\right) = \int_\Omega (\lambda u^{q-1} + \sigma_1)\left(\frac{\phi}{\phi_\varepsilon} - \frac{\psi}{\psi_\varepsilon}\right)(\phi_\varepsilon^2 - \psi_\varepsilon^2), \tag{2.9}$$

where the test functions $\widetilde{\phi}_\varepsilon, \widetilde{\psi}_\varepsilon$ are defined as

$$\widetilde{\phi}_\varepsilon = \frac{\phi_\varepsilon^2 - \psi_\varepsilon^2}{\phi_\varepsilon}, \qquad \widetilde{\psi}_\varepsilon = \frac{\phi_\varepsilon^2 - \psi_\varepsilon^2}{\psi_\varepsilon}.$$

It follows that

$$I_\varepsilon = \int_\Omega \left|\nabla\phi_\varepsilon - \frac{\phi_\varepsilon}{\psi_\varepsilon}\nabla\psi_\varepsilon\right|^2 + \left|\nabla\psi_\varepsilon - \frac{\psi_\varepsilon}{\phi_\varepsilon}\nabla\phi_\varepsilon\right|^2. \tag{2.10}$$

Taking $\liminf_{\varepsilon\to 0} I_\varepsilon$ in the second integral in (2.9), using Fatou's lemma and taking into account (2.10), we obtain

$$\int_\Omega \left|\nabla\phi - \frac{\phi}{\psi}\nabla\psi\right|^2 + \left|\nabla\psi - \frac{\psi}{\phi}\nabla\phi\right|^2 = 0.$$

Thus, ψ is a scalar multiple of ϕ. This completes the proof of the lemma. □

3. L^∞ estimates

The purpose of this section is to obtain a priori L^∞ bounds for all possible positive solutions of (1.1), with the additional assumption (1.6). These bounds will be required in the proof of (d) in Theorem 1.1. For technical reasons, we

need also that $q > 1$ on $\partial\Omega_+ \cap \partial\Omega$ in case Ω_+ meets the boundary $\partial\Omega$ (otherwise we have $\overline{\Omega}_+ \subset \Omega$).

THEOREM 3.1. *Assume that $q \in C^\gamma(\overline{\Omega})$ satisfies $(\mathrm{H})_q$ and (1.6), and either $\Omega_+ \subset \overline{\Omega}_+ \subset \Omega$ or $q > 1$ on $\partial\Omega_+ \cap \partial\Omega$ in case that the latter set is nonempty. Let $\{(\lambda_n, u_n)\}$ be a family of nonnegative solutions to (1.1) with $\lambda_n > 0$ varying in a compact interval $I \subset (0, \infty)$. Then, there exists a positive constant M such that*

$$\|u_n\|_{L^\infty(\Omega)} \le M \quad \text{for every } n \in \mathbb{N}. \tag{3.1}$$

The proof of this theorem relies on some lemmas which we state and prove first.

LEMMA 3.2. *Assume $q \in C^\gamma(\overline{\Omega})$ satisfies $(H)_q$. Let $\{(\lambda_n, u_n)\}$ be a family of weak solutions to*

$$\begin{cases} -\Delta u = \lambda_n |u|^{q(x)-1} u & \text{if } x \in \Omega, \\ u = 0 & \text{if } x \in \partial\Omega, \end{cases} \tag{3.2}$$

such that $\lambda_n > 0$ is bounded and

$$\|u_n\|_{L^\infty(\Omega_+)} \le M \quad \text{for all } n, \tag{3.3}$$

and a certain constant $M > 0$. Then, there exists $M_1 > 0$ such that

$$\|u_n\|_{L^\infty(\Omega)} \le M_1 \quad \text{for all } n \in \mathbb{N}.$$

PROOF. By (3.3) it follows that u_n is uniformly bounded on $\partial\Omega_+$. Thus, when restricted to Ω_-, u_n satisfies

$$\begin{cases} -\Delta u_n = \lambda_n |u_n|^{q(x)-1} u_n & \text{if } x \in \Omega_-, \\ u_n = \varphi_n & \text{if } x \in \partial\Omega_-, \end{cases} \tag{3.4}$$

with a family φ_n of Dirichlet data which are uniformly bounded on $\partial\Omega_-$. Since $q < 1$ in Ω_- this implies that the sequence u_n remains uniformly bounded in Ω_-, therefore in Ω. □

LEMMA 3.3. *Let $q \in C^\gamma(\overline{\Omega})$ be a positive function in $\overline{\Omega}$ satisfying the growth condition (1.6),*

$$0 < q(x) < \frac{N}{N-2}, \quad x \in \overline{\Omega}.$$

Assume that for every $n \in \mathbb{N}$, $u_n \in H_0^1(\Omega)$ is a weak solution to problem (3.2), where $\lambda_n > 0$ is a bounded sequence. If for some $D \Subset \Omega$ and some $p > \max\left\{ \sup_\Omega q, 1 \right\}$, we have $\|u_n\|_{L^p(D_{2\varepsilon})} \le M$, for all n and a certain $M > 0$, where $D_{2\varepsilon} = B(D, 2\varepsilon)$, then $\|u_n\|_{L^\infty(D_\varepsilon)}$ is also bounded.

PROOF. Choose a family of domains D_k verifying $D_\varepsilon \subset D_k \subset D_{2\varepsilon}$ and $\overline{D_{k+1}} \subset D_k$. Taking $\gamma > 1$ such that $\sup_\Omega q \le \gamma < N/(N-2)$, together with $\gamma < p$, we observe that if $f(x,u) = |u|^{q(x)-1}u$ then

$$|\lambda_n f(x,u)| \le C(1+|u|^\gamma) \quad \text{for } x \in \Omega \text{ and } u \in \mathbb{R}.$$

Keeping the notation of Lemma 2.1, for a function u defined in Ω we set

$$N_f(u)(\,\cdot\,) = f(\,\cdot\,, u(\,\cdot\,)).$$

Suppose that $\|u_n\|_{L^{p_k}(D_k)}$ is bounded for some $k \ge 1$ and a certain $p_k > \gamma$. Then, $\|N_f(u_n)\|_{L^{r_k}(D_k)} \le C$, for some $C > 0$, with $r_k = p_k/\gamma$. By using interior W^{2,r_k} estimates and the Sobolev embedding we find that $\|u_n\|_{L^{p_{k+1}}(D_{k+1})} \le C$, for a certain constant C and $p_{k+1} = Nr_k/(N-2r_k)$. Starting the iteration with $p_1 = p$ and observing that

$$\frac{p_{k+1}}{p_k} \ge 1+\delta \quad \text{with } \delta = \frac{1}{\gamma}\frac{N}{N-2} - 1,$$

we achieve a bound for u_n in $L^\infty(D_\varepsilon)$ after finitely many steps. □

LEMMA 3.4. *Suppose that $q \in C^\gamma(\overline{\Omega})$ satisfies $(\mathrm{H})_q$ and (1.6) and let $Q \subset \Omega_+$ be an open subset such that $Q \subset \overline{Q} \subset \Omega$. Let $\{(\lambda_n, u_n)\}$ be a sequence of weak solutions to (3.2) with $\lambda_n > 0$ bounded, u_n positive and*

$$\|u_n\|_{L^\infty(Q)} \to \infty. \tag{3.5}$$

Then, there exits $x_0 \in Q$ such that

$$\lim_n u_n(x_0) = \infty. \tag{3.6}$$

PROOF. A small $\varepsilon > 0$ can be found such that $\overline{Q} \subset D := B_1 \cup \ldots \cup B_M$, where $B_i = B_i(x_i, \varepsilon)$, $1 \le i \le M$, is an open ball with the center $x_i \in Q$ such that $\overline{B}_i(x_i, 2\varepsilon) \subset \Omega$. Since $\|u_n\|_{L^\infty(Q_{2\varepsilon})} \to \infty$, by Lemma 3.3, an exponent $p > 1$ can be found so that $\sup_\Omega q < p < N/(N-2)$, together with $\lim_n \|u_n\|_{L^p(D)} = \infty$. Thus, there exists $1 \le i \le M$ satisfying $\lim_n \|u_n\|_{L^p(B_i)} = \infty$.

Now, Theorem 8.18 in [23] implies that

$$\|u_n\|_{L^p(B_i)} \le \frac{C}{(N/(N-2)-p)^2} \inf_{B_i/2} u_n \to \infty,$$

for a certain $C > 0$, where $B_i/2 = B(x_i, \varepsilon/2)$. The existence of $x_0 \in Q$ satisfying (3.6) is therefore shown. □

We finally consider a result which will be used when obtaining local bounds for scaled solutions in the proof of Theorem 3.1.

Lemma 3.5. *Assume that $u > 0$ is a weak solution to $-\Delta u = \lambda a(x) u^{q(x)}$, in a ball B_{4R} with radius $4R$, $a \in L^\infty(B_{4R})$, $\lambda > 0$, $a(x) \geq a_0 > 0$, and the exponent q satisfies $1 < q_0 \leq q(x) \leq q_1$ for $x \in B_{4R}$. Then, a positive constant $C = C(R, \lambda, q_0, q_1, a_0, N)$ exists such that the following Harnack inequality holds*:

$$\sup_{B_R} u \leq C \inf_{B_R} u, \tag{3.7}$$

where B_R is the ball concentric with B_{4R} and radius R. Moreover, the constant C can be chosen independent of λ when λ varies in a compact interval $I \subset (0, \infty)$.

For the proof of Lemma 3.5, we need an auxiliary estimates for solutions, which we state and prove first.

Lemma 3.6. *Assume that $u > 0$ is a weak solution to the inequality*

$$-\Delta u \geq \lambda a(x) u^{q(x)}, \tag{3.8}$$

in a ball B_{2R} with radius $2R$, $a \in L^\infty(B_{3R})$, $a(x) \geq a_0 > 0$, and $q(x) \geq q_0 > 1$. Then, for all $\gamma > q_0 - 1$, there exists a constant $C = C(\lambda, a_0, q_0, \gamma) > 0$ such that

$$\int_{B_R} u^\gamma \leq C\big(R^N + R^{N - 2\gamma/(q_0 - 1)}\big),$$

where C can be taken independent of λ varying in a compact interval $I \subset (0, \infty)$.

Proof. We assume the center of B_{2R} to be the origin. Suppose also $\lambda_0 \leq \lambda \leq \lambda_1$. A function $\xi \in C_0^\infty(B(0,2))$, $0 \leq \xi \leq 1$, is chosen so that $\xi \equiv 1$ in $\overline{B(0,1)}$, $|\nabla \xi| \leq 2$. For k to be determined, the test function

$$\phi(x) = \xi\left(\frac{x}{R}\right)^k u(x)^{-\theta}, \quad \theta = q_0 - \gamma,$$

is introduced in the weak differential inequality (3.8) to obtain

$$\lambda_0 a_0 \int_{B_{2R}} u^{q(x)} \phi \leq \int_{B_{2R}} \nabla u \nabla \phi,$$

and so,

$$\int_{B_{2R}} \theta \xi^k u^{-\theta - 1} |\nabla u|^2 + \lambda_0 a_0 \xi^k u^{q(x) - \theta} \leq \frac{2k}{R} \int_{B_{2R}} \xi^{k-1} u^{-\theta} |\nabla u|.$$

Young's inequality allows us to write the estimates

$$\xi^{k-1} u^{-\theta} |\nabla u| \leq \frac{\theta R}{4k} \xi^k u^{-\theta - 1} |\nabla u|^2 + \frac{k}{\theta R} \xi^{k-2} u^{-\theta + 1}.$$

Thus,

$$\int_{B_{2R}} \frac{\theta}{2} \xi^k u^{-\theta - 1} |\nabla u|^2 + \lambda_0 a_0 \int_{B_{2R}} \xi^k u^{q(x) - \theta} \leq \frac{2k^2}{\theta R^2} \int_{B_{2R}} \xi^{k-2} u^{-\theta + 1}.$$

In particular, for a certain positive constant C,

$$\int_{B_{2R}} \xi^k u^{q(x) - \theta} \leq \frac{C}{R^2} \int_{B_{2R}} \xi^{k-2} u^{-\theta + 1}. \tag{3.9}$$

We now observe that,

$$\int_{B_{2R}} \xi^k u^{q(x)-\theta} \geq \int_{B_{2R}\cap\{u\geq 1\}} \xi^k u^{q_0-\theta} = \int_{B_{2R}\cap\{u\geq 1\}} \xi^k u^{\gamma}. \tag{3.10}$$

Fixing k and p so that $(1-\theta)p = \gamma$ and $p(k-2) = k$, Young's inequality implies that

$$\frac{1}{R^2}\xi^{k-2}u^{1-\theta} \leq \frac{\varepsilon^p}{p}\xi^k u^{\gamma} + \frac{1}{p'\varepsilon^{p'}R^{2p'}}$$

for every $\varepsilon > 0$, with $p = \gamma/(1-\theta)$ and $p' = \gamma/(q_0-1)$. A suitable choice of $\varepsilon > 0$ gives

$$\frac{C}{R^2}\int_{B_{2R}} \xi^{k-2}u^{-\theta+1} \leq \frac{1}{3}\int_{B_{2R}} \xi^k u^{\gamma} + CR^{N-2\gamma/(q_0-1)}. \tag{3.11}$$

Combining (3.9)–(3.11) we obtain

$$\int_{B_{2R}\cap\{u\geq 1\}} \xi^k u^{\gamma} \leq \frac{1}{3}\int_{B_{2R}\cap\{u\geq 1\}} \xi^k u^{\gamma} + \frac{1}{3}\int_{B_{2R}\cap\{u<1\}} \xi^k u^{\gamma} + CR^{N-2\gamma/(q_0-1)}.$$

Thus,

$$\int_{B_{2R}\cap\{u\geq 1\}} \xi^k u^{\gamma} \leq CR^N + CR^{N-2\gamma/(q_0-1)}.$$

The desired inequality follows from this estimate. □

PROOF OF LEMMA 3.5. The assertion in the lemma is a consequence of Lemma 4.2 in [38] (see also Theorem 5 in [37]) according to the following arguments: the equation satisfied by u has the form

$$-\Delta u = f(x,u),$$

where $|f(x,u)| \leq d(x)|u|$, with $d = \lambda|a(x)||u|^{q(x)-1}$. We are showing that for every $p \in (N/2, N)$ the norm $\|d\|_{L^p(B_{2R})}$ can be estimated solely in terms of R, N and q. Therefore, the above mentioned result in [38] ensures the existence of a constant C depending on N, q and $R^{2-N/p}\|d\|_{L^p(B_{2R})}$ such that (3.7) holds.

To show the assertion on the estimate of $\|d\|_{L^p(B_{2R})}$ let $\lambda = \|a\|_{L^\infty(\Omega)} = 1$ for simplicity, and observe that

$$\begin{aligned}\|d\|^p_{L^p(B_{2R})} &= \int_{B_{2R}} u^{p(q(x)-1)} \\ &\leq \int_{B_{2R}\cap\{u>1\}} u^{p(q_1-1)} + CR^N \leq \int_{B_{2R}} u^{p(q_1-1)} + CR^N.\end{aligned}$$

Since $\gamma := p(q_1-1) > q_0 - 1$ for every $p > N/2$ then Lemma 3.6 implies that

$$\int_{B_{2R}} u^{\gamma} \leq C(R^N + R^{N-2\gamma/(q_0-1)}),$$

for a certain positive constant $C > 0$. Therefore, the proof is concluded. □

We end this section with the proof of our main result, Theorem 3.1.

Proof of Theorem 3.1. If (3.1) fails then Lemma 3.2 implies that

$$\lim_n \|u_n\|_{L^\infty(\Omega_+)} = \infty.$$

Accordingly, a sequence $x_n \in \overline{\Omega}_+$ exists so that

$$\lim_n u_n(x_n) = \infty.$$

After passing to a subsequence if necessary there exist $\lambda_0 > 0$ and $x_0 \in \overline{\Omega}_+$ verifying $\lambda_n \to \lambda_0$ and $x_n \to x_0$. Now, three options are possible:

(1) $x_0 \in \Omega_+$.
(2) $x_0 \in \partial\Omega_+ \cap \partial\Omega$.
(3) $x_0 \in \Gamma_+ = \partial\Omega_+ \cap \Omega$.

As can be checked next, cases (1) and (2) can be handled according to the general arguments in [22]. However, case (3) is less standard and will be separately treated later.

To proceed in case (1) we set (see [22])

$$v_n(y) = \frac{1}{M_n} u_n(x_n + \theta_n y),$$

where we assumed that $u_n(x_n) = \sup_{\Omega_+} u_n =: M_n$ and choose $\theta_n = 1/M_n^{(q(x_n)-1)/2}$. Then, v_n satisfies

$$-\Delta v_n = \lambda_n M_n^{\Delta_n q} v_n^{q(x_n+\theta_n y)}$$

with $\Delta_n q := q(x_n + \theta_n y) - q(x_n)$, where we are assuming that

$$|y| < R_n := \frac{\varepsilon - |x_n - x_0|}{\theta_n}$$

and it is supposed that $\overline{B}(x_0, \varepsilon) \subset \Omega$. Observe that, since $q(x_n) \to q(x_0) > 1$ then $\theta_n \to 0$ meanwhile $R_n \to \infty$.

On the other hand, $|\Delta_n q| \le CR\theta_n^\gamma$, for $|y| \le R$ and a certain constant $C > 0$. This means that, as a function of y, $M_n^{\Delta_n q} \to 1$ uniformly on compact sets of $\mathbb{R}^N$. Now we fix $R > 0$. Let $B_R = B(0, R)$, take n large and observe that

$$\|v_n\|_{L^\infty(B_R)} \le 1,$$

allows us to obtain, via the standard L^p estimates, a uniform bound of the form,

$$\|v_n\|_{C^{1,\beta}(\overline{B}_R)} \le M,$$

for a certain positive M. This furnishes a subsequence converging in $C^1(\overline{B}_R)$ to a nonnegative function v satisfying

$$-\Delta v = \lambda_0 v^{q(x_0)},$$

in both weak and classical senses in B_R. A further diagonal procedure leads to a nonnegative solution v of such equation in $\mathbb{R}^N$ fulfilling $v(0) = 1$. Since this is not possible (see [21]), case (1) can be ruled out.

As for case (2) we also proceed following [22]. The argument there is next reviewed for the sake of completeness. By means of a translation and a rotation it can be assumed that $x_0 = 0$ and that $U \cap \Omega = \{(x', x_N) : x_N > \varphi(x')\}$ with $U = [B(0,\eta) \cap \{x_N = 0\}] \times (-\delta, \delta)$ a neighbourhood of $x = 0$, $x' = (x_1, \dots, x_{N-1})$, and the function $\varphi \in C^1(B(0,\eta) \cap \{x_N = 0\})$ satisfies $\varphi(0) = 0$, $\nabla\varphi(0) = 0$. The mapping $x \mapsto y = (y', y_N)$ with $y' = x'$, $y_N = x_N - \varphi(x')$ transforms the equation in (1.1) in

$$
\begin{aligned}
(3.12) \quad & -\Delta_{y'} u + (1 + |\nabla\varphi(y')|^2)\partial^2_{y_N y_N} u \\
& \qquad - \sum_{i=1}^{N-1} \partial_{y_i}\varphi(y')\partial^2_{y_i y_N} u + \Delta\varphi(y')\partial_{y_N} u = \lambda u^{q(y)},
\end{aligned}
$$

where $y \in V := \{y : y' \in [B(0,\eta) \cap \{y_N = 0\}],\ 0 < y_N < \delta - \varphi(y')\}$ and u is now regarded as a function of y. By setting $y_n = (x'_n, (x_n)_N - \varphi(x'_n))$, $d_n = (x_n)_N - \varphi(x'_n)$ and performing the scaling

$$
v_n(z) = \frac{1}{M_n} u(y_n + \theta_n z),
$$

with $\theta_n = 1/M_n^{(q(y_n)-1)/2}$, (3.12) is transformed in

$$
\begin{aligned}
(3.13) \quad & -\Delta_{z'} v + (1 + |\nabla\varphi(y'_n + \theta_n z')|^2)\partial^2_{z_N z_N} v - \sum_{i=1}^{N-1} \partial_{y_i}\varphi(y'_n + \theta_n z')\partial^2_{z_i z_N} v \\
& \qquad + \theta_n \Delta\varphi(y'_n + \theta_n z')\partial_{z_N} v = \lambda_n M_n^{\Delta_n q}\ v^{q(y_n + \theta_n z)},
\end{aligned}
$$

where $z \in V_n/\theta_n$, $V_n := V - y_n$ and $\Delta_n q = q(y_n + \theta_n z) - q(y_n)$. Since we are assuming that $y_n \to 0$ together with $q(0) > 1$, we conclude, as in case (1), that $M_n^{\Delta_n q} \to 1$. Observe that $v_n(z) = 0$, for $z_N = -d_n/\theta_n$. Thus, by arguing as in [22], it can be shown that

$$
s := \varliminf_n \frac{d_n}{\theta_n} > 0.
$$

Two options are now possible: $s = \infty$ and $s < \infty$. In the former case and just as in (1) we obtain by means of a suitable limit process a nontrivial nonnegative solution to the equation

$$
-\Delta v = \lambda_0 v^{q(0)},
$$

in $\mathbb{R}^N$, which is verified to be not possible as before. In case $s < \infty$, by using the L^p estimates up to the boundary we obtain a nontrivial nonnegative solution to the problem

$$
\begin{cases}
-\Delta v = \lambda_0 v^{q(0)} & \text{if } z_N > -s, \\
v = 0 & \text{if } z_N = -s.
\end{cases}
$$

As proved in [22] this is also impossible. Therefore, case (2) can be discarded.

In the discussion of the final case (3), the approach in [22] cannot be employed since, as $q(x_0)=1$, the asymptotic behavior of $M_n^{\Delta_n q}$ cannot be properly determined. We are following instead the ideas developed in [35] (see [2] for a related approach).

As a first remark observe that it can be assumed that

$$\lim_n \|u_n\|_{L^\infty(Q)} = \infty,$$

where $Q=\{x\in\Omega_+ : \operatorname{dist}(x,\partial\Omega_+\cap\partial\Omega)>\delta\}$, for some $\delta>0$. Otherwise we can apply case (2) which has been already ruled out. Thus, by Lemma 3.4 we can find some $x_0\in Q$ such that $\lim_n u_n(x_0)=\infty$.

We introduce now the alternative scaling,

$$v_n(y)=\frac{1}{u_n(x_0)}u_n(x_0+\theta_n y),$$

where $y\in(1/\theta_n)B(0,\eta)$, $\eta>0$ being chosen so that $\overline{B}(x_0,\eta)\subset\Omega_+$. By setting $\theta_n^2 u_n^{q(x_0)-1}=1$, $v=v_n(y)$ solves the equation

$$-\Delta v=\lambda_n u_n(x_0)^{q(x_0+\theta_n y)-q(x_0)}v^{q(x_0+\theta_n y)}. \tag{3.14}$$

In addition (see the proof of case (1)),

$$\lim_n u_n(x_0)^{q(x_0+\theta_n y)-q(x_0)}=1, \tag{3.15}$$

uniformly on compact sets of $\mathbb{R}^N$.

Fixing $R>0$ we next observe equation (3.14) in $B(0,R)$. Since $q(x_0+\theta_n y)\to q(x_0)>1$ uniformly in $B(0,R)$ then $1<q_0\le q(x_0+\theta_n y)\le q_1$ for suitable q_0,q_1. In view of (3.15), Lemma 3.5 implies

$$\sup_{B_{R/4}} v_n \le C \inf_{B_{R/4}} v_n \le C,$$

for large n, since $\inf\limits_{B_{R/4}} v_n \le 1$. Thus we obtain a bound for $\|v_n\|_{L^\infty(B_{R/4})}$ and passing through a subsequence we conclude that $v_n\to v$ in $C^1(\overline{B}_{R/4})$, v being a weak, and hence a classical nonnegative solution to the equation

$$-\Delta v=\lambda_0 v^{q(x_0)},\quad x\in B_{R/4},$$

satisfying $v(0)=1$. Moreover, by a diagonal procedure, a subsequence v_n can be extracted so that $v_n\to v$ in $C^1(\mathbb{R}^N)$, v nonnegative, $v(0)=1$ and v solving the previous equation in the whole $\mathbb{R}^N$. This is not possible, as before, and the proof of Theorem 3.1 is finished. □

4. Proof of Theorem 1.1

PROOF OF THEOREM 1.1. First, we prove that there exists $\lambda^{**} > 0$ such that no positive solutions to (1.1) are possible when $\lambda \geq \lambda^{**}$.

To begin with, assume $u \in H_0^1(\Omega)$, $u \neq 0$, is any nonnegative weak solution to (1.1). Then, Lemma 2.1 implies that u becomes a classical positive solution $u \in C^{2,\alpha}(\overline{\Omega})$. In addition $\overline{u} = u$ defines a strict supersolution to the auxiliary problem (2.2). As noticed in the proof of Theorem 2.2, this problem admits subsolutions as small as desired, and so a particular one $\underline{u}$ can be found so that $\underline{u} \leq \overline{u}$. Therefore, by the method of sub and supersolutions and the uniqueness of the positive solution to (2.2) we obtain

$$u \geq \widetilde{u}_\lambda. \tag{4.1}$$

Take now an arbitrary subdomain $D \subset \Omega_+$ having a first Dirichlet eigenvalue $\lambda_{1,D}$ with an associated positive eigenfunction $\phi_{1,D}$. We will show that no positive solutions are possible if $\lambda \geq \lambda^{**} := \max\{\lambda_0, \lambda_{1,D}\}$ where λ_0 is fixed so that $\widetilde{u}_\lambda(x) \geq 1$, for $x \in D$, when $\lambda \geq \lambda_0$ (recall that $\widetilde{u}_\lambda \to +\infty$ uniformly in compact sets of Ω by Theorem 2.2 (c)). In fact, multiplying the equation in (1.1) by $\phi_{1,D}$ and integrating by parts we find that

$$\lambda \int_D \phi_{1,D} u^{q(x)} \leq \lambda_{1,D} \int_D \phi_{1,D} u.$$

If $\lambda \geq \lambda_0$ then $u \geq \widetilde{u}_{\lambda_0} \geq 1$ in D. Thus $u < u^{q(x)}$ in D and the previous inequality implies that $\lambda < \lambda_{1,D}$. Therefore, (1.1) does not admit positive solutions if $\lambda \geq \lambda^{**}$.

To characterize the existence regime of positive solutions we set, as customary, $\mathcal{S} = \{\lambda > 0 : \text{a positive solution to (1.1) exists}\}$ and define $\lambda^* = \sup \mathcal{S}$. To see that the set $\mathcal{S}$ is nonempty, we take $\overline{u} = M\psi$ where ψ is the solution to the problem $-\Delta\psi = 1$ in Ω, $\psi = 0$ on $\partial\Omega$, and M is a positive constant. It is not hard to see that $\overline{u}$ becomes a supersolution to (1.1) if λ is small enough to have $M^{1-q(x)} \geq \lambda$ in Ω. A comparable subsolution $\underline{u}$ is obtained by setting $\underline{u} = \tilde{u}_{\lambda'}$ with $0 < \lambda' \leq \lambda$ small enough. As a byproduct, we obtain that $[0, \lambda] \subset \mathcal{S}$, and in particular $0 < \lambda^* < \infty$. A similar argument shows that actually $\mathcal{S} = [0, \lambda^*)$ or $\mathcal{S} = [0, \lambda^*]$. Thus it is clear that a positive solution to (1.1) exists if $0 < \lambda < \lambda^*$ while solutions do not exist if $\lambda > \lambda^*$.

On the other hand, notice that, by (4.1), since $\widetilde{u}_\lambda$ lies below any positive solution to (1.1) it is well-known that there exists a minimal positive solution for every $\lambda \in (0, \lambda^*)$.

Let us prove now the monotonicity of u_λ in λ. In fact if $\lambda < \lambda'$ then $\overline{u} = u_{\lambda'}$ defines a supersolution to (1.1) while $\underline{u} = \widetilde{u}_\lambda$ defines a subsolution to that problem with $\underline{u} \leq \overline{u}$. Therefore $\widetilde{u}_\lambda < u_\lambda < u_{\lambda'}$ in Ω, as was wanted to prove.

Consider next any family v_λ of positive solutions such that

$$\|v_\lambda\|_{L^\infty(\Omega)} = O(1) \quad \text{as } \lambda \to 0.$$

Choosing a sequence v_{λ_n} with $\lambda_n \to 0$ and using L^p and Schauder estimates we conclude that, modulus a subsequence, $v_{\lambda_n} \to 0$ in, say, $C^2(\overline{\Omega})$ (observe that $v = 0$ is the only solution to (1.1) if $\lambda = 0$). Thus, the whole family $v_\lambda \to 0$ in $C^2(\overline{\Omega})$. In particular, $v = v_\lambda$ satisfies $-\Delta v \le \lambda v^{q_-}$, $x \in \Omega$, for small λ and so $v_\lambda \le \lambda^{1/(1-q_-)}\varphi(x)$, for small λ, where $q_- = \inf_\Omega q$ and φ is the positive solution to $-\Delta\varphi = \varphi^{q_-}$ in Ω together with $\varphi_{|\partial\Omega} = 0$.

Concerning the continuity assertion suppose $\lambda_n \to \overline{\lambda}$ in $(0, \lambda^*)$ with λ_n increasing. Since $u_n := u_{\lambda_n} \le u_{\overline{\lambda}}$ and $u_{\overline{\lambda}} \in L^\infty(\Omega)$ we first obtain a $C^{1,\beta}(\overline{\Omega})$ uniform estimate for u_n and thus, we achieve a uniform estimate in $C^{2,\alpha}(\overline{\Omega})$. Therefore, $u_n \to \widehat{u}$ in $C^2(\overline{\Omega})$ where $\widehat{u} \le u_{\overline{\lambda}}$ solves (1.1) at $\lambda = \overline{\lambda}$. Since $u_{\overline{\lambda}}$ is the minimal solution then $\widehat{u} = u_{\overline{\lambda}}$ and we are done. This completes the proofs of (a) and (b).

To show (c) we follow the argument in [31] and set $\mathcal{K} = \{u \in H_0^1(\Omega) : \widetilde{u}_\lambda \le u \le u_\lambda\}$. Then, the functional

$$J(u) = \int_\Omega \left(\frac{1}{2}|\nabla u|^2 - \frac{\lambda}{q+1}|u|^{q+1}\right)$$

achieves a minimum $\widehat{u}$ in $\mathcal{K}$ which turns out to be a solution to (1.1) (see [40]). Thus, $\widehat{u} = u_\lambda$.

Choose now $\varphi \in C_0^1(\overline{\Omega})$ nonnegative. Then,

$$D^2J(u_\lambda)[\varphi] = \int_\Omega |\nabla\varphi|^2 - \lambda q u_\lambda^{q-1}\varphi^2 \ge 0. \tag{4.2}$$

Since an arbitrary nonnegative $\varphi \in H_0^1(\Omega)$ can be approached in $H_0^1(\Omega)$ by a sequence of nonnegative functions $\varphi_n \in C_0^1(\overline{\Omega})$, then (4.2) holds for nonnegative functions in $H_0^1(\Omega)$. Moreover, for arbitrary $\varphi \in H_0^1(\Omega)$, writing $\varphi = \varphi_+ - \varphi_-$, we have

$$D^2J(u_\lambda)[\varphi] = D^2J(u_\lambda)[\varphi_+] + D^2J(u_\lambda)[\varphi_-] \ge 0.$$

This proves (c).

The proof of (d) involves a topological degree argument. To settle the abstract framework we set $X = C_0^1(\overline{\Omega}) = \{u \in C^1(\overline{\Omega}) : u_{|\partial\Omega} = 0\}$, and, denoting as usual the dual space of $H_0^1(\Omega)$ as $H^{-1}(\Omega)$, we define the solution operator $G\colon H^{-1}(\Omega) \to H_0^1(\Omega)$ as the continuous (linear) operator mapping $g \in H^{-1}(\Omega)$, to the weak solution $u \in H_0^1(\Omega)$ of the problem $-\Delta u = g$ in Ω, $u_{|\partial\Omega} = 0$. By $N_f\colon L^\infty(\Omega) \to L^\infty(\Omega)$ we denote the Nemytskiĭ operator associated to $f(x,u) = |u|^{q(x)-1}u$, mapping u to the function $|u|^{q(\,\cdot\,)-1}u$ (Lemma 2.1). Then, the operator

$$H\colon [0,\infty) \times X \to X, \quad (\lambda, u) \mapsto H(\lambda, u) = \lambda G(N_f(u)),$$

exhibits the following features:

(1) H is completely continuous, i.e., is continuous and maps bounded sets into relatively compact sets of X.

(2) H is strongly order preserving, i.e. $u \le v$ and $u \ne v$ implies

$$\widetilde{u}(x) < \widetilde{v}(x) \quad \text{for } x \in \Omega \quad \text{and} \quad \frac{\partial \widetilde{u}}{\partial \nu}(x) > \frac{\partial \widetilde{v}}{\partial \nu}(x) \quad \text{for each } x \in \partial\Omega,$$

where $\widetilde{u} = H(\lambda, u(\,\cdot\,))$, $\widetilde{v} = H(\lambda, v(\,\cdot\,))$.

(3) A function $u \in X$ positive in Ω defines a weak solution to (1.1) if and only if $u = H(\lambda, u)$.

A proof of the previous assertions is more or less standard and is therefore omitted.

Now fix $0 < \overline{\lambda}_1 < \lambda^*$. We already know that problem (1.1) admits a minimal solution $u_1 := u_{\overline{\lambda}_1}$ and we are looking for a second positive solution. To this end we choose $0 < \underline{\lambda} < \overline{\lambda}_1 < \overline{\lambda} < \lambda^*$, set $\underline{u} = u_{\underline{\lambda}}$, $\overline{u} = u_{\overline{\lambda}}$ and observe that $u_1 \in \mathcal{C} := \{u \in X : \underline{u}(x) \le u(x) \le \overline{u}(x),\ x \in \Omega\}$. Moreover, the strong maximum principle ([23]) implies that $H_{\overline{\lambda}_1}(\mathcal{C}) \subset \operatorname{int}(\mathcal{C})$, where for $\lambda > 0$, $H_\lambda \colon X \to X$ stands for the operator mapping u to the function $H(\lambda, u(\,\cdot\,))$, and $\operatorname{int}(\mathcal{C})$ designates the interior of $\mathcal{C}$ relative to the topology of X. In particular, a ball $B_\varepsilon := B(u_1, \varepsilon)$ in X, $\varepsilon > 0$ small, can be found so that $B_\varepsilon \subset \overline{B}_\varepsilon \subset \mathcal{C}$.

The proof of existence of an additional positive solution to (1.1), $\lambda = \overline{\lambda}_1$, is obtained by reaching a contradiction if it is assumed that u_1 is the unique positive solution. In fact, under the latter assumption the Leray–Schauder degree

$$d(I - H_{\overline{\lambda}_1}, B_\varepsilon, 0), \tag{4.3}$$

is well-defined. To compute its value we observe that the fixed point index $i(H_{\overline{\lambda}_1}, \mathcal{C}, \mathcal{C})$ of $H_{\overline{\lambda}_1}$ relative to the closed and convex set $\mathcal{C}$ is well-defined while

$$i(H_{\overline{\lambda}_1}, \mathcal{C}, \mathcal{C}) = 1 \tag{4.4}$$

(see [1], Section 11). The isolation of u_1 together with the excision property of the index ([1]) implies that

$$i(H_{\overline{\lambda}_1}, \mathcal{C}, \mathcal{C}) = i(H_{\overline{\lambda}_1}, B_\varepsilon, \mathcal{C}).$$

Since $\mathcal{C}$ is a retract of the whole space X, a retraction $r \colon X \to \mathcal{C}$ exists such that the restriction $r_{|\mathcal{C}}$ is the identity in $\mathcal{C}$. Accordingly, $r(u) = u$ for all $u \in B_\varepsilon$. Moreover, $r^{-1}(B_\varepsilon) = B_\varepsilon$. Thus, the definition of the fixed point index in terms of the Leray–Schauder degree ([1]) permits us writing

$$i(H_{\overline{\lambda}_1}, B_\varepsilon, \mathcal{C}) = d(I - H_{\overline{\lambda}_1}, r^{-1}(B_\varepsilon), 0) = d(I - H_{\overline{\lambda}_1}, B_\varepsilon, 0).$$

Thus (see Remark 4.1 below),

$$d(I - H_{\overline{\lambda}_1}, B_\varepsilon, 0) = 1. \tag{4.5}$$

We introduce now the open set Q in X given by

$$Q = \{u \in X : \underline{u}(x) < u(x) < M + 1 \text{ for all } x \in \Omega\},$$

where M is a uniform L^∞ estimate of the positive solutions to (1.1) corresponding to the interval of the parameter $I = [\overline{\lambda}_1, \lambda^* + 1]$ (Theorem 3.1). Since no positive solutions u to (1.1) on the boundary ∂Q exist for all $\lambda \in I$, the Leray–Schauder degree $d(I - H_\lambda, Q, 0)$ is well-defined for $\lambda \in I$. Moreover, under such conditions the homotopy invariance of the degree permits concluding that $d(I - H_\lambda, Q, 0)$ is constant. Therefore,

$$d(I - H_\lambda, Q, 0) = d(I - H_{\lambda^*+1}, Q, 0) = 0,$$

the last identity coming from the fact that no solutions to (1.1) exist in Q for $\lambda > \lambda^*$. Finally, the excision property implies

$$d(I - H_{\overline{\lambda}_1}, B_\varepsilon, 0) = d(I - H_{\overline{\lambda}_1}, Q, 0) = 0.$$

Since this contradicts (4.5) problem (1.1) must admit a second positive solution $v_{\overline{\lambda}_1}$ satisfying $v_{\overline{\lambda}_1} \geq u_{\overline{\lambda}_1}$.

The assertion in (d) concerning the minimal solution at $\lambda = \lambda^*$ is a consequence of the L^∞ estimate in Theorem 3.1. The family $\{u_\lambda\}$ is uniformly bounded in $C^{1,\beta}(\overline{\Omega})$ for all $0 < \beta < 1$. Thus, the limit

$$u_{\lambda^*} := \lim_{\lambda \to \lambda^*} u_\lambda = \sup_{0<\lambda<\lambda^*} u_\lambda$$

holds in $C^{1,\beta}(\overline{\Omega})$ and defines a weak positive solution to (1.1) for $\lambda = \lambda^*$. Schauder theory and the properties of N_f as a Nemytskiĭ operator taking values in $C^\theta(\overline{\Omega})$ for some $0 < \theta < 1$ (Lemma 2.1) permits upgrading the limit up to (at least) $C^2(\overline{\Omega})$. It is clear that u_{λ^*} is the minimal solution to (1.1) at $\lambda = \lambda^*$. □

Remark 4.1. It will be shown in the next section that u_λ is asymptotically stable for all $\lambda \in (0, \lambda^*)$ with the possible exception of $\lambda = \lambda_n$, with λ_n a decreasing sequence satisfying $\lambda_n \to 0$. Therefore,

$$d(I - H_\lambda, B(u_\lambda, \varepsilon), 0) = 1,$$

for $\varepsilon > 0$ small and each $\lambda \neq \lambda_n$. This fact is coherent with (4.5).

5. Further properties of the branch of minimal solutions u_λ

The main purpose of this section is to prove Theorem 1.3. Our first result depicts the structure of the set of solutions to (1.1) in the space $X \times (0, \infty)$ near a solution $(\overline{u}, \overline{\lambda})$, when $\overline{u}$ is positive and linearly stable, but in the critical case when the associated eigenvalue $\sigma_1 = 0$ (recall that $X = C_0^1(\overline{\Omega})$). As customary

we set $\mathcal{P} = \{u \in X : u(x) \geq 0\ x \in \Omega\}$, the positive cone in X, and $\overset{\circ}{\mathcal{P}}$ its interior relative to the C^1 topology. It is well-known that

$$\overset{\circ}{\mathcal{P}} = \left\{ u \in X : u(x) > 0,\ x \in \Omega \text{ and } \frac{\partial u}{\partial \nu}(x) < 0,\ x \in \partial\Omega \right\}.$$

The first eigenvalue to problem (2.7) will be designated, if necessary, by $\sigma_1(u)$ to stress its dependence on the function u.

LEMMA 5.1. *Let $(\overline{u}, \overline{\lambda})$ be a positive solution to (1.1) with $\sigma_1(\overline{u}) = 0$. Then there exist positive numbers ε, η and C^∞ mappings,*

$$\lambda\colon (-\varepsilon, \varepsilon) \to \mathbb{R}, \quad s \mapsto \lambda(s), \qquad u\colon (-\varepsilon, \varepsilon) \to \overset{\circ}{\mathcal{P}}, \quad s \mapsto u_s = u(\,\cdot\,, s),$$

verifying $u(\,\cdot\,, 0) = \overline{u}$, $\lambda(0) = \overline{\lambda}$ and such that:

(a) *$(u_s, \lambda(s))$ solves (1.1) for every $-\varepsilon < s < \varepsilon$,*

(b) *if (u, λ) solves (1.1) with $\|u - \overline{u}\|_X < \eta$, $|\lambda - \overline{\lambda}| < \eta$, then a unique $s \in (-\varepsilon, \varepsilon)$ exists such that $u = u_s$ and $\lambda = \lambda(s)$.*

Moreover, $\lambda'(0) = 0$, $\lambda''(0) < 0$ while $\sigma_1(u_s) > 0$ (respectively, $\sigma_1(u_s) < 0$) for $-\varepsilon < s < 0$ $(0 < s < \varepsilon)$.

REMARK 5.2. A solution $(\overline{u}, \overline{\lambda})$ exhibiting the properties in Lemma 5.1 is referred to as a "turning point" in the bifurcation diagram of solutions to (1.1) (see [24], [25]).

PROOF OF LEMMA 5.1. According to the notation in Section 4 define

$$F(u, \lambda) = u - H(u, \lambda).$$

We now observe that the Nemytskiĭ operator N_f when considered as $N_f\colon \overset{\circ}{\mathcal{P}} \to C(\overline{\Omega})$, $\overset{\circ}{\mathcal{P}}$ endowed with the topology of X, defines a mapping of class C^∞ such that

$$D_u^k N_f(u)(\widehat{v}_1, \ldots, \widehat{v}_k) = q(q-1)\ldots(q-k+1)u^{q-k}\widehat{v}_1 \ldots \widehat{v}_k,$$

$\widehat{v}_1, \ldots, \widehat{v}_k \in X$, where $D_u^k N_f(u)$ stands for the k-th order Fréchet differential of N_f at u, acting as a k-linear operator. To show this assertion notice that a fixed function $u \in \overset{\circ}{\mathcal{P}}$ satisfies estimates (2.6) while

$$\left| \frac{v(x)}{d(x)} \right| \leq C\|v\|_X, \quad x \in \Omega,$$

for all $v \in X$ and a constant C not depending on v (recall the definition of the function d introduced in Section 2). Therefore $F\colon \overset{\circ}{\mathcal{P}} \times \mathbb{R} \to X$ defines a C^∞ mapping.

As a consequence of Lemma 2.3 we know that $N(D_u F(\overline{u}, \overline{\lambda})) = \operatorname{span}\{\phi\}$, where $\phi \in X$ is a positive eigenfunction associated to $\sigma_1(\overline{u})$, normalized in

some way. Moreover, suppose that $\widehat{w} = D_u F(\overline{u}, \overline{\lambda})\widehat{v}$ with $\widehat{v}, \widehat{w} \in X$. Then, $\widehat{v} - \widehat{w} = \overline{\lambda} G(q\overline{u}^{q-1}\widehat{v})$, which means that

$$\int_\Omega \nabla\widehat{v}\nabla\varphi - \int_\Omega \nabla\widehat{w}\nabla\varphi = \overline{\lambda}\int_\Omega q\overline{u}^{q-1}\widehat{v}\varphi,$$

for each $\varphi \in H_0^1(\Omega)$. Taking into account that $\sigma_1(\overline{u}) = 0$, i.e.

$$\int_\Omega \nabla\phi\nabla\varphi = \overline{\lambda}\int_\Omega q\overline{u}^{q-1}\phi\varphi,$$

for each $\varphi \in H_0^1(\Omega)$, we conclude that

$$\int_\Omega \nabla\widehat{w}\nabla\phi = 0. \tag{5.1}$$

Thus, relation (5.1) characterizes the fact that $\widehat{w}$ lies in the range of the operator $D_u F(\overline{u}, \overline{\lambda})$. Since $\widehat{w} = D_\lambda F(\overline{u}, \overline{\lambda}) = -G(\overline{u}^q)$ does not satisfy such relation, the existence of functions $u_s = u(\,\cdot\,, s)$ and $\lambda(s)$ verifying assertions (a) and (b) together with $\lambda'(0) = 0$ follows from [25] (see Chapter 1). Moreover,

$$u(\,\cdot\,, s) = \overline{u} + \phi s + o(s),$$

as $s \to 0$ in X. Accordingly, $u(\,\cdot\,, s)$ can be chosen increasing in $(-\varepsilon, \varepsilon)$. As for the assertion concerning the sign of $\lambda''(0)$ we follow ideas from [24]. By differentiating the equality

$$F(u_s, \lambda(s)) = 0$$

twice with respect to s and then setting $s = 0$ we obtain

$$D_u^2 F(\overline{u}, \overline{\lambda})(\phi, \phi) + D_u F(\overline{u}, \overline{\lambda})(u''(0)) + D_\lambda F(\overline{u}, \overline{\lambda})\lambda''(0) = 0,$$

with $u''(0)$ the second derivative of u_s at $s = 0$. Using (5.1) we find

$$\lambda''(0) = -\frac{\displaystyle\int_\Omega \nabla\{D_u^2 F(\overline{u}, \overline{\lambda})(\phi, \phi)\}\nabla\phi}{\displaystyle\int_\Omega \nabla\{D_\lambda F(\overline{u}, \overline{\lambda})\}\nabla\phi}.$$

Now observe that $\int_\Omega \nabla\{D_\lambda F(\overline{u}, \overline{\lambda})\}\nabla\phi = -\int_\Omega \overline{u}^q\phi$ while

$$\int_\Omega \nabla\{D_u^2 F(\overline{u}, \overline{\lambda})(\phi, \phi)\}\nabla\phi = -\overline{\lambda}\int_\Omega q(q-1)\overline{u}^{q-2}\phi^3.$$

Accordingly,

$$\lambda''(0) = -\frac{\displaystyle\overline{\lambda}\int_\Omega q(q-1)\overline{u}^{q-2}\phi^3}{\displaystyle\int_\Omega \overline{u}^q\phi}.$$

On the other hand, Picone's identity ([24]) states that

$$\int_\Omega h\left(\frac{\phi}{\overline{u}}\right)(\phi\Delta\overline{u} - \overline{u}\Delta\phi) = \int_\Omega \overline{u}^2 h'\left(\frac{\phi}{\overline{u}}\right)\left|\nabla\left(\frac{\phi}{\overline{u}}\right)\right|^2,$$

with h an arbitrary C^1 function. Choosing $h(t) = t^2$ we get that

$$\overline{\lambda} \int_\Omega (q-1)\overline{u}^{q-2}\phi^3 > 0.$$

That $\lambda''(0) < 0$ then follows from the equality $q(q-1) = (q-1)^2 + (q-1)$. In particular, the function $\lambda(s)$ can be chosen increasing in $(-\varepsilon, 0)$ and decreasing in $(0, \varepsilon)$.

Concerning the sign of $\sigma_1(s)$, it is classical (cf. [11]) that both $\sigma_1(\overline{u})$ and ϕ can be perturbed to obtain smooth mappings $\sigma_1(s) := \sigma_1(u_s)$, $\phi(s)$ such that

$$D_u(u_s, \lambda(s))(\phi(s)) = \sigma_1(s)\phi(s),$$

for $|s| < \varepsilon$, $\varepsilon > 0$ small. Taking derivatives with respect to s, setting $s = 0$ and using that $\lambda'(0) = 0$, $\sigma_1(0) = \sigma_1(\overline{u}) = 0$ we obtain

$$D_u^2 F(\overline{u}, \overline{\lambda})(\phi, \phi) + D_u F(\overline{u}, \overline{\lambda})(\phi'(0)) = \sigma_1'(0)\phi.$$

Taking gradients, performing a scalar product with $\nabla\phi$ and integrating in Ω gives

$$\sigma_1'(0) = \frac{\displaystyle\int_\Omega q\overline{u}^q\phi}{\overline{\lambda}\displaystyle\int_\Omega q\overline{u}^{q-1}\phi^2}\lambda''(0) < 0.$$

Thus the assertions concerning the sign of $\sigma_1(s)$ hold, provided $\varepsilon > 0$ is suitably reduced. □

LEMMA 5.3. *Suppose* $(\overline{u}, \overline{\lambda})$ *is a positive solution to* (1.1) *with* $\sigma_1(\overline{u}) \geq 0$. *Then, there exists an increasing family* $\{(\widehat{u}_\lambda, \lambda)\}$, $0 < \lambda \leq \overline{\lambda}$, *of positive solutions to* (1.1) *such that* $\lambda \mapsto \widehat{u}_\lambda$ *is smooth in* $(0, \overline{\lambda})$, $\sigma_1(\widehat{u}_\lambda) > 0$ *for each* $\lambda \in (0, \overline{\lambda})$ *and* $\widehat{u}_\lambda \to 0$ *as* $\lambda \to 0+$.

PROOF. Let us first assume that $\sigma_1(\overline{u}) > 0$. Then, $D_u F(\overline{u}, \overline{\lambda})$ admits an order preserving inverse. The implicit function theorem then implies that (1.1) is uniquely solvable near $(\overline{u}, \overline{\lambda})$ in $\overset{\circ}{\mathcal{P}} \times (0, \infty)$ by a smooth family $\{(\widehat{u}_\lambda, \lambda)\}$, with $\overline{\lambda} - \delta < \lambda < \overline{\lambda} + \delta$, being $\sigma_1(\widehat{u}_\lambda) > 0$. Therefore, $\widehat{u}_\lambda$ is increasing in λ.

Assume that $\overline{\lambda} - \delta > 0$. Then, the limit

$$\lim_{\lambda \to (\overline{\lambda}-\delta)+} \widehat{u}_\lambda := \widehat{u},$$

is finite and satisfies $\widehat{u} \geq \widetilde{u}_{\overline{\lambda}-\delta} > 0$, where $\widetilde{u}_\lambda$ is the solution to (2.2). Thus $\widehat{u}$ defines a positive solution to (1.1) at $\lambda = \overline{\lambda} - \delta$. In addition $\sigma_1(\widehat{u}) \geq 0$. The case $\sigma_1(\widehat{u}) = 0$ can be ruled out since Lemma 5.1 would imply that $(\widehat{u}, \overline{\lambda} - \delta)$ is isolated from the right of $\overline{\lambda}$ that is not true. Therefore $\sigma_1(\widehat{u}) > 0$. A new iteration of the process permits a further continuation of $\widehat{u}_\lambda$ to the left of $\overline{\lambda} - \delta$. In this way, $\lambda \mapsto \widehat{u}_\lambda$ can be defined as an increasing smooth mapping in the

whole interval $(0, \overline{\lambda} + \delta)$ which fulfils condition $\sigma_1(\widehat{u}_\lambda) > 0$. The convergence $\widehat{u}_\lambda \to 0$ as $\lambda \to 0$ is a consequence of (b) in Theorem 1.1.

As for the case $\sigma_1(\overline{u}) = 0$, Lemma 5.1 provides the existence of a family $\widehat{u}_\lambda$ of positive solutions which are smooth in $\overline{\lambda} - \varepsilon < \lambda < \overline{\lambda}$, continuous in $\overline{\lambda} - \varepsilon < \lambda \leq \overline{\lambda}$, increasing in λ, satisfy $\sigma_1(\widehat{u}_\lambda) > 0$ for $\lambda < \overline{\lambda}$ and $\widehat{u}_\lambda = \overline{u}$ at $\lambda = \overline{\lambda}$. The previous argument can now be employed to extend $\widehat{u}_\lambda$ up to $\lambda = 0$. □

Lemma 5.4. *The branch of minimal solutions u_λ is discontinuous at $\lambda = \overline{\lambda} \in (0, \lambda^*)$ if and only if $\sigma_1(u_{\overline{\lambda}}) = 0$.*

Proof. If $\sigma_1(u_{\overline{\lambda}}) = 0$ then $(u_{\overline{\lambda}}, \overline{\lambda})$ defines a turning point and therefore (Lemma 5.1) it is isolated from the right of $\overline{\lambda}$, i.e. there exists U, a neighbourhood of $u_{\overline{\lambda}}$ in $\overset{\circ}{\mathcal{P}}$ and $\varepsilon > 0$ so that there exist no solutions $(u, \lambda) \in U \times (\overline{\lambda}, \overline{\lambda} + \varepsilon)$ to (1.1). Thus, $\lim_{\lambda \to \overline{\lambda}+} u_\lambda > u_{\overline{\lambda}}$ and u_λ is discontinuous at $\lambda = \overline{\lambda}$.

Now assume that u_λ is discontinuous at $\lambda = \overline{\lambda}$. Since $\sigma_1(u_{\overline{\lambda}}) \geq 0$ we rule out the case $\sigma_1(u_{\overline{\lambda}}) > 0$. In fact, if such fact holds then Lemma 5.3 furnishes a smooth family $\widehat{u}_\lambda$ of positive solutions such that $\widehat{u}_\lambda = u_{\overline{\lambda}}$ for $\lambda = \overline{\lambda}$. Since u_λ is continuous at $\overline{\lambda}$ from the left we obtain that $\widehat{u}_\lambda = u_\lambda$ for $\overline{\lambda} - \varepsilon < \lambda \leq \overline{\lambda}$. On the other hand, discontinuity of u_λ from the left of $\overline{\lambda}$ implies that $u^+_{\overline{\lambda}} := \lim_{\lambda \to \overline{\lambda}+} u_\lambda > u_{\overline{\lambda}}$, in the sense that even $u^+_{\overline{\lambda}} - u_{\overline{\lambda}} \in \overset{\circ}{\mathcal{P}}$. This entails that $\widehat{u}_\lambda < u_\lambda$ for $\overline{\lambda} < \lambda < \overline{\lambda} + \varepsilon$ and a certain small $\varepsilon > 0$. This is not possible since u_λ is the minimal solution. □

Lemma 5.5. *For every $\lambda_0 \in (0, \lambda^*)$ the branch of minimal solutions u_λ is discontinuous at most at a finite number of values $\lambda_0 \leq \lambda < \lambda^*$.*

Proof. Assume, with no loss of generality, the existence of a decreasing sequence λ_n such that $\lambda_n \to \lambda_0$ for some $\lambda_0 > 0$, being $\lambda = \lambda_n$ a discontinuity point of u_λ. Since every $(\lambda_n, u_{\lambda_n})$ is a turning point, Lemmas 5.3 and 5.4 assert the existence of a smooth increasing family $\widehat{u}^{(n)}_\lambda$ of positive solutions connecting $(\lambda_n, u_{\lambda_n})$ with $(0, 0)$ in $\overset{\circ}{\mathcal{P}} \times [0, \lambda^*)$. Let us set now $u_n = \widehat{u}^{(n)}_{\lambda_0}$. Then, u_n defines a family of positive solutions to (1.1), $\lambda = \lambda_0$, satisfying

$$\widetilde{u}_{\lambda_0} \leq u_n \leq u_{\lambda_n} \leq u_{\lambda_1},$$

where $\widetilde{u}_\lambda$ is the family of positive solutions to (2.2). Since $\|u_n\|_{L^\infty(\Omega)} \leq M$, u_n converges to a positive solution u_0 to (1.1) with $\lambda = \lambda_0$. On the other hand, $\sigma_1(u_n) > 0$. Thus $\sigma_1(u_0) \geq 0$. However, if either $\sigma_1(u_0) > 0$ or $\sigma_1(u_0) = 0$, u_0 becomes an isolated positive solution to (1.1). This is not compatible with the definition of u_0. Therefore, u_λ exhibits at most a finite number of discontinuities for $\lambda \geq \lambda_0 > 0$. □

Proof of Theorem 1.3. As a first remark, Lemma 5.5 implies that u_λ is at most discontinuous in a decreasing sequence λ_n in the interval $(0, \lambda^*)$.

Fix now n. Lemma 5.3 ensures the existence of a smooth increasing family of positive solutions $\widehat{u}_\lambda$, connecting $u = 0$ at $\lambda = 0$ with u_{λ_n} at $\lambda = \lambda_n$ and fulfilling $\sigma_1(\widehat{u}_\lambda) > 0$. Since u_λ is continuous from the left at $\lambda = \lambda_n$, it follows from the uniqueness assertion in Lemma 5.1 that $\widehat{u}_\lambda = u_\lambda$ for $\lambda_{n+1} \leq \lambda_n - \varepsilon < \lambda \leq \lambda_n$ for a small $\varepsilon > 0$.

Set $\widehat{\lambda} = \inf\{\lambda : \lambda_{n+1} < \lambda < \lambda_n,\ \widehat{u}_\lambda = u_\lambda\}$. The uniqueness assertion in the implicit function theorem then implies that $\sigma_1(u_{\widehat{\lambda}}) = 0$. But then u_λ is discontinuous at $\lambda = \widehat{\lambda}$ (Lemma 5.4) and thus $\widehat{\lambda} = \lambda_{n+1}$. This shows (b) and the proof is finished. □

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