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FILIPPOV–WAŻEWSKI THEOREM FOR CERTAIN SECOND ORDER DIFFERENTIAL INCLUSIONS

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Abstract. In the paper we give a generalization of the Filippov–Ważewski Theorem to the second order differential inclusions

$$\mathcal{D}y = y'' - A^2 y \in F(t,y), \tag{$*$}$$

with the initial conditions

$$y(0)=\alpha, \quad y'(0)=\beta, \tag{$**$}$$

where $A \in \mathbb{R}^{d\times d}$ and $F\colon [0,T]\times\mathbb{R}^d \rightsquigarrow c(\mathbb{R}^d)$ is a multifunction satisfying for each $t\in[0,T]$ the Lipschitz condition in y

$$d_H(F(t,y_1),F(t,y_2)) \le l(t)|y_1-y_2|,$$

where $l(\,\cdot\,)$ is integrable. The main result is the following:

Theorem 5.1. *Assume that $F\colon [0,T]\times\mathbb{R}^d \rightsquigarrow c(\mathbb{R}^d)$ is measurable in t, Lipschitz continuous in $x\in\mathbb{R}^d$ (with integrable constant) and integrably bounded. Let $r\in W^{2,1}$ be a solution of the relaxed problem*

$$\mathcal{D}y = y'' - A^2 y \in \operatorname{cl}\operatorname{co} F(t,y), \tag{$***$}$$

with ($**$). Then, for each $\varepsilon>0$, there exists a solution $y\in W^{2,1}$ of ($*$) with ($**$) such that

$$\|y-r\|_{C^1[0,T]} < \varepsilon.$$

The proof goes via a version of the Fillipov Lemma (Theorem 4.4) for inclusions ($*$).

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1. Introduction

In the differential inclusion theory one of seminal results is the Filippov–Ważewski Theorem. In the classical statement it concerns the set of all absolutely continuous solutions of differential inclusions of the first order

$$y' \in F(t,y), \qquad y(0) = y_0, \tag{1.1}$$

where $F\colon [0,T] \times \mathbb{R}^d \rightsquigarrow \mathbb{R}^d$ is a Lipschitzean in y multifunction. It states that the solution set is dense in the uniform convergence topology on $[0,T]$ in the solution set of the so-called relaxed differential inclusion

$$y' \in \operatorname{cl}\operatorname{co} F(t,y), \qquad y(0) = y_0, \tag{1.2}$$

where $\operatorname{cl}\operatorname{co} A$ means the closed convex hull of a set $A \subset \mathbb{R}^d$.

The importance of the Filippov–Ważewski Theorem follows not only from its purely mathematical elegance. The celebrated theorem also gives the wide spectrum of applications in optimal control theory and differential inclusions (see cf. [1], [2], [4]–[13], [15]–[19], [21]–[24] and many others). It can be generalized in many ways. In particular, lately there is observed increase of interest in the field of ordinary differential inclusions of higher order in the form

$$\mathcal{D}y \in F(t,y), \tag{1.3}$$

where $\mathcal{D}$ is an ordinary differential operator. For example there have been examined initial value problems for certain evolution inclusions [8], [9], [17], [3], [20] and n-th order of the form $y^{(n)} - \lambda y \in F(t,y)$ in [7].

In this paper our attention is focused on the differential inclusions in the form

$$\mathcal{D}y = y'' - A^2 y \in F(t,y), \tag{1.4}$$

where $F\colon [0,T] \times \mathbb{R}^d \rightsquigarrow \mathbb{R}^d$ is a multifunction and $\mathcal{D}y = y'' - A^2 y$ is a matrix differential operator with a nonsigular matrix $A \in \mathbb{R}^{d\times d}$. For (1.4) we impose initial conditions

$$y(a) = \alpha, \qquad y'(a) = \beta, \tag{1.5}$$

where $a \in [0,T]$ and $\alpha, \beta \in \mathbb{R}^d$. By a solution of (1.4) with initial conditions (1.5) we mean a function $y \in W^{2,1}[0,T]$ satisfying (1.4) almost everywhere in $[0,T]$ and (1.5). Our considerations are based on the convolution form of solutions of the differential equation $\mathcal{D}y = f$, where $f \in L^1([0,T],\mathbb{R}^d)$. We present them in Section 2, while in Section 3 we give a Gronwall type inequality for a sequence of iterations. In Section 4 we present (Theorem 4.4) an analogue of the Filippov Lemma for (1.4) with an arbitrary initial condition (1.5). It usually plays the crucial role in the proofs of relaxation results (see cf. [1]–[3], [5]–[10], [13], [15], [17]–[19], [21], [22], [24]). That result generalizes Theorem 3 from [3], where conditions concerning the multifunction $F\colon [0,T]\times\mathbb{R}^d \rightsquigarrow \mathbb{R}^d$ were stronger. Section 5 is devoted to the proof of our relaxation result:

THEOREM 5.1. *Assume that* $F\colon [0,T]\times\mathbb{R}^d \rightsquigarrow c(\mathbb{R}^d)$ *is*:

(a) *for each* $y\in\mathbb{R}^d$ *measurable in* t,

(b) *for each* $t\in[0,T]$ *Lipschitz continuous in* $x\in\mathbb{R}^d$ *with a constant* $l(t)$ *and* $l\in L^1([0,T],\mathbb{R}_+)$,

(c) *integrably bounded by* $\gamma\in L^1([0,T],\mathbb{R}_+)$, *i.e.*

$$\sup\{|z| : z\in(t,y)\}\le\gamma(t)\quad \text{a.e. in } [0,T].$$

Let $r\in W^{2,1}$ *be a solution of the relaxed problem*

$$\mathcal{D}y = y'' - A^2 y\in \operatorname{cl}\operatorname{co} F(t,y)\quad \text{with } y(0)=\alpha,\ y'(0)=\beta.$$

Then, for each $\varepsilon>0$, *there exists a solution* $y\in W^{2,1}$ *of* (1.4) *with* $y(0)=\alpha$, $y'(0)=\beta$ *such that* $\|y-r\|_{C^1[0,T]}<\varepsilon$.

2. An IVP for matrix second order ODE

Let $(\mathbb{R}^d,|\cdot|)$ be a Euclidean space. By $L^1([a,b],\mathbb{R}^d)$ we mean the Banach space of Lebesgue integrable functions $u\colon [a,b]\to\mathbb{R}^d$ with the norm $\|u\|_1=\int_{[a,b]}|u(t)|\,dt$ and by $V=V[a,b]=\{y\in W^{2,1}([a,b],\mathbb{R}^d) : y(a)=y'(a)=0\}$ with the norm $\|y\|_V=\|y''\|_1$.

For a given function $f\in L^1([a,b],\mathbb{R}^d)$ and a nonsingular matrix $A\in\mathbb{R}^{d\times d}$ we shall consider the differential equation

$$\mathcal{D}y = y''-A^2y=f, \tag{2.1}$$

with initial conditions

$$y(a)=0,\qquad y'(a)=0. \tag{2.2}$$

By a solution of (2.1) with (2.2) we mean a function $y\in V$ satisfying (2.1) almost everywhere (a.e.) in $[a,b]$.

Let us observe first that the solution of (2.1) has the convolution form

$$y(t)=(\mathcal{R}_Af)(t)=(A^{-1}\sinh(Ax)*_a f)(t),$$

where $(\varphi *_a f)(t)=\int_a^t\varphi(t-x)f(x)\,dx$. Indeed, for each function $\varphi\in C^1(\mathbb{R},\mathbb{R}^{d\times d})$ we have

$$(\varphi *_a f)'(t)=(\varphi' *_a f)(t)+\varphi(0)f(t). \tag{2.3}$$

Thus evaluating the derivatives, we obtain

$$y'(t)=(\cosh(Ax)*_a f)(t)\quad\text{and}\quad y''(t)=(A\sinh(Ax)*_a f)(t)+f(t).$$

Hence we get $y''(t)=(A\sinh(Ax)*f)(t)+f(t)=A^2y(t)+f(t)$. Checking the IC is straightforward.

Denote

$$z_0(t) = A^{-1}\sinh(A(t-a))\beta + \cosh(A(t-a))\alpha. \tag{2.4}$$

Then the function $y(t) = (\mathcal{R}_A f)(t) + z_0(t)$ is the unique solution of (2.1) with initial conditions (2.2). Note that $y \in W = z_0 + V$.

Let us recall that by the norm $\|A\|$ of the matrix $A = [a_{ij}] \in \mathbb{R}^{d\times d}$ we mean the number $\|A\| = \max_i \left[\sum_{j=1}^{d} |a_{ij}|\right]$. We shall need the following pointwise inequalities:

$$\|A^n\| \le \|A\|^n, \qquad \|\sinh(At)\| \le \sinh(\|A\|t), \tag{2.5}$$

$$\|A^{-1}\sinh(At)\| = \left\| \sum_{n=0}^{\infty} \frac{A^{2n}t^{2n+1}}{(2n+1)!} \right\| \le \sum_{n=0}^{\infty} \frac{\|A\|^{2n}t^{2n+1}}{(2n+1)!} = \frac{\sinh(\|A\|t)}{\|A\|},$$

which hold for all $t \ge 0$.

In what follows we shall assume that $[a,b] \subset [0,T]$. Observe that for any $f \in L^1([a,b],\mathbb{R}^d)$, for all $t \in [a,b]$ we have the inequality

$$|A^{-1}(\sinh(Ax) *_a f)(t)| \le \left(\frac{\sinh(\|A\|x)}{\|A\|} *_a |f|\right)(t). \tag{2.6}$$

As a consequence of the later estimate we have for all $t \in [a,T]$

$$|\mathcal{R}_A u - \mathcal{R}_A v|(t) \le (\mathcal{R}_{\|A\|}|u-v|)(t).$$

3. A Gronwall type lemma

In the case $d = 1$ take $p_0, l \in L^1([0,T],\mathbb{R}_+)$ and $\omega > 0$ and consider the sequence $(p_n)_{n\ge 0} \subset L^1([a,b],\mathbb{R}_+)$ satisfying for $n = 0,1,\ldots$ the inequalities

$$p_{n+1}(t) \le l(t)\left(\frac{\sinh(\omega x)}{\omega} *_a p_n\right)(t), \tag{3.1}$$

where $[a,b] \subset [0,T]$. Let

$$\varphi(t) = \sum_{n=0}^{\infty} p_n(t). \tag{3.2}$$

Then we have the following estimates:

Lemma 3.1. *Assume that the sequence* $(p_n)_{n\ge 0} \subset L^1([a,b],\mathbb{R}_+)$ *satisfies for* $n = 0,1,\ldots$ *inequality* (3.1). *Then*:

(a) *For* $n \ge 1$

$$p_n(t) \le l(t)\int_a^t \left(\frac{\sinh\omega(t-z)}{\omega}\right)^n \frac{(m(t)-m(z))^{n-1}}{(n-1)!} p_0(z)\,dz, \tag{3.3}$$

where $m(t) = \int_0^t l(z)\,dz$.

(b) *The function* φ, *given by* (3.2), *is integrable and for almost all* $t \in [a,b]$ *we have*

$$\varphi(t) \le p_0(t) + l(t) \int_a^t \frac{\sinh \omega(t-z)}{\omega} K(t,z) p_0(z) dz,$$

where $K(t,z) = \exp(\sinh \omega(t-z)(m(t)-m(z))/\omega)$. *Then,*

$$\frac{\sinh(\omega x)}{\omega} *_a \varphi \le \int_a^t \frac{\sinh \omega(t-z)}{\omega} K(t,z) p_0(z)\, dz, \tag{3.4}$$

$$\cosh(\omega x) *_a \varphi \le \int_a^t \cosh(\omega(t-z)) K(t,z) p_0(z)\, dz. \tag{3.5}$$

(c) *If for* $l > 0$ *and* $n \ge 0$ *we have*

$$p_{n+1}(t) = l(t) \left(\frac{\sinh(\omega x)}{\omega} *_a p_n \right)(t).$$

Moreover, $\psi = (\varphi - p_0)/l$ *is the solution of the* IVP

$$\psi'' - (\omega^2 + l)\psi = p_0, \qquad \psi(0) = \psi'(0) = 0.$$

Proof. (a) We shall use the induction argument. For $n=1$ the statement is obvious. Assume that (3.3) holds for n. Then for $n+1$ we proceed as follows:

$$\begin{aligned} p_{n+1}(t) &\le l(t) \left(\frac{\sinh(\omega x)}{\omega} *_a p_n \right)(t) = l(t) \int_a^t \frac{\sinh(\omega(t-z))}{\omega}\, p_n(z)\, dz \\ &\le l(t) \int_a^t \frac{\sinh(\omega(t-z))}{\omega}\, l(z) \\ &\quad \cdot \left(\int_a^z \left(\frac{\sinh(\omega(z-x))}{\omega} \right)^n \frac{(m(z)-m(x))^{n-1}}{(n-1)!}\, p_0(x)\, dx \right) dz. \end{aligned}$$

Applying the Fubini Theorem, we get

$$\begin{aligned} &p_{n+1}(t) \le \frac{l(t)}{\omega^{n+1}} \\ &\cdot \int_a^t \left(\int_x^t \sinh(\omega(t-z)) l(z) (\sinh(\omega(z-x)))^n \frac{(m(z)-m(x))^{n-1}}{(n-1)!}\, dz \right) p_0(x)\, dx. \end{aligned}$$

But $d(m(z)-m(x))/dz = l(z)$ and $\sinh(\omega t)$ is increasing. Hence for all $0 \le x \le z \le t \le T$ we have

$$\sinh(\omega(t-z)) \le \sinh(\omega(t-x)) \quad \text{and} \quad \sinh(\omega(z-x)) \le \sinh(\omega(t-x)).$$

Thus

$$\begin{aligned}p_{n+1}(t) &\le \frac{l(t)}{\omega^{n+1}}\int_a^t\bigg(\int_x^t \frac{d}{dz}(m(z)-m(x))\frac{(m(z)-m(x))^{n-1}}{(n-1)!}\,dz\bigg)\\ &\quad\cdot\sinh^{n+1}(\omega(t-x))p_0(x)\,dx\\ &=\frac{l(t)}{\omega^{n+1}}\int_a^t\bigg(\int_x^t \frac{d}{dz}\bigg(\frac{(m(z)-m(x))^{n}}{n!}\bigg)\,dz\bigg)\sinh^{n+1}(\omega(t-x))p_0(x)\,dx\\ &=l(t)\int_a^t\bigg(\frac{\sinh(\omega(t-x))}{\omega}\bigg)^{n+1}\frac{(m(t)-m(x))^n}{n!}p_0(x)\,dx,\end{aligned}$$

what ends the induction step and the proof of (a).

(b) By (a) we have

$$\begin{aligned}\varphi(t) &\le p_0(t)+\sum_{n=1}^{\infty} l(t)\int_a^t\bigg(\frac{\sinh\omega(t-z)}{\omega}\bigg)^n\frac{(m(t)-m(z))^{n-1}}{(n-1)!}p_0(z)\,dz\\ &=p_0(t)+l(t)\int_a^t\bigg(\frac{\sinh\omega(t-z)}{\omega}\exp\bigg(\frac{\sinh\omega(t-z)(m(t)-m(z))}{\omega}\bigg)\bigg)p_0(z)\,dz\\ &=p_0(t)+l(t)\int_a^t\frac{\sinh\omega(t-z)}{\omega}K(t,z)p_0(z)\,dz.\end{aligned}$$

To see the next inequalities we can proceed as follows:

- for (3.4)

$$\begin{aligned}l(t)\bigg(\frac{\sinh(\omega x)}{\omega}*_a\varphi\bigg)(t) &= \sum_{n=0}^{\infty} l(t)\bigg(\frac{\sinh(\omega x)}{\omega}*_a p_n\bigg)(t)\\ &\le\sum_{n=0}^{\infty}p_{n+1}(t)=\varphi(t)-p_0(t)\le l(t)\int_a^t\frac{\sinh\omega(t-z)}{\omega}K(t,z)p_0(z)\,dz;\end{aligned}$$

- for (3.5)

$$\begin{aligned}(\cosh(\omega x)*_a\varphi)(t) &= \sum_{n=0}^{\infty}\int_a^t\cosh(\omega(t-z))p_n(z)\,dz\le\int_a^t\cosh(\omega(t-z))p_0(z)\,dz\\ &+\sum_{n=1}^{\infty}\int_a^t\cosh(\omega(t-z))l(z)\int_a^z\bigg(\frac{\sinh\omega(z-x)}{\omega}\bigg)^n\frac{(m(z)-m(x))^{n-1}}{(n-1)!}\,p_0(x)\,dx\,dz.\end{aligned}$$

Now the same Fubini Theorem argument yields

$$\begin{aligned}&\int_a^t\cosh(\omega(t-z))l(z)\int_a^z\bigg(\frac{\sinh\omega(z-x)}{\omega}\bigg)^n\frac{(m(z)-m(x))^{n-1}}{(n-1)!}p_0(x)\,dx\,dz\\ &=\int_a^t\bigg(\int_x^t\cosh(\omega(t-z))\bigg(\frac{\sinh\omega(z-x)}{\omega}\bigg)^n\bigg(\frac{(m(z)-m(x))^n}{n!}\bigg)'\,dz\bigg)p_0(x)\,dx.\end{aligned}$$

But for all $0\le x\le z\le t\le T$ we have

$$\cosh(\omega(t-z))\le\cosh(\omega(t-x))\quad\text{and}\quad\sinh(\omega(z-x))\le\sinh(\omega(t-x)).$$

Therefore

$$
\begin{aligned}
(\cosh(\omega x) *_a \varphi)(t) \le{} & \int_a^t \cosh(\omega(t-z))p_0(z)\,dz \\
& + \sum_{n=1}^{\infty} \int_a^t \bigg(\int_x^t \bigg(\frac{(m(z)-m(x))^n}{n!} \bigg)' dz \bigg) \\
& \cdot \bigg(\frac{\sinh \omega(t-x)}{\omega} \bigg)^n \cosh(\omega(t-x))p_0(x)\,dx \\
={} & \sum_{n=0}^{\infty} \int_a^t \frac{(m(t)-m(x))^n}{n!} \bigg(\frac{\sinh \omega(t-x)}{\omega} \bigg)^n \cosh(\omega(t-x)p_0(x)\,dx \\
={} & \int_a^t \exp \frac{\sinh \omega(t-x)(m(t)-m(x))}{\omega} \cosh(\omega(t-x))p_0(x)\,dx,
\end{aligned}
$$

what gives (3.5).

(c) For almost all $t \in [a,b]$ we have

$$
\begin{aligned}
\varphi(t) &= p_0(t) + \sum_{n=0}^{\infty} p_{n+1}(t) \\
&= p_0(t) + \sum_{n=0}^{\infty} l(t) \bigg(\frac{\sinh(\omega x)}{\omega} *_a p_n \bigg)(t) = p_0(t) + l(t) \bigg(\frac{\sinh(\omega x)}{\omega} *_a \varphi \bigg)(t).
\end{aligned}
$$

Hence $((\varphi - p_0)/l)'' - \omega^2(\varphi - p_0)/l = \varphi$, what gives the claim. $\square$

4. A Filippov Lemma on $[a,b]$

Consider an IVP problem

$$\mathcal{D}y \in F(t,y), \tag{4.1}$$

$$y(a) = \alpha, \qquad y'(a) = \beta, \tag{4.2}$$

where $\alpha, \beta \in \mathbb{R}^d$ are arbitrary but fixed. By a solution of (1.4) with initial conditions (4.2) we mean a function $y \in W = z_0 + V$ satisfying (4.1), where z_0 is given by (2.4).

We shall pose the following assumptions on $F\colon [0,T] \times \mathbb{R}^d \rightsquigarrow c(\mathbb{R}^d)$, where $c(\mathbb{R}^d)$ stands for the family of all nonempty compact subsets of $\mathbb{R}^d$:

Condition 4.1. *For every $y \in \mathbb{R}^d$ the multifunction $F(\,\cdot\,,y)$ is Lebesgue measurable in t.*

Condition 4.2. *The multifunction $F(t,\,\cdot\,)$ is Lipschitz continuous in y with an integrable function $l(\,\cdot\,)$, i.e. for every $y_1, y_2 \in \mathbb{R}^d$ the inequality*

$$d_H(F(t,y_1),F(t,y_2)) \le l(t)\,|y_1 - y_2|$$

holds for almost all $t \in [0,T]$, where $d_H(K,L)$ stands for the Hausdorff distance between sets $K, L \in c(\mathbb{R}^d)$.

Condition 4.3. *The multivalued mapping $t \mapsto F(t,y)$ is integrably bounded by some $\gamma \in L^1[0,T]$, i.e. for each $y \in \mathbb{R}^d$*

$$\sup\{|z| : z \in F(t,y)\} \leq \gamma(t) \quad \text{a.e. in } [0,T].$$

Below we present a version of the Filippov Lemma for (4.1) with (4.2). It generalizes our result from [3], where in Condition 4.2 we have assumed that there is a constant $l > 0$ such that

$$d_H(F(t,y_1),F(t,y_2)) \leq l|y_1 - y_2|$$

holds for almost all $t \in [0,T]$. The main result of this section is the following:

Theorem 4.4. *Assume that $F\colon [0,T]\times\mathbb{R}^d \rightsquigarrow c(\mathbb{R}^d)$ satisfies Conditions* 4.1–4.3. *Fix $[a,b] \subset [0,T]$ and let $y_0 \in W[a,b] = z_0 + V[a,b] = W$ be an arbitrary function with* (4.2) *such that*:

$$2\mathrm{d}(\mathcal{D}y_0(t),F(t,y_0(t))) \leq p_0(t) \quad \text{a.e. in } [a,b],$$

where $p_0 \in L^1[0,T]$ and $\mathrm{d}(y,A) = \inf\{|y-x| : x \in A\}$. Then there exists a solution $y \in W$ of (4.1) *with* (4.2) *such that*

$$\begin{aligned} |\mathcal{D}y(t) - \mathcal{D}y_0(t)| &\leq l(t)\int_a^t \frac{\sinh((t-z)\|A\|)}{\|A\|}\Phi(t,z)p_0(z)\,dz + p_0(t), \\ |y(t)-y_0(t)| &\leq \int_a^t \frac{\sinh((t-z)\|A\|)}{\|A\|}\Phi(t,z)p_0(z)\,dz \end{aligned}$$

and

$$|y'(t) - y_0'(t)| \leq \int_a^t \cosh((t-z)\|A\|)\Phi(t,z)p_0(z)\,dz,$$

where $\Phi(t,z) = \exp[((m(t)-m(z))\sinh((t-z)\|A\|))/\|A\|]$.

Proof. Observe first that for any $y \in L^\infty([a,b],\mathbb{R}^d)$ the multivalued mapping $t \mapsto F(t,y(t))$ is measurable and, by Condition 3, integrably bounded by $\gamma(t)$, i.e.

$$\sup\{|z| : z \in F(t,y(t))\} \leq \gamma(t) \quad \text{a.e. in } [a,b]. \tag{4.3}$$

For each $u \in L^1([a,b],\mathbb{R}^d)$ denote

$$\mathcal{K}(u) = \{f \in L^1([a,b],\mathbb{R}^d) : f(t) \in F(t,(\mathcal{R}_A u)(t) + z_0(t)) \text{ a.e. in } [a,b]\}.$$

Since $\mathcal{R}_A u + z_0 \in W \subset L^\infty([a,b],\mathbb{R}^d)$ then, by (4.3), each $\mathcal{K}(u)$ is nonempty. Observe now that for every $u,v \in L^1([a,b],\mathbb{R}^d)$ and any $f \in \mathcal{K}(u)$ there is $g \in \mathcal{K}(v)$ such that

$$|f(t)-g(t)| \leq l(t)(\mathcal{R}_{\|A\|}|u-v|)(t) \quad \text{a.e. in } [a,b]. \tag{4.4}$$

Indeed, let $g \in \mathcal{K}(v)$ be such that

$$|f(t)-g(t)| = \mathrm{d}(f(t),F(t,(\mathcal{R}_A v)(t)) + z_0(t)).$$

But for such g, by Condition 4.2, we have

$$\begin{aligned}
&\mathrm{d}(f(t), F(t, (\mathcal{R}_A v)(t)) + z_0(t)) \\
&\qquad \le d_H(F(t, (\mathcal{R}_A u)(t) + z_0(t)), F(t, (\mathcal{R}_A v)(t) + z_0(t))) \\
&\qquad \le l(t)(|\mathcal{R}_A u - \mathcal{R}_A v|)(t) \le l(t)(\mathcal{R}_{\|A\|}|u - v|)(t)
\end{aligned}$$

almost everywhere in $[a, b]$, what shows our claim.

In what follows we shall adopt the Filippov technique with some necessary changes. Starting with $y_0 = \mathcal{R}_A u_0 + z_0$ ($\mathcal{D} y_0 = u_0$), we may choose, by (4.4), $u_1 \in \mathcal{K}(u_0)$ such that

$$|u_1(t) - u_0(t)| \le p_0(t) \quad \text{a.e. in } [a, b],$$

where $y_1 = \mathcal{R}_A u_1 + z_0$. Hence for all $t \in [a, b]$ we have

$$|y_1(t) - y_0(t)| = |(\mathcal{R}_A(u_1 - u_0))(t)| \le \left(\frac{\sinh(x\|A\|)}{\|A\|} *_a p_0\right)(t) \tag{4.5}$$

and

$$|y_1'(t) - y_0'(t)| = |(\cosh(xA) *_a (u_1 - u_0))(t)| \le (\cosh(x\|A\|) *_a p_0)(t).$$

Now (4.5) and Condition 4.2 yield

$$\mathrm{d}((\mathcal{D} y_1)(t), F(t, y_1(t))) \le l(t)\left(\frac{\sinh(x\|A\|)}{\|A\|} *_a p_0\right)(t) = p_1(t) \quad \text{a.e. in } [a, b].$$

Therefore, by (4.4), we may select $y_2 = \mathcal{R} u_2 + z_0 \in W$ such that $u_2 = \mathcal{D} y_2 \in \mathcal{K}(u_1)$ and

$$|u_2(t) - u_1(t)| \le p_1(t) \quad \text{a.e. in } [a, b].$$

Observe that, for all $t \in [a, b]$

$$\begin{aligned}
|y_2(t) - y_1(t)| &= |(\mathcal{R}_A(u_2 - u_1))(t)| \\
&= |A^{-1}(\sinh(Ax) *_a (u_2 - u_1))(t)| \le \left(\frac{\sinh(x\|A\|)}{\|A\|} *_a p_1\right)(t)
\end{aligned}$$

and

$$|y_2'(t) - y_1'(t)| = |(\cosh(Ax) *_a (u_2 - u_1))(t)| \le (\cosh(x\|A\|) *_a p_1).$$

The latter together with (4.4) yields

$$\mathrm{d}((\mathcal{D} y_2)(t), F(t, y_2(t))) \le l(t)\left(\frac{\sinh(x\|A\|)}{\|A\|} *_a p_1\right)(t) = p_2(t) \quad \text{a.e. in } [a, b].$$

Continuing this procedure we obtain, by mathematical induction, the sequences $(p_n) \in L^1([a, b])$, $(u_n) \in L^1([a, b], \mathbb{R}^d)$ and $(y_n) = (\mathcal{R} u_n + z_0) \in W$ such that for $n = 0, 1, \ldots$

$$p_{n+1}(t) = l(t)\left(\frac{\sinh(x\|A\|)}{\|A\|} *_a p_n\right)(t), \qquad u_{n+1} \in \mathcal{K}(u_n), \qquad y_n = \mathcal{R}_A u_n + z_0,$$

and

$$|u_{n+1}(t)-u_n(t)|=|(\mathcal{D}y_{n+1})(t)-(\mathcal{D}y_n)(t)|\leq p_n(t)\quad\text{a.e. in } [a,b].$$

Then

$$\begin{aligned}|y_{n+1}(t)-y_n(t)|&\leq\left(\frac{\sinh(x\|A\|)}{\|A\|}*_a p_n\right)(t),\\ |y'_{n+1}(t)-y'_n(t)|&\leq(\cosh(x\|A\|)*_a p_n)(t)\end{aligned}$$

and thus almost everywhere in $[a,b]$

$$d_H((\mathcal{D}y_{n+1})(t),F(t,y_{n+1}(t)))\leq l(t)(\mathcal{R}_{\|A\|}p_n)(t)=p_{n+1}(t).$$

We are going to show that $(u_n)=(\mathcal{D}y_n)$, (y_n) and (y'_n) are Cauchy sequences. By Lemma 3.1 (a), for $\omega=\|A\|$ we conclude first that for $n\geq 1$

$$p_n(t)\leq l(t)\int_a^t\left(\frac{\sinh((t-z)\|A\|)}{\|A\|}\right)^n\frac{(m(t)-m(z))^{n-1}}{(n-1)!}p_0(z)\,dz, \tag{4.6}$$

where $m(t)=\int_0^t l(z)\,dz$. Moreover, from Lemma 3.1 (b) we know that $\varphi(t)=\sum_{n=0}^{\infty}p_n(t)$ is integrable and we have the estimate

$$\varphi(t)\leq p_0(t)+l(t)\int_a^t\frac{\sinh(t-z)\|A\|}{\|A\|}\Phi(t,z)p_0(z)\,dz,$$

where $\Phi(t,z)=\exp(\sinh((t-z)\|A\|)(m(t)-m(z))/\|A\|)$. Therefore $\varphi_n(t)=\sum_{i=n}^{\infty}p_i(t)$, with $\varphi_0=\varphi$, are integrable and $\varphi_n\to 0$ pointwisely in L^1 as $n\to\infty$. Observe now that for $n=0,1,\ldots$ and $k=1,2,\ldots$ we have

$$|\mathcal{D}y_{n+k}(t)-\mathcal{D}y_n(t)|\leq\sum_{i=n}^{n+k-1}p_i(t)\leq\varphi_n(t)\quad\text{a.e. in } [a,b]. \tag{4.7}$$

So, for almost all $t\in[a,b]$

$$|y_{n+k}(t)-y_n(t)|\leq\sum_{i=n}^{\infty}\left(\frac{\sinh(x\|A\|)}{\|A\|}*_a p_i\right)(t)=\left(\frac{\sinh(x\|A\|)}{\|A\|}*_a\varphi_n\right)(t) \tag{4.8}$$

and

$$|y'_{n+k}(t)-y'_n(t)|\leq(\cosh(x\|A\|)*_a\varphi_n)(t). \tag{4.9}$$

Therefore the sequences $\{u_n\}\subset L^1([a,b],\mathbb{R}^d)$, $\{y_n\}=\{\mathcal{R}_A u_n+z_0\}\subset W$ and $\{y'_n\}\subset W_0^{1,1}[a,b]+z'_0$ are convergent pointwisely and thus, by the Lebesgue Dominated Convergence Theorem, strongly.

Denote $\lim\mathcal{D}y_n=\mathcal{D}y$. Thus $\lim y_n=y$ and $\lim y'_n=y'$. Since, for each $n=0,1,\ldots$, $(\mathcal{D}y_{n+1})(t)\in F(t,y_n(t))$ almost everywhere in $[a,b]$ and each $F(t,\,\cdot\,)$ is Lipschitz continuous, then y is a solution of (4.1). We shall check that it is the required one. Indeed, taking $n = 0$ in (4.8), (4.9) and passing to the limit with $k \to \infty$, we have

$$|\mathcal{D}y(t) - \mathcal{D}y_0(t)| \le \varphi(t) \quad \text{a.e. in } [a,b].$$

So, for almost all $t \in [a,b]$

$$|y(t) - y_0(t)| \le \left(\frac{\sinh(x\|A\|)}{\|A\|} *_a \varphi\right)(t)$$

and

$$|y'(t) - y_0'(t)| \le (\cosh(x\|A\|) *_a \varphi)(t).$$

Using Lemma 3.1 (b) again, we obtain that almost everywhere in $[a,b]$

$$|\mathcal{D}y(t) - \mathcal{D}y_0(t)| \le p_0(t) + l(t)\int_a^t \frac{\sinh(t-z)\|A\|}{\|A\|}\Phi(t,z)p_0(z)\,dz,$$

$$|y(t) - y_0(t)| \le \int_a^t \frac{\sinh((t-z)\|A\|)}{\|A\|}\Phi(t,z)p_0(z)\,dz$$

and

$$|y'(t) - y_0'(t)| \le \int_a^t \cosh((t-z)\|A\|)\Phi(t,z)p_0(z)\,dz.$$

This ends the proof. □

5. A Filippov–Ważewski Theorem on $[0,T]$

We are going to give a version of the Filippov–Ważewski result concerning the relation between the solution sets of the problem:

$$\mathcal{D}y \in F(t,y), \tag{5.1}$$

$$\mathcal{D}y \in \operatorname{cl}\operatorname{co} F(t,y), \tag{5.2}$$

with the same IC's

$$y(0) = \alpha, \qquad y'(0) = \beta, \tag{5.3}$$

where $F\colon [0,T] \times \mathbb{R}^d \rightsquigarrow c(\mathbb{R}^d)$ satisfies Conditions 4.1–4.3. Namely, we have the following:

THEOREM 5.1. *Let r be a solution of* (5.2) *with* (5.3). *Then, for each $\varepsilon > 0$, there exists a solution y of* (5.1) *with* (5.3) *such that $\|y - r\|_{C^1[0,T]} < \varepsilon$.*

PROOF. Fix $\varepsilon > 0$ and denote

$$M = 1 + \sup_{t\in[0,T]}\left(\int_0^t \frac{\sinh((t-z)\|A\|)}{\|A\|}\Phi(t,z)l(z)\,dz,\right.$$
$$\left.\int_0^t \cosh((t-z)\|A\|)\Phi(t,z)l(z)\,dz\right),$$

where $\Phi(t,z) = \exp[\sinh((t-z)\|A\|)(m(t) - m(z))/\|A\|]$.

Take a partition $0 = t_0 < t_1 < \ldots < t_{N+1} = T$ such that for each $k = 0, \ldots, N$ the following inequalities hold:

$$\begin{aligned}\int_{[t_k,t_{k+1}]} \gamma(x)\,dx &< \frac{\varepsilon}{2M},\\ \int_{[t_k,t_{k+1}]} \frac{\sinh(\|A\|(t_{k+1}-x))}{\|A\|}\gamma(x)\,dx &< \frac{\varepsilon}{2M},\\ \int_{[t_k,t_{k+1}]} \cosh(\|A\|(t_{k+1}-x))\gamma(x)\,dx &< \frac{\varepsilon}{2M}.\end{aligned}$$

Let $\mathcal{D}r = v$, where $v(t) \in \operatorname{cl}\operatorname{co} F(t, r(t))$ almost everywhere in t. For $t \in [t_k, t_{k+1}]$ denote by $z_k = z_k(t)$ the unique solution of

$$\mathcal{D}z = 0, \quad \text{with IC's } z(t_k) = r(t_k),\ z'(t_k) = r'(t_k).$$

Then, for $t \in [t_k, t_{k+1}]$, we have

$$\begin{bmatrix} r(t) - z_k(t) \\ r'(t) - z_k'(t) \end{bmatrix} = \int_{t_k}^{t} \begin{bmatrix} A^{-1}\sinh(A(t-x)) \\ \cosh(A(t-x)) \end{bmatrix} v(x)\,dx.$$

Therefore

$$\begin{bmatrix} r(t_{k+1}) - z_k(t_{k+1}) \\ r'(t_{k+1}) - z_k'(t_{k+1}) \end{bmatrix} \in \int_{[t_k,t_{k+1}]} \begin{bmatrix} A^{-1}\sinh(A(t_{k+1}-x)) \\ \cosh(A(t_{k+1}-x)) \end{bmatrix} \operatorname{cl}\operatorname{co} F(x, r(x)).$$

But by the properties of the Aumann integral (see cf. [12] and [15]) we have

$$\int_{[t_k,t_{k+1}]} \Psi(x)\operatorname{cl}\operatorname{co} F(x, r(x)) = \int_{[t_k,t_{k+1}]} \Psi(x) F(x, r(x)),$$

where $\Psi(x)$ are $n \times d$-matrices with essentially bounded entries. Thus

$$\begin{bmatrix} r(t_{k+1}) - z_k(t_{k+1}) \\ r'(t_{k+1}) - z_k'(t_{k+1}) \end{bmatrix} \in \int_{[t_k,t_{k+1}]} \begin{bmatrix} A^{-1}\sinh(A(t_{k+1}-x)) \\ \cosh(A(t_{k+1}-x)) \end{bmatrix} F(x, r(x))$$

and this means that, for each $k = 0, \ldots, N$, there exists an integrable selection $u_k(t) \in F(t, r(t))$ almost everywhere in $[t_k, t_{k+1}]$ such that

$$\begin{bmatrix} r(t_{k+1}) - z_k(t_{k+1}) \\ r'(t_{k+1}) - z_k'(t_{k+1}) \end{bmatrix} = \int_{[t_k,t_{k+1}]} \begin{bmatrix} A^{-1}\sinh(A(t_{k+1}-x)) \\ \cosh(A(t_{k+1}-x)) \end{bmatrix} u_k(x)\,dx.$$

Take $u = \sum_{k=0}^{N} u_k \cdot \chi_{[t_k,t_{k+1}]}$ and let y_0 be a solution of $\mathcal{D}y = u$ with (5.3). Then, for $t \in [0, t_1]$, we have

$$\begin{aligned}\begin{bmatrix} r(t_1) - z_1(t_1) \\ r'(t_1) - z_1'(t_1) \end{bmatrix} &= \int_{[0,t_1]} \begin{bmatrix} A^{-1}\sinh(A(t_1-x)) \\ \cosh(A(t_1-x)) \end{bmatrix} u_0(x)\,dx \\ &= \begin{bmatrix} y_0(t_1) - z_1(t_1) \\ y_0'(t_1) - z_1'(t_1) \end{bmatrix}.\end{aligned}$$

So $y_0(t_1) = r(t_1)$, $y_0'(t_1) = r'(t_1)$. Moreover,

$$\begin{aligned} |r(t) - y_0(t)| &\leq \int_{[0,t_1]} \|A^{-1}\sinh((t_1 - x)A)\||v(x) - u_0(x)|\,dx \\ &\leq \int_{[0,t_1]} \frac{2\sinh(\|A\|(t_1 - x))}{\|A\|}\gamma(x)\,dx \leq \frac{\varepsilon}{M} \end{aligned}$$

and

$$\begin{aligned} |r'(t) - y_0'(t)| &\leq \int_{[0,t_1]} \cosh(\|A\|(t_{k+1} - x))|v(x) - u_0(x)|\,dx \\ &\leq \int_{[0,t_1]} 2\cosh(\|A\|(t_{k+1} - x))\gamma(x)\,dx \leq \frac{\varepsilon}{M}. \end{aligned}$$

Hence, for almost all $t \in [0, t_1]$

$$d((\mathcal{D}y_0)(t), F(t, y_0(t))) = d(u(t), F(t, y_0(t))) \leq l(t)|r(t) - y_0(t)| \leq \frac{\varepsilon l(t)}{M}.$$

Similarly, for $t \in [t_1, t_2]$, we conclude that

$$\begin{bmatrix} r(t) - z_2(t) \\ r'(t) - z_2'(t) \end{bmatrix} = \int_{t_1}^{t} \begin{bmatrix} A^{-1}\sinh(A(t - x)) \\ \cosh(A(t - x)) \end{bmatrix} u_1(x)\,dx,$$

where $u_1(t) \in F(t, r(t))$ almost everywhere in $[t_1, t_2]$. Hence

$$\begin{aligned} \begin{bmatrix} r(t_2) - z_2(t_2) \\ r'(t_2) - z_2'(t_2) \end{bmatrix} &= \int_{t_1}^{t_2} \begin{bmatrix} A^{-1}\sinh(A(t_2 - x)) \\ \cosh(A(t_2 - x)) \end{bmatrix} u_1(x)\,dx \\ &= \begin{bmatrix} y(t_2) - z_1(t_2) \\ y'(t_2) - z_1'(t_2) \end{bmatrix}. \end{aligned}$$

Thus $r(t_2) = y_0(t_2)$ and $r'(t_2) = y_0'(t_2)$. Moreover, for almost all $t \in [t_1, t_2]$

$$\begin{aligned} |r(t) - y_0(t)| &\leq \int_{[t_1,t_2]} \|A^{-1}\sinh((t_2 - x)A)\||v(x) - u_1(x)|\,dx \\ &\leq \int_{[t_1,t_2]} 2\,\frac{\sinh(\|A\|(t_2 - x))}{\|A\|}\gamma(x)\,dx \leq \frac{\varepsilon}{M} \end{aligned}$$

and

$$\begin{aligned} |r'(t) - y_0'(t)| &\leq \int_{[t_1,t_2]} \cosh(\|A\|(t_2 - x))|v(x) - u_0(x)|\,dx \\ &\leq \int_{[t_1,t_2]} 2\cosh(\|A\|(t_2 - x))\gamma(x)\,dx \leq \frac{\varepsilon}{M}. \end{aligned}$$

Hence, for almost all $t \in [t_1, t_2]$, $d((\mathcal{D}y_0)(t), F(t, y_0(t))) \leq \varepsilon l(t)M$.

Continuing this procedure from $[t_k, t_{k+1}]$ to $[t_{k+1}, t_{k+2}]$, we conclude that for the function y_0 we have, for all $t \in [0, T]$

$$|r(t) - y_0(t)| \leq \frac{\varepsilon}{M}, \qquad |r'(t) - y_0'(t)| \leq \frac{\varepsilon}{M}, \qquad d((\mathcal{D}y_0)(t), F(t, y_0(t))) \leq \frac{\varepsilon l(t)}{M}$$

and, for $k = 1, \ldots, N + 1$ $r(t_k) = y_0(t_k)$ and $r'(t_k) = y_0'(t_k)$.

Now from Theorem 4.4 for $p_0(t) = \varepsilon l(t)/M$ we conclude the existence of the solution y of (5.1) with (5.3) such that

$$|\mathcal{D}y(t) - \mathcal{D}y_0(t)| \leq \frac{\varepsilon l(t)}{M} \int_0^t \frac{\sinh((t-z)\|A\|)}{\|A\|} \Phi(t,z) l(z)\, dz + \frac{\varepsilon l(t)}{M},$$

$$|y(t) - y_0(t)| \leq \frac{\varepsilon}{M} \int_0^t \frac{\sinh((t-z)\|A\|)}{\|A\|} \Phi(t,z) l(z)\, dz$$

and

$$|y'(t) - y_0'(t)| \leq \frac{\varepsilon}{M} \int_0^t \cosh((t-z)\|A\|) \Phi(t,z) l(z)\, dz.$$

Hence

$$|y'(t) - r'(t)| \leq \frac{\varepsilon}{M} \bigg(1 + \int_0^t \cosh((t-z)\|A\|)(\Phi(t,z)) l(z)\, dz\bigg) \leq \varepsilon,$$

$$|y(t) - r(t)| \leq \frac{\varepsilon}{M} \bigg(1 + \frac{1}{\|A\|} \int_0^t \frac{\sinh((t-z)\|A\|)}{\|A\|} \Phi(t,z) l(z)\, dz\bigg) \leq \varepsilon.$$

This ends the proof. □

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