POSITIVE SOLUTIONS
OF A DIFFUSIVE PREDATOR-PREY MUTUALIST MODEL
WITH CROSS-DIFFUSION

Jun Zhou

ABSTRACT. In this paper, a competitor-competitor-mutualist model with cross-diffusion is studied by means of the Leray–Schauder degree theory and global bifurcation theory. The conditions for the existence and multiplicity of positive solutions are established.

1. Introduction

The competitor-competitor-mutualist model is the following ODE system:

$$
(1.1) \quad
\begin{cases}
\dfrac{du_1}{dt} = \alpha u_1 \left(1 - \dfrac{u_1}{K_1} - \dfrac{\delta u_2}{1 + m u_3} \right), & t > 0, \\[2ex]
\dfrac{du_2}{dt} = \beta u_2 \left(1 - \dfrac{u_2}{K_2} - \eta u_1 \right), & t > 0, \\[2ex]
\dfrac{du_3}{dt} = \gamma u_3 \left(1 - \dfrac{u_3}{L_0 + l u_1} \right), & t > 0,
\end{cases}
$$

where u_1, u_2 and u_3 represent the population densities of two competitors and a mutualist. Model (1.1) was proposed and studied by Rai et al. in [17], where the explanations of the ecological background of this model can be found as well.

Zheng [24] introduced diffusion to (1.1) to get the following reaction-diffusion system:

$$(1.2) \quad \begin{cases} \dfrac{\partial u_1}{\partial t} - d_1 \Delta u_1 = \alpha u_1 \left(1 - \dfrac{u_1}{K_1} - \dfrac{\delta u_2}{1 + m u_3} \right), & x \in \Omega,\ t > 0, \\[2mm] \dfrac{\partial u_2}{\partial t} - d_2 \Delta u_2 = \beta u_2 \left(1 - \dfrac{u_2}{K_2} - \eta u_1 \right), & x \in \Omega,\ t > 0, \\[2mm] \dfrac{\partial u_3}{\partial t} - d_3 \Delta u_3 = \gamma u_3 \left(1 - \dfrac{u_3}{L_0 + l u_1} \right), & x \in \Omega,\ t > 0, \\[2mm] \dfrac{\partial u_i}{\partial \nu}(x,t) = 0 \text{ o } r u_i(x,t) = 0,\ i = 1,2,3, & x \in \partial\Omega,\ t > 0, \\[2mm] u_i(x,0) = u_{i0}(x),\ i = 1,2,3, & x \in \Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$ ($N \geq 1$ is an integer) with smooth boundary $\partial\Omega$, $\partial/\partial\nu$ is the outward normal derivative on $\partial\Omega$. The homogeneous Neumann boundary $\frac{\partial u_i}{\partial \nu}\big|_{\partial\Omega\times(0,\infty)} = 0$ means there is no migration of all species across the boundary of their habitat. While the homogeneous Dirichlet boundary condition $u_i|_{\partial\Omega\times(0,\infty)} = 0$ can be considered as such a condition under which neither of the three species can exist on the boundary. The positive constants d_1, d_2 and d_3 are called diffusion coefficients, the initial data u_{i0} are nonnegative continuous functions. Zheng discussed the stability of nonnegative constant solutions of (1.2) with Neumann boundary and the existence and stabilities of trivial and nontrivial nonnegative equilibrium solutions with Dirichlet boundary.

The steady-states of (1.2) with Dirichlet boundary were studied by Chen and Wang in [5], and Hei in [12], and the conditions for the existence of coexistence states and the corresponding parameter regions were established by. While the steady-states of (1.2) with Neumann boundary were investigated by Cheng and Wang in [4], and Xu in [22], and the global stability of the unique positive constant steady-state and the existence and non-existence of non-constant positive steady-state were established.

Taking into account the inter-specific population pressure between two competitors, Chen and Peng [3] introduced a cross-diffusion into (1.2) and considered the following elliptic system after scaling:

$$(1.3) \quad \begin{cases} - d_1 \Delta u_1 = u_1 \left(1 - u_1 - \dfrac{\sigma u_2}{1 + u_3} \right), & x \in \Omega, \\[2mm] - d_2 \Delta [(1 + d_4 u_1) u_2] = u_2 (1 - u_2 - u_1), & x \in \Omega, \\[2mm] - d_3 \Delta u_3 = u_3 \left(1 - \dfrac{u_3}{1 + u_1} \right), & x \in \Omega, \\[2mm] \dfrac{\partial u_i}{\partial \nu}(x) = 0,\ i = 1,2,3, & x \in \partial\Omega. \end{cases}$$

System (1.3) is a quasi-linear elliptic system and the term $d_4 u_1 u_2$ is called the cross-diffusion term. The existence and non-existence results concerning non-constant positive steady-states were proved in that paper.

Based on the above consideration, a natural question is how about the stationary patterns of (1.2) with cross-diffusion and Dirichlet boundary. In this paper, we will investigate this problem. After scaling, we consider the following problem:

(1.4)
$$
\begin{cases}
-\Delta u = u\left(a - u - \dfrac{rv}{1 + mw}\right), & x \in \Omega, \\[2mm]
-\Delta[(1 + \kappa u)v] = v(b - v + du), & x \in \Omega, \\[2mm]
-\Delta w = w\left(c - \dfrac{w}{1 + nu}\right), & x \in \Omega, \\[2mm]
u = v = w = 0, & x \in \partial\Omega,
\end{cases}
$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $N \geq 1$ is an integer. The parameters a, c, d, r, m, n are positive, $b \in \mathbb{R}$, and κ is nonnegative.

Analysis of the existence and multiplicity of positive solutions to (1.4) is the main goal of this paper. Positive solutions here refer to those solutions being component-wise strictly positive in Ω. Our method is based on the bifurcation theory and the Leray–Schauder degree theory.

The rest of this paper is organized as follows. In Section 2, we give some preliminary results that are needed in later discussions. In Section 3, we obtain sufficient conditions for the existence of positive solutions to (1.4). In Section 4, we give some multiplicity results about the positive solutions to (1.4).

2. Preliminaries

In this section we list some notation, definitions and well-known facts which will be used in the sequel. We use $\|\cdot\|_X$ as the norm of Banach space X, $\langle\,\cdot\,,\,\cdot\,\rangle$ as the duality pair of a Banach space X and its dual space X^*. For a linear operator L, we use $\mathcal{N}(L)$ as the null space of L and $\mathcal{R}(L)$ as the range space of L, and we use $L[w]$ to denote the image of w under the linear mapping L. For a multilinear operator L, we use $L[w_1, \ldots, w_k]$ to denote the image of $(w_1, \ldots, w_k)$ under L, and when $w_1 = \ldots = w_k$, we use $L[w_1]^k$ instead of $L[w_1, \ldots, w_1]$. For a nonlinear operator F, we use F_u as the partial derivative of F with respect to the argument u.

First we recall some well-known abstract bifurcation theorems. Consider an abstract equation

$$ F(\lambda, u) = 0, $$

where $F\colon \mathbb{R} \times X \to Y$ is a nonlinear differential mapping, and X, Y are Banach spaces such that X is continuously embedded in Y. The following bifurcation and stability theorems were obtained in [6], [7], [16] (see also [19], [20]).

THEOREM 2.1. *Let U be a neighbourhood of (λ_0, u_0) in $\mathbb{R} \times X$, and $F\colon U \to Y$ be a twice continuously differentiable mapping. Assume that $F(\lambda, u_0) = 0$ for all $(\lambda, u_0) \in U$. At (λ_0, u_0), F satisfies*

$$\operatorname{dim}\mathcal{N}(F_u(\lambda_0, u_0)) = \operatorname{codim}\mathcal{R}(F_u(\lambda_0, u_0)) = 1$$

and

$$F_{\lambda u}(\lambda_0, u_0)[w_0] \notin \mathcal{R}(F_u(\lambda_0, u_0)).$$

Here $\mathcal{N}(F_u(\lambda_0, u_0)) = \operatorname{span}\{w_0\}$. Let Z be the complement of $\operatorname{span}\{w_0\}$ in X. Then the solution set of $F(\lambda, u) = 0$ near (λ_0, u_0) consists precisely of the curves $u = u_0$ and $\Gamma := \{(\lambda(s), u(s)) : s \in I = (-\varepsilon, \varepsilon)\}$, where $\lambda\colon I \to \mathbb{R}$, $z\colon I \to Z$ are C^1 functions such that $u(s) = u_0 + sw_0 + sz(s)$, $\lambda(0) = \lambda_0$, $z(0) = 0$, and

$$\lambda'(0) = -\frac{\langle \ell, F_{uu}(\lambda_0, u_0)[w_0, w_0]\rangle}{2\langle \ell, F_{\lambda u}(\lambda_0, u_0)[w_0]\rangle},$$

where $\ell \in Y^*$ satisfies $\mathcal{R}(F_u(\lambda_0, u_0)) = \{\phi \in Y : \langle \ell, \phi\rangle = 0\}$. Moreover, if in addition, $F_u(\lambda, u)$ is a Fredholm operator for all $(\lambda, u) \in U$, then the bifurcation curve Γ is contained in Σ, which is a connected component of $\overline{S}$, where $\overline{S} := \{(\lambda, u) \in U : F(\lambda, u) = 0, \ u \neq u_0\}$; and either Σ is not compact in U, or Σ contains a point (λ_*, u_0) with $\lambda_* \neq \lambda_0$.

THEOREM 2.2. *Assume that all assumptions in Theorem 2.1 are satisfied, and let $\{\lambda(t), u(t)\}$ be the solution curve in Theorem 2.1. Then there exist C^2 functions $m\colon (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon) \to \mathbb{R}$, $z\colon (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon) \to X$, $\mu\colon (-\delta, \delta) \to \mathbb{R}$, and $w\colon (-\delta, \delta) \to X$ such that*

$$F_u(\lambda, u_0)z(\lambda) = m(\lambda)z(\lambda), \quad \lambda \in (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon),$$

$$F_u(\lambda(t), u(t))w(t) = \mu(t)w(t), \quad t \in (-\delta, \delta),$$

where $m(\lambda_0) = \mu(0) = 0$, $z(\lambda_0) = w(0) = w_0$. Moreover, near $t = 0$ the functions $\mu(t)$ and $-t\lambda'(t)m'(\lambda_0)$ have the same zeros and, whenever $\mu(t) \neq 0$, the same sign. More precisely,

$$\lim_{t \to 0} \frac{-t\lambda'(t)m'(\lambda_0)}{\mu(t)} = 1.$$

For each $q(x) \in C(\overline{\Omega})$, the eigenvalues of

$$-\Delta u + qu = \lambda u, \quad x \in \Omega, \qquad u = 0, \quad x \in \partial\Omega,$$

are denoted by $\lambda_i(q)$, $i = 1, 2, \ldots$ Then $\lambda_1(q) < \lambda_2(q) \leq \ldots$ and $\lim_{i \to \infty} \lambda_i(q) = \infty$. It is well known that $\lambda_i(q)$ is strictly increasing in the sense that $q_1(x) \leq q_2(x)$ and $q_1(x) \not\equiv q_2(x)$ implies $\lambda_i(q_1) < \lambda_i(q_2)$. Moreover, the eigenfunction corresponding to $\lambda_1(q)$ can be chosen positive in Ω. We denote $\lambda_i(0)$ by λ_i and let

$\phi_1(x)$ be the eigenfunction corresponding to λ_1 with normalization $\|\phi_1\|_{L^2(\Omega)} = 1$ (see [23, Proposition 1.1]).

Now we give two results for comparison of eigenvalues (see [1], [9], [14], [21]).

THEOREM 2.3. *Let $q \in C(\overline{\Omega})$ and M be a sufficiently large number such that $M > q(x)$ for all $x \in \overline{\Omega}$, define a positive and compact operator*

$$\mathfrak{L} := (-\Delta + M)^{-1}(M - q(x)) \colon C_0(\overline{\Omega}) \to C_0(\overline{\Omega}),$$

where $C_0(\overline{\Omega}) \colon \{\varphi \in C(\overline{\Omega}) : \varphi = 0 \text{ on } \partial\Omega\}$. Denote the spectral radius of $\mathfrak{L}$ by $r(\mathfrak{L})$. Then we have:

(a) $\lambda_1(q) > 0$ *if and only if $r(\mathfrak{L}) < 1$;*
(b) $\lambda_1(q) < 0$ *if and only if $r(\mathfrak{L}) > 1$;*
(c) $\lambda_1(q) = 0$ *if and only if $r(\mathfrak{L}) = 1$.*

THEOREM 2.4. *Let $q(x) \in C(\overline{\Omega})$ and $\phi \in C(\overline{\Omega}) \cap C^2(\Omega)$ such that $\phi \geq 0$, $\phi \not\equiv 0$ in Ω, and $\phi = 0$ on $\partial\Omega$. Then:*

(a) $\lambda_1(q) < 0$ *if $0 \not\equiv -\Delta\phi + q(x)\phi \leq 0$;*
(b) $\lambda_1(q) = 0$ *if $-\Delta\phi + q(x)\phi \equiv 0$;*
(c) $\lambda_1(q) > 0$ *if $0 \not\equiv -\Delta\phi + q(x)\phi \geq 0$.*

Consider the following single equation:

$$(2.1) \qquad -\Delta u = uf(x, u), \quad x \in \Omega, \qquad u = 0, \quad x \in \partial\Omega,$$

where $f \colon \overline{\Omega} \times [0, \infty) \mapsto \mathbb{R}$ satisfies:

(1) $f(x, u)$ is a C^α-function in x, where $0 < \alpha < 1$;
(2) $f(x, u)$ is a C^1-function in u with $f_u(x, u) < 0$ for all $(x, u) \in \overline{\Omega} \times [0, \infty)$;
(3) $f(x, u) \leq 0$ in $\overline{\Omega} \times [C, \infty)$ for some positive constant C.

THEOREM 2.5 (see [2], [15]). *Under assumptions (1)–(3). The following conclusions hold:*

(a) *The nonnegative solution u of (2.1) satisfies $u(x) \leq C$ for all $x \in \overline{\Omega}$.*
(b) *If $\lambda_1(-f(x, 0)) \geq 0$, then (2.1) has no positive solutions. Moreover, the trivial solution $u = 0$ is globally asymptotically stable.*
(c) *If $\lambda_1(-f(x, 0)) < 0$, then (2.1) has a unique positive solution which is globally asymptotically stable. In this case, the trivial solution $u = 0$ is unstable.*

Due to the above theorem, we denote by θ_a the unique positive solution of the following problem:

$$-\Delta u = u(a - u), \quad x \in \Omega, \qquad u = 0, \quad x \in \partial\Omega,$$

under the assumption $a > \lambda_1$. It is well known that the mapping $a \mapsto \theta_a$ is strictly increasing, continuously differentiable on (λ_1, ∞), and $\theta_a \to 0$ uniformly on $\overline{\Omega}$ as $a \to \lambda_1$. Moreover, $0 < \theta_a < a$ in Ω.

Now, let us recall a result of the fixed point index theory, which is a fundamental tool in our proofs.

Let E be a Banach space and $\mathcal{W} \subset E$ be a closed convex set. The set $\mathcal{W}$ is called a total wedge if $\gamma \mathcal{W} \subset \mathcal{W}$ for all $\gamma \geq 0$ and $\overline{\mathcal{W} - \mathcal{W}} = E$. For $y \in \mathcal{W}$, define $\mathcal{W}_y = \{x \in E : y + \gamma x \in \mathcal{W} \text{ for some } \gamma > 0\}$ and $S_y = \{x \in \overline{\mathcal{W}}_y : -x \in \overline{\mathcal{W}}_y\}$. Then $\overline{\mathcal{W}}_y$ is a wedge containing $\mathcal{W}, y, -y$, while S_y is a closed subset of E containing y. Let T be a compact linear operator on E which satisfies $T(\overline{\mathcal{W}}_y) \subset \overline{\mathcal{W}}_y$. We say that T has property $\mathfrak{a}$ on $\overline{\mathcal{W}}_y$ if there are $t \in (0,1)$ and $\omega \in \overline{\mathcal{W}}_y \setminus S_y$ such that $(I - tT)\omega \in S_y$. Let $A \colon \mathcal{W} \to \mathcal{W}$ be a compact operator with a fixed point $y \in \mathcal{W}$ and A be Fréchet differentiable at y. Let $\mathcal{B} = A'(y)$ be the Fréchet derivative of A at y. Then $\mathcal{B}$ maps $\overline{\mathcal{W}}_y$ into itself. We denote by $\deg_{\mathcal{W}}(I - A, D)$ the degree of $I - A$ in D relative to $\mathcal{W}$, by $\mathrm{index}_{\mathcal{W}}(A, y)$ the fixed point index of A at y relative to $\mathcal{W}$.

THEOREM 2.6 (see [8], [14], [18]). *Assume that* $I - \mathcal{B}$ *has no non-trivial kernel in* $\overline{\mathcal{W}}_y$. *Then*:

(a) *If* $\mathcal{B}$ *has property* $\mathfrak{a}$ *on* $\overline{\mathcal{W}}_y$, *then* $\mathrm{index}_{\mathcal{W}}(A, y) = 0$.
(b) *If* $\mathcal{B}$ *does not have property* $\mathfrak{a}$ *on* $\overline{\mathcal{W}}_y$, *then* $\mathrm{index}_{\mathcal{W}}(A, y) = (-1)^{\sigma}$, *where* σ *is the sum of multiplicities of all eigenvalues of* $\mathcal{B}$ *which is greater than* 1.

Finally, we recall a result of Dancer and Du in [10]. Suppose E_1 and E_2 are ordered Banach spaces with positive cones $\mathcal{W}_1$ and $\mathcal{W}_2$, respectively. Let $E = E_1 \oplus E_2$ and $\mathcal{W} = \mathcal{W}_1 \oplus \mathcal{W}_2$. Then E is an ordered Banach space with positive cone $\mathcal{W}$. Let D be an open set in $\mathcal{W}$ containing 0 and $A_i \colon \overline{D} \to \mathcal{W}_i$ be completely continuous operators, $i = 1, 2$. Denote by (u, v) a general element in $\mathcal{W}$ with $u \in \mathcal{W}_1$ and $v \in \mathcal{W}_2$. Let $A \colon \overline{D} \to \mathcal{W}$ be defined by $A(u, v) = (A_1(u, v), A_2(u, v))$. Also we define $\mathcal{W}_2(\varepsilon) = \{v \in \mathcal{W}_2 : \|v\|_{E_2} < \varepsilon\}$.

THEOREM 2.7. *Suppose* $\widetilde{U} \subset \mathcal{W}_1 \cap D$ *is relatively open and bounded, and* $A_1(u, 0) \neq u$ *for* $u \in \partial \widetilde{U}$, $A_2(u, 0) \equiv 0$ *for* $u \in \overline{\widetilde{U}}$. *Suppose* $A_2 \colon D \to \mathcal{W}_2$ *extends to a continuously differentiable mapping of a neighbourhood of* D *into* E_2, $\mathcal{W}_2 - \mathcal{W}_2$ *is dense in* E_2 *and* $\Phi = \{u \in U : u = A_1(u, 0)\}$. *Then the following conclusions are true*:

(a) $\deg_{\mathcal{W}}(I - A, \widetilde{U} \times \mathcal{W}_2(\varepsilon), 0) = 0$ *for* $\varepsilon > 0$ *small, if for any* $u \in \Phi$, *the spectral radius* $r(A_2'(u, 0)|_{\mathcal{W}_2}) > 1$ *and* 1 *is not an eigenvalue of* $A_2'(u, 0)|_{\mathcal{W}_2}$ *corresponding to a positive eigenvector*;
(b) $\deg_{\mathcal{W}}(I - A, \widetilde{U} \times \mathcal{W}_2(\varepsilon), 0) = \deg_{\mathcal{W}_1}(I - A_1|_{\mathcal{W}_1}, \widetilde{U}, 0)$ *for* $\varepsilon > 0$ *small, if for any* $u \in \Phi$, *the spectral radius* $r(A_2'(u, 0)|_{\mathcal{W}_2}) < 1$.

3. Existence of positive solutions

In this section, we will derive some sufficient conditions for the existence of positive solutions to (1.4). For (1.4), only nonnegative solutions are of practical interest. Obviously, (1.4) has a trivial nonnegative solution $(0, 0, 0)$. As in [10], [11], the other nonnegative solutions of (1.4) can be classified by three types:

(1) nonnegative solutions with exactly two components identically zero;
(2) nonnegative solutions with exactly one component identically zero;
(3) nonnegative solutions with no component identically zero.

We call solutions of types 1–3 weakly semi-trivial solutions, strongly semi-trivial solutions and positive solutions, respectively. It is obvious that (1.4) has weakly semi-trivial solutions $(\theta_a, 0, 0)$ if and only if $a > \lambda_1$, $(0, \theta_b, 0)$ if and only if $b > \lambda_1$, $(0, 0, \theta_c)$ if and only if $c > \lambda_1$.

Next, we analyze the strong semi-trivial solutions of (1.4). When $u = 0$, (1.4) has a strongly semi-trivial solution $(0, \theta_b, \theta_c)$ if and only if $b, c > \lambda_1$.

When $v = 0$, by virtue of Theorem 2.5, (1.4) has a strong semi-trivial solution $(\theta_a, 0, \theta_{(a,c)})$ if and only if $a, c > \lambda_1$, where $\theta_{(a,c)}$ is the unique positive solution of the problem

$$(3.1) \qquad -\Delta w = w\left(c - \frac{w}{1 + n\theta_a}\right), \quad x \in \Omega, \qquad w = 0, \quad x \in \partial\Omega.$$

If $w = 0$, (1.4) degenerates to

$$(3.2) \qquad \begin{cases} -\Delta u = u(a - u - rv), & x \in \Omega, \\ -\Delta[(1 + \kappa u)v] = v(b - v + du), & x \in \Omega, \\ u = v = 0, & x \in \partial\Omega. \end{cases}$$

Then it follows from [23, Theorem 3.1] that (3.2) admits one positive solution if one of the following conditions holds true:

(C$_1$) $\lambda_1\left(-\dfrac{b + d\theta_a}{1 + \kappa\theta_a}\right) < 0$ and $a > \begin{cases} \lambda_1(r\theta_b), & \text{if } b > \lambda_1; \\ \lambda_1, & \text{if } b \le \lambda_1. \end{cases}$

(C$_2$) $\lambda_1\left(-\dfrac{b + d\theta_a}{1 + \kappa\theta_a}\right) > 0$, $b > \lambda_1$, and $\lambda_1 < a < \lambda_1(r\theta_b)$.

So, (1.4) has a strongly semi-trivial solution $(\overline{u}, \overline{v}, 0)$ if (C$_1$) or (C$_2$) holds, where $(\overline{u}, \overline{v})$ is a positive solution of (3.2).

By using the transformation $z = (1 + \kappa u)v$, i.e.

$$(3.3) \qquad\qquad\qquad v = z/(1 + \kappa u),$$

(1.4) can be rewritten as follows:

$$(3.4) \quad \begin{cases} -\Delta u = F(u,z,w) := u\left(a - u - \dfrac{rz}{(1+\kappa u)(1+mw)}\right), & x \in \Omega, \\[2mm] -\Delta z = G(u,z,w) := \dfrac{z}{1+\kappa u}\left(b + du - \dfrac{z}{1+\kappa u}\right), & x \in \Omega, \\[2mm] -\Delta w = H(u,z,w) := w\left(c - \dfrac{w}{1+nu}\right), & x \in \Omega, \\[2mm] u = z = w = 0, & x \in \partial\Omega. \end{cases}$$

It is obvious that (1.4) has a positive solution if and only if (3.4) has a positive solution. So, we will study the existence of positive solutions of (1.4) through (3.4). Let (u,z,w) be a positive solution of (3.4), then $u < a$ and

$$(3.5) \quad \begin{cases} -\Delta u < au, & x \in \Omega, \\[2mm] -\Delta z < \dfrac{b+du}{1+\kappa u}z \le \begin{cases} \dfrac{b+ad}{1+a\kappa}z, & d > \kappa b; \\[2mm] bz, & d \le \kappa b, \end{cases} & x \in \Omega, \\[2mm] -\Delta w < cw, & x \in \Omega, \\[2mm] u = z = w = 0, & x \in \partial\Omega, \end{cases}$$

and it follows from the Krein–Rutman theorem that

$$(H) \qquad a, c > \lambda_1, \quad b > b_* := \begin{cases} \lambda_1(1 + a\kappa) - ad, & d > \kappa b; \\ \lambda_1, & d \le \kappa b. \end{cases}$$

Since we are interested in positive solutions, we assume that (H) holds throughout this section.

By the maximum principle, one can get that any nonnegative solution (u, z, w) of (3.4) satisfies

$$(3.6) \quad u(x) < a, \ z(x) < M_1 := (b+ad)(1+\kappa a), \ w(x) < M_2 := c(1+na), \ x \in \overline{\Omega}.$$

In order to use the functional analytic framework of the degree theory, we introduce

- $E := C_0(\overline{\Omega}) \times C_0(\overline{\Omega}) \times C_0(\overline{\Omega})$, where $C_0(\overline{\Omega}) := \{\varphi \in C(\overline{\Omega}) : \varphi = 0 \text{ on } \partial\Omega\}$;
- $\mathcal{W} := K \times K \times K$, where $K := \{\varphi \in C_0(\overline{\Omega}) : \varphi \ge 0 \text{ on } \overline{\Omega}\}$;
- $D := \{(u, z, w) \in \mathcal{W} : u < a, \ z < M_1, \ w < M_2 \text{ on } \overline{\Omega}\}$, where M_1 and M_2 are defined in (3.6).

Then $\mathcal{W}$ is a cone of E and D is a bounded open set in $\mathcal{W}$. For any $\tau \in [0,1]$, we define $\mathcal{A}_\tau \colon \overline{D} \mapsto \mathcal{W}$ by

$$\mathcal{A}_\tau(u, z, w) = (-\Delta + M)^{-1} \begin{pmatrix} \tau F + Mu \\ \tau G + Mz \\ \tau H + Mw \end{pmatrix},$$

where M is a sufficiently large constant with $M > \max\{rM_1, M_1, M_2\}$.

It follows from the standard elliptic regularity theory that $\mathcal{A}_\tau$ is a completely continuous operator. Moreover, $\mathcal{A}_\tau$ has no fixed point on ∂D. Let $\mathcal{A} = \mathcal{A}_1$. Then (u, z, w) is a solution of (3.4) in $\mathcal{W}$ if and only if it is a fixed point of $\mathcal{A}$ in D.

Now we start to calculate the indices of the trivial and semi-trivial fixed points of $\mathcal{A}$. It follows from the relationship between v and z that

(1) $\mathcal{A}$ has a trivial fixed point $(0, 0, 0)$;

(2) $\mathcal{A}$ has a weak semi-trivial fixed point $(\theta_a, 0, 0)$ if and only if $a > \lambda_1$;

(3) $\mathcal{A}$ has a weak semi-trivial fixed point $(0, \theta_b, 0)$ if and only if $b > \lambda_1$;

(4) $\mathcal{A}$ has a weak semi-trivial fixed point $(0, 0, \theta_c)$ if and only if $c > \lambda_1$;

(5) $\mathcal{A}$ has a strong semi-trivial fixed point $(0, \theta_b, \theta_c)$ if and only if $b, c > \lambda_1$;

(6) $\mathcal{A}$ has a strong semi-trivial fixed point $(\theta_a, 0, \theta_{(a,c)})$ if and only if $a, c > \lambda_1$, where $\theta_{(a,c)}$ is defined in (3.1);

(7) $\mathcal{A}$ has a strong semi-trivial fixed point $(\overline{u}, (1 + \kappa\overline{u})\overline{v}, 0)$ if (C_1) or (C_2) holds, where $(\overline{u}, \overline{v})$ is a positive solution of (3.2).

LEMMA 3.1. *Assume $a, c > \lambda_1$ and $b \neq \lambda_1$. Then:*

(a) $\deg_\mathcal{W}(I - \mathcal{A}, D) = 1$,

(b) $\text{index}_\mathcal{W}(\mathcal{A}, (0, 0, 0)) = 0$.

PROOF. By homotopic invariance of the degree and as $(0, 0, 0)$ is the only fixed point of A_0 in D, we obtain

$$\deg_\mathcal{W}(I - \mathcal{A}, D) = \deg_\mathcal{W}(I - \mathcal{A}_0, D) = \text{index}_\mathcal{W}(A_0, (0, 0, 0)).$$

It is obvious that $\text{index}_\mathcal{W}(A_0, (0, 0, 0)) = 1$ and the first conclusion holds.

For the second result. We let $y = (0, 0, 0)$ and $\mathcal{B} = \mathcal{A}'(0, 0, 0)$, then $\overline{\mathcal{W}}_y = \overline{\mathcal{W}}$, $S_y = \{(0, 0, 0)\}$, and

$$\mathcal{B} = (-\Delta + M)^{-1} \begin{pmatrix} M + a & 0 & 0 \\ 0 & M + b & 0 \\ 0 & 0 & M + c \end{pmatrix}.$$

Since $a, c > \lambda_1$ and $b \neq \lambda_1$, it is easy to see that $I - \mathcal{B}$ has no non-trivial kernel in $\overline{\mathcal{W}}_y$, and $r_a := r((-\Delta + M)^{-1}(M + a)) > 1$, by Theorems 2.3 and 2.4. Furthermore, r_a is the principal eigenvalue of the operator $(-\Delta + M)^{-1}(M + a)$ with a corresponding eigenfunction $\phi_a > 0$ and $\phi_a|_{\partial\Omega} = 0$. Set $t_a = 1/r_a \in (0, 1)$, then $(\phi_a, 0, 0) \in \overline{\mathcal{W}}_y \setminus S_y$, but $(I - t_a\mathcal{B})(\phi_a, 0, 0) = (0, 0, 0) \in S_y$. This shows that $\mathcal{B}$ has property $\mathfrak{a}$, and the second conclusion follows from Theorem 2.6. $\square$

LEMMA 3.2. *Assume $a, c > \lambda_1$. Then we have $\text{index}_\mathcal{W}(\mathcal{A}, (\theta_a, 0, 0)) = 0$ if $\lambda_1\left(-\frac{b + d\theta_a}{1 + \kappa\theta_a}\right) \neq 0$; $\text{index}_\mathcal{W}(\mathcal{A}, (0, \theta_b, 0)) = 0$ if $b > \lambda_1$ and $a \neq \lambda_1(r\theta_b)$; $\text{index}_\mathcal{W}(\mathcal{A}, (0, 0, \theta_c)) = 0$ if $b \neq \lambda_1$.*

PROOF. We only prove that $\text{index}_\mathcal{W}(\mathcal{A}, (\theta_a, 0, 0)) = 0$ since the proofs of the other two are similar. Let $y = (\theta_a, 0, 0)$. Then we have $\overline{\mathcal{W}}_y = C_0(\overline{\Omega}) \times K \times K$,

$S_y = C_0(\overline{\Omega}) \times \{0\} \times \{0\}$. Let $\mathcal{B} = \mathcal{A}'(\theta_a, 0, 0)$, then

$$\mathcal{B} = (-\Delta + M)^{-1} \begin{pmatrix} M + a - 2\theta_a & -\dfrac{r\theta_a}{1 + \kappa\theta_a} & 0 \\ 0 & M + \dfrac{b + d\theta_a}{1 + \kappa\theta_a} & 0 \\ 0 & 0 & M + c \end{pmatrix}.$$

Let $\mathcal{B}(\xi, \eta, \zeta) = (\xi, \eta, \zeta)$ for some $(\xi, \eta, \zeta) \in \overline{\mathcal{W}}_y$. Since $b \neq g(a)$ and $c > \lambda_1$, $\eta = \zeta = 0$, then $\xi = 0$ as $\lambda_1(2\theta_a - a) > \lambda_1(\theta_a - a) = 0$. It follows from Theorems 2.3 and 2.4 that $r_c = (-\Delta + M)^{-1}(M + c) > 1$. Then we can show that $\mathcal{B}$ has property $\mathfrak{a}$ by a similar argument as in the proof of Lemma 3.1, and the conclusion follows. $\qquad\square$

Lemma 3.3. *The following conclusions hold true:*

(a) *Assume $b, c > \lambda_1$, then*
  (i) $\text{index}_{\mathcal{W}}(\mathcal{A}, (0, \theta_b, \theta_c)) = 1$ *if $a < \lambda_1(r\theta_b/(1 + m\theta_c))$;*
  (ii) $\text{index}_{\mathcal{W}}(\mathcal{A}, (0, \theta_b, \theta_c)) = 0$ *if $a > \lambda_1(r\theta_b/(1 + m\theta_c))$.*

(b) *Assume $a, c > \lambda_1$, then*
  (i) $\text{index}_{\mathcal{W}}(\mathcal{A}, (\theta_a, 0, \theta_{(a,c)})) = 1$ *if $\lambda_1(-(b + d\theta_a)/(1 + \kappa\theta_a)) > 0$;*
  (ii) $\text{index}_{\mathcal{W}}(\mathcal{A}, (\theta_a, 0, \theta_{(a,c)})) = 0$ *if $\lambda_1(-(b + d\theta_a)/(1 + \kappa\theta_a)) < 0$.*

(c) *Assume $c > \lambda_1$, then $\text{index}_{\mathcal{W}}(\mathcal{A}, S) = 0$, where $S = \emptyset$ or $S = \{(\overline{u}, (1 + \kappa\overline{u})\overline{v}, 0)\}$ with $(\overline{u}, \overline{v})$ a positive solution of (3.2).*

Proof. (a) Let $y = (0, \theta_b, \theta_c)$. Then we have $\overline{\mathcal{W}}_y = K \times C_0(\overline{\Omega}) \times C_0(\overline{\Omega})$, $S_y = \{0\} \times C_0(\overline{\Omega}) \times C_0(\overline{\Omega})$. Let $\mathcal{B} = \mathcal{A}'(0, \theta_b, \theta_c)$, then

$$\mathcal{B} = (-\Delta + M)^{-1} \begin{pmatrix} M + a - \dfrac{r\theta_b}{1 + m\theta_c} & 0 & 0 \\ (d - \kappa b)\theta_b 2\kappa\theta_b^2 & M + b - 2\theta_b & 0 \\ n\theta_c^2 & 0 & M + c - 2\theta_c \end{pmatrix}.$$

Let $\mathcal{B}(\xi, \eta, \zeta) = (\xi, \eta, \zeta)$ for some $(\xi, \eta, \zeta) \in \overline{\mathcal{W}}_y$. Since $a \neq \lambda_1(r\theta_b/(1 + m\theta_c))$ and $\xi \in K$, $\xi = 0$. Then $\eta = \zeta = 0$ as $\lambda_1(2\theta_b - b) > \lambda_1(\theta_b - b) = 0$ and $\lambda_1(2\theta_c - c) > \lambda_1(\theta_c - c) = 0$.

Firstly, we prove conclusion (i). We claim that $\mathcal{B}$ does not have property $\mathfrak{a}$. On the contrary, suppose that $\mathcal{B}$ has property $\mathfrak{a}$. Then there exist $t \in (0, 1)$ and $(\phi_1, \phi_2, \phi_3) \in \overline{\mathcal{W}}_y \setminus S_y$ such that $(I - tB)(\phi_1, \phi_2, \phi_3)^T \in S_y$. Since $S_y = \{0\} \times C_0(\overline{\Omega}) \times C_0(\overline{\Omega})$, $\phi_1 \neq 0$ must hold. Then

$$r\left((-\Delta + M)^{-1}\left(M + a - \dfrac{r\theta_b}{1 + m\theta_c}\right)\right) \geq \dfrac{1}{t} > 1.$$

Then it follows from Theorem 2.3 that $\lambda_1\left(-a + \frac{r\theta_b}{1+m\theta_c}\right) < 0$, i.e. $a > \lambda_1\left(\frac{r\theta_b}{1+m\theta_c}\right)$, which is a contradiction. So the claim is correct. By Theorem 2.6, we have

$$\text{index}_{\mathcal{W}}(\mathcal{A}, (\theta_a, 0, 0)) = (-1)^{\sigma},$$

where σ is the sum of multiplicities of all eigenvalues of $\mathcal{B}$ that are greater than 1. Next, we will show $\sigma = 0$. To this end, suppose that $1/\varrho > 1$ is an eigenvalue of $\mathcal{B}$ with the corresponding eigenfunction $(\xi, \eta, \zeta) \in \overline{\mathcal{W}}_y$, i.e.,

$$(3.7) \quad \begin{cases} -\Delta\xi + \left(M - \varrho\left(M + a - \frac{r\theta_b}{1+m\theta_c}\right)\right)\xi = 0, & x \in \Omega, \\ -\Delta\eta + (M - \varrho(M + b - 2\theta_b))\eta = \varrho(d - \kappa b)\theta_b 2\kappa\theta_b^2\xi, & x \in \Omega, \\ -\Delta\zeta + (M - \varrho(M + c - 2\theta_c))\zeta = n\theta_c^2\xi, & x \in \Omega, \\ \xi = \eta = \zeta = 0, & x \in \partial\Omega. \end{cases}$$

Since $0 < \varrho < 1$,

$$\lambda_1\left(M - \varrho\left(M + a - \frac{r\theta_b}{1+m\theta_c}\right)\right) > -a + \lambda_1\left(\frac{r\theta_b}{1+m\theta_c}\right) > 0.$$

We have $\xi = 0$, and then $\eta = \zeta = 0$, what contradicts the fact that (ξ, η, ζ) is an eigenvalue. Consequently, $\sigma = 0$ and conclusion (i) holds.

Secondly, we prove conclusion (ii). It follows from Theorems 2.3 and 2.4 that

$$r_a = r\left((\Delta + M)^{-1}\left(M + a - \frac{r\theta_b}{1+m\theta_c}\right)\right) > 1.$$

Then we can show that $\mathcal{B}$ has property $\mathfrak{a}$ by a similar argument as in the proof of Lemma 3.1, and the conclusion follows.

The proof of (b) is similar to the proof of (a), and we omit it. Now we give the proof of (c). It is obvious that $\text{index}_{\mathcal{W}}(\mathcal{A}, S) = 0$ if $S = \emptyset$. So we assume that $S \neq \emptyset$. Let $E_1 = C_0(\overline{\Omega}) \times C_0(\overline{\Omega})$, $E_2 = C_0(\overline{\Omega})$, $\mathcal{W}_1 = K \times K$, $\mathcal{W}_2 = K$. Define

$$\mathcal{A}_1(u, z, w) = (-\Delta + M)^{-1}\begin{pmatrix} F + Mu \\ G + Mz \end{pmatrix},$$

$$\mathcal{A}_2(u, z, w) = (-\Delta + M)^{-1}(H + Mz),$$

then $\mathcal{A} = (\mathcal{A}_1, \mathcal{A}_2)$. A direct calculation shows that $\mathcal{A}_2'(\overline{u}, (1 + \kappa\overline{u})\overline{v}, 0)|_{\mathcal{W}_2} = (-\Delta + M)^{-1}(M + c)$. Since $c > \lambda_1$, it follows from Theorems 2.3 and 2.4 that $r(\mathcal{A}_2'(\overline{u}, (1 + \kappa\overline{u})\overline{v}, 0)|_{\mathcal{W}_2}) > 1$, and the conclusion follows from Theorem 2.7. $\square$

Now we establish the existence of positive solution to (1.4).

THEOREM 3.4. *Assume $c > \lambda_1$. Then (1.4) (or equivalently (3.4)) admits at least one positive solution if:*

(a) $\lambda_1\left(-\frac{b + d\theta_a}{1 + \kappa\theta_a}\right) < 0$ *and* $a > \begin{cases} \lambda_1\left(\frac{r\theta_b}{1+m\theta_c}\right), & \text{if } b > \lambda_1; \\ \lambda_1, & \text{if } b \leq \lambda_1, \end{cases}$ *or*

(b) $\lambda_1\left(-\dfrac{b+d\theta_a}{1+\kappa\theta_a}\right) > 0,\ \lambda_1 < a < \lambda_1\left(\dfrac{r\theta_b}{1+m\theta_c}\right),\ and\ b > \lambda_1.$

PROOF. First we consider the case $\lambda_1\left(-\frac{b+d\theta_a}{1+\kappa\theta_a}\right) < 0$, $a > \lambda_1\left(\frac{r\theta_b}{1+m\theta_c}\right)$ and $b > \lambda_1$. Assume, on the contrary, that (3.4) has no positive solution. Then it follows from the properties of degree that

$$
(3.8) \qquad \deg_{\mathcal{W}}(I - \mathcal{A}, D) = \mathrm{index}_{\mathcal{W}}(A, (0,0,0))
$$
$$
+ \mathrm{index}_{\mathcal{W}}(A, (\theta_a, 0, 0)) + \mathrm{index}_{\mathcal{W}}(A, (0, \theta_b, 0))
$$
$$
+ \mathrm{index}_{\mathcal{W}}(A, (0, 0, \theta_c)) + \mathrm{index}_{\mathcal{W}}(A, (0, \theta_b, \theta_c))
$$
$$
+ \mathrm{index}_{\mathcal{W}}(A, (\theta_a, 0, \theta_{(a,c)})) + \mathrm{index}_{\mathcal{W}}(A, S).
$$

By Lemmas 3.1–3.3, the left hand side of (3.8) is equal to one, while the right hand side of (3.8) is equal to zero. This is a contradiction, and thus (3.4) has a positive solution.

Similarly, we can prove that (3.4) has a positive solution if $\lambda_1\left(-\frac{b+d\theta_a}{1+\kappa\theta_a}\right) < 0$, $a > \lambda_1$ and $b < \lambda_1$ or $\lambda_1(-\frac{b+d\theta_a}{1+\kappa\theta_a}) > 0$, $\lambda_1 < a < \lambda_1\left(\frac{r\theta_b}{1+m\theta_c}\right)$, and $b > \lambda_1$.

Now we prove that (3.4) has a positive solution if $\lambda_1\left(-\frac{b+d\theta_a}{1+\kappa\theta_a}\right) < 0$, $a > \lambda_1$ and $b = \lambda_1$. For fixed $a > \lambda_1$, there exists a sequence $\{(b_n, u_n, z_n, w_n)\}$ such that

$$
\lambda_1\left(-\frac{b_n + d\theta_a}{1+\kappa\theta_a}\right) < 0, \quad b_n < \lambda_1, \quad \lim_{n\to\infty} b_n = \lambda_1,
$$

and (u_n, z_n, w_n) is a positive solution of (1.4) with $b = b_n$. By the maximum principle, there exists a constant C independent of n such that $\|u_n\|_{L^\infty(\Omega)} + \|z_n\|_{L^\infty(\Omega)} + \|w_n\|_{L^\infty(\Omega)} < \infty$. It follows from the regularity of elliptic equations that (u_n, z_n, w_n) converges to (u_0, z_0, w_0) in $C^2(\overline{\Omega})$, and obviously (u_0, z_0, w_0) is a nonnegative solution of (3.4) with $b = \lambda_1$. Assume, on the contrary, that (3.4) has no positive solution when $b = \lambda_1$. Then $u_0 \equiv 0$ or $z_0 \equiv 0$ or $w_0 \equiv 0$. First we assume that $u_0 \equiv 0$. Let $\phi_n := z_n/\|z_n\|_{L^\infty(\Omega)}$, then (ϕ_n, u_n, z_n) satisfies

$$
\begin{cases}
-\Delta\phi_n = \dfrac{\phi_n}{1+\kappa u_n}\left(b_n + du_n - \dfrac{z_n}{1+\kappa u_n}\right), & x \in \Omega, \\
\phi_n = 0, & x \in \partial\Omega.
\end{cases}
$$

Since

$$
\left\| \frac{\phi_n}{1+\kappa u_n}\left(b_n + du_n - \frac{z_n}{1+\kappa u_n}\right) \right\|_{L^\infty(\Omega)} \le C
$$

for some constant C independent of n, then the regularity results of elliptic equations imply that there exists a nonnegative function $\phi_0 \in C^2(\overline{\Omega})$ with $\|\phi_0\|_{L^\infty(\Omega)} = 1$ such that $\lim_{n\to\infty} \phi_n = \phi_0$ in $C^2(\overline{\Omega})$, and it satisfies

$$
\begin{cases}
-\Delta\phi_0 = \phi_0(\lambda_1 - z_0), & x \in \Omega, \\
\phi_0 = 0, & x \in \partial\Omega.
\end{cases}
$$

Since ϕ_0 is nonnegative and nontrivial, then $\phi_0 > 0$ in Ω by the strong maximum principle, and so $\lambda_1 = \lambda_1(z_0)$, which means $z_0 \equiv 0$. Let $\psi_n = u_n / \|u_n\|_{L^\infty(\Omega)}$. It follows from the first equation of (1.4) that

$$\begin{cases} -\Delta \psi_n = \psi_n \left(a - u_n - \dfrac{r z_n}{(1 + \kappa u_n)(1 + m w_n)} \right), & x \in \Omega, \\ \psi_n = 0, & x \in \partial\Omega. \end{cases}$$

Then by a similar argument, one can show that there exists a positive function $\psi_0 \in C^2(\overline{\Omega})$ such that

$$-\Delta \psi_0 = a \psi_0, \quad x \in \Omega, \qquad \psi_0 = 0, \quad x \in \partial\Omega,$$

which means $a = \lambda_1$, a contradiction. If $z_0 = 0$, one can also show that $a = \lambda_1$ by a similar argument as above. Finally, if $w_0 = 0$, then we can obtain $c = \lambda_1$ by the last equation of (1.4) which contradicts $c > \lambda_1$. □

Next, we make some comments on Theorem 3.4. For fixed $c > \lambda_1$, we denote

$$S_1 = \left\{ (a,b) \in \mathbb{R} \times \mathbb{R} : \lambda_1 \left(-\frac{b + d\theta_a}{1 + \kappa\theta_a} \right) = 0, \ a \geq \lambda_1 \right\}$$

and

$$S_2 = \left\{ (a,b) \in \mathbb{R} \times \mathbb{R} : a = \lambda_1 \left(\frac{r\theta_b}{1 + m\theta_c} \right), \ b \geq \lambda_1 \right\}.$$

Then it follows from [23, Lemma 1.6] that S_1 can be expressed as

$$S_1 = \{(a,b) : b = \varpi(a) \text{ for } a \geq \lambda_1\},$$

where $\varpi(\cdot)$ is a C^1-function with respect to $a \in [\lambda_1, \infty)$ with the following properties:

(1) ϖ is strictly monotone decreasing if $\kappa\lambda_1 < d$, while ϖ is strictly monotone increasing if $\kappa\lambda_1 > d$.

(2) $\varpi(\lambda_1) = \lambda_1$, $\lim\limits_{a \to \infty} \varpi(a) = -\infty$ if $\kappa\lambda_1 < d$ and $\lim\limits_{a \to \infty} \varpi(a) = \infty$ if $\kappa\lambda_1 > d$.

(3) $\varpi'(\lambda_1) = \kappa\lambda_1 - d$.

Similarly, S_2 can be expressed as

$$S_2 = \{(a,b) : a = \tau(b) \text{ for } b \geq \lambda_1\},$$

where $\tau(\cdot)$ is a C^1-function with respect to $b \in [\lambda_1, \infty)$ with the following properties:

(1) τ is strictly monotone increasing.

(2) $\tau(\lambda_1) = \lambda_1$, $\lim\limits_{b \to \infty} \tau(b) = \infty$.

(3) $\tau'(\lambda_1) = \displaystyle\int_\Omega \frac{r\phi_1^3}{1 + m\theta_c}\, dx \Big/ \int_\Omega \phi_1^3 \, dx$, where $\phi_1(x)$ is the positive eigenfunction corresponding to λ_1 such that $\|\phi_1\|_{L^2(\Omega)} = 1$.

We denote

$$\mathcal{S} := \{(a,b) \in \mathbb{R} \times \mathbb{R} : \tau^{-1}(a) < b < \varpi(a), \ a > \lambda_1\}$$
$$\cup \{(a,b) : \tau^{-1}(a) < b < \varpi(a), \ a > \lambda_1\}.$$

Then it follows from the properties of ϖ and τ that

(1) if $\kappa\lambda_1 < d$, then $\varpi(a) < \tau^{-1}(a)$ for all $a > \lambda_1$;

(2) if $\kappa\lambda_1 > d$ and $\kappa\lambda_1 - d \neq \int_\Omega \phi_1^3 dx \Big/ \int_\Omega \dfrac{r\phi_1^3}{1+m\theta_c} dx$, then $\mathcal{S} \neq \emptyset$.
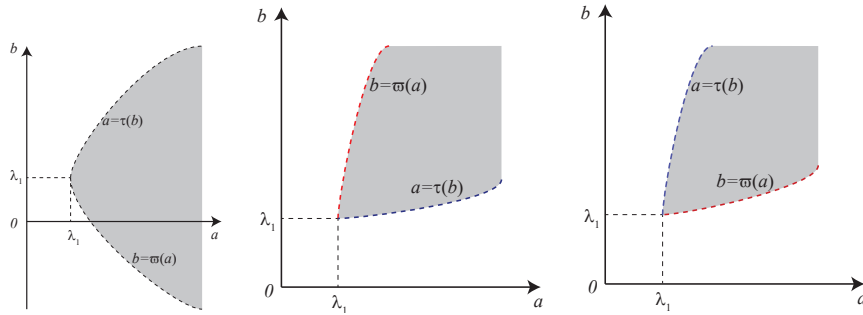

FIGURE 1. Possible coexistence regions. Left: $\kappa\lambda_1 < d$, middle: $\kappa\lambda_1 > d$ and $\kappa\lambda_1 - d > \int_\Omega \phi_1^3 \, dx / \int_\Omega (r\phi_1^3/(1+m\theta_c)) \, dx$, right: $\kappa\lambda_1 > d$ and $\kappa\lambda_1 - d < \int_\Omega \phi_1^3 \, dx / \int_\Omega (r\phi_1^3/(1+m\theta_c)) \, dx$.

Using the notations introduced above, we get the following results from Theorem 3.4.

COROLLARY 3.5. *Assume $c > \lambda_1$. Then (1.4) (or equivalently (3.4)) admits at least one positive solution if*

(a) $\kappa\lambda_1 < d$ *and* $\{(a,b) \in \mathbb{R} \times \mathbb{R} : \varpi(a) < b < \tau^{-1}(a), \ a > \lambda_1\}$ *(see the first graph of Figure 1); or*

(b) $\kappa\lambda_1 > d$ *and* $(a,b) \in \mathcal{S}$ *(see the second and third graphs of Figure 1).*

4. Global bifurcation and multiplicity of positive solutions

In this section we continue to study positive solutions of (1.4) (or equivalently (3.4)) by means of the bifurcation theory. Assume $b, c > \lambda_1$, then (3.4) has a strongly semi-trivial solution $(0, \theta_b, \theta_c)$. By linearizing (3.4) at $(0, \theta_b, \theta_c)$, we obtain the following eigenvalue problem:

(4.1)
$$\begin{cases} \Delta\phi + \left(a - \dfrac{r\theta_b}{1+m\theta_c}\right)\phi = \lambda\phi, & x \in \Omega, \\ \Delta\varphi + ((d-\kappa b)\theta_b + 2\kappa\theta_b^2)\phi + (b-2\theta_b)\varphi = \lambda\varphi, & x \in \Omega, \\ \Delta\psi + n\theta_c^2\phi + (c-2\theta_c)\psi = \lambda\psi,, & x \in \Omega, \\ \phi = \varphi = \psi = 0, & x \in \partial\Omega. \end{cases}$$

A necessary condition for bifurcation is that the principal eigenvalue of (4.1) is zero, which occurs if $a = \lambda_1\left(\frac{r\theta_b}{1+m\theta_c}\right)$. Let Φ be the positive eigenfunction corresponding to $a = \lambda_1\left(\frac{r\theta_b}{1+m\theta_c}\right)$. We assume that Φ is normalized so that $\|\Phi\|_{L^2(\Omega)} = 1$. Since $\lambda_1(2\theta_b - b) > 0$, $-\Delta + 2\theta_b - b$ is invertible. Define

$$\Psi = (-\Delta + 2\theta_b - b)^{-1}[((d - \kappa b)\theta_b + 2\kappa\theta_b^2)\Phi].$$

Similarly, define

$$\Upsilon = (-\Delta + 2\theta_c - c)^{-1}[n\theta_c^2\Phi].$$

With the functions defined above, we have the following result regarding the bifurcation solutions of (3.4) from $(0, \theta_b, \theta_c)$ at $a = \lambda_1\left(\frac{r\theta_b}{1+m\theta_c}\right)$.

LEMMA 4.1. *Let $b, c > \lambda_1$ be fixed. Then $a = \lambda_1\left(\frac{r\theta_b}{1+m\theta_c}\right)$ is a bifurcation value of (3.4) where positive solutions bifurcate from the line of semi-trivial solutions $\{(a, 0, \theta_b, \theta_c) : a > 0\}$; near $\left(\lambda_1\left(\frac{r\theta_b}{1+m\theta_c}\right), 0, \theta_b, \theta_c\right)$, there exists $\delta > 0$ such that all positive solutions of (3.4) lie on a smooth curve $\Gamma_1 = \{(a(s), u(s), z(s), w(s)) : 0 < s < \delta\}$ and*

$$\begin{cases} a(s) = \lambda_1\left(\dfrac{r\theta_b}{1+m\theta_c}\right) + sa_1 + sa_2(s), \\ u(s) = s\Phi + su_1(s, x), \\ z(s) = \theta_b + s\Psi + sz_1(s, x), \\ w(s) = \theta_c + s\Upsilon + sw_1(s, x). \end{cases}$$

Here $s \mapsto (a_2(s), u_1(s, x), z_1(s, x), w_1(s, x))$ is a smooth function from $(0, \delta)$ to $\mathbb{R} \times X \times X \times X$ for $X = C^{1+\sigma}(\Omega) \cap C_0(\overline{\Omega})$ with $\sigma \in (0, 1)$ such that $a_2(0) = 0$, $u_1(0, x) = z_1(0, x) = w_1(0, x) = 0$ and

$$(4.2) \quad a_1 = \int_\Omega \left(1 - \frac{r\kappa\theta_b}{1+m\theta_c}\right)\Phi^3\, dx + \int_\Omega \frac{r}{1+m\theta c}\left(\Psi - \frac{m\theta_b}{1+m\theta_c}\Upsilon\right)\Phi^2\, dx.$$

PROOF. Denote $\widetilde{a} := \lambda_1\left(\frac{r\theta_b}{1+m\theta_c}\right)$. Let $X = C^{1+\sigma}(\Omega) \cap C_0(\overline{\Omega})$ and $Y = C^\sigma(\Omega) \cap C(\overline{\Omega})$. Define a nonlinear $\mathfrak{F}: \mathbb{R} \times X \times X \times X \mapsto Y \times Y \times Y$ by

$$\mathfrak{F}(a, u, z, w) = \begin{pmatrix} \Delta u + F(a, u, z, w) \\ \Delta z + G(u, z, w) \\ \Delta w + H(u, z, w) \end{pmatrix},$$

where F, G, H are defined in (1.4). By a straightforward calculation, we obtain

$$\mathfrak{F}_{(u,z,w)}(a, u, z, w)[\xi, \eta, \zeta] = \begin{pmatrix} \Delta\xi + F_u\xi + F_z\eta + F_w\zeta \\ \Delta\eta + G_u\xi + G_x\eta + G_w\zeta \\ \Delta\zeta + H_u\xi + H_z\eta + H_w\zeta \end{pmatrix},$$

$$\mathfrak{F}_a(a, u, z, w) = \begin{pmatrix} u \\ 0 \\ 0 \end{pmatrix}, \qquad \mathfrak{F}_{a(u,z,w)}[\xi, \eta, \zeta] = \begin{pmatrix} \xi \\ 0 \\ 0 \end{pmatrix},$$

$$\mathfrak{F}_{(u,z,w)(u,z,w)}(a,u,z,w)[\xi,\eta,\zeta]^2$$
$$= \begin{pmatrix} F_{uu}\xi^2 + F_{zz}\eta^2 + F_{ww}\zeta^2 + 2(F_{uz}\xi\eta + F_{uw}\xi\zeta + F_{zw}\eta\zeta) \\ G_{uu}\xi^2 + G_{zz}\eta^2 + G_{ww}\zeta^2 + 2(G_{uz}\xi\eta + G_{uw}\xi\zeta + G_{zw}\eta\zeta) \\ H_{uu}\xi^2 + H_{zz}\eta^2 + H_{ww}\zeta^2 + 2(H_{uz}\xi\eta + H_{uw}\xi\zeta + H_{zw}\eta\zeta) \end{pmatrix}.$$

At $(a,u,z,w) = (\widetilde{a}, 0, \theta_b, \theta_c)$, it is easy to see that the kernel

$$\mathcal{N}(\mathfrak{F}_{(u,z,w)}(\widetilde{a}, 0, \theta_b, \theta_c)) = \operatorname{span}\{(\Phi, \Psi, \Upsilon)\}$$

and the range

$$\mathcal{R}(\mathfrak{F}_{(u,z,w)}(\widetilde{a}, 0, \theta_b, \theta_c)) = \left\{ (\alpha, \beta, \gamma) \in Y \times Y \times Y : \int_\Omega \alpha(x)\Phi(x) = 0 \right\}.$$

Then,

$$\mathfrak{F}_{a(u,z,w)}(\widetilde{a}, 0, \theta_b, \theta_c)[\Phi, \Psi, \Upsilon] = (\Phi, 0, 0) \notin \mathcal{R}(\mathfrak{F}_{(u,z,w)}(\widetilde{a}, 0, \theta_b, \theta_c))$$

by the fact that $\int_\Omega \Phi^2\, dx = 1 \neq 0$. Thus we can apply Theorem 2.1 to conclude that the set of positive solutions to (3.4) near $(\widetilde{a}, 0, \theta_b, \theta_c)$ is a smooth curve

$$\Gamma_1 = \{a(s), u(s), z(s), w(s) : s \in (0, \delta)\},$$

where δ is a small positive constant, such that $a(0) = \widetilde{a}$, $u(s) = s\Phi + o(s)$, $z(s) = \theta_b + s\Psi + o(s)$ and $w(s) = \theta_c + s\Upsilon + o(s)$. Moreover, by Theorem 2.1,

$$a_1 = a'(0) = -\frac{\langle \ell, \mathfrak{F}_{(u,z,,w)(u,z,w)}(\widetilde{a}, 0, \theta_b, \theta_c)[\Phi, \Psi, \Upsilon]^2 \rangle}{2\langle \ell, \mathfrak{F}_{a(u,z,w)}(\widetilde{a}, 0, \theta_b, \theta_c)[\Phi, \Psi, \Upsilon] \rangle},$$

where ℓ is a liner functional on $Y \times Y \times Y$ defined as

$$\langle \ell, (\alpha, \beta, \gamma) \rangle = \int_\Omega \alpha(x)\Phi(x)\, dx.$$

Thus,

$$a_1 = \int_\Omega \left(1 - \frac{r\kappa\theta_b}{1 + m\theta_c} \right)\Phi^3\, dx + \int_\Omega \frac{r}{1 + m\theta c}\left(\Psi - \frac{m\theta_b}{1 + m\theta_c}\Upsilon \right)\Phi^2\, dx. \qquad \square$$

By Theorem 2.1, the curve Γ_1 of bifurcating positive solutions is contained in a connected component Σ of the set of positive solutions of (3.4). Our next result is about Σ. To this end, denote

$$P_1 := \{\phi \in X : \phi(x) > 0,\ x \in \Omega,\ \partial\phi/\partial\nu < 0,\ x \in \partial\Omega\},$$
$$P := \{(a, u, z, w) \in \mathbb{R}_+ \times X \times X \times X\},$$

where X is defined in Lemma 4.1.

THEOREM 4.2. *Assume* $b, c > \lambda_1$ *and* $\lambda_1\left(-\frac{b + d\theta_a}{1 + \kappa\theta_a} \right) \neq 0$, *the global bifurcation curve* Σ *tends to* ∞ *in* P.

PROOF. Denote $\widetilde{a} := \lambda_1\left(\frac{r\theta_b}{1+m\theta_c}\right)$ and $\widetilde{\Sigma} := \Sigma \setminus \{(\widetilde{a}, 0, \theta_b, \theta_c)\}$. We need to prove only $\widetilde{\Sigma} \subset P$. On the contrary, suppose that $\widetilde{\Sigma}$ is not contained in P, then there exists a point $(\widehat{a}, \widetilde{u}, \widetilde{z}, \widetilde{w}) \in \widetilde{\Sigma} \cap \partial P$, which is the limit of a sequence of points $\{(a_n, u_n, z_n, w_n)\}_{n=1}^{\infty} \subset \Sigma \cap P$. It is obvious that $\widetilde{u} \in \partial P_1$, or $\widetilde{z} \in \partial P_1$, or $\widetilde{w} \in \partial P_1$.

Suppose $\widetilde{u} \in \partial P_1$, then $\widetilde{u} \geq 0$ for $x \in \Omega$ and either $\widetilde{u}(x_0) = 0$ for some $x_0 \in \Omega$ or $\partial\widetilde{u}/\nu(x_0) = 0$ for some $x_0 \in \partial\Omega$. Since $\widetilde{u}$ satisfies

$$-\Delta\widetilde{u} = \widetilde{u}\left(\widehat{a} - \widetilde{u} - \frac{r\widetilde{z}}{(1+\kappa\widetilde{u})(1+m\widetilde{w})}\right), \quad x \in \Omega, \qquad \widetilde{u} = 0, \quad x \in \partial\Omega,$$

it follows from the maximum principle that $\widetilde{u} \equiv 0$. Similarly, we can show that $\widetilde{z} \equiv 0$ if $\widetilde{v} \in \partial P_1$ and $\widetilde{w} \equiv 0$ if $\widetilde{w} \in \partial P$. Since $(\widehat{a}, \widetilde{u}, \widetilde{z}, \widetilde{w}) \in \Sigma$ and $c > \lambda_1$, we have $\widetilde{w} \geq \theta_c$. Thus we have the following three cases:

(i) $(\widetilde{u}, \widetilde{z}, \widetilde{w}) = (0, \theta_b, \theta_c)$;
(ii) $(\widetilde{u}, \widetilde{z}, \widetilde{w}) = (0, 0, \theta_c)$;
(iii) $(\widetilde{u}, \widetilde{z}, \widetilde{w}) = (\theta_a, 0, \theta_{(a,c)})$.

Next, we will prove that all of the above three cases cannot happen. Suppose (i) holds. Let $U_n = u_n/\|u_n\|_{L^\infty(\Omega)}$, then U_n satisfies

$$(4.3) \qquad \begin{aligned} -\Delta U_n &= U_n\left(a_n - u_n - \frac{rz_n}{(1+\kappa u_n)(1+mw_n)}\right), \quad x \in \Omega, \\ U_n &= 0, \quad x \in \partial\Omega. \end{aligned}$$

Thanks to the L^p-estimate and Sobolev embedding theorem, there exists a convergent subsequence of $\{U_n\}_{n=1}^{\infty}$, which we relabel as the original one, such that $U_n \to U$ in $C^2(\overline{\Omega})$ as $n \to \infty$, and $U \geq 0, \not\equiv 0$ in Ω, which satisfies $\|U\|_{L^\infty(\Omega)} = 1$. Taking limit in (4.3) as $n \to \infty$, we obtain

$$-\Delta U = \left(\widehat{a} - \frac{r\theta_b}{1+m\theta_c}\right)U, \quad x \in \Omega, \qquad U = 0, \quad x \in \partial\Omega.$$

It follows from Theorem 2.4 that $\widehat{a} = \widetilde{a}$, which contradicts $\widehat{a} \neq \widetilde{a}$.

Suppose $\widetilde{z} = 0$ holds. Let $Z_n = z_n/\|z_n\|_{L^\infty(\Omega)}$, then Z_n satisfies

$$(4.4) \quad -\Delta Z_n = \frac{Z_n}{1+\kappa u_n}\left(b + du_n - \frac{z_n}{1+\kappa u_n}\right), \quad x \in \Omega, \qquad Z_n = 0, \quad x \in \partial\Omega.$$

Similarly, by the L^p-estimate and Sobolev embedding theorem, there exists a convergent subsequence of $\{Z_n\}_{n=1}^{\infty}$, which we relabel as the original one, such that $Z_n \to Z$ in $C^2(\overline{\Omega})$ as $n \to \infty$, and $Z \geq 0, \not\equiv 0$ in Ω, which satisfies $\|Z\|_{L^\infty(\Omega)} = 1$. Suppose (ii) holds. Taking limit in (4.4) as $n \to \infty$, we obtain $b = \lambda_1$, which contradicts $b > \lambda_1$. Suppose (iii) holds. Taking limit in (4.4) as $n \to \infty$, we obtain $\lambda_1\left(-\frac{b+d\theta_a}{1+\kappa\theta_a}\right) = 0$, a contradiction. $\square$

Next, we discuss the stability of the positive solutions obtained im Lemma 4.1.

THEOREM 4.3. *Assume the assumptions of Lemma 4.1 hold, and let a_1 be defined as in (4.2). If $a_1 \neq 0$, then there exists $\widetilde{\delta} \in (0, \delta]$ such that for $s \in (0, \widetilde{\delta})$, the positive solution $(a(s), u(s), z(s), w(s))$ got in Lemma 4.1 is not degenerate, where δ is the constant from Lemma 4.1. Moreover, $(u(s), z(s), w(s))$ is unstable if $a_1 < 0$, and it is stable if $a_1 > 0$.*

PROOF. We use the notations of the proof of Lemma 4.1. In order to study the stability, we consider the following eigenvalue problem:

$$
\begin{cases}
\mathcal{L}(s) \begin{pmatrix} \xi(s) \\ \eta(s) \\ \zeta(s) \end{pmatrix} = \mu(s) \begin{pmatrix} \xi(s) \\ \eta(s) \\ \zeta(s) \end{pmatrix}, & x \in \Omega, \\
\xi(s) = \eta(s) = \zeta(s) = 0, & x \in \partial\Omega,
\end{cases}
$$

where

$$
\mathcal{L}(s) = -\mathfrak{F}_{(u,z,w)}(a(s), u(s), z(s), w(s)) =
$$

$$
\begin{pmatrix}
-\Delta - a(s) + 2u(s) + \frac{rz(s)}{(1+mw(s))(1+\kappa u(s))^2} & \frac{ru(s)}{(1+\kappa u(s))(1+mw(s))} & -\frac{rmu(s)z(s)}{(1+\kappa u(s))(1+mw(s))^2} \\
\frac{\kappa bz(s) - dz(s)}{(1+\kappa u(s))^2} - \frac{2\kappa z(s)^2}{(1+\kappa u(s))^3} & -\Delta - \frac{b+du(s)}{1+\kappa u(s)} + \frac{2z(s)}{(1+\kappa u(s))^2} & 0 \\
-\frac{nw(s)^2}{(1+nu(s))^2} & 0 & -\Delta - c + \frac{2w(s)}{1+nu(s)}
\end{pmatrix}.
$$

Furthermore,

$$
\lim_{s \to 0} \mathcal{L}(s) = \begin{pmatrix}
-\Delta - \widetilde{a} + \frac{r\theta_b}{1+m\theta_c} & 0 & 0 \\
(\kappa b - d)\theta_b - 2\kappa\theta_b^2 & -\Delta - b + 2\theta_b & 0 \\
-n\theta_c^2 & 0 & -\Delta - c + 2\theta_c
\end{pmatrix} := \mathcal{L}_0.
$$

Since $\lambda_1\big(-\widetilde{a} + \frac{r\theta_b}{1+m\theta_c}\big) = 0$, $\lambda_1(-b + 2\theta_b) > 0$, and $\lambda_1(-c + 2\theta_c) > 0$, 0 is the first eigenvalue of $\mathcal{L}_0$ with the corresponding eigenfunction (Φ, Ψ, Υ). Moreover, the real parts of all other eigenvalues of $\mathcal{L}_0$ are positive and are apart from 0. By perturbation of linear operator [13], we know that for $s > 0$ small $\mathcal{L}(s)$ has a unique eigenvalue $\mu(s)$ such that $\lim_{s \to 0} \mu(s) = 0$ and all other eigenvalues of $\mathcal{L}(s)$ have positive real part and apart from 0.

Now we determine the sign of $\mu(s)$ for $s > 0$ small, applying Theorem 2.2. Consider the following eigenvalue problem:

$$
\begin{cases}
-\mathfrak{F}_{(u,z,w)}(a, 0, \theta_b, \theta_c) \begin{pmatrix} \phi(a) \\ \varphi(a) \\ \psi(a) \end{pmatrix} = \gamma(a) \begin{pmatrix} \phi(a) \\ \varphi(a) \\ \psi(a) \end{pmatrix}, & x \in \Omega, \\
\phi(a) = \varphi(a) = \psi(a) = 0, & x \in \partial\Omega.
\end{cases}
$$

Then $\phi(a)$ satisfies

$$(4.5) \qquad \begin{cases} -\Delta\phi(a) + \dfrac{r\theta_b}{1+m\theta_c}\phi(a) - a\phi(a) = \gamma(a)\phi(a), & x \in \Omega, \\ \phi(a) = 0, & x \in \partial\Omega. \end{cases}$$

Since $\gamma(\widetilde{a}) = 0$ and $\phi(\widetilde{a}) = \Phi$, then differentiating (4.5) with respect to a at $a = \widetilde{a}$ we obtain that

$$(4.6) \qquad \begin{cases} -\Delta\chi + \dfrac{r\theta_b}{1+m\theta_c}\chi - \widetilde{a}\chi - \Phi = \gamma'(\widetilde{a})\Phi, & x \in \Omega, \\ \chi = 0, & x \in \partial\Omega, \end{cases}$$

where $\chi = \phi'(\widetilde{a})$. Multiplying both sides of (4.6) by Φ and integrating it over Ω, we obtain

$$\gamma'(\widetilde{a})\int_\Omega \Phi^2\,dx = -\int_\Omega \Phi^2\,dx + \int_\Omega \left(-\Delta\chi\Phi + \frac{r\theta_b}{1+m\theta_c}\chi\Phi - \widetilde{a}\chi\Phi \right)dx$$

$$= -\int_\Omega \Phi^2\,dx + \int_\Omega \chi\left(-\Delta\Phi + \frac{r\theta_b}{1+m\theta_c}\Phi - \widetilde{a}\Phi \right)dx = -\int_\Omega \Phi^2\,dx,$$

that is $\gamma'(\widetilde{a}) = -1$. Since $a_1 \neq 0$, then it follows from Theorem 2.2 that $\mu(s) \neq 0$ for $s > 0$ small and

$$(4.7) \qquad \lim_{s \to 0} \frac{\mu(s)}{s} = -\gamma'(\widetilde{a})a'(0) = a_1.$$

Since all other eigenvalues of $\mathcal{L}(s)$ have positive real parts, then the conclusion follows from (4.7). $\qquad\square$

Combining the above preparations, we get the following multiplicity result.

THEOREM 4.4. *Assume* $b, c > \lambda_1$ *and* $\lambda_1\left(-\frac{b+d\theta_a}{1+\kappa\theta_a} \right) < 0$. *Let* $\widetilde{a} = \lambda_1\left(\frac{r\theta_b}{1+m\theta_c} \right)$ *and* a_1 *be defined as in* (4.2). *If* $a_1 < 0$, *then there exists a positive constant* $\varepsilon < \widetilde{a} - \lambda_1$ *such that* (1.4) (*or equivalently* (3.4)) *has at least two positive solutions if* $\lambda_1 - \varepsilon < a < \widetilde{a}$, *and it has at least one positive solution if* $a \geq \widetilde{a} - \varepsilon$.

PROOF. From Lemma 4.1, (3.4) has a curve $\Gamma_1 = \{(a(s), u(s), z(s), w(s)) : 0 < s < \delta\}$ of positive solutions near $(\widetilde{a}, o, \theta_b, \theta_c)$. Since $a_1 < 0$, $a(s) < \widetilde{a}$ for $s > 0$ small. Assume, on the contrary, that (3.4) has a unique positive solution $(\widehat{u}, \widehat{z}, \widehat{w})$ when $a < \widetilde{a}$ but near $\det a$. Then it is obvious that $(\widehat{u}, \widehat{z}, \widehat{w})$ is the positive solution bifurcating from $(\widetilde{a}, o, \theta_b, \theta_c)$, which was obtained in Lemma 4.1, and it is not degenerate by Theorem 4.3. Thus $I - \mathcal{A}_{(u,z,w)}(\widehat{u}, \widehat{z}, \widehat{w}) \colon \overline{\mathcal{W}_{(\widehat{u},\widehat{z},\widehat{w})}} \mapsto \overline{\mathcal{W}_{(\widehat{u},\widehat{z},\widehat{w})}}$ is invertible, where $\mathcal{A}$ is the operator defined in Section 4. Since $(\widehat{u}, \widehat{z}, \widehat{w})$ is an isolated interior point of D, $\operatorname{index}_W(\mathcal{A}, (\widehat{u}, \widehat{z}, \widehat{w})) = \pm 1$. Notice that $\lambda_1 < a < \widetilde{a}$ for $s > 0$ small, $b, c > \lambda_1$ and $\lambda_1\left(-\frac{b+d\theta_a}{1+\kappa\theta_a} \right) < 0$. It follows from Lemmas 3.1–3.3

that

$$1 = \deg_{\mathcal{W}}(\mathcal{A}, D)$$
$$= \text{index}_{\mathcal{W}}(\mathcal{A}, (0,0,0)) + \text{index}_{\mathcal{W}}(\mathcal{A}, (\theta_a, 0, 0)) + \text{index}_{\mathcal{W}}(\mathcal{A}, (0, \theta_b, 0))$$
$$+ \text{index}_{\mathcal{W}}(\mathcal{A}, (0, 0, \theta_c)) + \text{index}_{\mathcal{W}}(\mathcal{A}, (0, \theta_b, \theta_c)) + \text{index}_{\mathcal{W}}(\mathcal{A}, (\theta_a, 0, \theta_{(a,c)}))$$
$$+ \text{index}_{\mathcal{W}}(\mathcal{A}, S) + \text{index}_{\mathcal{W}}(\mathcal{A}, (\widehat{u}, \widehat{z}, \widehat{w}))$$
$$= 0 + 0 + 0 + 0 + 1 + 0 + 0 \pm 1,$$

which is a contradiction. Thus if $a < \widetilde{a}$ and near $\widetilde{a}$, then there exist at least two positive solutions of (3.4). Since $u(s)(x) < a$ for all $x \in \overline{\Omega}$ and there are no positive solutions if $a \leq \lambda_1$, it follows from Theorem 4.2 that there exists $\varepsilon \in (0, \widetilde{a} - \lambda_1)$ such that the projection of the closure of the global bifurcation curve Σ is $[a - \varepsilon, \infty)$ (see Figure 2), and the conclusion follows. $\square$
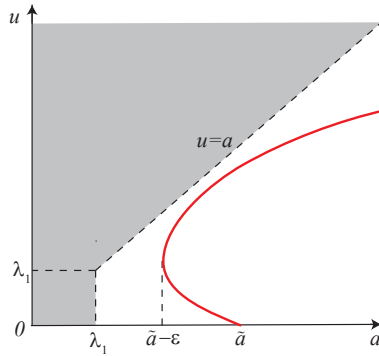

FIGURE 2. Backward bifurcation of u when $a_1 < 0$.

REMARK 4.5. If $a, c, d, r, m > 0$, $b \in \mathbb{R}$ and $\kappa \geq 0$ are fixed, then Φ, Ψ are fixed and $\Upsilon \to \infty$ uniformly on any compact subset of Ω as $n \to \infty$. Hence, there exists a constant $n_* > 0$ large enough such that $a_1 < 0$ if $n > n_*$.

REFERENCES

[1] H. AMANN AND J. LÓPEZ-GÓMEZ, *A priori bounds and multiple solutions for superlinear indefinite elliptic problems*, J. Differential Equations **146** (1998), no. 2, 336–374.

[2] S. CANO-CASANOVA, *Existence and structure of the set of positive solutions of a general class of sublinear elliptic non-classical mixed boundary value problems*, Nonlinear Anal. **49** (2002), no. 3, 361–430.

[3] W. CHEN AND R. PENG, *Stationary patterns created by cross-diffusion for the competitor-competitor-mutualist model*, J. Math. Anal. Appl. **291** (2004), no. 2, 550–564.

[4] W. CHEN AND M. WANG, *Non-constant positive steady-states of a predator-prey-mutualist model*, Chinese Ann. Math. Ser. B **25** (2004), no. 2, 243–254.

[5] ———, *Positive steady states of a competitor-competitor-mutualist model*, Acta Math. Appl. Sin. Engl. Ser. **20** (2004), no. 1, 53–57.

[6]   M.G. Crandall and P.H. Rabinowitz, *Bifurcation from simple eigenvalues*, J. Funct. Anal. **8** (1971), 321–340.

[7]   ———, *Bifurcation, perturbation of simple eigenvalues and linearized stability*, Arch. Rational Mech. Anal. **52** (1973), 161–180.

[8]   E.N. Dancer, *On the indices of fixed points of mappings in cones and applications*, J. Math. Anal. Appl. **91** (1983), no. 1, 131–151.

[9]   ———, *On positive solutions of some pairs of differential equations*, Trans. Amer. Math. Soc. **284** (1984), no. 2, 729–743.

[10] E.N. Dancer and Y.H. Du, *Positive solutions for a three-species competition system with diffusion.* I. *General existence results*, Nonlinear Anal. **24** (1995), no. 3, 337–357.

[11] ———, *Positive solutions for a three-species competition system with diffusion.* II. *The case of equal birth rates*, Nonlinear Anal. **24** (1995), no. 3, 359–373.

[12] L. Hei, *Global bifurcation of co-existence states for a predator-prey-mutualist model with diffusion*, Nonlinear Anal. Real World Appl. **8** (2007), no. 2, 619–635.

[13] T. Kato, *Perturbation theory for linear operators*, Classics in Mathematics, Springer-Verlag, Berlin, 1995, Reprint of the 1980 edition.

[14] L. Li, *Coexistence theorems of steady states for predator-prey interacting systems*, Trans. Amer. Math. Soc. **305** (1988), no. 1, 143–166.

[15] C.V. Pao, *Nonlinear parabolic and elliptic equations*, Plenum Press, New York, 1992.

[16] P.H. Rabinowitz, *Some global results for nonlinear eigenvalue problems*, J. Functional Analysis **7** (1971), 487–513.

[17] B. Rai, H.I. Freedman and J.F. Addicott, *Analysis of three-species models of mutualism in predator-prey and competitive systems*, Math. Biosci. **65** (1983), no. 1, 13–50.

[18] W. Ruan and W. Feng, *On the fixed point index and multiple steady-state solutions of reaction-diffusion systems*, Differential Integral Equations **8** (1995), no. 2, 371–391.

[19] J. Shi, *Persistence and bifurcation of degenerate solutions*, J. Funct. Anal. **169** (1999), no. 2, 494–531.

[20] J. Shi and X. Wang, *On global bifurcation for quasilinear elliptic systems on bounded domains*, J. Differential Equations **246** (2009), no. 7, 2788–2812.

[21] J. Smoller, *Shock waves and reaction-diffusion equations*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Science], vol. 258, Springer-Verlag, New York, 1983.

[22] S. Xu, *Global stability of a reaction-diffusion system of a competitor-competitor-mutualist model*, Taiwanese J. Math. **15** (2011), no. 4, 1617–1627.

[23] Y. Yamada, *Positive solutions for Lotka–Volterra systems with cross-diffusion*, Handbook of differential equations: stationary partial differential equations. Vol. VI, Handb. Differ. Equ., Elsevier/North-Holland, Amsterdam, 2008, pp. 411–501.

[24] S.N. Zheng, *A reaction-diffusion system of a competitor-competitor-mutualist model*, J. Math. Anal. Appl. **124** (1987), no. 1, 254–280.

Jun Zhou
School of Mathematics and Statistics
Southwest University
Chongqing, 400715, P.R. CHINA

E-mail address: jzhouwm@163.com