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MIXED BOUNDARY CONDITION FOR THE MONGE–KANTOROVICH EQUATION

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Abstract. In this work we give some equivalent formulations for the optimization problem

$$\max\bigg\{\int_\Omega \xi\,d\mu+\int_{\Gamma_N}\xi\,d\nu;\ \xi\in W^{1,\infty}(\Omega)\text{ such that } \xi_{/\Gamma_D}=0,\ |\nabla\xi(x)|\leq 1\text{ a.e. } x\in\Omega\bigg\},$$

where the boundary of Ω is $\Gamma=\Gamma_N\cup\Gamma_D$.

1. Introduction and main result

In this paper, we study the equivalence between the Monge–Kantorovich equation in a bounded domain and weak formulations with mixed boundary condition. Recall that the Monge–Kantorovich equation found its origin in the Monge–Kantorovich optimal mass transport problem (cf. [1], [10]) as well as in the optimal mass transfer problem (cf. [5]). Then, the equation was extensively used in the description of the dynamics of granular matter like the sandpile (cf. [10] and [9]) and also in the deformation of polymer plastic during compression molding (cf. [2]).

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Given two Radon measures f^+ and f^- such that $f^+(\mathbb{R}^N) = f^-(\mathbb{R}^N) < \infty$, it is known (cf. [1]) that the problem

$$\max\bigg\{ \int_{\mathbb{R}^N} \xi\, df,\ \xi \in \mathrm{Lip}_1(\mathbb{R}^N) \bigg\}, \tag{1.1}$$

is closely related to the optimal transportation problem associated with f^+ and f^- where $f = f^+ - f^-$ and the cost function is given by $c(x,y) = |x-y|$. Here, $|\cdot|$ denotes the Euclidean norm of $\mathbb{R}^N$. Problem (1.1) is called the dual Monge–Kantorovich problem in the literature. Formally, by the standard convex duality argument, the dual formulation associated with (1.1) is given by

$$\min\bigg\{ \int_{\mathbb{R}^N} d|\lambda| : \lambda \in (\mathcal{M}_b(\mathbb{R}^N))^N : -\mathrm{div}(\lambda) = f \bigg\}, \tag{1.2}$$

where $|\lambda|$ denotes the total variation of λ and $(\mathcal{M}_b(\mathbb{R}^N))^N$ the space of $\mathbb{R}^N$-valued Radon measures of $\mathbb{R}^N$ with bounded total variation (see the following section). Under some additional regularity conditions on f^+ and f^-, Evans and Gangbo in [11] showed that the Euler–Lagrange equation associated with (1.1) is given by

$$\begin{cases} -\nabla.(m\nabla u) = f^+ - f^-, \\ m \geq 0, \quad |\nabla u| \leq 1 \quad \text{and} \quad m(|\nabla u| - 1) = 0. \end{cases} \tag{1.3}$$

In connection with the optimal mass transport problem, the unknown function m is the transport density, $-\nabla u$ is the given direction of the optimal transport, $m\nabla u$ represents the flux transportation and u is the Kantorovich potential. In connection with the granular matter and the deformation of polymer plastic during compression molding, this equation appears in the definition of the main differential operator governing the dynamics. In this case, the parameter m is connected to the Lagrange multiplayer associated with the gradient constraint connected to the subgradient flux phenomena. In general, m is not a Lebesgue function but is a nonegative Radon measure. In this case, problem (1.3) may be written as

$$\begin{cases} -\mathrm{div}(m\nabla_m u) = f, \\ |\nabla_m u| = 1 \quad m\text{-a.e.} \end{cases} \tag{1.4}$$

where ∇_m denotes the tangential gradient with respect to m (see the following section for preliminaries and references). This is the so called Monge–Kantorovich equation (cf. [4]). In connection with the optimal mass transport problem, (1.4) is well studied in $\mathbb{R}^N$ which is equivalent to homogenous Neumann boundary conditions (cf. [5]). It is clear that the case of non-homogeneous Neumann boundary condition falls into the scope of the homogenous one with the Radon measure source term f and maybe handled by the results of [5]. In the case of the Dirichlet boundary condition, the problem was studied in [12]. Our aim here is to study the case of mixed boundary condition; Dirichlet and nonhomogeneous Neumann boundary conditions. In particular, this kind of boundary conditions appears in the study of the movement of the sand dunes. Roughly speaking, the dynamic in this case can be split into a free boundary problem with two regions. The first region facing the wind that could be handled by a transport equation and a second region sheltered from the wind and governed by the gravity. This last region maybe handled by an evolution equation with a differential operator of type (1.4) subject to Dirichlet and nonhomegeneous boundary condition, modeling the transfer of sand across the crest. The associated evolution problem as well as the application to the modelization of traveling sand dunes will be treated in forthcoming papers.

Let Ω be a bounded open Lipschitz domain of $\mathbb{R}^N$ with $\mathcal{C}^1$ smooth boundary Γ. We assume that Γ is divided into two parts Γ_N, Γ_D such that $\Gamma_D \cap \Gamma_N = \emptyset$ and the measure of Γ_D is nonzero. We set

$$\begin{gathered} W^{1,\infty}_D(\Omega) = \{z \in W^{1,\infty}(\Omega);\ z = 0 \text{ on } \Gamma_D\}, \\ K = \{z \in W^{1,\infty}_D(\Omega);\ |\nabla z(x)| \leq 1 \text{ a.e. } x \in \Omega\}. \end{gathered}$$

In our situation, problem (1.1) reads

$$\max\bigg\{ \int_\Omega \xi\, d\mu + \int_{\Gamma_N} \xi\, d\nu\ \xi \in K \bigg\}, \tag{1.5}$$

where μ and ν are bounded Radon measures concentrated respectively in Ω and on Γ_N.

Our aim here is to study the equivalence between (1.5) and the following problems:

- Find $\phi \in (\mathcal{M}_b(\Omega))^N$ and $v \in K$, solution of the PDE

$$\begin{cases} -\operatorname{div}(\phi) = \mu, \quad \phi = |\phi|\nabla_{|\phi|} v & \text{in } \Omega, \\ \phi \cdot \eta = \nu & \text{on } \Gamma_N, \\ u = 0 & \text{on } \Gamma_D, \end{cases} \tag{1.6}$$

  where η is the outward normal to Γ.
- Find $\phi \in (\mathcal{M}_b(\Omega))^N$ and $v \in K$ such that

$$|\phi|(\Omega) = \min\{|\Phi|(\Omega) : \Phi \text{ satisfies PDE (1.6)}\} = \int_\Omega v\, d\mu + \int_{\Gamma_N} v\, d\nu. \tag{1.7}$$

THEOREM 1.1. *Let $\mu \in \mathcal{M}_b(\Omega)$, $\nu \in \mathcal{M}_b(\Gamma)$ and $v \in K$. Then, problem* (1.5) *has a solution. Moreover, we have*:

(a) *v is a solution of* (1.5)*, i.e.*

$$\int_\Omega (v - \xi)\, d\mu + \int_{\Gamma_N} (v - \xi)\, d\nu \geq 0 \quad \textit{for any } \xi \in K,$$

if and only if there exists $\phi \in (\mathcal{M}_b(\Omega))^N$ *such that* (v, ϕ) *satisfies* (1.6) *in the following equivalent sense:*

(a1) $\phi \in (\mathcal{M}_b(\Omega))^N$, $v \in K$ *and satisfies*

$$\begin{cases} \displaystyle\int_\Omega \nabla\xi \, d\phi = \int_\Omega \xi \, d\mu + \int_{\Gamma_N} \xi \, d\nu & \textit{for any } \xi \in K, \\ \displaystyle|\phi|(\Omega) = \int_\Omega v \, d\mu + \int_{\Gamma_N} v \, d\nu. \end{cases} \tag{1.8}$$

(a2) $\phi \in (\mathcal{M}_b(\Omega))^N$, $v \in K$ *and satisfies*

$$\begin{cases} \displaystyle\int_\Omega \nabla\xi \, d\phi = \int_\Omega \xi \, d\mu + \int_{\Gamma_N} \xi \, d\nu & \textit{for any } \xi \in K, \\ \phi = |\phi| \nabla_{|\phi|} v. \end{cases} \tag{1.9}$$

(b) v *is a solution of* (1.5) *if and only if there exists* $\phi \in (\mathcal{M}_b(\Omega))^N$ *such that the couple* (ϕ, v) *satisfies* (1.7).

The rest of paper is organized as follows: in the next section we give some preliminaries and recall some technical lemmas. Section 3 is devoted to the proof of the main theorem.

2. Preliminaries

In this section we introduce some notations and lemmas that will be useful later on. Let Ω be a bounded open subset of $\mathbb{R}^N$ ($N \geq 2$) equipped with the N-dimensional Lebesgue measure. The space of Radon measure and the set of continuous functions with compact support in Ω will be denoted by $\mathcal{M}(\Omega)$ and $\mathcal{C}_c(\Omega)$, respectively. We recall that each Radon measure μ can be interpreted as an element of the dual of the space $\mathcal{C}_c(\Omega)$. This result can be extended to the space $\mathcal{C}(\overline{\Omega})$, i.e. $\mathcal{M}(\Omega) = (\mathcal{C}(\overline{\Omega}))^*$, in the sense that, for every $\mu \in \mathcal{M}(\Omega)$ there exists $\widetilde{\mu} \in (\mathcal{C}(\overline{\Omega}))^*$ such that $\langle \widetilde{\mu}, \xi \rangle = \int_\Omega \xi \, d\mu$, for all $\xi \in \mathcal{C}(\overline{\Omega})$.

For $\mu \in \mathcal{M}(\Omega)$, we denote by μ^+, μ^-, and $|\mu|$ the positive part, negative part and the total variation measure associated with μ, respectively. Then we denote by $\mathcal{M}_b(\Omega)$ the space of Radon measures with bounded total variation $|\mu|(\Omega)$. Recall that $\mathcal{M}_b(\Omega)$ equipped with the norm $|\mu|(\Omega)$ is a Banach space.

We denote by $(\mathcal{M}(\Omega))^N$ the space of $\mathbb{R}^N$-valued Radon measures of Ω, i.e. $\mu \in (\mathcal{M}(\Omega))^N$ if and only if $\mu = (\mu_1, \dots, \mu_n)$ with $\mu_i \in \mathcal{M}(\Omega)$. We recall that the total variation measure associated with $\mu \in (\mathcal{M}(\Omega))^N$, denoted again by $|\mu|$, is defined by

$$|\mu|(B) = \sup \left\{ \sum_{i=1}^{\infty} |\mu(B_i)|; \ B = \bigcup_{i=1}^{\infty} B_i, \ B_i \text{ a Borelean set} \right\}$$

and belongs to $\mathcal{M}^+(\Omega)$, the set of nonnegative Radon measure. The subspace $(\mathcal{M}_b(\Omega))^N$ equipped with the norm $\|\mu\| = |\mu|(\Omega)$ is a Banach space. The space

$(\mathcal{M}(\Omega))^N$ endowed with the norm $\|\cdot\|$ is isometric to the dual space of $(\mathcal{C}_c(\Omega))^N$. The duality is given by

$$\langle \mu, \xi \rangle = \sum_{i=1}^{N} \int_\Omega \xi_i \, d\mu_i,$$

for any $\mu = (\mu_1, \dots, \mu_N) \in (\mathcal{M}(\Omega))^N$ and $\xi = (\xi_1, \dots, \xi_N) \in (\mathcal{C}_c(\Omega))^N$.

For any $\mu \in (\mathcal{M}_b(\Omega))^N$ and $\nu \in \mathcal{M}_b(\Omega)^+$, μ is absolutely continuous with respect to ν and denoted by $\mu \ll \nu$, provided $\nu(A) = 0$ implies $|\mu|(A) = 0$, for any $A \subset \Omega$. Thanks to the Radon–Nikodym Decomposition Theorem, we know that for any $\mu \in (\mathcal{M}_b(\Omega))^N$ and $\nu \in \mathcal{M}_b(\Omega)$, such that $\mu \ll \nu$, there exists a unique bounded $\mathbb{R}^N$-valued measure denoted by $D_\nu \mu$, such that

$$\mu(A) = \int_A D_\nu \mu \, d\nu \quad \text{for any } A \subseteq \Omega;$$

$D_\nu \mu \in (\mathcal{M}_b(\Omega))^N$ is the density of μ with respect to ν which can be computed by differentiating. Since $|\mu(A)| \le |\mu|(A)$, for all $\mu \in (\mathcal{M}(\Omega))^N$, we have $\mu \ll |\mu|$, $D_{|\mu|}\mu \in (L^1_{|\mu|}(\Omega))^N$ and $|D_{|\mu|}\mu| = 1$, $|\mu|$-almost everywhere in Ω. In the literature $D_{|\mu|}\mu$ is denoted by $\mu/|\mu|$. So, for any $\mu \in (\mathcal{M}_b(\Omega))^N$, we have

$$\mu(A) = \int_A \frac{\mu}{|\mu|} \, d|\mu| \quad \text{for any Borel set } A \subseteq \Omega.$$

Hence, every $\mu \in (\mathcal{M}_b(\Omega))^N$ can be identified with the linear application

$$\xi \in (\mathcal{C}_c(\Omega))^N \mapsto \int_\Omega \frac{\mu}{|\mu|} \, \xi \, d|\mu|.$$

In what follows, we will denote the integral $\int_\Omega \nabla\xi \, d\mu$ by $\int_\Omega (\mu/|\mu|)\xi \, d|\mu|$.

We recall the following sets used in the definition of tangential gradient with respect to $\widetilde{\nu} \in \mathcal{M}_b(\Omega)^+$ (see [4]).

$$\begin{aligned} \mathcal{N}_{\widetilde{\nu}} := \{\xi \in (L^\infty_{\widetilde{\nu}}(\Omega))^N;\ \exists\, u_n \in \mathcal{C}^\infty(\Omega),\ u_n \to 0 \text{ in } \mathcal{C}(\Omega) \\ \text{and } Du_n \to \xi \text{ in } \sigma((L^\infty_{\widetilde{\nu}}(\Omega))^N, (L^1_{\widetilde{\nu}}(\Omega))^N)\} \end{aligned}$$

and

$$\mathcal{N}^\perp_{\widetilde{\nu}} := \bigg\{ \eta \in (L^1_{\widetilde{\nu}}(\Omega))^N;\ \int_\Omega \eta \cdot \xi \, d\widetilde{\nu} = 0, \text{ for all } \xi \in \mathcal{N}_{\widetilde{\nu}} \bigg\}.$$

For $\widetilde{\nu}$-almost every $x \in \Omega$, we define the tangent space $T_{\widetilde{\nu}}(x)$ to the measure $\widetilde{\nu}$, as the subspace of $\mathbb{R}^N$:

$$T_{\widetilde{\nu}}(x) = \{A \in \mathbb{R}^N; \exists\, \xi \in \mathcal{N}^\perp_{\widetilde{\nu}},\ A = \xi(x)\}.$$

Then (cf. Proposition 3.2 of [6]) the operator $\nabla_{\widetilde{\nu}} \colon \mathrm{Lip}(\Omega) \to (L^\infty_{\widetilde{\nu}}(\Omega))^N$ is the continuous operator such that for any $u \in \mathcal{C}^1(\Omega)$,

$$\nabla_{\widetilde{\nu}} u(x) = P_{T_{\widetilde{\nu}(x)}} \nabla u(x) \quad \widetilde{\nu}\text{-p.p.} \quad x \in \Omega,$$

where $P_{T_{\widetilde{\nu}(x)}}$ is the orthogonal projection on $T_{\widetilde{\nu}}(x)$, $\mathrm{Lip}(\Omega)$ the set of Lipschitz continuous functions equipped with the uniform convergence and $L^\infty_{\widetilde{\nu}}(\Omega)$ is equipped with the weak star topology. An $\mathbb{R}^N$-valued Radon measure ϕ is said to be a tangential measure on Ω provided there exist $\widetilde{\nu} \in \mathcal{M}_b(\Omega)^+$ and $\sigma \in (L^1_{\widetilde{\nu}}(\Omega))^N$, such that $\sigma(x) \in T_{\widetilde{\nu}}(x)$, $\widetilde{\nu}$-almost every $x \in \Omega$ and $\phi = \sigma\widetilde{\nu}$. Thanks to Proposition 3.5 of [6], we know that for any tangential measure $\phi = \sigma\widetilde{\nu}$ on Ω, such that $-\nabla \cdot \phi = \widetilde{\mu} \in \mathcal{M}_b(\Omega)$, we have the following integration by parts:

$$\int_\Omega u\, d\widetilde{\mu} = \int_\Omega \sigma \cdot \nabla_{\widetilde{\nu}} u\, d\widetilde{\nu},$$

for any $u \in \mathrm{Lip}(\Omega)$ null on the boundary of Ω.

In the sequel, we need the following two lemmas.

Lemma 2.1. *For any $z \in K$, there exists $(z_\varepsilon)_{\varepsilon>0}$, a sequence in $C^1(\Omega) \cap K$, such that*

$$z_\varepsilon \to z \quad \text{uniformly in } \overline{\Omega}.$$

Proof. For a given $\varepsilon > 0$, we consider the application $I_\varepsilon \colon \mathbb{R} \to \mathbb{R}$, defined by

$$I_\varepsilon(r) = \begin{cases} 0 & \text{if } |r| \leq \varepsilon, \\ r - \mathrm{sign}(r)\varepsilon & \text{if } |r| > \varepsilon. \end{cases}$$

Then, for a given $z \in K$, we take $d_\varepsilon = I_\varepsilon(z)$. One sees that $\mathrm{support}(d_\varepsilon) \subseteq \Omega_\varepsilon$, where $\Omega_\varepsilon := \{x \in \Omega : d(x, \Gamma_D) \geq r_\varepsilon\}$ with $r_\varepsilon > 0$ depending on ε and $d_\varepsilon \in K$.

Now we introduce the sequences $(\widetilde{z}_\varepsilon)_{\varepsilon>0}$ by

$$\widetilde{z}_\varepsilon(x) = \begin{cases} d_\varepsilon(x) & \text{if } x \in \Omega_\varepsilon, \\ 0 & \text{if } x \in \mathbb{R}^N \setminus \Omega_\varepsilon. \end{cases} \tag{2.1}$$

It is not difficult to see that, $\widetilde{z}_\varepsilon \in K$ and $\widetilde{z}_\varepsilon$ is supported in Ω_ε. Let $(\rho_\varepsilon)_{\varepsilon>0}$ be the standard sequence of mollifiers, then there exists $0 < \alpha < 1$, such that

$$z_\varepsilon = \widetilde{z}_\varepsilon * \rho_{\alpha\varepsilon} \in K \cap \mathcal{C}^\infty(\overline{\Omega}), \quad \text{for any } \varepsilon > 0.$$

Moreover, for any $p \geq 1$, z_ε is bounded in $W^{1,p}(\Omega)$ and the results of the lemma follows. $\square$

Then, similarly as in [12], the following result can be proved.

Lemma 2.2. *For any $v \in K$ and $\lambda \in \mathcal{M}_b(\Omega)^+$, we have $|\nabla_\lambda v| \leq 1$, λ-almost everywhere in Ω.*

3. Proof of Theorem 1.1

First we introduce a set of lemmas.

Lemma 3.1. *Let $\phi \in (\mathcal{M}_b(\Omega))^N$ and $v \in K$. If (v, ϕ) satisfies* (1.8)*, then v is a solution of* (1.5)*.*

Proof. Let $\xi \in K$, thanks to Lemma 2.1, there exists $\xi_\varepsilon \in C^1(\Omega) \cap K$ such that $\xi_\varepsilon \to \xi$, uniformly in Ω. Taking ξ_ε as a test function in (1.8), we have

$$\int_\Omega \nabla \xi_\varepsilon \cdot \frac{\phi}{|\phi|}\, d|\phi| = \int_\Omega \xi_\varepsilon\, d\mu + \int_{\Gamma_N} \xi_\varepsilon\, d\nu.$$

Using the fact that $\xi_\varepsilon \in K$, we get

$$\int_\Omega \nabla \xi_\varepsilon \cdot \frac{\phi}{|\phi|}\, d|\phi| \le |\phi|(\Omega).$$

Thus,

$$\int_\Omega \xi\, d\mu + \int_{\Gamma_N} \xi\, d\nu = \lim_{\varepsilon\to 0} \int_\Omega \xi_\varepsilon\, d\mu + \int_{\Gamma_N} \xi_\varepsilon\, d\nu = \lim_{\varepsilon\to 0} \int_\Omega \nabla \xi_\varepsilon \cdot \frac{\phi}{|\phi|}\, d|\phi| \le |\phi|(\Omega),$$

i.e.

$$\int_\Omega \xi\, d\mu + \int_{\Gamma_N} \xi\, d\nu \le |\phi|(\Omega). \tag{3.1}$$

Since

$$|\phi|(\Omega) = \int_\Omega v\, d\mu + \int_{\Gamma_N} v\, d\nu, \tag{3.2}$$

then

$$\int_\Omega v\, d\mu + \int_{\Gamma_N} v\, d\nu \ge \int_\Omega \xi\, d\mu + \int_{\Gamma_N} \xi\, d\nu \quad \text{for all } \xi \in K. \qquad \square$$

Lemma 3.2. *Let $\phi \in (\mathcal{M}_b(\Omega))^N$ and $v \in K$. Then (v,ϕ) satisfies* (1.8) *if and only if (v,ϕ) satisfies* (1.9) *and* (1.7).

Proof. Assume that (v,ϕ) satisfies (1.8) and taking $v_\varepsilon \in C^1(\Omega) \cap K$, the approximation of v given by Lemma 2.1, we have

$$\begin{aligned} |\phi|(\Omega) &= \int_\Omega v\, d\mu + \int_{\Gamma_N} v\, d\nu = \lim_{\varepsilon\to 0} \left(\int_\Omega v_\varepsilon\, d\mu + \int_{\Gamma_N} v_\varepsilon\, d\nu \right) \\ &= \lim_{\varepsilon\to 0} \int_\Omega \nabla_{|\phi|} v_\varepsilon \cdot \frac{\phi}{|\phi|}\, d|\phi| = \int_\Omega \nabla_{|\phi|} v \cdot \frac{\phi}{|\phi|}\, d|\phi|. \end{aligned}$$

So

$$\int_\Omega \left(1 - \nabla_{|\phi|} v \cdot \frac{\phi}{|\phi|} \right) d|\phi| = 0. \tag{3.3}$$

Since by Lemma 2.2, we have

$$\left| \nabla_{|\phi|} v \cdot \frac{\phi}{|\phi|} \right| \le |\nabla_{|\phi|} v| \le 1 \quad |\phi|\text{-a.e. in } \Omega,$$

then, by (3.3), we deduce that

$$\nabla_{|\phi|} v \cdot \frac{\phi}{|\phi|} = 1 \quad |\phi|\text{-a.e. in } \Omega.$$

This implies that

$$\nabla_{|\phi|} v = \frac{\phi}{|\phi|} \quad |\phi|\text{-a.e. in } \Omega.$$

Therefore,

$$\phi = |\phi|\frac{\phi}{|\phi|} = |\phi|\nabla_{|\phi|}v \quad |\phi|\text{-a.e. in } \Omega. \tag{3.4}$$

Moreover, if $\Phi \in (\mathcal{M}_b(\Omega))^N$ is such that Φ satisfies the first equality of (1.8), we have

$$\begin{aligned}|\phi|(\Omega) &= \int_\Omega v\,d\mu + \int_{\Gamma_N} v\,d\nu = \lim_{\varepsilon\to 0}\int_\Omega v_\varepsilon\,d\mu + \int_{\Gamma_N} v_\varepsilon\,d\nu \\ &= \lim_{\varepsilon\to 0}\int_\Omega \nabla v_\varepsilon\cdot\frac{\Phi}{|\Phi|}d|\Phi| \le \int_\Omega d|\Phi|.\end{aligned}$$

Hence (1.8) implies (1.7) and (1.9). It is clear that (1.7) implies (1.8); now suppose that (v,ϕ) satisfies (1.9), then $\nabla_{|\phi|}v\cdot\phi/|\phi| = 1$ $|\phi|$-almost everywhere in Ω and we have

$$\begin{aligned}|\phi|(\Omega) &= \int_\Omega \nabla_{|\phi|}v\cdot\frac{\phi}{|\phi|}\,d|\phi| = \lim_{\varepsilon\to 0}\int_\Omega \nabla v_\varepsilon\cdot\frac{\phi}{|\phi|}\,d|\phi| \\ &= \lim_{\varepsilon\to 0}\int_\Omega v_\varepsilon\,d\mu + \int_{\Gamma_N} v_\varepsilon\,d\nu = \int_\Omega v\,d\mu + \int_{\Gamma_N} v\,d\nu.\end{aligned}$$

Thus,

$$|\phi|(\Omega) = \int_\Omega v\,d\mu + \int_{\Gamma_N} v\,d\nu. \qquad \square$$

As a consequence of Lemmas 3.1 and 3.2, we have (1.8) $\Rightarrow$ (1.5) and (1.8) $\Leftrightarrow$ (1.9) $\Leftrightarrow$ (1.7). To prove that (1.5) $\Rightarrow$ (1.8), we consider the following system:

$$\begin{cases} -\nabla\cdot\phi_\varepsilon(\nabla v_\varepsilon) = \mu & \text{in } \Omega, \\ v_\varepsilon = 0 & \text{on } \Gamma_D, \\ \phi_\varepsilon(\nabla v_\varepsilon)\cdot\eta = \nu & \text{on } \Gamma_N, \end{cases} \tag{S$_\varepsilon$}$$

where η is the outward normal to $\partial\Omega$, for any $\varepsilon > 0$ and $x \in \Omega$, $\phi_\varepsilon\colon \mathbb{R}^N \to \mathbb{R}^N$ is given by

$$\phi_\varepsilon(r) = \frac{1}{\varepsilon}((|r|-1)^+)^{(p-1)}\frac{r}{|r|} \quad \text{for all } r \in \mathbb{R}^N \text{ and } x \in \Omega,$$

with $p > N$ fixed. It is not difficult to see that ϕ_ε satisfies the following properties:

- (i) For any $r_1, r_2 \in \mathbb{R}^N$ and $x \in \Omega$, $(\phi_\varepsilon(r_1) - \phi_\varepsilon(r_2))\cdot(r_1 - r_2) \ge 0$.
- (ii) There exist $\varepsilon_0 > 0$ and $C_0 > 1$ such that $\phi_\varepsilon(r)\cdot r \ge |r|^p$ for any $|r| \ge C_0$ and $\varepsilon < \varepsilon_0$.
- (iii) For any $\varepsilon > 0$, $r \in \mathbb{R}^N$ and $x \in \Omega$, $|\phi_\varepsilon(r)| \le \phi_\varepsilon(r)\cdot r$.

We define the following separable and reflexive Banach space for $W^{1,p}(\Omega)$-norm:

$$W^{1,p}_{\Gamma_D}(\Omega) = \{z \in W^{1,p}(\Omega);\ z_{|\Gamma_D} = 0\}.$$

LEMMA 3.3. *For any* $0 < \varepsilon < \varepsilon_0$, *problem* (S_ε) *has a unique solution* v_ε *in the sense that* $v_\varepsilon \in W^{1,p}_{\Gamma_D}(\Omega)$ *and*

$$\int_\Omega \phi_\varepsilon(\nabla v_\varepsilon)\cdot\nabla z\,dx = \int_\Omega z\,d\mu + \int_{\Gamma_N} z\,d\nu, \tag{3.5}$$

for all $z \in W^{1,p}_{\Gamma_D}(\Omega)$.

PROOF. We define the operator $A_\varepsilon\colon W^{1,p}_{\Gamma_D}(\Omega)\to (W^{1,p}_{\Gamma_D}(\Omega))'$ by

$$\langle A_\varepsilon v, z\rangle = \int_\Omega \phi_\varepsilon(\nabla v)\cdot\nabla z\,dx. \tag{3.6}$$

A_ε is monotone, coercive, hemi-continuous and bounded. Indeed, property (i) of ϕ_ε gives the monotonicity.

For any $v, z \in W^{1,p}_{\Gamma_D}(\Omega)$, we have

$$\begin{aligned} |\langle A_\varepsilon(v), z\rangle| &\le \frac{1}{\varepsilon}\int_\Omega |(|\nabla v|-1)^+|^{p-1}|\nabla z|\,dx \\ &\le \frac{1}{\varepsilon}\int_\Omega |(|\nabla v|-1)^+|^{p-1}|\nabla z|\,dx \le \frac{1}{\varepsilon}\int_\Omega |\nabla v|^{p-1}|\nabla z|\,dx \\ &\le \frac{1}{\varepsilon}\biggl(\int_\Omega |\nabla v|^p\,dx\biggr)^{1/p'}\biggl(\int_\Omega |\nabla z|^p\,dx\biggr)^{1/p} \\ &\le \frac{1}{\varepsilon}\|v\|^{p/p'}_{W^{1,p}(\Omega)}\|z\|_{W^{1,p}(\Omega)}, \end{aligned} \tag{3.7}$$

which implies that

$$\|A_\varepsilon(v)\|_{(W^{1,p}_{\Gamma_D}(\Omega))'} \le \frac{1}{\varepsilon}\|v\|^{p/p'}_{W^{1,p}(\Omega)}. \tag{3.8}$$

Let B be a bounded set of $W^{1,p}_{\Gamma_D}(\Omega)$, there exists $M>0$ such that

$$\|A_\varepsilon(v)\|_{(W^{1,p}_{\Gamma_D}(\Omega))'} \le \frac{1}{\varepsilon}M^{p/p'}, \quad \text{for all } v\in B. \tag{3.9}$$

Hence, A_ε is a bounded operator. Moreover, using properties (ii) and (iii) of ϕ_ε, we obtain

$$\begin{aligned} \langle A_\varepsilon(v), v\rangle &= \int_\Omega \phi_\varepsilon(\nabla v)\cdot\nabla v\,dx \\ &= \int_{[|\nabla v|<C_0]} \phi_\varepsilon(\nabla v)\cdot\nabla v\,dx + \int_{[|\nabla v|\ge C_0]} \phi_\varepsilon(\nabla v)\cdot\nabla v\,dx \\ &\ge \int_{[|\nabla v|<C_0]} |\phi_\varepsilon(\nabla v)|\,dx + \int_{[|\nabla v|\ge C_0]} |\nabla v|^p\,dx \\ &\ge \int_{[|\nabla v|\ge C_0]} |\nabla v|^p\,dx. \end{aligned} \tag{3.10}$$

Consequently,

$$\langle A_\varepsilon(v), v\rangle + \int_{[|\nabla v|<C_0]} |\nabla v|^p\,dx \ge \int_\Omega |\nabla v|^p\,dx. \tag{3.11}$$

Thus,

$$\begin{aligned}\langle A_\varepsilon(v), v\rangle &\geq -\int_{[|\nabla v|<C_0]} |\nabla v|^p \mathrm{d}x + \int_\Omega |\nabla v|^p\,dx \\ &\geq -\int_{[|\nabla v|<C_0]} C_0^p\,dx + \int_\Omega |\nabla v|^p\,dx \\ &\geq -C_0^p \mathrm{meas}([|\nabla v|<C_0]) + \int_\Omega |\nabla v|^p\,dx \\ &\geq -C_0^p \mathrm{meas}(\Omega) + \|v\|^p_{W^{1,p}(\Omega)}.\end{aligned} \tag{3.12}$$

Hence,

$$\frac{\langle A_\varepsilon(v), v\rangle}{\|v\|_{W^{1,p}(\Omega)}} \geq -\frac{C_0^p \mathrm{meas}(\Omega)}{\|v\|_{W^{1,p}(\Omega)}} + \|v\|^{p-1}_{W^{1,p}(\Omega)}. \tag{3.13}$$

Since $p>1$, letting $\|v\|_{W^{1,p}(\Omega)} \to +\infty$ in (3.13), it follows that A_ε is coercive.

Now, consider the map $F\colon \mathbb{R}\to\mathbb{R}$ defined by

$$F(\lambda) = \langle A(u+\lambda v), w\rangle = \int_\Omega \phi_\varepsilon(\nabla u + \lambda\nabla v)\cdot\nabla w\,dx, \tag{3.14}$$

with u, v, w in $W^{1,p}_{\Gamma_D}(\Omega)$. We will prove that F is continuous. The functions $x\mapsto \phi_\varepsilon(\nabla u+\lambda\nabla v)\cdot\nabla w$, $\lambda\mapsto\phi_\varepsilon(\nabla u+\lambda\nabla v)\cdot\nabla w$ are measurable almost everywhere in Ω and continuous in $\mathbb{R}$, respectively. Let $(\lambda_n)_{n\in\mathbb{N}}$ be such that $\lambda_n\to\lambda$, so that there exists a constant $c>0$ with $|\lambda_n|\leq c$. Therefore,

$$\begin{aligned}|\phi_\varepsilon(\nabla u+\lambda_n\nabla v)\cdot\nabla w| &\leq |\phi_\varepsilon(\nabla u+\lambda_n\nabla v)||\nabla w| \\ &\leq \frac{1}{\varepsilon}(|\nabla u| + |\lambda_n||\nabla v| + 1)^{p-1}|\nabla w| \\ &\leq \frac{1}{\varepsilon}(|\nabla u| + c|\nabla v| + 1)^{p-1}|\nabla w|.\end{aligned} \tag{3.15}$$

Letting $n\to+\infty$ in (3.15) and using the fact that the function $\lambda\mapsto|\phi_\varepsilon(\nabla u + \lambda\nabla v)\cdot\nabla w|$ is continuous, we obtain

$$|\phi_\varepsilon(\nabla u+\lambda\nabla v)\cdot\nabla w| \leq \frac{1}{\varepsilon}(|\nabla u| + c|\nabla v| + 1)^{p-1}|\nabla w| \in L^1(\Omega). \tag{3.16}$$

Therefore, thanks to the Lebesgue theorem, we can say that F is continuous. Hence, the operator A is hemi-continuous.

Since $p>N$, we have $W^{1,p}_{\Gamma_D}(\Omega)\subset C(\overline{\Omega})$ and the linear form $G\colon W^{1,p}_{\Gamma_D}(\Omega)\to\mathbb{R}$ defined by

$$\langle G, v\rangle = \int_\Omega v\,d\mu + \int_{\Gamma_N} v\,d\nu \tag{3.17}$$

belongs to the dual space of $W^{1,p}_{\Gamma_D}(\Omega)$. So (see for instance [14]), for any $0<\varepsilon<\varepsilon_0$ and $p>N$, there exists $v_\varepsilon\in W^{1,p}_{\Gamma_D}(\Omega)$ such that $A(v_\varepsilon)=G$, i.e.

$$\int_\Omega \phi_\varepsilon(\nabla v_\varepsilon)\cdot\nabla z\,dx = \int_\Omega z\,d\mu + \int_{\Gamma_N} z\,d\nu \quad \text{for all } z\in W^{1,p}_{\Gamma_D}(\Omega). \tag{3.18}$$

Now, suppose that v_ε and $\widetilde{v}_\varepsilon$ are two solutions of (S_ε). For v_ε and $\widetilde{v}_\varepsilon$, we take $z = v_\varepsilon - \widetilde{v}_\varepsilon$ in (3.18) to get

$$\int_\Omega (\phi_\varepsilon(\nabla v_\varepsilon) - \phi_\varepsilon(\nabla \widetilde{v}_\varepsilon)) \cdot (\nabla v_\varepsilon - \nabla \widetilde{v}_\varepsilon)\, dx = 0. \tag{3.19}$$

It follows that there exists a constant $\widetilde{c}$ such that $v_\varepsilon - \widetilde{v}_\varepsilon = \widetilde{c}$ almost everywhere in Ω. Using the fact that $v_\varepsilon = \widetilde{v}_\varepsilon = 0$ on Γ_D, we get $\widetilde{c} = 0$. Thus $v_\varepsilon = \widetilde{v}_\varepsilon$ almost everywhere in Ω. □

LEMMA 3.4. *Let $(v_\varepsilon)_{0<\varepsilon<\varepsilon_0}$ be the sequence of solutions of* (S_ε). *Then*:

(a) $(v_\varepsilon)_{0<\varepsilon<\varepsilon_0}$ *is bounded in* $W^{1,p}_{\Gamma_D}(\Omega)$.
(b) $(\phi_\varepsilon(\nabla v_\varepsilon))_{0<\varepsilon<\varepsilon_0}$ *is bounded in* $(L^1(\Omega))^N$.
(c) *For any Borel set* $B \subseteq \Omega$,

$$\liminf_{\varepsilon\to 0} \bigg(\int_B |\nabla v_\varepsilon|^{p-1}\, dx\bigg)^{1/(p-1)} \le |B|^{1/(p-1)}.$$

PROOF. (a) Taking v_ε as a test function in (3.19) and using the fact that $W^{1,p}(\Omega) \subset C(\overline{\Omega})$, we get the following estimate:

$$\begin{aligned} &\frac{1}{\varepsilon}\int_\Omega (|\nabla v_\varepsilon| - 1)^{+(p-1)} |\nabla v_\varepsilon|\, dx \\ &\quad= \int_\Omega \phi_\varepsilon(\nabla v_\varepsilon)\cdot \nabla v_\varepsilon\, dx = \int_\Omega v_\varepsilon\, d\mu + \int_{\Gamma_N} v_\varepsilon\, d\nu \\ &\quad\le (|\mu|(\Omega) + |\nu|(\Gamma_N))\|v_\varepsilon\|_\infty \le C(|\mu|(\Omega) + |\nu|(\Gamma_N))\|v_\varepsilon\|_{W^{1,p}(\Omega)}. \end{aligned} \tag{3.20}$$

Combining (3.20) and property (ii) of ϕ_ε, for any $0 < \varepsilon < \varepsilon_0$, we get

$$\begin{aligned} \int_\Omega |\nabla v_\varepsilon|^p\, dx &\le \int_{[|\nabla v_\varepsilon| \le C_0]} |\nabla v_\varepsilon|^p\, dx + \int_{[|\nabla v_\varepsilon| > C_0]} |\nabla v_\varepsilon|^p\, dx \\ &\le \int_{[|\nabla v_\varepsilon| \le C_0]} |\nabla v_\varepsilon|^p\, dx + \frac{1}{\varepsilon}\int_\Omega (|\nabla v_\varepsilon| - 1)^{+(p-1)}|\nabla v_\varepsilon|\, dx \\ &\le \int_{[|\nabla v_\varepsilon| \le C_0]} |\nabla v_\varepsilon|^p\, dx + C(|\mu|(\Omega) + |\nu|(\Gamma_N))\|\nabla v_\varepsilon\|_{L^p(\Omega)} \\ &\le C_0^p|\Omega| + C(|\mu|(\Omega) + |\nu|(\Gamma_N))\|\nabla v_\varepsilon\|_{L^p(\Omega)}. \end{aligned} \tag{3.21}$$

Thus, according to the Young inequality, we deduce that

$$\|\nabla v_\varepsilon\|^p_{L^p(\Omega)} \le p' C_0^p|\Omega| + [C(|\mu|(\Omega) + |\nu|(\Gamma_N))]^{p'}, \tag{3.22}$$

which implies that $(\nabla v_\varepsilon)_{0<\varepsilon<\varepsilon_0}$ is bounded in $(L^p(\Omega))^N$. Hence, $(v_\varepsilon)_{0<\varepsilon<\varepsilon_0}$ is bounded in $W^{1,p}_{\Gamma_D}(\Omega)$.

(b) Using (3.20) and property (iii) of ϕ_ε, we deduce that

$$\begin{aligned} \int_\Omega |\phi_\varepsilon(\nabla v_\varepsilon)|\, dx &\le \int_\Omega \phi_\varepsilon(\nabla v_\varepsilon)\cdot\nabla v_\varepsilon\, dx \\ &\le \frac{1}{\varepsilon}\int_\Omega (|\nabla v_\varepsilon| - 1)^{(p-1)}|\nabla v_\varepsilon|\, dx \le C(|\mu|(\Omega) + |\nu|(\Gamma_N))\|\nabla v_\varepsilon\|_{L^p(\Omega)}. \end{aligned} \tag{3.23}$$

So, by (3.22) we deduce that $\phi_\varepsilon(\nabla v_\varepsilon)$ is bounded in $(L^1(\Omega))^N$.

(c) Now, let $B \subseteq \Omega$ be a fixed Borel set. We have

$$\begin{aligned}
(3.24)\quad \|\nabla v_\varepsilon\|_{L^{p-1}(B)} &\le \|(\nabla v_\varepsilon - 1)^+ + 1\|_{L^{p-1}(B)} \\
&\le \|(\nabla v_\varepsilon - 1)^+\|_{L^{p-1}(B)} + |B|^{1/(p-1)} \\
&\le \bigg(\int_B (\nabla v_\varepsilon - 1)^{+(p-1)}|\nabla v_\varepsilon|\,dx\bigg)^{1/(p-1)} + |B|^{1/(p-1)} \\
&\le [\varepsilon C(|\mu|(\Omega) + |\nu|(\Gamma_N))\|\nabla v_\varepsilon\|_{L^p(\Omega)}]^{1/(p-1)} + |B|^{1/(p-1)}.
\end{aligned}$$

Letting $\varepsilon \to 0$ and using the fact that v_ε is bounded in $W^{1,p}(\Omega)$, we obtain

$$\liminf_{\varepsilon\to 0} \bigg(\int_\Omega |\nabla v_\varepsilon|^{p-1}\,dx\bigg)^{1/(p-1)} \le |B|^{1/(p-1)}. \tag{3.25}$$ □

Lemma 3.5. *Under the assumptions of Lemma* 3.4*, there exists a subsequence denoted again by v_ε, such that, as $\varepsilon \to 0$,*

$$v_\varepsilon \to \widetilde{v} \quad \text{uniformly in } \overline{\Omega} \text{ and in } W^{1,\infty}(\Omega)\text{-weak}, \tag{3.26}$$

$$\phi_\varepsilon(\nabla v_\varepsilon) \to \phi \quad \text{in } (\mathcal{M}_b(\Omega))^N\text{-weak}^* \tag{3.27}$$

and

$$\int_\Omega |\phi_\varepsilon(\nabla v_\varepsilon)|\,dx \to |\phi|(\Omega). \tag{3.28}$$

Moreover, $\widetilde{v} \in K$ and $(\widetilde{v}, \phi)$ satisfies (1.8).

Proof. Thanks to Lemma 3.4, there exist $\widetilde{v} \in W^{1,p}_{\Gamma_D}(\Omega)$, $\phi \in \mathcal{M}_b(\Omega)$ and a subsequence denoted again by v_ε, such that (3.26) and (3.27) are fulfilled.

For any $\xi \in \mathcal{C}^1(\Omega) \cap W^{1,\infty}_D(\Omega)$, we have

$$\int_\Omega \phi_\varepsilon(\nabla v_\varepsilon)\cdot\nabla\xi\,dx = \int_\Omega \xi\,d\mu + \int_{\Gamma_N} \xi\,d\nu.$$

Thus, letting $\varepsilon \to 0$, we deduce that

$$\int_\Omega \nabla\xi\cdot\frac{\phi}{|\phi|}\,d|\phi| = \int_\Omega \xi\,d\mu + \int_{\Gamma_N}\xi\,d\nu. \tag{3.29}$$

To prove that $\widetilde{v} \in K$, let us consider $A_\delta = [|\nabla\widetilde{v}| \ge 1 + \delta]$, with arbitrary $\delta > 0$. Since as $\varepsilon \to 0$, $\nabla v_\varepsilon \to \nabla\widetilde{v}$ in $(L^1(\Omega))^N$-weak,

$$\begin{aligned}
(3.30)\qquad (1+\delta)|A_\delta| &\le \int_{A_\delta} |\nabla\widetilde{v}|\,dx \le \liminf_{\varepsilon\to 0}\int_{A_\delta}|\nabla v_\varepsilon|\,dx \\
&\le \liminf_{\varepsilon\to 0}\bigg(\int_{A_\delta}|\nabla v_\varepsilon|^{p-1}\,dx\bigg)^{1/(p-1)}|A_\delta|^{(p-2)/(p-1)}.
\end{aligned}$$

So that, by using the third part of Lemma 3.4, we deduce that $(1+\delta)|A_\delta| \le |A_\delta|$, which implies that $|A_\delta| = 0$. Since δ is arbitrary, we deduce that $|\nabla\widetilde{v}| \le 1$ almost everywhere in Ω. Therefore, $\widetilde{v} \in K$.

To prove (3.28), we see that according to property (iii) of ϕ_ε and (3.26), we have

$$\begin{aligned}(3.31)\quad \limsup_{\varepsilon\to 0}\int_\Omega |\phi_\varepsilon(\nabla v_\varepsilon)|\,dx &\le \limsup_{\varepsilon\to 0}\int_\Omega \phi_\varepsilon(\nabla v_\varepsilon)\cdot\nabla v_\varepsilon\,dx\\ &\le \limsup_{\varepsilon\to 0}\left(\int_\Omega v_\varepsilon\,d\mu+\int_{\Gamma_N} v_\varepsilon\,d\nu\right)\le \int_\Omega \widetilde{v}\,d\mu+\int_{\Gamma_N}\widetilde{v}\,d\nu.\end{aligned}$$

In addition, we have

$$(3.32)\quad \int_\Omega \widetilde{v}\,d\mu+\int_{\Gamma_N}\widetilde{v}\,d\nu = \lim_{\varepsilon\to 0}\int_\Omega \phi_\varepsilon(\nabla v_\varepsilon)\cdot\nabla\widetilde{v}\,dx \le \lim_{\varepsilon\to 0}\int_\Omega |\phi_\varepsilon(\nabla v_\varepsilon)|\,dx.$$

So, (3.31) and (3.32) imply that

$$(3.33)\quad \lim_{\varepsilon\to 0}\int_\Omega |\phi_\varepsilon(\nabla v_\varepsilon)|\,dx = \int_\Omega \widetilde{v}\,d\mu+\int_{\Gamma_N}\widetilde{v}\,d\nu$$

and by (3.27), we get

$$(3.34)\quad |\phi|(\Omega)=\int_\Omega d|\phi| \le \liminf_{\varepsilon\to 0}\int_\Omega |\phi_\varepsilon(\nabla v_\varepsilon)|\,dx = \int_\Omega \widetilde{v}\,d\mu+\int_{\Gamma_N}\widetilde{v}\,d\nu.$$

Using $\widetilde{v}_\varepsilon$, the approximation of $\widetilde{v}$ given by Lemma 2.1, we see that

$$(3.35)\quad \int_\Omega \widetilde{v}\,d\mu+\int_{\Gamma_N}\widetilde{v}\,d\nu = \lim_{\varepsilon\to 0}\int_\Omega \nabla\widetilde{v}_\varepsilon\cdot\frac{\phi}{|\phi|}\,d|\phi| \le \int_\Omega d|\phi| = |\phi|(\Omega).$$

Combining the above inequality with (3.34), we obtain

$$|\phi|(\Omega)=\int_\Omega \widetilde{v}\,d\mu+\int_{\Gamma_N}\widetilde{v}\,d\nu \quad\text{and}\quad \lim_{\varepsilon\to 0}\int_\Omega |\phi(\nabla v_\varepsilon)|\,dx = |\phi|(\Omega). \qquad \square$$

LEMMA 3.6. *Let* $v\in K$ *be a solution of* (1.5). *Then, there exists* $\phi\in(\mathcal{M}_b(\Omega))^N$ *such that* (v,ϕ) *satisfies* (1.8).

PROOF. Let $\widetilde{v}=\lim_{\varepsilon\to 0}\widetilde{v}_\varepsilon$, where $\widetilde{v}_\varepsilon$ is a solution of (S_ε). According to Lemma 3.5, there exists ϕ in $(\mathcal{M}_b(\Omega))^N$ such that

$$(3.36)\quad \int_\Omega \nabla\xi\cdot\frac{\phi}{|\phi|}\,d|\phi| = \int_\Omega \xi\mathrm{d}\mu+\int_{\Gamma_N}\xi\,d\nu \quad \text{for all } \xi\in\mathcal{C}^1(\Omega)\cap W_D^{1,\infty}(\Omega)$$

and

$$(3.37)\quad |\phi|(\Omega)=\int_\Omega \widetilde{v}\,d\mu+\int_{\Gamma_N}\widetilde{v}\,d\nu.$$

Let $v_\varepsilon\in\mathcal{C}^1(\Omega)\cap K$ be the approximation of v given by Lemma 2.1, we have

$$\begin{aligned}(3.38)\quad \int_\Omega v\,d\mu+\int_{\Gamma_N} v\,d\nu &= \lim_{\varepsilon\to 0}\int_\Omega v_\varepsilon\,d\mu+\int_{\Gamma_N} v_\varepsilon\,d\nu\\ &= \lim_{\varepsilon\to 0}\int_\Omega \nabla v_\varepsilon\cdot\frac{\phi}{|\phi|}\,d|\phi| \le \int_\Omega d|\phi| = |\phi|(\Omega).\end{aligned}$$

Since v is a solution of (1.5), we have

$$\int_\Omega v\,d\mu + \int_{\Gamma_N} v\,d\nu \geq \int_\Omega z\,d\mu + \int_{\Gamma_N} z\,d\nu \quad \text{for all } z \in K. \tag{3.39}$$

In particular, taking $z = \widetilde{v}$, we deduce that

$$\int_\Omega v\,d\mu + \int_{\Gamma_N} v\,d\nu \geq \int_\Omega \widetilde{v}\,d\mu + \int_{\Gamma_N} \widetilde{v}\,d\nu = |\phi|(\Omega). \tag{3.40}$$

Consequently, (3.38) and (3.40) imply that

$$|\phi|(\Omega) = \int_\Omega v\,d\mu + \int_{\Gamma_N} v\,d\nu. \qquad \square$$

Proof of Theorem 1.1. Since K is bounded, problem (1.5) admits at least one solution. Thanks to Lemma 3.1, we have that (1.8) implies (1.5). As a consequence of Lemma 3.6, we have that (1.5) implies (1.8). The equivalence between (1.8), (1.9) and (1.7) is given by Lemma 3.2. $\square$

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