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MULTIPLE SOLUTIONS TO THE BAHRI–CORON PROBLEM IN A BOUNDED DOMAIN WITHOUT A THIN NEIGHBORHOOD OF A MANIFOLD

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Abstract. We show that the critical problem

$$-\Delta u = |u|^{4/(N-2)}u \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega,$$

has at least

$$\max\{\mathrm{cat}(\Theta, \Theta \setminus B_rM), \mathrm{cupl}(\Theta, \Theta \setminus B_rM) + 1\} \geq 2$$

pairs of nontrivial solutions in every domain Ω obtained by deleting from a given bounded smooth domain $\Theta \subset \mathbb{R}^N$ a thin enough tubular neighborhood B_rM of a closed smooth submanifold M of Θ of dimension $\leq N-2$, where "cat" is the Lusternik–Schnirelmann category and "cupl" is the cup-length of the pair.

1. Introduction

Let Θ be a bounded smooth domain in $\mathbb{R}^N$, $N \geq 3$, and let M be a compact smooth submanifold of $\mathbb{R}^N$, without boundary, contained in Θ. Consider the problem

$$\begin{cases} -\Delta u = |u|^{2^*-2}u & \text{in } \Theta_r, \\ u = 0 & \text{on } \partial\Theta_r, \end{cases} \tag{1.1}$$

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where $2^* := 2N/(N-2)$ is the critical Sobolev exponent and

$$\Theta_r := \{x \in \Theta : \operatorname{dist}(x, M) > r\}, \quad r > 0.$$

Our aim is to establish multiplicity of solutions for r small.

If M is a point and r is small enough, Coron showed in [9] that this problem has at least one positive solution. The existence of at least two solutions was established by Clapp and Weth in [8]. More recently, Ge, Musso and Pistoia [14] proved that the number of sign changing solutions becomes arbitrarily large as r goes to zero. Their solutions are bubble-towers, i.e. they look like superpositions of standard bubbles with alternating signs concentrating at the point M. Under additional assumptions, positive and sign changing solutions which look like a sum of standard bubbles one of which concentrates at the point M and the others at some points in $\Theta \setminus M$ were constructed in [6]. There are also various results on the existence and shape of solutions to this problem when M is a finite set of points and r is small enough, see e.g. [16], [17], [18], [19].

In contrast to this, if M has positive dimension only few results are known. Hirano and Shioji established the existence of two solutions in an annular domain with a thin straight tunnel in [15]. Some multiplicity results were recently obtained by Clapp, Grossi and Pistoia in [5] when both Θ and M are invariant under the action of some group of symmetries. They also showed that, without any symmetry assumption, this problem has at least $\operatorname{cat}(\Theta, \Theta_r)$ positive solutions for small enough r, where $\operatorname{cat}(\Theta, \Theta_r)$ is the Lusternik-Schnirelmann category of the pair (Θ, Θ_r).

Here we show that for some domains there is an additional solution. We write $\operatorname{cupl}(\Theta, \Theta_r)$ for the cup-length of the pair (Θ, Θ_r). The definitions of category and cup-length are given in appendix A. We prove the following result.

Theorem 1.1. *Assume that* $\dim M \leq N-2$. *Then there exists* $r_0 > 0$ *such that, if* Ω *is a bounded smooth domain in* $\mathbb{R}^N$ *which satisfies*

$$M \cap \overline{\Omega} = \emptyset \quad \textit{and} \quad \Theta_r \subset \Omega \subset \Theta,$$

for some $r \in (0, r_0)$*, then problem*

$$\begin{cases} -\Delta u = |u|^{2^*-2}u & \textit{in } \Omega, \\ u = 0 & \textit{on } \partial\Omega, \end{cases} \tag{1.2}$$

has at least $\max\{\operatorname{cat}(\Theta, \Theta_r), \operatorname{cupl}(\Theta, \Theta_r) + 1\} \geq 2$ *pairs of nontrivial solutions.*

It is well known that $\operatorname{cat}(\Theta, \Theta_r) \geq \operatorname{cupl}(\Theta, \Theta_r)$ (see Lemma A.3). So Theorem 1.1 improves Corollary 1.2 in [5] when $\operatorname{cat}(\Theta, \Theta_r) = \operatorname{cupl}(\Theta, \Theta_r)$. There are some interesting situations in which this occurs. For example, the following ones.

EXAMPLE 1.2. If M is contractible in Θ, then $\mathrm{cat}(\Theta, \Theta_r) = \mathrm{cupl}(\Theta, \Theta_r) = 1$ for r small enough.

EXAMPLE 1.3. If Θ is a tubular neighborhood of M and $\mathrm{cat}(M) = \mathrm{cupl}(M)$, then $\mathrm{cat}(\Theta, \Theta_r) = \mathrm{cupl}(\Theta, \Theta_r) = \mathrm{cupl}(M)$ for r small enough.

Examples of manifolds M such that $\mathrm{cat}(M) = \mathrm{cupl}(M)$ are those having the homotopy type of a sphere $\mathbb{S}^k$, of a real $\mathbb{R}P^k$, a complex $\mathbb{C}P^k$ or a quaternionic $\mathbb{H}P^k$ projective space, or of a product of such spaces.

Note that both $\mathrm{cat}(\Theta, \Theta_r)$ and $\mathrm{cupl}(\Theta, \Theta_r)$ depend on the embedding of M into Θ. For example, if M is the circle $C := \{(x_1, x_2, 0) \in \mathbb{R}^3 : x_1^2 + x_2^2 = 1\}$ and Θ is the torus $\{x \in \mathbb{R}^3 : \mathrm{dist}(x, C) < 1/2\}$, then Θ is a tubular neighborhood of M and Example 1.3 gives

$$\mathrm{cat}(\Theta, \Theta_r) = \mathrm{cupl}(\Theta, \Theta_r) = \mathrm{cupl}(\mathbb{S}^1) = 2$$

for $r \in (0, 1/2)$. On the other hand, if M is the circle $\{(x_1, 0, x_3) \in \mathbb{R}^3 : (x_1 - 1)^2 + x_3^2 = 1/4\}$, then Example 1.2 gives $\mathrm{cat}(\Theta, \Theta_r) = \mathrm{cupl}(\Theta, \Theta_r) = 1$ for $r \in (0, 1/4)$. Theorem 1.1 asserts the existence of three solutions in the first case, and two solutions in the second one.

As we shall show in Proposition 4.1, at least $\mathrm{cat}(\Theta, \Theta_r) \geq 1$ solutions are positive. Our methods do not allow us to conclude whether the additional solution is sign changing or not.

We wish to stress the fact that multiplicity results for problem (1.2) are only available for some particular types of domains. A remarkable result obtained by Bahri and Coron [1] establishes the existence of at least one positive solution to (1.2) in every domain Ω having nontrivial reduced homology with $\mathbb{Z}/2$-coefficients. One expects to have multiple solutions in every domain of this type, but the proof of this fact remains open. Classical variational methods cannot be applied to establish multiplicity due to the lack of compactness of the associated energy functional. Under suitable symmetry assumptions compactness is restored: if Ω is invariant under the action of a group G of linear isometries of $\mathbb{R}^N$ and every G-orbit in Ω is infinite, problem (1.2) is known to have infinitely many G-invariant solutions [3]. Recently, Clapp and Faya [4] considered domains having finite G-orbits and gave conditions for the existence of a prescribed number of solutions. In a non-symmetric setting, the Lyapunov–Schmidt reduction method has been successfully applied to obtain multiplicity results for problem (1.1) when M is a point or a finite set (see [14] and the references therein), but this method becomes very hard to apply when M has positive dimension.

Our proof of Theorem 1.1 uses variational methods and some tools from algebraic topology which include the fixed point transfer introduced by Dold in [10]. Key elements of our variational approach are a refinement of the deformation lemma which was proved in [8] and a lower bound for the energy of sign changing solutions to the limit problem in $\mathbb{R}^N$ obtained by Weth in [23]. These results are stated in Section 2. Section 3 is devoted to the construction of two auxiliary maps which play an important role in the proof of Theorem 1.1. The proof of this theorem is given in Section 4. Finally, in Appendix A we recall the definition and properties of the Lusternik–Schnirelmann category and the cup-length, and prove the statements of Examples 1.2 and 1.3.

2. Preliminaries and notation

Let Ω be a bounded smooth domain in $\mathbb{R}^N$. We consider the Sobolev space $H_0^1(\Omega)$ with the norm

$$\|u\| := \bigg(\int_\Omega |\nabla u|^2\bigg)^{1/2}.$$

We write $|u|_p$ for the L^p-norm of u, $1 \le p \le \infty$.

The solutions to problem (1.2) are the critical points of the energy functional $J\colon H_0^1(\Omega) \to \mathbb{R}$ given by

$$J(u) := \frac{1}{2}\|u\|^2 - \frac{1}{2^*}|u|_{2^*}^{2^*}.$$

The nontrivial solutions are the critical points of the restriction of J to the Nehari manifold

$$\mathcal{N} := \{u \in H_0^1(\Omega) : u \neq 0,\ \|u\|^2 = |u|_{2^*}^{2^*}\},$$

which is a $\mathcal{C}^2$-manifold, radially diffeomorphic to the unit sphere in $H_0^1(\Omega)$.

Recall that J is said to satisfy the Palais–Smale condition $(\mathrm{PS})_c$ on $\mathcal{N}$ at the level $c \in \mathbb{R}$ if every sequence (u_k) in $\mathcal{N}$ such that $J(u_k) \to c$ and $\nabla_{\mathcal{N}} J(u_k) \to 0$ contains a convergent subsequence. Here $\nabla_{\mathcal{N}} J$ denotes the gradient of the restriction of J to $\mathcal{N}$, i.e. $\nabla_{\mathcal{N}} J(u)$ is the orthogonal projection of $\nabla J(u)$ onto the tangent space to $\mathcal{N}$ at u.

We write $\mathcal{C}_0^1(\overline{\Omega})$ for the Banach space of $\mathcal{C}^1$-functions on $\overline{\Omega}$ which vanish on $\partial\Omega$, endowed with the norm

$$\|u\|_{\mathcal{C}^1} := |u|_\infty + |\nabla u|_\infty.$$

For $d \in \mathbb{R}$ we write $\mathcal{N}^d := \{u \in \mathcal{N} : J(u) \le d\}$. The following refinement of the deformation lemma was proved in [8, Lemma 1].

Lemma 2.1. *Assume that J has no critical values in the interval $[b,d]$ and that it satisfies $(\mathrm{PS})_c$ for every $b \le c \le d$. Then there exists a continuous map $\eta\colon [0,1] \times \mathcal{N}^d \to \mathcal{N}^d$ with the following properties*:

(a) *$\eta(0,u) = u$ and $\eta(1,u) \in \mathcal{N}^b$ for every $u \in \mathcal{N}^d$, and $\eta(t,v) = v$ for every $v \in \mathcal{N}^b$, $t \in [0,1]$.*

(b) *If $u \in \mathcal{N}^d \cap \mathcal{C}_0^1(\overline{\Omega})$, then $\eta(t,u) \in \mathcal{C}_0^1(\overline{\Omega})$ for every $t \in [0,1]$.*

(c) *If* $B \subset \mathcal{N}^d \cap \mathcal{C}_0^1(\overline{\Omega})$ *is bounded in* $\mathcal{C}_0^1(\overline{\Omega})$, *then* $\widehat{B} := \{\eta(t,u) : u \in B,\ t \in [0,1]\}$ *is bounded in* $\mathcal{C}_0^1(\overline{\Omega})$.

(d) *If* $u \in \mathcal{N}^d \cap \mathcal{C}_0^1(\overline{\Omega})$ *and* $u \geq 0$, *then* $\eta(t,u) \geq 0$ *for every* $t \in [0,1]$.

Next, we consider the limit problem

$$-\Delta u = |u|^{2^*-2}u, \qquad u \in D^{1,2}(\mathbb{R}^N), \tag{2.1}$$

and write $\|u\| := (\int_{\mathbb{R}^N} |\nabla u|^2)^{1/2}$ for the norm in $D^{1,2}(\mathbb{R}^N)$. The energy functional $J_\infty : D^{1,2}\left(\mathbb{R}^N\right) \to \mathbb{R}$ associated to (2.1) is given by

$$J_\infty(u) := \frac{1}{2}\|u\|^2 - \frac{1}{2^*}|u|_{2^*}^{2^*},$$

and the Nehari manifold is

$$\mathcal{N}_\infty := \{u \in D^{1,2}(\mathbb{R}^N) : u \neq 0,\ \|u\|^2 = |u|_{2^*}^{2^*}\}.$$

As usual we consider $H_0^1(\Omega)$ as a Hilbert subspace of $D^{1,2}(\mathbb{R}^N)$ via trivial extensions. Then J is the restriction of J_∞ to $H_0^1(\Omega)$ and $\mathcal{N} = \mathcal{N}_\infty \cap H_0^1(\Omega)$. The radial projection $\rho\colon D^{1,2}(\mathbb{R}^N) \setminus \{0\} \to \mathcal{N}_\infty$ onto the Nehari manifold is given by

$$\rho(u) = \left(\frac{\|u\|^2}{|u|_{2^*}^{2^*}}\right)^{(N-2)/4} u.$$

Set $c_\infty := \inf\limits_{\mathcal{N}_\infty} J_\infty$. The standard bubbles

$$U_{\lambda,y}(x) = [N(N-2)]^{(N-2)/4}\frac{\lambda^{(N-2)/2}}{(\lambda^2 + |x-y|^2)^{(N-2)/2}}, \quad \lambda \in (0,\infty), \quad y \in \mathbb{R}^N,$$

are the only positive solutions to problem (2.1). They satisfy $J(U_{\lambda,y}) = c_\infty$. It is a well known fact that

$$\inf_{\mathcal{N}} J = \inf_{\mathcal{N}_\infty} J_\infty = c_\infty,$$

independently of Ω, and that c_∞ is not attained by J on $\mathcal{N}$ if Ω is bounded, see e.g. [22], [24].

We consider the barycenter map $\beta\colon H_0^1(\Omega) \setminus \{0\} \to \mathbb{R}^N$, given by

$$\beta(u) := \frac{\displaystyle\int_{\mathbb{R}^N} x|u(x)|^{2^*}\,dx}{\displaystyle\int_{\mathbb{R}^N} |u(x)|^{2^*}\,dx}.$$

The following fact will be used below.

Lemma 2.2. *Let* X *be a closed subset of* $\mathbb{R}^N$ *such that* $\overline{\Omega} \cap X = \emptyset$. *Then*

$$c_X := \inf\{J(u) : u \in \mathcal{N},\ \beta(u) \in X\} > c_\infty.$$

Proof. Arguing by contradiction, assume there exist $u_k \in \mathcal{N}$ with $\beta(u_k) \in X$ and $J(u_k) \to c_\infty$. Using Ekeland's variational principle [12], [24], we may assume that (u_k) is a $(\mathrm{PS})_{c_\infty}$ sequence. Then, by Struwe's global compactness theorem [21], [24], there exist $y_k \in \Omega$ and $\lambda_k > 0$ such that, after passing to a subsequence,

$$\|u_k - U_{\lambda_k,y_k}\| \to 0 \quad \text{as} \quad k \to \infty.$$

It follows that $|\beta(u_k) - y_k| \to 0$ as $k \to \infty$ and, hence, that $\operatorname{dist}(\beta(u_k),\Omega) \to 0$ as $k \to \infty$. This is a contradiction. $\square$

We shall also use the following result, which was proved by Weth in [23].

Theorem 2.3. *There exists an $\varepsilon_0 > 0$ such that $J_\infty(u) > 2c_\infty + 3\varepsilon_0$ for every sign changing solution u of problem* (2.1).

3. Two auxiliary maps

Let Θ be a bounded smooth domain in $\mathbb{R}^N$ and let M be a compact smooth submanifold of Θ, without boundary, such that $\dim M \le N-2$. We write

$$\mathrm{d}(x) := \operatorname{dist}(x, M).$$

For $a > 0$ we set $B_aM := \{x \in \mathbb{R}^N : \mathrm{d}(x) < a\}$, and write $\overline{B}_aM$ for its closure and S_aM for its boundary in $\mathbb{R}^N$.

We fix $R > 0$ such that $\overline{B}_RM$ is a tubular neighborhood of M in $\mathbb{R}^N$ and $\overline{B}_RM \subset \Theta$. Then, the following statement holds true.

Lemma 3.1. *For any given $a \in (0,R)$ and $\varepsilon > 0$ there exists $a_0 \in (0,a)$ such that, for every $\delta > 0$, there is a continuous function $h_\delta\colon \mathbb{R}^N \to D^{1,2}(\mathbb{R}^N)$ with the following properties*:

(a) $h_\delta(y) \in \mathcal{N}_\infty \cap \mathcal{C}_c^\infty(\mathbb{R}^N)$ *and* $h_\delta(y) \ge 0$ *for all* $y \in \mathbb{R}^N$,

(b) *The $\mathcal{C}^1$-norm of $h_\delta(y)$ is uniformly bounded on $\mathbb{R}^N$, i.e.*

$$\sup\{|h_\delta(y)|_\infty + |\nabla(h_\delta(y))|_\infty : y \in \mathbb{R}^N\} < \infty.$$

(c) $J_\infty(h_\delta(y)) \le c_\infty + \varepsilon$ *for all* $y \in \mathbb{R}^N$,

(d) $J_\infty(h_\delta(y)) \le c_\infty + \delta$ *for all* $y \in \mathbb{R}^N \setminus B_aM$.

(e) $\operatorname{supp}(h_\delta(y)) \subset \overline{B}_{a+\delta}M \setminus B_{a_0}M$ *for all* $y \in \overline{B}_aM$,

(f) $\operatorname{supp}(h_\delta(y)) \subset B_\delta(y)$ *for all* $y \in \mathbb{R}^N \setminus B_aM$,

(g) $\beta(h_\delta(y)) = y$ *for all* $y \in \mathbb{R}^N \setminus B_aM$,

(h) $\beta(h_\delta(y)) \in \overline{B}_aM$ *for all* $y \in \overline{B}_aM$, *and there is a continuous map* $\vartheta\colon [0,1] \times \overline{B}_aM \to \overline{B}_aM$ *such that* $\vartheta(0,y) = y$, $\vartheta(1,y) = \beta(h_\delta(y))$ *for all* $y \in \overline{B}_aM$, *and* $\vartheta(t,z) = z$ *for all* $z \in S_aM$.

Proof. Let $\chi \in \mathcal{C}_c^\infty(\mathbb{R}^N)$ be a radial function such that $\chi(x) = 1$ if $|x| \le a/8$, $\chi(x) \in (0,1]$ if $|x| < a/4$ and $\chi(x) = 0$ if $|x| \ge a/4$. Fix $\mu > 0$ so that the function

$$w := \rho(\chi U_{\mu,0})$$

satisfies $J_\infty(w) \le c_\infty + \min\{\delta, \varepsilon/2\}$, were $U_{\mu,0}$ is the standard bubble and ρ is the radial projection onto $\mathcal{N}_\infty$. For $\lambda > 0$ and $y \in \mathbb{R}^N$ we define

$$w_{\lambda,y}(x) := \lambda^{(2-N)/2} w\left(\frac{x-y}{\lambda}\right).$$

Then $w_{\lambda,y} \in \mathcal{N}_\infty \cap \mathcal{C}_c^\infty(\mathbb{R}^N)$, $J_\infty(w_{\lambda,y}) = J_\infty(w) \le c_\infty + \min\{\delta, \varepsilon/2\}$, $\operatorname{supp}(w_{\lambda,y}) \subset B_{\lambda a/4}(y)$ and $\beta(w_{\lambda,y}) = y$.

Set $\gamma := \min\{\delta, a/4\} > 0$ and choose a nonincreasing function $\Lambda \in \mathcal{C}^\infty[0,\infty)$ such that $\Lambda(t) = 1$ if $t \le a/2$, $\Lambda(t) = 4\gamma/a$ if $t \ge a$ and $\Lambda(t) \le 4(a+\gamma-t)/a$ for all $t \le a$. Define $\widetilde{h}\colon \mathbb{R}^N \to \mathcal{N}_\infty \cap \mathcal{C}_c^\infty(\mathbb{R}^N)$ as

$$\widetilde{h}(y) := w_{\Lambda(\mathrm{d}(y)),y}.$$

Note that $\widetilde{h}(y) = w_{1,y}$ if $\mathrm{d}(y) \le a/2$, $\operatorname{supp}(\widetilde{h}(y)) \subset B_\gamma(y)$ if $\mathrm{d}(y) \ge a$ and $\operatorname{supp}(\widetilde{h}(y)) \subset B_{a-\mathrm{d}(y)+\gamma}(y) \subset \overline{B}_{a+\delta}M$ for all $y \in \overline{B}_a M$.

Fix $a_1 \in (0, a/4)$. Since $\dim M \le N-2$, the 2-capacity of M in $\mathbb{R}^N$ is zero, see [13, 4.7 Theorem 3]. Hence, for $k > 1/a_1$, there are functions $\psi_k \in \mathcal{C}_c^\infty(\mathbb{R}^N)$ such that $\psi_k(x) = 1$ if $\mathrm{d}(x) \le 1/k$, $\psi_k(x) = 0$ if $\mathrm{d}(x) \ge a_1$, and $\|\psi_k\| \to 0$ as $k \to \infty$. Then, $[1-\psi_k]\widetilde{h}(y) = \widetilde{h}(y) \ne 0$ if $\mathrm{d}(y) \ge a/2$ and $[1-\psi_k]\widetilde{h}(y) = [1-\psi_k]w_{1,y} \ne 0$ if $\mathrm{d}(y) \le a/2$, because $w_{1,y} > 0$ in $B_{a/4}(y)$ and $B_{a/4}(y) \setminus \overline{B}_{a_1}M \ne \emptyset$. Therefore, the function $\widetilde{h}_k\colon \mathbb{R}^N \to \mathcal{N}_\infty \cap \mathcal{C}_c^\infty(\mathbb{R}^N)$ given by

$$\widetilde{h}_k(y) := \rho([1-\psi_k]\widetilde{h}(y))$$

is well defined. It satisfies that $\operatorname{supp}(\widetilde{h}_k(y)) \subset \overline{B}_{a+\delta}M \setminus B_{1/k}M$ for all $y \in \overline{B}_a M$. Moreover, since

$$\begin{aligned}\|[1-\psi_k]w_{1,y} - w_{1,y}\|^2 = \|\psi_k w_{1,y}\|^2 &= \int_{\mathbb{R}^N} |w_{1,y}\nabla\psi_k - \psi_k \nabla w_{1,y}|^2 \\ &\le 4\int_{\mathbb{R}^N} |w_{1,y}|^2|\nabla\psi_k|^2 + |\psi_k|^2|\nabla w_{1,y}|^2 \le C(\|\psi_k\|^2 + |\psi_k|_2^2) \to 0\end{aligned}$$

as $k \to \infty$, the set

$$\mathcal{K} = \{w_{1,y} : y \in \overline{B}_{a/2}M\} \cup \{[1-\psi_k]w_{1,y} : y \in \overline{B}_{a/2}M,\ k > 1/a_1\}$$

is compact in $D^{1,2}(\mathbb{R}^N)$ and the functions $J_\infty \circ \rho$ and $\beta \circ \rho$ are uniformly continuous on $\mathcal{K}$. This implies that there exists $k_0 > 1/a_1$ such that

$$\left|J_\infty(\widetilde{h}_k(y)) - J_\infty(\widetilde{h}(y))\right| < \frac{\varepsilon}{2} \quad \text{and} \quad \left|\beta(\widetilde{h}_k(y)) - y\right| < \frac{a}{2}$$

for all $k \ge k_0$ and all $y \in \mathbb{R}^N$. Set

$$h_\delta(y) := \widetilde{h}_{k_0}(y) \quad \text{and} \quad a_0 := \frac{1}{k_0}.$$

Then, $\operatorname{supp}(\widetilde{h}_\delta(y)) \subset \overline{B}_{a+\delta}M \setminus B_{a_0}M$ for all $y \in \overline{B}_a M$ and, since $J_\infty(\widetilde{h}(y)) \le c_\infty + \varepsilon/2$, we have that

$$J_\infty(h_\delta(y)) \le c_\infty + \varepsilon \quad \text{for all } y \in \mathbb{R}^N,$$

Clearly, $h_\delta(y)$ satisfies (a) and (b) and, since $h_\delta(y) = \widetilde{h}(y)$ if $\mathrm{d}(y) \geq a/2$, it also satisfies properties (d), (f) and (g). The map $\vartheta(t,y) := (1-t)y + t\beta(h_\delta(y))$ is well defined and has the properties stated in (h). $\square$

Since $\overline{B}_R M$ is a tubular neighborhood of M in $\mathbb{R}^N$, for every $x \in \overline{B}_R M$ there is a unique point $q(x) \in M$ such that

$$\mathrm{d}(x) = |x - q(x)|. \tag{3.1}$$

The map $q\colon \overline{B}_R M \to M$ is well defined and smooth. It is convenient sometimes to write the elements of $\overline{B}_R M$ as

$$[\zeta, t] := q(\zeta) + \frac{t}{R}(\zeta - q(\zeta)) \quad \text{with } \zeta \in S_R M \text{ and } t \in [0,R]. \tag{3.2}$$

Define $\mathcal{E} := \{u \in \mathcal{N} : u^+, u^- \in \mathcal{N}\}$, where $u^+ := \max\{u, 0\}$ and $u^- := \min\{u, 0\}$. We prove the following statement.

Lemma 3.2. *Given $\varepsilon \in (0, c_\infty)$ there exist $b_0, b_1, b_2 \in (0,R)$ such that, if Ω is a bounded smooth domain in $\mathbb{R}^N$ which satisfies*

$$M \cap \overline{\Omega} = \emptyset \quad \text{and} \quad (\Theta \setminus B_r M) \subset \Omega \subset \Theta,$$

for some $r \in (0, b_0)$, and $c_\infty < c_0 < c_1 < c_\infty + \varepsilon$ are such that J has no critical values in $(c_1, c_\infty + \varepsilon]$, then there exists a continuous map $G : \overline{B}_{b_1} M \times \overline{B}_{b_2} M \to \mathcal{E}$ with the following properties:

(a) $J(G(x,y)) \leq 2c_\infty + 2\varepsilon$ *for all* $(x,y) \in \overline{B}_{b_1} M \times \overline{B}_{b_2} M$,
(b) $J(G(x,y)) \leq c_0 + c_1$ *for all* $(x,y) \in (S_{b_1} M \times \overline{B}_{b_2} M) \cup (\overline{B}_{b_1} M \times S_{b_2} M)$,
(c) $J(G(x,y)^+) \leq c_0$ *and* $\beta(G(x,y)^+) = x$ *for all* $(x,y) \in (S_{b_1} M \times \overline{B}_{b_2} M)$,
(d) $J(G(x,y)^-) \leq c_0$ *and* $\beta(G(x,y)^-) = y$ *for all* $(x,y) \in (\overline{B}_{b_1} M \times S_{b_2} M)$.

Proof. Fix $a_1 \in (0,R)$. For $a := a_1$ and the given ε, let $a_{1,0} \in (0, a_1)$ be as in Lemma 3.1. Now fix $a_2 \in (0, a_{1,0})$ and, for $a := a_2$ and the same ε, let $a_{2,0} \in (0, a_2)$ be as in Lemma 3.1. Set $b_0 := a_{2,0}$ and $r \in (0, b_0)$. Let

$$\delta := \frac{1}{4} \min\{R - a_1,\ a_{1,0} - a_2, a_2 - b_0,\ b_0 - r,\ 2\varepsilon,\ c_0 - c_\infty,\ c_\infty + \varepsilon - c_1\}$$

and choose a function $h^i_\delta : \mathbb{R}^N \to D^{1,2}(\mathbb{R}^N)$ with the properties stated in Lemma 3.1 for a_i, ε and $a_{i,0}$. Since

$$\mathrm{supp}(h^i_\delta(x)) \subset (\overline{B}_{a_i + \delta} M \setminus B_{a_{i,0}} M) \subset (\overline{B}_R M \setminus B_{a_{i,0}} M) \quad \text{for all } x \in \overline{B}_{a_i} M,$$

we have that

$$h^i_\delta\left(\overline{B}_{a_i} M\right) \subset \mathcal{N} \cap \mathcal{C}^1_0(\overline{\Omega}) \quad \text{if } r \in (0, b_0), \quad i = 1, 2,$$

In addition, $h^i_\delta(x) \geq 0$ and $J(h^i_\delta(x)) \leq c_\infty + \varepsilon$ for all $x \in \overline{B}_{a_i} M$; $J(h^i_\delta(y)) \leq c_\infty + \delta$, $\mathrm{supp}(h^i_\delta(y)) \subset B_\delta(y)$ and $\beta(h^i_\delta(y)) = y$ for all $y \in S_{a_i} M$; and the set $\{h^i_\delta(x) : x \in \overline{B}_{a_i} M\}$ is bounded in $\mathcal{C}^1_0(\overline{\Omega})$.

Since J satisfies $(\mathrm{PS})_c$ at every $c \in (c_\infty, 2c_\infty)$ and J has no critical values in $(c_1, c_\infty + \varepsilon]$, Lemma 2.1 yields a deformation

$$\eta\colon [0,1] \times \mathcal{N}^{c_\infty+\varepsilon} \to \mathcal{N}^{c_\infty+\varepsilon}$$

such that $\eta(0,u) = u$ and $\eta(1,u) \in \mathcal{N}^{c_1+\delta}$ for every $u \in \mathcal{N}^{c_\infty+\varepsilon}$; $\eta(t,v) = v$ for every $v \in \mathcal{N}^{c_1+\delta}$, $t \in [0,1]$; and $\eta(t, h^i_\delta(x)) \geq 0$ for every $x \in \overline{B}_{a_i}M$. Moreover, the sets

$$\mathcal{K}_i := \{\eta(t, h^i_\delta(x)) : x \in \overline{B}_{a_i}M,\ t \in [0,1]\}, \quad i = 1,2,$$

are compact in $H^1_0(\Omega)$ and, by statement (c) in Lemma 2.1, they are bounded in the $\mathcal{C}^1$-norm.

Fix a radial function $\phi \in \mathcal{C}^\infty(\mathbb{R}^N)$ such that $\phi(x) = 1$ if $|x| \geq 2$ and $\phi(x) = 0$ if $|x| \leq 1$. Then, there is a $\gamma \in (0, \delta/2)$ such that the function $\phi_x(y) := \phi((y-x)/\gamma)$ satisfies

$$\phi_x u \neq 0 \quad \text{and} \quad |J(\rho(\phi_x u)) - J(u)| < \delta, \quad \text{for all } (x,u) \in \mathbb{R}^N \times (\mathcal{K}_1 \cup \mathcal{K}_2),$$

see [8, Lemma 2]. Note that $\phi_x u \equiv 0$ in $\overline{B}_\gamma(x)$ for every $u \in \mathcal{K}_1 \cup \mathcal{K}_2$, $x \in \mathbb{R}^N$.

Next, we choose $\lambda_i \in \mathcal{C}^\infty[0,\infty)$ nonincreasing and such that $\lambda_i(t) = 1$ if $t \leq a_i$ and $\lambda_i(t) = \gamma/\delta$ if $t \geq a_i + \delta$. Using the notation introduced in (3.2), we define $G_i\colon \overline{B}_{a_i+2\delta}M \to \mathcal{N}$ as follows:

$$G_i([\zeta,t]) := \begin{cases} h^i_\delta([\zeta,t]) & \text{if } t \in [0,a_i],\ \zeta \in S_R M, \\ \big(h^i_\delta([\zeta,a_i])\big)_{\lambda(t),[\zeta,t]-\lambda(t)[\zeta,a_i]} & \text{if } t \in [a_i, a_i+2\delta],\ \zeta \in S_R M, \end{cases}$$

where $u_{\lambda,x}(y) := \lambda^{(2-N)/2} u((y-x)/\lambda)$. Then, $J(G_i(y)) \leq c_\infty + \delta$ and $\beta(G_i(y)) = y$ if $a_i \leq \mathrm{d}(y) \leq a_i + 2\delta$, and $\operatorname{supp}(G_i(y)) \subset \overline{B}_\gamma(y)$ if $a_i + \delta \leq \mathrm{d}(y) \leq a_i + 2\delta$.

We define $G\colon \overline{B}_{a_1+2\delta}M \times \overline{B}_{a_2+2\delta}M \to \mathcal{E}$ in the following way: for $x \in \overline{B}_{a_1+2\delta}$, $y \in \overline{B}_{a_2+2\delta}M$, let

$$G(x,y) := \begin{cases} G_1(x) - G_2(y) & \text{if } \mathrm{d}(x) \leq a_1+\delta,\ \mathrm{d}(y) \leq a_2+\delta, \\ \rho\Big[\phi_y \eta\Big(\dfrac{\mathrm{d}(y)-a_2-\delta}{\delta}, G_1(x)\Big)\Big] - G_2(y) & \\ & \text{if } \mathrm{d}(x) \leq a_1+\delta,\ a_2+\delta \leq \mathrm{d}(y), \\ G_1(x) - \rho\Big[\phi_x \eta\Big(\dfrac{\mathrm{d}(x)-a_1-\delta}{\delta}, G_2(y)\Big)\Big] & \\ & \text{if } a_1+\delta \leq \mathrm{d}(x),\ \mathrm{d}(y) \leq a_2+\delta, \\ G_1(x) - G_2(y) & \text{if } a_1+\delta \leq \mathrm{d}(x),\ a_2+\delta \leq \mathrm{d}(y). \end{cases}$$

Note that in all four cases, the first summand and the second summand in the definition of $G(x,y)$ have disjoint supports. Therefore, the first summand is $G(x,y)^+$ and the second one is $G(x,y)^-$. Since both summands belong to $\mathcal{N}$ we conclude that $G(x,y) \in \mathcal{E}$. Moreover,

$$J(G(x,y)) = J(G(x,y)^+) + J(G(x,y)^-).$$

Setting $b_i := a_i + 2\delta$, one can easily check that G has the desired properties. $\square$

4. Proof of Theorem 1.1

As before, we fix R small enough so that $\overline{B}_R M$ is a tubular neighborhood of M contained in Θ. Fix $\varrho \in (0, \mathrm{dist}(\overline{B}_R M, \partial\Theta))$ small enough so that $\overline{B}_\varrho(\partial\Theta)$ is a tubular neighborhood of $\partial\Theta$, and set $\Theta^- := \Theta \backslash B_\varrho(\partial\Theta)$ and $\Theta^+ := \Theta \cup B_\varrho(\partial\Theta)$. Define

$$d^* := \inf\{J_\infty(u) : u \in \mathcal{N}_\infty \cap H_0^1(\Theta),\ \beta(u) \notin \Theta^+\}.$$

By Lemma 2.2 we have that $d^* > c_\infty$.

Choose $\varepsilon_0 \in (0, c_\infty/3)$ as in Theorem 2.3 and such that $c_\infty + \varepsilon_0 < d^*$. For $\varepsilon := 3\varepsilon_0/2$ fix $b_0, b_1, b_2 \in (0, R)$ as in Lemma 3.2, and for $a := b_1$ and $\varepsilon := \varepsilon_0$ fix $a_0 \in (0, a)$ as in Lemma 3.1.

Set $r_0 := \min\{a_0, b_0\}$ and let Ω be a bounded smooth domain in $\mathbb{R}^N$ which satisfies

$$M \cap \overline{\Omega} = \emptyset \quad \text{and} \quad (\Theta \setminus \overline{B}_r M) \subset \Omega \subset \Theta$$

for some $r \in (0, r_0)$. Set $r_1 := \mathrm{dist}(\overline{\Omega}, M)/2$ and define

$$c^* := \inf\{J(u) : u \in \mathcal{N},\ \beta(u) \in \overline{B}_{r_1} M\}.$$

By Lemma 2.2 we have that $c^* > c_\infty$. Now fix a regular value c_0 of J such that

$$c_\infty < c_0 < \min\{c^*, c_\infty + \varepsilon_0\}.$$

Note that

$$\begin{aligned} &\beta(u) \in \Theta^+ && \text{for all } u \in \mathcal{N}^c \text{ with } c \in (0, d^*), \\ &\beta(u) \in \Theta^+ \setminus \overline{B}_{r_1} M && \text{for all } u \in \mathcal{N}^{c_0}. \end{aligned}$$

Let $\mathcal{H}^*$ be Čech cohomology with $\mathbb{Z}/2$-coefficients and define

$$c_1 := \inf\{c \in [c_0, d^*) : \beta^* \colon \mathcal{H}^*(\Theta^+, \Theta^+ \setminus B_{r_1} M) \to \mathcal{H}^*(\mathcal{N}^c, \mathcal{N}^{c_0}) \\ \text{is a monomorphism}\}.$$

For these data all statements below hold true.

Proposition 4.1. *$c_0 < c_1 \le c_\infty + \varepsilon_0$, and problem* (1.2) *has at least*

$$\mathrm{cat}(\Theta, \Theta \setminus B_r M) \ge 1$$

positive solutions with energy in $[c_0, c_\infty + \varepsilon_0]$, and at least

$$\mathrm{cupl}(\Theta, \Theta \setminus B_r M) \ge 1$$

positive solutions with energy in $[c_0, c_1]$. Moreover, c_1 is a critical value of J.

Proof. Since $c_0 \in (c_\infty, 2c_\infty)$ and $(\mathrm{PS})_c$ holds true for every $c \in (c_\infty, 2c_\infty)$, there exist $\alpha > 0$ and a deformation of $\mathcal{N}^{c_0+\alpha}$ into $\mathcal{N}^{c_0}$ which keeps $\mathcal{N}^{c_0}$ fixed. Hence $\mathcal{H}^*(\mathcal{N}^{c_0+\alpha}, \mathcal{N}^{c_0}) = 0$. On the other hand, the inclusion $i\colon (\overline{B}_{r_1}M, S_{r_1}M) \hookrightarrow (\Theta^+, \Theta^+ \setminus B_{r_1}M)$ induces an isomorphism in cohomology

$$i^*\colon \mathcal{H}^*(\Theta^+, \Theta^+ \setminus B_{r_1}M) \cong \mathcal{H}^*(\overline{B}_{r_1}M, S_{r_1}M)$$

by excision. $\mathcal{H}^{N-m}(\overline{B}_{r_1}M, S_{r_1}M)$ contains a nontrivial element: the Thom class of the disk bundle $q\colon B_{r_1}M \to M$, where $m := \dim M$. Therefore, $\mathcal{H}^{N-m}(\Theta^+, \Theta^+ \setminus B_{r_1}M) \neq 0$. This implies that $c_1 \geq c_0 + \alpha > c_0$. Note that it also implies that

$$\operatorname{cupl}(\Theta, \Theta \setminus B_rM) = \operatorname{cupl}(\Theta^+, \Theta^+ \setminus B_{r_1}M) \geq 1. \tag{4.1}$$

Set $\delta := \min\{c_0 - c_\infty, \varrho\}$. Then, Lemma 3.1 yields a map $h_\delta\colon \mathbb{R}^N \to D^{1,2}(\mathbb{R}^N)$ with $\operatorname{supp}(h_\delta(x)) \subset \Theta \setminus B_{a_0}M \subset \Omega$ for all $x \in \Theta^-$, which restricts to a map of pairs

$$h_\delta\colon (\Theta^-, \Theta^- \setminus B_aM) \to (\mathcal{N}^{c_\infty+\varepsilon_0}, \mathcal{N}^{c_0}) \tag{4.2}$$

such that the composition

$$(\Theta^-, \Theta^- \setminus B_aM) \xrightarrow{h_\delta} (\mathcal{N}^{c_\infty+\varepsilon_0}, \mathcal{N}^{c_0}) \xrightarrow{\beta} (\Theta^+, \Theta^+ \setminus B_{r_1}M)$$

is homotopic to the inclusion $\iota\colon (\Theta^-, \Theta^- \setminus B_aM) \hookrightarrow (\Theta^+, \Theta^+ \setminus B_{r_1}M)$. Since

$$\iota^* = h_\delta^* \circ \beta^*\colon \mathcal{H}^*(\Theta^+, \Theta^+ \setminus B_{r_1}M) \to \mathcal{H}^*(\Theta^-, \Theta^- \setminus B_rM)$$

is an isomorphism, we have that

$$\beta^*\colon \mathcal{H}^*(\Theta^+, \Theta^+ \setminus B_{r_1}M) \to \mathcal{H}^*(\mathcal{N}^{c_\infty+\varepsilon_0}, \mathcal{N}^{c_0})$$

is a monomorphism. Hence, $c_1 \leq c_\infty + \varepsilon_0$.

If $c \in (c_0, 2c_\infty)$, the number of pairs $\pm u$ of critical points of J on $\mathcal{N}$ with critical values in $[c_0, c]$ is at least $\operatorname{cat}(\widetilde{\mathcal{N}}^c, \widetilde{\mathcal{N}}^{c_0})$, where $\widetilde{\mathcal{N}}^c$ is the quotient space of $\mathcal{N}^c$ obtained by identifying u with $-u$ (see [2], [7]). Note that $\beta(u) = \beta(-u)$. Hence, there is a map $\widetilde{\beta}\colon (\widetilde{\mathcal{N}}^c, \widetilde{\mathcal{N}}^{c_0}) \to (\Theta^+, \Theta^+ \setminus B_{r_1}M)$ such that $\widetilde{\beta} \circ \kappa = \beta$, where $\kappa\colon (\mathcal{N}^c, \mathcal{N}^{c_0}) \to (\widetilde{\mathcal{N}}^c, \widetilde{\mathcal{N}}^{c_0})$ is the quotient map.

Set $c := c_\infty + \varepsilon_0$ and let h_δ be the map given in (4.2). Since $\widetilde{\beta} \circ \kappa \circ h_\delta = \beta \circ h_\delta$ is homotopic to ι, and each of the inclusions $(\Theta^-, \Theta^- \setminus B_aM) \hookrightarrow (\Theta, \Theta \setminus B_rM) \hookrightarrow (\Theta^+, \Theta^+ \setminus B_{r_1}M)$ is a homotopy equivalence of pairs, using Lemma A.4 we conclude that

$$\operatorname{cat}(\widetilde{\mathcal{N}}^{c_\infty+\varepsilon_0}, \widetilde{\mathcal{N}}^{c_0}) \geq \operatorname{cat}(\Theta, \Theta \setminus B_rM).$$

Hence, problem (1.2) has at least $\operatorname{cat}(\Theta, \Theta \setminus B_rM)$ pairs of solutions $\pm u$ with $J(u) \in [c_0, c_\infty + \varepsilon_0]$. Moreover, Lemma A.3 and inequality (4.1) allow us to conclude that

$$\operatorname{cat}(\Theta, \Theta \setminus B_rM) \geq \operatorname{cupl}(\Theta, \Theta \setminus B_rM) \geq 1.$$

Now set $c := c_1$. Since $\beta^*\colon \mathcal{H}^*(\Theta^+, \Theta^+ \setminus B_{r_1}M) \to \mathcal{H}^*(\mathcal{N}^{c_1}, \mathcal{N}^{c_0})$ is a monomorphism, $\widetilde{\beta}^*\colon \mathcal{H}^*(\Theta^+, \Theta^+ \setminus B_{r_1}M) \to \mathcal{H}^*(\widetilde{\mathcal{N}}^{c_1}, \widetilde{\mathcal{N}}^{c_0})$ is also a monomorphism. Lemmas A.3 to A.5 imply that

$$\operatorname{cat}(\widetilde{\mathcal{N}}^{c_1}, \widetilde{\mathcal{N}}^{c_0}) \geq \operatorname{cupl}(\widetilde{\mathcal{N}}^{c_1}, \widetilde{\mathcal{N}}^{c_0}) \geq \operatorname{cupl}(\Theta^+, \Theta^+ \setminus B_{r_1}M) = \operatorname{cupl}(\Theta, \Theta \setminus B_r M).$$

Hence, problem (1.2) has at least $\operatorname{cupl}(\Theta, \Theta \setminus B_r M) \geq 1$ pairs of solutions $\pm u$ with $J(u) \in [c_0, c_1]$.

Note that c_1 must be a critical value. Otherwise, for $\alpha > 0$ small enough, we would be able to deform $\mathcal{N}^{c_1+\alpha}$ into $\mathcal{N}^{c_1-\alpha}$ keeping $\mathcal{N}^{c_0}$ fixed. Since $\beta^*\colon \mathcal{H}^*(\Theta^+, \Theta^+ \setminus B_{r_1}M) \to \mathcal{H}^*(\mathcal{N}^{c_1+\alpha}, \mathcal{N}^{c_0})$ is a monomorphism, this would imply that $\beta^*\colon \mathcal{H}^*(\Theta^+, \Theta^+ \setminus B_{r_1}M) \to \mathcal{H}^*(\mathcal{N}^{c_1-\alpha}, \mathcal{N}^{c_0})$ is also a monomorphism, contradicting the definition of c_1.

Finally, recall that every critical point u of J with $J(u) \in (c_\infty, 2c_\infty)$ does not change sign. Otherwise, we would have that $u^+ \neq 0$ and $u^- \neq 0$ and, hence, that $u^\pm \in \mathcal{N}$, because $\|u^\pm\|^2 - |u^\pm|_{2^*}^{2^*} = J'(u)u^\pm = 0$. But then $J(u) = J(u^+) + J(u^-) \geq 2c_\infty$, which is a contradiction. □

To conclude the proof of Theorem 1.1 we shall show next that J has a critical value in $(c_1, 2c_\infty + 3\varepsilon_0]$. We need the following two lemmas.

Lemma 4.2. *The connecting homomorphism*

$$\delta^*\colon \widetilde{\mathcal{H}}^{*-1}(\Theta^+ \setminus B_{r_1}M) \to \mathcal{H}^*(\Theta^+, \Theta^+ \setminus B_{r_1}M)$$

of the reduced cohomology exact sequence of the pair $(\Theta^+, \Theta^+ \setminus B_{r_1}M)$ *is an epimorphism.*

Proof. Since the sequence

$$\widetilde{\mathcal{H}}^{*-1}(\Theta^+ \setminus B_{r_1}M) \xrightarrow{\delta^*} \mathcal{H}^*(\Theta^+, \Theta^+ \setminus B_{r_1}M) \xrightarrow{j^*} \widetilde{\mathcal{H}}^*(\Theta^+)$$

is exact, we need only to show that the homomorphism j^*, induced by the inclusion, is trivial. The diagram

$$\begin{array}{ccc} \mathcal{H}^*(\mathbb{R}^N, \mathbb{R}^N \setminus B_{r_1}M) & \longrightarrow & \widetilde{\mathcal{H}}^*(\mathbb{R}^N) \\ \cong\big\downarrow & & \big\downarrow \\ \mathcal{H}^*(\Theta^+, \Theta^+ \setminus B_{r_1}M) & \xrightarrow[j^*]{} & \widetilde{\mathcal{H}}^*(\Theta^+) \end{array}$$

induced by inclusions, commutes. The left vertical arrow is an isomorphism by excision. Since $\widetilde{\mathcal{H}}^*(\mathbb{R}^N) = 0$, we conclude that $j^* = 0$. □

The next lemma is a consequence of Struwe's global compactness theorem [21], [24] and Theorem 2.3.

Lemma 4.3. *If J does not have a critical value in $(c_1, 2c_\infty)$, then J satisfies* $(\mathrm{PS})_c$ *for every* $c \in (c_\infty + c_1, 2c_\infty + 3\varepsilon_0]$.

PROOF. See [8, Lemma 6]. $\square$

PROPOSITION 4.4. *J has a critical value in $(c_1, 2c_\infty + 3\varepsilon_0]$.*

PROOF. Arguing by contradiction, assume that J does not have a critical value in $(c_1, 2c_\infty + 3\varepsilon_0]$. By Lemma 3.2 there is a continuous map $G\colon \overline{B}_{b_1}M \times \overline{B}_{b_2}M \to \mathcal{E}$ such that

$$\begin{aligned}
&J(G(x,y)) \le 2c_\infty + 3\varepsilon_0 \quad \text{for all } (x,y) \in \overline{B}_{b_1}M \times \overline{B}_{b_2}M,\\
&J(G(x,y)) \le c_0 + c_1 \qquad \text{for all } (x,y) \in (S_{b_1}M \times \overline{B}_{b_2}M) \cup (\overline{B}_{b_1}M \times S_{b_2}M),\\
&J(G(x,y)^+) \le c_0 \quad \text{and} \quad \beta(G(x,y)^+) = x \quad \text{for all } (x,y) \in (S_{b_1}M \times \overline{B}_{b_2}M),\\
&J(G(x,y)^-) \le c_0 \quad \text{and} \quad \beta(G(x,y)^-) = y \quad \text{for all } (x,y) \in (\overline{B}_{b_1}M \times S_{b_2}M),
\end{aligned}$$

where b_1, b_2 were chosen at the beginning of this section. By Lemma 4.3 there is a continuous map

$$\eta\colon [0,1] \times \mathcal{N}^{2c_\infty+3\varepsilon_0} \to \mathcal{N}^{2c_\infty+3\varepsilon_0}$$

such that $\eta(0,u) = u$ and $\eta(1,u) \in \mathcal{N}^{c_0+c_1}$ for every $u \in \mathcal{N}^{2c_\infty+3\varepsilon_0}$, and $\eta(t,v) = v$ for every $v \in \mathcal{N}^{c_0+c_1}$, $t \in [0,1]$.

For $t \in [0,1]$ we define $g_t\colon \overline{B}_{b_1}M \times \overline{B}_{b_2}M \times [-1,1] \to \mathcal{N}^{2c_\infty+3\varepsilon_0}$ by

$$g_t(x,y,\lambda) := \eta(t, \rho((1+\lambda)G(x,y)^+ + (1-\lambda)G(x,y)^-)),$$

where ρ is the radial projection onto $\mathcal{N}$. Then,

$$g_t(x,y,\lambda) = \rho((1+\lambda)G(x,y)^+ + (1-\lambda)G(x,y)^-)$$

for all $(x,y) \in \partial(\overline{B}_{b_1}M \times \overline{B}_{b_2}M \times [-1,1])$. Consider the sets

$$\begin{aligned}
\mathcal{E}^* &:= \{u \in \mathcal{E} : \beta(u^-) \in M\},\\
K &:= \{\mathbf{z} \in \overline{B}_{b_1}M \times \overline{B}_{b_2}M \times [-1,1] : g_1(\mathbf{z}) \in \mathcal{E}^*\}.
\end{aligned}$$

Since K is compact and

$$c_0 + c_1 \ge J(g_1(\mathbf{z})) = J(g_1(\mathbf{z})^+) + J(g_1(\mathbf{z})^-) > J(g_1(\mathbf{z})^+) + c_0 \quad \text{for all } \mathbf{z} \in K,$$

we have that

$$d := \max_{\mathbf{z}\in K} J(g_1(\mathbf{z})^+) < c_1.$$

We claim that $\beta\colon (\mathcal{N}^d, \mathcal{N}^{c_0}) \to (\Theta^+, \Theta^+ \setminus B_{r_1}M)$ induces a monomorphism

$$\beta^*\colon \mathcal{H}^*(\Theta^+, \Theta^+ \setminus B_{r_1}M) \to \mathcal{H}^*(\mathcal{N}^d, \mathcal{N}^{c_0}). \tag{4.3}$$

This contradicts the definition of c_1, and proves the proposition by contradiction.

The rest of the argument is devoted to the proof of this claim. Let $\gamma_0\colon H_0^1(\Omega) \to \mathbb{R}$ be given by

$$\gamma_0(u) := \begin{cases} \dfrac{|u|_{2^*}^{2^*}}{\|u\|^2} - 1 & \text{if } u \ne 0,\\ -1 & \text{if } u = 0.\end{cases}$$

Then γ_0 is continuous, and $\gamma_0(u) = 0$ if and only if $u \in \mathcal{N}$.

Define $\gamma\colon \mathcal{N} \to \mathbb{R}$ as $\gamma(u) := \gamma_0(u^+) - \gamma_0(u^-)$. Note that

$$\gamma(u) = -1 \quad \text{iff} \quad u \leq 0, \qquad \gamma(u) = 1 \quad \text{iff} \quad u \geq 0, \qquad \gamma(u) = 0 \quad \text{iff} \quad u \in \mathcal{E}.$$

Denote by $\mathbf{z} := (x, y, \lambda) \in \overline{B}_{b_1}M \times \overline{B}_{b_2}M \times [-1, 1]$ and, for each $t \in [0, 1]$, define $\widetilde{\beta}_t\colon \overline{B}_{b_1}M \times \overline{B}_{b_2}M \times [-1, 1] \to \mathbb{R}^N$ by

$$\widetilde{\beta}_t(\mathbf{z}) := \begin{cases} [1 - \gamma(g_t(\mathbf{z}))]\beta(g_t(\mathbf{z})^-) + \gamma(g_t(\mathbf{z}))y & \text{if } g_t(\mathbf{z})^- \neq 0, \\ y & \text{if } g_t(\mathbf{z})^- = 0. \end{cases}$$

This function is continuous and depends continuously on t.

Next, consider the map $\theta_t\colon \overline{B}_{b_1}M \times \overline{B}_{b_2}M \times [-1, 1] \to \mathbb{R}^N \times \mathbb{R}$ defined by

$$\theta_t(\mathbf{z}) := \begin{cases} (\widetilde{\beta}_t(\mathbf{z}),\, \gamma(g_t(\mathbf{z}))) & \text{if } t \in [0, 1], \\ -t(y, \lambda) + (1 + t)\theta_0(\mathbf{z}) & \text{if } t \in [-1, 0]. \end{cases}$$

We write $\theta_t(\mathbf{z}) = (\theta_{t,1}(\mathbf{z}), \theta_{t,2}(\mathbf{z})) \in \mathbb{R}^N \times \mathbb{R}$. It is easy to check that θ_t has the following properties (cf. [8, Lemma 7]):

(a) If $\theta_t(\mathbf{z}) \in M \times \{0\}$ then $g_t(\mathbf{z}) \in \mathcal{E}^*$ for all $t \in [0, 1]$.
(b) If $\lambda \in \{-1, 1\}$ then $\theta_{t,2}(\mathbf{z}) = \lambda$ for all $t \in [-1, 1]$.
(c) If $y \in S_{b_2}M$ then $\theta_{t,1}(\mathbf{z}) = y$ for all $t \in [-1, 1]$.
(d) If $(y, \lambda) \in \partial\big(\overline{B}_{b_2}M \times [-1, 1]\big)$ then $\theta_t(\mathbf{z}) \notin M \times \{0\}$ for all $t \in [-1, 1]$.

Performing a translation, if necessary, we may assume that $0 \in M$. Now, for each $t \in [-1, 1]$, we define the map $f_t\colon \overline{B}_{b_1}M \times \overline{B}_{b_2}M \times [-1, 1] \to \mathbb{R}^N \times \mathbb{R}$ by

$$f_t(x, y, \lambda) := (x, (y, \lambda) - \theta_t(x, y, \lambda)).$$

This is a map over $\overline{B}_{b_1}M$, i.e. $p \circ f_t = p$ where $p\colon \overline{B}_{b_1}M \times \mathbb{R}^N \times \mathbb{R} \to \overline{B}_{b_1}M$ is the projection. Its set of fixed points,

$$\mathrm{Fix}(f_t) := \{(x, y, \lambda) \in \overline{B}_{b_1}M \times \overline{B}_{b_2}M \times [-1, 1] : f_t(x, y, \lambda) = (x, y, \lambda)\},$$

is the set of zeroes of θ_t. Thus, by property (d), $\mathrm{Fix}(f_t) \subset \overline{B}_{b_1}M \times B_{b_2}M \times (-1, 1)$ and, since $\mathrm{Fix}(f_t)$ is compact, the restriction $p\colon \mathrm{Fix}(f_t) \to \overline{B}_{b_1}M$ of the projection is a proper map. Hence, f_t is compactly fixed in the sense of Dold [10], and there exist transfer homomorphisms

$$\begin{aligned} \mathrm{t}_{f_t}\colon \mathcal{H}^*(\mathrm{Fix}(f_t), \mathrm{Fix}(f_t) \cap p^{-1}(S_{b_1}M)) &\to \mathcal{H}^*(\overline{B}_{b_1}M, S_{b_1}M), \\ \mathrm{t}_{f_t}\colon \mathcal{H}^*(\mathrm{Fix}(f_t) \cap p^{-1}(S_{b_1}M)) &\to \mathcal{H}^*(S_{b_1}M), \end{aligned}$$

for each $t \in [-1, 1]$. The definition and properties of the fixed point transfer were introduced in [10]. A brief account may be found in [8].

Observe that the map $g_1^+\colon (\mathrm{Fix}(f_1), \mathrm{Fix}(f_1) \cap p^{-1}(S_{b_1}M)) \to (\mathcal{N}^d, \mathcal{N}^{c_0})$ is well defined, and consider the diagram

$$\begin{tikzcd}
\mathcal{H}^{*-1}(\Theta^+ \setminus B_{r_1}M) \arrow[r, "\delta^*"] \arrow[d, "\beta^*"'] & \mathcal{H}^*(\Theta^+, \Theta^+ \setminus B_{r_1}M) \arrow[d, "\beta^*"] \\
\mathcal{H}^{*-1}(\mathcal{N}^{c_0}) \arrow[r, "\delta^*"] \arrow[d, "(g_1^+)^*"'] & \mathcal{H}^*(\mathcal{N}^d, \mathcal{N}^{c_0}) \arrow[d, "(g_1^+)^*"] \\
\mathcal{H}^{*-1}(\mathrm{Fix}(f_1) \cap p^{-1}(S_{b_1}M)) \arrow[r, "\delta^*"'] \arrow[d, "\mathrm{t}_{f_1}"'] & \mathcal{H}^*(\mathrm{Fix}(f_1), \mathrm{Fix}(f_1) \cap p^{-1}(S_{b_1}M)) \arrow[d, "\mathrm{t}_{f_1}"] \\
\mathcal{H}^{*-1}(S_{b_1}M) \arrow[r, "\delta^*"'] & \mathcal{H}^*(\overline{B}_{b_1}M, S_{b_1}M)
\end{tikzcd} \tag{4.4}$$

Due to the naturality property of the transfer, this diagram commutes.

Note that $f_{-1} = s \circ p$, where $s\colon \overline{S}_{b_1}M \to S_{b_1}M \times \overline{B}_{b_2}M \times [-1, 1]$ is the zero section $s(x) := (x, 0, 0)$. So the units property of the transfer [10, (3.11)] gives

$$\mathrm{t}_{f_{-1}} = s^*\colon \mathcal{H}^*(s(S_{b_1}M)) \to \mathcal{H}^*(S_{b_1}M),$$

and the homotopy property [10, (3.13)] yields

$$\mathrm{t}_{f_1} \circ p^* = \mathrm{t}_{f_{-1}} \circ p^* = s^* \circ p^* = \mathrm{id}\colon \mathcal{H}^*(S_{b_1}M) \to \mathcal{H}^*(S_{b_1}M).$$

Note also that

$$\beta(g_1(\mathbf{z})^+) = \beta(G(x, y)^+) = x = p(\mathbf{z})$$

for all $\mathbf{z} = (x, y, \lambda) \in \mathrm{Fix}(f_1) \cap p^{-1}(S_{b_1}M)$. Therefore,

$$i^* = \mathrm{t}_{f_1} \circ (g_1^+)^* \circ \beta^*\colon \mathcal{H}^*(\Theta^+ \setminus B_{r_1}M) \to \mathcal{H}^*(S_{b_1}M), \tag{4.5}$$

where $i\colon (B_{b_1}M, S_{b_1}M) \hookrightarrow (\Theta^+, \Theta^+ \setminus B_{r_1}M)$ is the inclusion. The commutativity of the diagram (4.4) and equality (4.5) yield

$$\mathrm{t}_{f_1} \circ (g_1^+)^* \circ \beta^* \circ \delta^* = \delta^* \circ \mathrm{t}_{f_1} \circ (g_1^+)^* \circ \beta^* = \delta^* \circ i^* = i^* \circ \delta^*.$$

Since δ^* is an epimorphism (see Lemma 4.2), we conclude that

$$\mathrm{t}_{f_1} \circ (g_1^+)^* \circ \beta^* = i^*\colon \mathcal{H}^*(\Theta^+, \Theta^+ \setminus B_{r_1}M) \to \mathcal{H}^*(\overline{B}_{b_1}M, S_{b_1}M).$$

But $i^*\colon \mathcal{H}^*(\Theta^+, \Theta^+ \setminus B_{r_1}M) \to \mathcal{H}^*(\overline{B}_{b_1}M, S_{b_1}M)$ is an isomorphism. Hence, $\beta^*\colon \mathcal{H}^*(\Theta^+, \Theta^+ \setminus B_{r_1}M) \to \mathcal{H}^*(\mathcal{N}^d, \mathcal{N}^{c_0})$ is a monomorphism, and claim (4.3) is proved. □

Proof of Theorem 1.1. It follows immediately from Propositions 4.1 and 4.4. □

Appendix A. Category and cup-length

A pair consisting of a topological space X and a subset A of X is denoted by (X,A). A map of pairs $f\colon (X,A)\to (Y,B)$ is a continuous function $f\colon X\to Y$ such that $f(a)\in B$ for every $a\in A$. Two maps of pairs $f_0,f_1\colon (X,A)\to (Y,B)$ are homotopic if there exists a map of pairs $F\colon ([0,1]\times X,\ [0,1]\times A)\to (Y,B)$ such that $F(0,x)=f_0(x)$ and $F(1,x)=f_1(x)$ for every $x\in X$.

Definition A.1. The Lusternik–Schnirelmann category of the pair (X,A) is the smallest number $k =: \operatorname{cat}(X,A)$ such that there exists an open cover $U_0,U_1,\dots,U_k$ of X with the following properties:

(LS_1) $A\subset U_0$ and there exists a homotopy $F:([0,1]\times U_0,\ [0,1]\times A)\to (X,A)$ such that $F(0,x)=x$ and $F(1,x)\in A$ for every $x\in U_0$,

(LS_2) U_j is contractible in X for every $j=1,\dots,k$.

If no such cover exists we set $cat(X,A):=\infty$.

If $A=\emptyset$ we write $\operatorname{cat}(X)$ instead of $\operatorname{cat}(X,\emptyset)$.

Let $\mathcal{H}^*$ be Čech cohomology with $\mathbb{Z}/2$-coefficients. We write $\widetilde{\mathcal{H}}^*$ for reduced Čech cohomology. The cup-product endows $\mathcal{H}^*(X,A)$ with a (graded right) $\mathcal{H}^*(X)$-module structure

$$\smile\colon \mathcal{H}^i(X,A)\times \mathcal{H}^j(X)\to \mathcal{H}^{i+j}(X,A),$$

see e.g. [11].

Definition A.2. The cup-length of (X,A) is the smallest number $k\in\mathbb{N}\cup\{0\}$ such that

$$\xi_0\smile\zeta_1\smile\cdots\smile\zeta_k=0 \quad \text{for all } \xi_0\in\mathcal{H}^*(X,A),\ \text{for all } \zeta_1,\dots,\zeta_k\in\widetilde{\mathcal{H}}^*(X).$$

We denote it by $cupl(X,A)$. If no such number exists we define $cupl(X,A):=\infty$.

We write $\operatorname{cupl}(X)$ instead of $\operatorname{cupl}(X,\emptyset)$. Note that $\operatorname{cupl}(X,A)\geq 1$ if an only if $\mathcal{H}^*(X,A)\neq 0$.

The category and the cup-length are related as follows.

Lemma A.3. $\operatorname{cat}(X,A)\geq \operatorname{cupl}(X,A)$.

Proof. See [7, Proposition 4.3]. □

Lemma A.4. *If $f\colon (X,A)\to (Y,B)$ and $h\colon (Y,B)\to (X,A)$ are maps of pairs whose composition $h\circ f\colon (X,A)\to (X,A)$ is homotopic to the identity of the pair (X,A) then*

$$\operatorname{cat}(X,A)\leq \operatorname{cat}(Y,B) \quad \textit{and} \quad \operatorname{cupl}(X,A)\leq \operatorname{cupl}(Y,B).$$

Proof. The proof is straightforward. □

PROOF OF EXAMPLE 1.2. Let $R > 0$ be such that $\overline{B}_R M$ is a tubular neighborhood of M contained in Θ. Since M is contractible in Θ, so is $B_R M$. For every $r \in (0, R)$ we have that $\Theta = \Theta_r \cup B_R M$. Thus $\operatorname{cat}(\Theta, \Theta_r) \leq 1$. Lemma A.3 and Proposition 4.1 yield $1 \leq \operatorname{cupl}(\Theta, \Theta_r) \leq \operatorname{cat}(\Theta, \Theta_r) \leq 1$. □

PROOF OF EXAMPLE 1.3. If $\Theta = B_R M$ is a tubular neighborhood of M and $r \in (0, R)$ then, for $s \in (r, R)$, the inclusion $(\overline{B}_s M, S_s M) \hookrightarrow (\Theta, \Theta_r)$ is a homotopy equivalence of pairs. The Thom isomorphism theorem asserts that

$$\Phi\colon \mathcal{H}^*(M) \to \mathcal{H}^{N-m+*}(\overline{B}_s M, S_s M), \qquad \Phi(\omega) = \tau \smile q^*(\omega),$$

is an isomorphism, where $\tau \in \mathcal{H}^{N-m}(\overline{B}_s M, S_s M)$ is the Thom class of the disk bundle $q\colon \overline{B}_s M \to M$, see e.g. [20]. Hence, $\operatorname{cupl}(M) = \operatorname{cupl}(\overline{B}_s M, S_s M)$. Clearly, $\operatorname{cat}(\overline{B}_s M, S_s M) \leq \operatorname{cat}(\overline{B}_s M) = \operatorname{cat}(M)$. Since we are assuming that $\operatorname{cat}(M) = \operatorname{cupl}(M)$, using Lemmas A.3 and A.4 we obtain

$$\begin{aligned}\operatorname{cat}(\Theta, \Theta_r) &= \operatorname{cat}(\overline{B}_s M, S_s M) = \operatorname{cat}(M) = \operatorname{cupl}(M) \\ &= \operatorname{cupl}(\overline{B}_s M, S_s M) = \operatorname{cupl}(\Theta, \Theta_r) \leq \operatorname{cat}(\Theta, \Theta_r),\end{aligned}$$

which proves our claim. □

LEMMA A.5. *If the map of pairs $f\colon (X, A) \to (Y, B)$ induces a monomorphism $f^*\colon \mathcal{H}^*(Y, B) \to \mathcal{H}^*(X, A)$, then*

$$\operatorname{cupl}(Y, B) \leq \operatorname{cupl}(X, A).$$

PROOF. Let $\xi_0 \in \mathcal{H}^*(Y, B)$, $\zeta_1, \ldots, \zeta_r \in \widetilde{\mathcal{H}}^*(Y)$ be such that $\xi_0 \smile \zeta_1 \smile \ldots \smile \zeta_r \neq 0$. Then, since $f^*\colon \mathcal{H}^*(Y, B) \to \mathcal{H}^*(X, A)$ is a monomorphism, we have that

$$0 \neq f^*(\xi_0 \smile \zeta_1 \smile \ldots \smile \zeta_r) = f^*(\xi_0) \smile f^*(\zeta_1) \smile \ldots \smile f^*(\zeta_r).$$

This proves our claim. □

If Θ is a bounded smooth domain in $\mathbb{R}^N$, M is an m-dimensional compact smooth manifold without boundary, and $\overline{B}_r M$ is a tubular neighborhood of M contained in Θ, then the inclusion $i\colon (\overline{B}_r M, S_r M) \hookrightarrow (\Theta, \Theta \setminus B_r M)$ induces an isomorphism in cohomology

$$i^*\colon \mathcal{H}^*(\Theta, \Theta \setminus B_r M) \cong \mathcal{H}^*(\overline{B}_r M, S_r M)$$

by excision. Let $\tau \in \mathcal{H}^{N-m}(\overline{B}_r M, S_r M)$ be the Thom class of the disk bundle $q\colon B_r M \to M$ and let $\widetilde{\tau} \in \mathcal{H}^{N-m}(\Theta, \Theta \setminus B_r M)$ be such that $i^*(\widetilde{\tau}) = \tau$. The cup-lenght of $(\Theta, \Theta \setminus B_r M)$ can be computed in terms of $\widetilde{\tau}$, as follows.

PROPOSITION A.6. *$\operatorname{cupl}(\Theta, \Theta \setminus B_r M)$ is the smallest number $k \in \mathbb{N}$ such that*

$$\widetilde{\tau} \smile \zeta_1 \smile \ldots \smile \zeta_k = 0 \quad \textit{for all } \zeta_1, \ldots, \zeta_k \in \widetilde{\mathcal{H}}^*(\Theta). \tag{A.1}$$

Proof. Let $k \in \mathbb{N}$ be such that (A.1) holds true, and let $\xi_0 \in \mathcal{H}^*(\Theta, \Theta \backslash B_r M)$ and $\zeta_1, \ldots, \zeta_k \in \widetilde{\mathcal{H}}^*(\Theta)$. By the Thom isomorphism theorem, $i^*(\xi_0) = \tau \smile q^*(\omega) = i^*(\widetilde{\tau}) \smile q^*(\omega)$ for some $\omega \in \mathcal{H}^*(M)$. Since $\widetilde{\tau} \smile \zeta_1 \smile \ldots \smile \zeta_k = 0$ we obtain that

$$\begin{aligned} i^*(\xi_0 \smile \zeta_1 \smile \ldots \smile \zeta_k) &= \tau \smile q^*(\omega) \smile i^*(\zeta_1 \smile \ldots \smile \zeta_k) \\ &= q^*(\omega) \smile i^*(\widetilde{\tau}) \smile i^*(\zeta_1 \smile \ldots \smile \zeta_k) \\ &= q^*(\omega) \smile i^*(\widetilde{\tau} \smile \zeta_1 \smile \ldots \smile \zeta_k) = 0 \end{aligned}$$

and, since $i^* \colon \mathcal{H}^*(\Theta, \Theta \setminus B_r M) \to \mathcal{H}^*(\overline{B}_r M, S_r M)$ is an isomorphism, we conclude that $\xi_0 \smile \zeta_1 \smile \ldots \smile \zeta_n = 0$. Hence $\operatorname{cupl}(\Theta, \Theta \setminus B_r M) \leq k$. The opposite inequality is trivial. $\square$

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