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EXISTENCE AND MULTIPLICITY OF POSITIVE SOLUTIONS FOR A SCHRÖDINGER–POISSON SYSTEM WITH A PERTURBATION

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Abstract. In this paper we study the nonlinear Schrödinger-Poisson system with a perturbation:

$$\begin{cases} -\Delta u + u + K(x)\phi u = |u|^{p-2}u + \lambda f(x)|u|^{q-2}u & \text{in } \mathbb{R}^3, \\ -\Delta \phi = K(x)u^2 & \text{in } \mathbb{R}^3, \end{cases}$$

where K and f are nonnegative functions, $2 < q \leq p < 6$ and $p > 4$, and the parameter $\lambda \in \mathbb{R}$. Under some suitable assumptions on K and f, the criteria of existence and multiplicity of positive solutions are established by means of the Lusternik–Schnirelmann category and minimax method.

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1. Introduction

In this paper we are concerned with the coupled system of Schrödinger–Poisson equations of the form:

$$\text{(SP)}\qquad \begin{cases} -\Delta u + u + K(x)\phi(x)u = h(x,u) & \text{in } \mathbb{R}^3, \\ -\Delta\phi = K(x)u^2 & \text{in } \mathbb{R}^3, \end{cases}$$

where K is a nonnegative function and $h\colon \mathbb{R}^3\times\mathbb{R}\to\mathbb{R}$ is a Carathédory function.

Such a system, also known as the nonlinear Schrödinger–Maxwell equations, have a strong physical meaning. It was first introduced in [8] as a model describing solitary waves for the nonlinear stationary Schrödinger equations interacting with the electrostatic field, and also in semiconductor theory, in nonlinear optics and in plasma physics. Indeed, in Problem (SP) the first equation is a nonlinear stationary Schrödinger equation (where, as usual, the nonlinear term simulates the interaction between many particles) that is coupled with a Poisson equation, to be satisfied by ϕ, meaning that the potential is determined by the charge of the wave function.

In recent years, problem (SP) has been studied widely via variational methods under the various hypotheses on K and f, see [3], [5], [6], [13], [14], [18], [20], [24], [25] and the references therein. Now we recall some of them as follows.

If $h(x,u)\equiv |u|^{p-2}u$ and $K(x)\equiv\mu>0$, Ruiz [24] gave existence and nonexistence results on positive radial solutions of problem (SP), depending on the parameters p and μ. It turned out that $p=3$ is a critical value for the existence of solutions. Later, the results in [24] were further improved in Ambrosetti and Ruiz [5] by showing the presence of multiple bound states when certain conditions on the parameters are satisfied.

If $h(x,u)\equiv a(x)|u|^{p-2}u$ and $K(x)\equiv\mu>0$, Chen et al. [14] studied the multiplicity of positive solutions for problem (SP) with $4\le p<6$. They showed that the number of positive solutions are dependent on the profile of the function a.

If $h(x,u)\equiv a(x)|u|^{p-2}u$ and K is a nonnegative L^2-function, Cerami and Vaira [13] obtained the existence of positive ground state and bound state solutions for problem (SP) with $4<p<6$ under some suitable assumptions, but not requiring any symmetry property on a and K.

Motivated by these findings, we now extend the analysis to the nonlinear Schrödinger–Poisson system with a perturbation. Our intension here is to illustrate the difference in the solution behavior which arises from the consideration of the perturbation. Here we consider the following Schrödinger–Poisson systems:

$$(\mathrm{SP}_\lambda)\qquad \begin{cases} -\Delta u + u + K(x)\phi(x)u = |u|^{p-2}u + \lambda f(x)|u|^{q-2}u & \text{in } \mathbb{R}^3, \\ -\Delta\phi = K(x)u^2 & \text{in } \mathbb{R}^3, \end{cases}$$

where K and f are nonnegative functions, $2 < q \leq p < 2^*$ ($2^* := 2 \times 3/(3-2) = 6$) and $p > 4$, and $\lambda \in \mathbb{R}$. We assume that the functions K and f satisfy the following conditions:

($\mathcal{K}$) $K \in L^2(\mathbb{R}^3)$ and there exists a positive number $r_K > 1$ such that

$$K(x) \leq \widehat{c}\exp(-r_K|x|) \quad \text{for some } \widehat{c} > 0 \text{ and for all } x \in \mathbb{R}^3;$$

($\mathcal{D}1$) $f \in C(\mathbb{R}^3) \cap L^{p/(p-q)}(\mathbb{R}^3)$ and there exist positive numbers $1 < r_f < \min\{r_K, 2\}$ and R_0 such that

$$f(x) \geq c_0 \exp(-r_f|x|) \text{ for some } c_0 > 0 \text{ and for all } x \in \mathbb{R}^3 \text{ with } |x| \geq R_0.$$

It is well known that problem (SP_λ) can be easily transformed in the Schrödinger equation with a non-local term (see [13], [24], [25] etc.). Briefly, the Poisson equation is solved by using the Lax–Milgram theorem, so, for all u in $H^1(\mathbb{R}^3)$, a unique $\phi_u \in D^{1,2}(\mathbb{R}^3)$ is obtained, such that $-\Delta\phi = K(x)u^2$ and that, inserted into the first equation, gives

$$(\mathrm{SP}'_\lambda) \qquad -\Delta u + u + K(x)\phi_u(x)u = |u|^{p-2}u + \lambda f(x)|u|^{q-2}u,$$

Moreover, wquation (SP'_λ) is variational and its solutions are the critical points of the functional defined in $H^1(\mathbb{R}^3)$ by

$$J_\lambda(u) = \frac{1}{2}\int_{\mathbb{R}^3} (|\nabla u|^2 + u^2)\,dx + \frac{1}{4}\int_{\mathbb{R}^3} K(x)\phi_u(x)u^2\,dx - \frac{1}{p}\int_{\mathbb{R}^3} |u|^p\,dx - \frac{\lambda}{q}\int_{\mathbb{R}^3} f|u|^q\,dx.$$

Furthermore, it is known that J_λ is a C^1 functional with derivative given by

$$\langle J'_\lambda(u), v\rangle = \int_{\mathbb{R}^3} (\nabla u \nabla v + uv + K\phi_u uv - |u|^{p-2}uv - \lambda f|u|^{q-2}uv)\,dx.$$

Note that $(u, \phi) \in H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$ is a solution of problem (SP_λ) if and only if u is a critical point of J_λ and $\phi = \phi_u$ (see [9], [15]). It is clear that for $u \in H^1(\mathbb{R}^3) \setminus \{0\}$, $\lim_{t\to\infty} J_\lambda(tu) = -\infty$ and so J_λ is not longer bounded below on $H^1(\mathbb{R}^3)$. In order to obtain the existence results, we introduce the Nehari manifold

$$\mathbf{N}_\lambda = \{u \in H^1(\mathbb{R}^3) \setminus \{0\} \mid \langle J'_\lambda(u), u\rangle = 0\}.$$

Thus, $u \in \mathbf{N}_\lambda$ if and only if

$$\int_{\mathbb{R}^3} (|\nabla u|^2 + u^2)\,dx + \int_{\mathbb{R}^3} K(x)\phi_u(x)u^2\,dx - \int_{\mathbb{R}^3} |u|^p\,dx - \lambda\int_{\mathbb{R}^3} f|u|^q\,dx = 0.$$

Clearly, $\mathbf{N}_\lambda$ contains every non-trivial critical point of J_λ on $H^1(\mathbb{R}^3)$.

Consider the following minimization problem

$$\alpha_\lambda = \inf_{u\in\mathbf{N}_\lambda} J_\lambda(u),$$

and we have the following definition.

Definition 1.1. $(u,\phi)\in H^1(\mathbb{R}^3)\times D^{1,2}(\mathbb{R}^3)$ is a ground state of problem (SP_λ) we mean that (u,ϕ) is such a solution of problem (SP_λ) which minimizes the action functional J_λ on the Nehari manifold $\mathbf{N}_\lambda$. If there exists a nontrivial solution (u,ϕ_u) of problem (SP_λ) such that $J_\lambda(u)>\alpha_\lambda$, then we called the solution (u,ϕ_u) is a higher energy solution of of problem (SP_λ).

Let

$$\eta_0=\frac{(p-2)(p-4)^{(q-2)/(p-2)}}{(4-q)(p-q)^{(p-q)/(p-2)}(q-2)^{(q-2)/(p-2)}S_p^{2(p-q)/(p-2)}|f|_{p/(p-q)}}>0, \tag{1.1}$$

it is easy to see that $\eta_0\to\infty$ as $q\to 4^-$. Define

$$\lambda(q)=\begin{cases}\infty & \text{if } 4\le q\le p,\\ \widehat{\lambda} & \text{if } 2<q<4,\end{cases} \tag{1.2}$$

where $\widehat{\lambda}=\min\{2^{(p-q)/(p-2)}\eta_0, qp^{(2-q)/(p-2)}\eta_0\}$. Then our main result is the following.

Theorem 1.2. *Suppose that the functions K, f satisfy the conditions $(\mathcal{K})$ and $(\mathcal{D}1)$, and $\lim_{|x|\to\infty} f(x)=0$. Then we have the following:*

(a) *Problem (SP_λ) has a positive higher energy solution and no any ground state solution for $\lambda=0$;*
(b) *Problem (SP_λ) has a positive ground state solution for $0<\lambda<\lambda(q)$;*
(c) *there exists a positive number $\Lambda_*<\lambda(q)$ such that problem (SP_λ) has at least three positive solutions for $0<\lambda<\Lambda_*$.*

Remark 1.3. (a) By a similar argument to that in the proof of [13, Proposition 6.1], problem (SP_λ) does not admit any ground state solutions for $\lambda=0$. Moreover, [13] showed that the existence of higher energy solution for problem (SP_λ) with $\lambda=0$ and $p=q$.

(b) Regarding the existence of higher energy solution, the main difference is the type of assumption on function K (see [13]) requires K being in L which is restricted within certain value, but K decays exponentially in our study.

Our analysis also makes use of the following result.

Theorem 1.4. *If in addition to the conditions $(\mathcal{K})$ and $(\mathcal{D}1)$, we still have $(\mathcal{D}2)$ there exists a positive number $1<\overline{r}_f\le r_f$ such that*

$$f(x)\le \overline{c}_0\exp(-\overline{r}_f|x|)\quad \textit{for some } \overline{c}_0>0 \textit{ and for all } x\in\mathbb{R}^3,$$

then problem (SP_λ) has a positive higher energy solution and no any ground state solution for $\lambda<0$.

Proof. The proof is similar to that of Theorem 1.2(a) (see Sections 3 and 4), so we leave the details to the reader. □

This paper is organized as follows. In Section 2, we give some notations and preliminaries. In Section 3, we give some estimates of the energy. In Section 4, we establish the existence of a positive solution. In Section 5, we establish the existence of two positive solutions for λ sufficiently small. In Section 6, we prove Theorem 1.2.

2. Notations and preliminaries

Hereafter we use the following notations:

- $H^1(\mathbb{R}^3)$ is the usual Sobolev space endowed withthe standard scalar product and norm

$$\langle u, v\rangle = \int_{\mathbb{R}^3} (\nabla u \nabla v + uv)\, dx, \qquad |u|^2 = \int_{\mathbb{R}^3} (|\nabla u|^2 + u^2)\, dx.$$

- $D^{1,2}(\mathbb{R}^3)$ is the completion of $C_0^\infty(\mathbb{R}^3)$ with respect to the norm

$$|u|^2_{D^{1,2}} = \int_{\mathbb{R}^3} |\nabla u|^2\, dx.$$

- H^{-1} denotes the dual space of $H^1(\mathbb{R}^3)$.
- $L^s(\mathbb{R}^3), 2 \le s \le +\infty$, a Lebesgue space, the norm in $L^s(\mathbb{R}^3)$ is denoted by $|\cdot|_s$.
- S_s is the best Sobolev constant for the embedding of $H^1(\mathbb{R}^3)$ in $L^s(\mathbb{R}^3)$, that is

$$S_s = \inf_{u \in H^1(\mathbb{R}^3)\setminus\{0\}} \frac{|u|}{|u|_s}.$$

- $\overline{S}$ is the best Sobolev constant for the embedding of $D^{1,2}(\mathbb{R}^3)$ in $L^6(\mathbb{R}^3)$, that is

$$\overline{S} = \inf_{u \in D^{1,2}(\mathbb{R}^3)\setminus\{0\}} \frac{|u|_{D^{1,2}}}{|u|_6}.$$

- C is various positive constants which may vary from line to line and are not essential to the problem.

Let us now define the operator $\Phi\colon H^1(\mathbb{R}^3) \to D^{1,2}(\mathbb{R}^3)$ as $\Phi[u] = \phi_u$.

In the following lemma we summarize some properties of Φ, useful to study our problem, the readers are referred to [13, Lemma 2.1] for a detailed proof.

Lemma 2.1. *We have the following*:

(a) Φ *is continuous*;
(b) Φ *maps bounded sets into bounded sets*;
(c) *if* $u_n \rightharpoonup u$ *weakly in* $H^1(\mathbb{R}^3)$, *then* $\Phi[u_n] \rightharpoonup \Phi[u]$ *weakly in* $D^{1,2}(\mathbb{R}^3)$;
(d) $\Phi[tu] = t^2\Phi[u]$ *for all* $t \in \mathbb{R}$.

Define

$$\psi_\lambda(u) = |u|^2 + \int_{\mathbb{R}^3} K(x)\phi_u(x)u^2\,dx - \int_{\mathbb{R}^3} |u|^p\,dx - \lambda\int_{\mathbb{R}^3} f|u|^q\,dx.$$

Then, for $u \in \mathbf{N}_\lambda$, if $4 \le q \le p < 6$,

$$\begin{aligned}\langle\psi_\lambda'(u),u\rangle &= 2|u|^2 + 4\int_{\mathbb{R}^3} K(x)\phi_u(x)u^2\,dx - p\int_{\mathbb{R}^3}|u|^p\,dx - q\lambda\int_{\mathbb{R}^3} f|u|^q\,dx \\ &= (2-q)|u|^2 + (4-q)\int_{\mathbb{R}^3} K(x)\phi_u(x)u^2\,dx - (p-q)\int_{\mathbb{R}^3}|u|^p\,dx < 0\end{aligned}$$

for all $\lambda \in \mathbb{R}$.

If $2 < q < 4 < p < 6$,

$$\begin{aligned}\langle\psi_\lambda'(u),u\rangle &= 2|u|^2 + 4\int_{\mathbb{R}^3} K(x)\phi_u(x)u^2\,dx - p\int_{\mathbb{R}^3}|u|^p\,dx - \lambda q\int_{\mathbb{R}^3} f|u|^q\,dx \\ &= -2|u|^2 + (4-p)\int_{\mathbb{R}^3}|u|^p\,dx + \lambda(4-q)\int_{\mathbb{R}^3} f|u|^q\,dx \\ &< \begin{cases} 0 & \text{if } \lambda \le 0, \\ -2S_p^{-2}|u|_p^2 - (p-4)|u|_p^p + \lambda(4-q)|f|_{p/(p-q)}|u|_p^q & \text{if } \lambda > 0.\end{cases}\end{aligned}$$

We can prove that for any $0 < \lambda < 2^{(p-q)/(p-2)}\eta_0$ and $u \in \mathbf{N}_\lambda$,

$$-2S_p^{-2}|u|_p^2 - (p-4)|u|_p^p + \lambda(4-q)|f|_{p/(p-q)}|u|_p^q < 0,$$

(see Lemma A.1 in Appendix), where $\eta_0 > 0$ is as in (1.1). Therefore, if $2 < q < 4 < p < 6$, then $\langle\psi_\lambda'(u),u\rangle < 0$ for any $\lambda < 2^{(p-q)/(p-2)}\eta_0$. These show that $\mathbf{N}_\lambda$ is a C^1 manifold and so the Nehari manifold $\mathbf{N}_\lambda$ is a natural constraint for the functional $\mathbf{N}_\lambda$. Furthermore, we have the following results.

Lemma 2.2. *The energy functional J_λ is coercive and bounded below on $\mathbf{N}_\lambda$ for any $\lambda < \lambda(q)$, where $\lambda(q)$ is as in* (1.2).

Proof. For any $u \in \mathbf{N}_\lambda$. We consider two cases: *Case 1.* $4 \le q \le p < 6$. Since

$$\begin{aligned}J_\lambda(u) &= \frac{1}{2}|u|^2 + \frac{1}{4}\int_{\mathbb{R}^3} K(x)\phi_u(x)u^2\,dx - \frac{1}{p}\int_{\mathbb{R}^3}|u|^p\,dx - \frac{\lambda}{q}\int_{\mathbb{R}^3} f|u|^q\,dx \\ &= \frac{q-2}{2q}|u|^2 + \frac{q-4}{4q}\int_{\mathbb{R}^3} K(x)\phi_u(x)u^2\,dx + \frac{p-q}{pq}\int_{\mathbb{R}^3}|u|^p\,dx > 0\end{aligned} \tag{2.1}$$

for all $\lambda \in \mathbb{R}$. Then J_λ is coercive and bounded below on $\mathbf{N}_\lambda$.

Case 2. $2 < q < 4 < p < 6$. Since

$$\begin{aligned}J_\lambda(u) &= \frac{1}{2}|u|^2 + \frac{1}{4}\int_{\mathbb{R}^3} K(x)\phi_u(x)u^2\,dx - \frac{1}{p}\int_{\mathbb{R}^3}|u|^p\,dx - \frac{\lambda}{q}\int_{\mathbb{R}^3} f|u|^q\,dx \\ &= \frac{1}{4}|u|^2 + \frac{p-4}{4p}\int_{\mathbb{R}^3}|u|^p\,dx - \frac{\lambda(4-q)}{4q}\int_{\mathbb{R}^3} f|u|^q\,dx\end{aligned} \tag{2.2}$$

$$\geq \frac{1}{4}|u|^2 + \frac{p-4}{4p}|u|_p^p - \frac{\lambda(4-q)}{4q}|f|_{p/(p-q)}|u|_p^q$$
$$\geq \frac{S_p^2}{4}|u|_p^2 + \frac{p-4}{4p}|u|_p^p - \frac{\lambda(4-q)}{4q}|f|_{p/(p-q)}|u|_p^q > 0,$$

for all $u \in \mathbf{N}_\lambda$ and $\lambda < qp^{(2-q)/(p-2)}\eta_0$ (see Lemma A.1 in Appendix). Set $\widehat{\lambda} = \min\{2^{(p-q)/(p-2)}\eta_0, qp^{(2-q)/(p-2)}\eta_0\}$. Then, by (2.2), for any $\lambda < \widehat{\lambda}$, J_λ is coercive and bounded below $\mathbf{N}_\lambda$. □

Lemma 2.3. *Suppose that u_0 is a local minimizer for J_λ on $\mathbf{N}_\lambda$. Then $J'_\lambda(u_0) = 0$ in $H^{-1}(\mathbb{R}^3)$.*

Proof. The proof is essentially the same as that in Brown and Zhang [12, Theorem 2.3] (or see Binding, Drábek and Huang [10]). □

To get a better understanding of the Nehari manifold, we consider the function $m_u : \mathbb{R}^+ \to \mathbb{R}$ defined by

$$m_u(t) = t^{-2}|u|^2 - t^{p-4}\int_{\mathbb{R}^3}|u|^p\,dx \quad \text{for } t > 0.$$

Clearly, $tu \in \mathbf{N}_0$ if and only if $m_u(t) + \int_{\mathbb{R}^3} K(x)\phi_u(x)u^2\,dx = 0$ and $m_u(\widehat{t}(u)) = 0$, where $\mathbf{N}_0 = \mathbf{N}_\lambda$ with $\lambda = 0$ and

$$\widehat{t}(u) = \left(\frac{|u|^2}{\int_{\mathbb{R}^3}|u|^p\,dx}\right)^{1/(p-2)} > 0. \tag{2.3}$$

Moreover,

$$m'_u(t) = -2t^{-3}|u|^2 - (p-4)t^{p-5}\int_{\mathbb{R}^3}|u|^p\,dx.$$

Thus, $m'_u(t) < 0$ for all $t > 0$, which implies that m_u is strictly decreasing on $(0,\infty)$ with $\lim_{t\to 0^+} m_u(t) = \infty$ and $\lim_{t\to\infty} m_u(t) = -\infty$. Then we have the following lemma.

Lemma 2.4. *Suppose that $\lambda < \lambda(q)$. Then, for each $u \in H^1(\mathbb{R}^3)\setminus\{0\}$ we have the following results*:

(a) *There exists a unique $t_\lambda(u) > 0$ such that $t_\lambda(u)u \in \mathbf{N}_\lambda$, and*

$$J_\lambda(t_\lambda(u)u) = \sup_{t\geq 0} J_\lambda(tu). \tag{2.4}$$

In particular, there exists a unique $t_0(u) \geq \widehat{t}(u)$ such that $t_0(u)u \in \mathbf{N}_0$, and

$$J_0(t_0(u)u) = \sup_{t\geq 0} J_0(tu) = \sup_{t\geq \widehat{t}(u)} J_0(tu), \tag{2.5}$$

where $J_0 = J_\lambda$ with $\lambda = 0$.

(b) *$t_\lambda(u)$ is a continuous function for $u \in H^1(\mathbb{R}^3)\setminus\{0\}$.*

(c) *$t_\lambda(u) = (1/|u|)t_\lambda(u/|u|)$.*

(d) $\mathbf{N}_\lambda = \{u \in H^1(\mathbb{R}^3) \setminus \{0\} \mid (1/|u|)t_\lambda(u/|u|) = 1\}$.

PROOF. (a) Fix $u \in H^1(\mathbb{R}^3) \setminus \{0\}$ and let

$$\begin{aligned} h_u(t) &= J_\lambda(tu) \\ &= \frac{t^2}{2}|u|^2 + \frac{t^4}{4}\int_{\mathbb{R}^3} K(x)\phi_u(x)u^2\,dx - \frac{t^p}{p}\int_{\mathbb{R}^3}|u|^p\,dx - \frac{\lambda t^q}{q}\int_{\mathbb{R}^3} f|u|^q\,dx \end{aligned}$$

for $t > 0$. Then

$$h_u'(t) = t|u|^2 + t^3\int_{\mathbb{R}^3} K(x)\phi_u(x)u^2\,dx - t^{p-1}\int_{\mathbb{R}^3}|u|^p\,dx - \lambda t^{q-1}\int_{\mathbb{R}^3} f|u|^q\,dx.$$

We distinguish two cases:

Case 1. $4 \le q \le p < 6$. Let

$$g_u(t) = t^2\int_{\mathbb{R}^3} K(x)\phi_u(x)u^2\,dx - t^{p-2}\int_{\mathbb{R}^3}|u|^p\,dx - \lambda t^{q-2}\int_{\mathbb{R}^3} f|u|^q\,dx$$

for $t > 0$. Clearly, $tu \in \mathbf{N}_\lambda$ if and only if $g_u(t) + \|u\|^2 = 0$.

$$\begin{aligned} g_u'(t) = {}& 2t\int_{\mathbb{R}^3} K(x)\phi_u(x)u^2\,dx - (p-2)t^{p-3}\int_{\mathbb{R}^3}|u|^p\,dx \\ &- \lambda(q-2)t^{q-3}\int_{\mathbb{R}^3} f|u|^q\,dx. \end{aligned}$$

If there exists $\widetilde{t} > 0$ such that $g_u'(\widetilde{t}) = 0$, that is

$$2\int_{\mathbb{R}^3} K(x)\phi_u(x)u^2\,dx - (p-2)\widetilde{t}^{p-4}\int_{\mathbb{R}^3}|u|^p\,dx - \lambda(q-2)\widetilde{t}^{q-4}\int_{\mathbb{R}^3} f|u|^q\,dx = 0.$$

Then

$$\begin{aligned} g_u''(\widetilde{t}) &= 2\int_{\mathbb{R}^3} K(x)\phi_u(x)u^2\,dx \\ &\qquad - (p-2)(p-3)\widetilde{t}^{p-4}\int_{\mathbb{R}^3}|u|^p\,dx - \lambda(q-2)(q-3)\widetilde{t}^{q-4}\int_{\mathbb{R}^3} f|u|^q\,dx \\ &= [2 - 2(q-3)]\int_{\mathbb{R}^3} K(x)\phi_u(x)u^2\,dx \\ &\qquad + [(p-2)(q-3) - (p-2)(p-3)]\widetilde{t}^{p-4}\int_{\mathbb{R}^3}|u|^p\,dx \\ &= 2(4-q)\int_{\mathbb{R}^3} K(x)\phi_u(x)u^2\,dx + (p-2)(q-p)\widetilde{t}^{p-4}\int_{\mathbb{R}^3}|u|^p\,dx < 0. \end{aligned}$$

Therefore, there exists a unique $t_\lambda(u) > 0$ such that $g_u(t_\lambda(u)) + \|u\|^2 = 0$, which implies that $h_u'(t_\lambda(u)) = 0$ and $t_\lambda(u)u \in \mathbf{N}_\lambda$. Moreover, by the profile of g_u, one has h_u is strictly increasing on $(0, t_\lambda(u))$ and strictly decreasing on $(t_\lambda(u), \infty)$. Therefore, (2.4) holds.

Case 2. $2 < q < 4 < p < 6$. Let

$$G_u(t) = t^{-2}|u|^2 - t^{p-4}\int_{\mathbb{R}^3}|u|^p\,dx - \lambda t^{q-4}\int_{\mathbb{R}^3} f|u|^q\,dx$$

for $t > 0$. Clearly, $G_u(t) \to +\infty$ as $t \to 0^+$ and $G_u(t) \to -\infty$ as $t \to +\infty$. Furthermore, for any $\lambda < \lambda_*$ and $t > 0$, we have

$$\begin{aligned} G_u'(t) =& -2t^{-3}|u|^2 - (p-4)t^{p-5}\int_{\mathbb{R}^3}|u|^p\,dx + \lambda(4-q)t^{q-5}\int_{\mathbb{R}^3} f|u|^q\,dx \\ =& \, t^{-5}(-2|tu|^2 - (p-4)\int_{\mathbb{R}^3}|tu|^p\,dx + \lambda(4-q)\int_{\mathbb{R}^3} f|tu|^q\,dx) < 0, \end{aligned}$$

(see Lemma A.1 in Appendix), which implies that G_u is decreasing on t for any $\lambda < \lambda(q)$. Therefore, there exists a unique $t_\lambda(u) > 0$ such that

$$G_u(t_\lambda(u)) + \int_{\mathbb{R}^3} K(x)\phi_u(x)u^2, dx = 0,$$

which implies that $h_u'(t_\lambda(u)) = 0$ and $t_\lambda(u)u \in \mathbf{N}_\lambda$. Moreover, it is easy to obtain that h_u is strictly increasing on $(0, t_\lambda(u))$ and strictly decreasing on $(t_\lambda(u), \infty)$. Therefore, (2.4) holds. Let

$$h_u^0(t) = J_0(tu) = \frac{t^2}{2}|u|^2 + \frac{t^4}{4}\int_{\mathbb{R}^3} K(x)\phi_u(x)u^2\,dx - \frac{t^p}{p}\int_{\mathbb{R}^3}|u|^p\,dx.$$

Thus

$$\begin{aligned} [h_u^0(t)]' =& \, t|u|^2 + t^3\int_{\mathbb{R}^3} K(x)\phi_u(x)u^2\,dx - t^{p-1}\int_{\mathbb{R}^3}|u|^p\,dx \\ =& \, t^3(t^{-2}|u|^2 - t^{p-4}\int_{\mathbb{R}^3}|u|^p\,dx + \int_{\mathbb{R}^3} K(x)\phi_u(x)u^2\,dx) \\ =& \, t^3\left(m_u(t) + \int_{\mathbb{R}^3} K(x)\phi_u(x)u^2\,dx\right). \end{aligned}$$

Since $\int_{\mathbb{R}^3} K(x)\phi_u(x)u^2\,dx > 0$ for any $u \in H^1(\mathbb{R}^3)\setminus\{0\}$, then the equation $m_u(t) + \int_{\mathbb{R}^3} K(x)\phi_u(x)u^2\,dx = 0$ has a unique solution $t_0(u) \ge \widehat{t}(u)$, which implies that $[h_u(t_0(u))]' = 0$ and $t_0(u)u \in \mathbf{N}_0$. Moreover, h_u is strictly increasing on $(0, t_0(u))$ and strictly decreasing on $(t_0(u), \infty)$. Therefore, (2.5) holds.

(b) By the uniqueness of $t_\lambda(u)$ and the extrema property of $t_\lambda(u)$, we have $t_\lambda(u)$ is a continuous function for $u \in H^1(\mathbb{R}^3)\setminus\{0\}$.

(c) Let $v = u/|u|$. Then, by parts (a) and (b), there is a unique $t_\lambda(v) > 0$ such that $t_\lambda(v)v \in \mathbf{N}_\lambda$ or $t_\lambda(u/|u|)u/|u| \in \mathbf{N}_\lambda$. Thus, by the uniqueness of $t_\lambda(v)$, we can conclude that $t_\lambda(u) = (1/|u|)t_\lambda(u/|u|)$.

(d) For $u \in \mathbf{N}_\lambda$. By parts (a)–(c), $t_\lambda(u/|u|)u/|u| \in \mathbf{N}_\lambda$. Since $u \in \mathbf{N}_\lambda$, we have $t_\lambda(u/|u|)/|u| = 1$, which implies that

$$\mathbf{N}_\lambda \subset \left\{u \in H^1(\mathbb{R}^3) \;\middle|\; \frac{1}{|u|}t_\lambda\left(\frac{u}{|u|}\right) = 1\right\}.$$

Conversely, let $u \in H^1(\mathbb{R}^3)$ such that $(1/|u|)t_\lambda(u/|u|) = 1$. Then, by part (c),

$$t_\lambda\left(\frac{u}{|u|}\right)\frac{u}{|u|} \in \mathbf{N}_\lambda.$$

Thus,

$$\mathbf{N}_\lambda = \left\{ u \in H^1(\mathbb{R}^3) \setminus \{0\} \;\middle|\; \frac{1}{|u|} t_\lambda\left(\frac{u}{|u|}\right) = 1 \right\}.$$

This completes the proof. □

We define the Palais–Smale (or simply (PS)-) sequences, (PS)-values, and (PS)-conditions in $H^1(\mathbb{R}^3)$ for J_λ as follows.

Definition 2.5. (a) For $\beta \in \mathbb{R}$, a sequence $\{u_n\}$ is a $(\mathrm{PS})_\beta$-sequence in $H^1(\mathbb{R}^3)$ for J_λ if $J_\lambda(u_n) = \beta + o(1)$ and $J'_\lambda(u_n) = o(1)$ strongly in $H^{-1}(\mathbb{R}^3)$ as $n \to \infty$.

(b) J_λ satisfies the $(\mathrm{PS})_\beta$-condition in $H^1(\mathbb{R}^3)$ if every $(\mathrm{PS})_\beta$-sequence in $H^1(\mathbb{R}^3)$ for J_λ contains a convergent subsequence.

Now we consider the following elliptic problem:

$$(\mathrm{E}^\infty) \qquad \begin{cases} -\Delta u + u = |u|^{p-2} u & \text{in } \mathbb{R}^3, \\ \lim\limits_{|x| \to \infty} u = 0. \end{cases}$$

Associated with equation (E^∞), we consider the energy functional J^∞ in $H^1(\mathbb{R}^3)$

$$J^\infty(u) = \frac{1}{2} \int_{\mathbb{R}^3} [|\nabla u|^2 + u^2] \, dx - \frac{1}{p} \int_{\mathbb{R}^3} |u|^p \, dx.$$

Consider the minimizing problem:

$$\inf_{u \in \mathbf{N}^\infty} J^\infty(u) = \alpha^\infty,$$

where $\mathbf{N}^\infty = \{u \in H^1(\mathbb{R}^3) \setminus \{0\} \mid \langle (J^\infty)'(u), u \rangle = 0\}$.

It is known that equation (E^∞) has a unique positive radially solution $w(x)$ such that $J^\infty(w) = \alpha^\infty$ and $w(0) = \max\limits_{x \in \mathbb{R}^3} w(x)$ (see [19]). Then we have the following results.

Proposition 2.6. *Let $\{u_n\}$ be a $(\mathrm{PS})_\beta$-sequence in $H^1(\mathbb{R}^3)$ for J_λ. Then there exist a subsequence $\{u_n\}$, $m \in \mathbb{N}$, sequences $\{x_n^i\}_{n=1}^\infty$ in $\mathbb{R}^3$, and functions $v_0 \in H^1(\mathbb{R}^3)$, and $0 \neq w^i \in H^1(\mathbb{R}^3)$, for $1 \le i \le m$ such that:*

(a) $|x_n^i| \to \infty$ *and* $|x_n^i - x_n^j| \to \infty$ *as* $n \to \infty$, *for* $1 \le i \neq j \le m$;

(b) $-\Delta v_0 + v_0 + K(x)\phi_{v_0}(x) v_0 = |v_0|^{p-2} v_0 + \lambda f(x) |v_0|^{q-2} v_0$ *in* $\mathbb{R}^3$;

(c) $-\Delta w^i + w^i = |w^i|^{p-2} w^i$ *in* $\mathbb{R}^3$;

(d) $u_n = v_0 + \sum\limits_{i=1}^m w^i(\,\cdot\, - x_n^i) + o(1)$ *strongly in* $H^1(\mathbb{R}^3)$;

(e) $J_\lambda(u_n) = J_\lambda(v_0) + \sum\limits_{i=1}^m J^\infty(w^i) + o(1)$.

In addition, if $u_n \ge 0$, then $v_0 \ge 0$ and $w^i \ge 0$ for each $1 \le i \le m$.

Proof. The proof is essentially the same as Lemma 4.1 in Cerami and Vaira [13] (or see Lions [21], [22]), and so we omit it here. □

Corollary 2.7. *Suppose that* $\{u_n\}$ *is a* $(\mathrm{PS})_\beta$*-sequence in* $H^1(\mathbb{R}^3)$ *for* J_λ *with* $0<\beta<\alpha^\infty+\min\{\alpha_\lambda,\alpha^\infty\}$ *and* $\beta\neq\alpha^\infty$. *Then there exists a subsequence* $\{u_n\}$ *and a non-zero* u_0 *in* $H^1(\mathbb{R}^3)$ *such that* $u_n\to u_0$ *strongly in* $H^1(\mathbb{R}^3)$ *and* $J_\lambda(u_0)=\beta$. *Furthermore,* (u_0,ϕ_{u_0}) *is a non-zero solution of problem* (SP_λ).

3. The estimate of energy

First, we let $w(x)$ be a positive radially solution of Equation (E^∞) such that $J^\infty(w)=\alpha^\infty$. Then by Gidas, Ni and Nirenberg [17], for any $\varepsilon>0$, there exist positive numbers A_ε and B_0 such that

$$A_\varepsilon\exp(-(1+\varepsilon)|x|)\le w(x)\le B_0\exp(-|x|)\quad\text{for all } x\in\mathbb{R}^3. \tag{3.1}$$

Let $e\in\mathbb{S}^2=\{x\in\mathbb{R}^3\mid |x|=1\}$ and let $z_0=(\delta_0,0,0)\in\mathbb{R}^3$, where

$$0<\delta_0=\frac{r_f-1}{2(r_f+1)}<1.$$

Clearly,

$$1-\delta_0\le|e-z_0|\le1+\delta_0\quad\text{for all } e\in\mathbb{S}^2. \tag{3.2}$$

Define

$$w_{e,l}(x)=w(x-le)\quad\text{for } l\ge0 \text{ and } e\in\mathbb{S}^2 \tag{3.3}$$

and

$$w_{z_0,l}(x)=w(x-lz_0)\qquad\text{for } l\ge0.$$

Note that $w_{e,l}$ and $w_{z_0,l}$ are also least energy positive solutions of equation (E^∞) for all $l\ge0$. Moreover, by Lemma 2.4 for each $u\in H^1(\mathbb{R}^3)\setminus\{0\}$ and $\lambda<\lambda_1(q)$, there is a unique $t_\lambda(u)>0$ such that $t_\lambda(u)u\in\mathbf{N}_\lambda$. Let $\widehat{t}$ be as in (2.3). Then we have the following results.

Lemma 3.1. *For each* $s_0\in(0,1)$ *there exist* $l(s_0)>0$ *and* $\sigma(s_0)>1$ *such that, for any* $l>l(s_0)$, *we have*

$$\widehat{t}^{\,p-2}(sw_{e,l}+(1-s)w_{z_0,l})>\frac{\sigma(s_0)}{s^{p-2}+(1-s)^{p-2}}$$

for all $e\in\mathbb{S}^2$ *and for all* $s\in(0,1)$ *with* $\min\{s,1-s\}\ge s_0$.

Proof. Since

$$\widehat{t}^{\,p-2}(sw_{e,l}+(1-s)w_{z_0,l})=\frac{s^2|w|^2+(1-s)^2|w|^2+2s(1-s)\langle w_{e,l},w_{z_0,l}\rangle}{\int_{\mathbb{R}^3}|sw_{e-z_0,l}+(1-s)w|^p\,dx} \tag{3.4}$$

for all $s\in[0,1]$ and for all $e\in\mathbb{S}^2$. Moreover, by

$$1-\delta_0\le|e-z_0|\le1+\delta_0\quad\text{for all } e\in\mathbb{S}^2, \tag{3.5}$$

we have

$$\begin{aligned}\langle w_{e,l}, w_{z_0,l}\rangle &= \int_{\mathbb{R}^3} w^{p-1} w_{e-z_0,l}\,dx \\ &\le B_0^p \int_{|x|<(1+\delta_0)l} \exp(-(|x|+|x-l(z_0-e)|))\,dx \\ &\quad + B_0^p \int_{|x|\ge(1+\delta_0)l} \exp(-(|x|+|x-l(z_0-e)|))\,dx \\ &\le c_0 B_0^p l^3 \int_{|x|<(1+\delta_0)} \exp(-(1-\delta_0)l)\,dx + C_0 B_0^p \exp(-(1+\delta_0)l) \\ &\le C_0 B_0^p l^3 \exp(-l(1-\delta_0))\end{aligned}$$

for all $l \ge 1$ and for all $e \in \mathbb{S}^2$, and implies that

$$\lim_{l\to\infty} \langle w_{e,l}, w_{z_0,l}\rangle = 0 \quad \text{uniformly in } e \in \mathbb{S}^2. \tag{3.6}$$

By (3.1), (3.5) and Brézis–Lieb lemma [11], for any $s \in [0,1]$ we also have

$$\lim_{l\to\infty} \int_{\mathbb{R}^3} |s w_{e-z_0,l} + (1-s)w|^p - |s w_{e-z_0,l}|^p\,dx = \int_{\mathbb{R}^3} |(1-s)w|^p\,dx \quad \text{uniformly in } e \in \mathbb{S}^2. \tag{3.7}$$

Thus, by (3.4), (3.6) and (3.7), for any $s \in [0,1]$,

$$\lim_{l\to\infty} \widehat{t}^{p-2}(s w_{e,l} + (1-s) w_{z_0,l}) = \frac{s^2+(1-s)^2}{s^p+(1-s)^p} \quad \text{uniformly in } e \in \mathbb{S}^2. \tag{3.8}$$

Since

$$\frac{(s^2+(1-s)^2)(s^{p-2}+(1-s)^{p-2})}{s^p+(1-s)^p} \ge 1 + \frac{s_0^2(1-s_0)^{p-2}+(1-s_0)^2 s_0^{p-2}}{2(1-s_0)^p} \tag{3.9}$$

for all $s \in (0,1)$ with $\min\{s, 1-s\} \ge s_0$, by (3.8) and (3.9), there exist $l(s_0) > 0$ and $\sigma(s_0) > 1$ such that for any $l > l(s_0)$, we have

$$\widehat{t}^{p-2}(s w_{e,l} + (1-s) w_{z_0,l}) > \frac{\sigma(s_0)}{s^{p-2}+(1-s)^{p-2}}$$

for all $e \in \mathbb{S}^2$ and for all $s \in (0,1)$ with $\min\{s, 1-s\} \ge s_0$. □

PROPOSITION 3.2. (a) *For each $0 < \lambda < \lambda(q)$, there exists $\widehat{l}_1 = \widehat{l}_1(\lambda) > 0$ such that, for any $l \ge \widehat{l}_1$, $\sup_{t\ge0} J_\lambda(t w_{e,l}) < \alpha^\infty$ for all $e \in \mathbb{S}^2$. Furthermore, there is a unique $t_\lambda(w_{e,l}) > 0$ such that $t_\lambda(w_{e,l}) w_{e,l} \in \mathbf{N}_\lambda$.*

(b) *If $\lambda = 0$, then there exists $l_1 > 0$ such that for any $l \ge l_1$*

$$\sup_{t\ge0} J_0(t[s w_{e,l} + (1-s) w_{z_0,l}]) < 2\alpha^\infty \quad \textit{for all } e \in \mathbb{S}^2.$$

Furthermore, there is a unique $t_0(s w_{e,l} + (1-s) w_{z_0,l}) > 0$ such that

$$t_0(s w_{e,l} + (1-s) w_{z_0,l})[s w_{e,l} + (1-s) w_{z_0,l}] \in \mathbf{N}_0.$$

PROOF. (a) Since

$$J_\lambda(tw_{e,l}) = \frac{t^2}{2}|w_{e,l}|^2 + \frac{t^4}{4}\int_{\mathbb{R}^3} K(x)\phi_{w_{e,l}}(x)w_{e,l}^2\,dx - \frac{t^p}{p}\int_{\mathbb{R}^3}|w_{e,l}|^p\,dx - \frac{\lambda t^q}{q}\int_{\mathbb{R}^3} f|w_{e,l}|^q\,dx$$
$$= \frac{t^2}{2}|w|^2 - \frac{t^p}{p}\int_{\mathbb{R}^3} w^p\,dx + \frac{t^4}{4}\int_{\mathbb{R}^3} K(x)\phi_{w_{e,l}}(x)w_{e,l}^2\,dx - \frac{\lambda t^q}{q}\int_{\mathbb{R}^3} f|w_{e,l}|^q\,dx$$
$$\le \frac{t^2}{2}|w|^2 - \frac{t^p}{p}\int_{\mathbb{R}^3} w^p\,dx + \frac{t^4}{4}\overline{S}^{-2}S^{-4}|K|_2^2\|w\|^4 - \frac{\lambda t^q}{q}\int_{\mathbb{R}^3} f|w_{e,l}|^q\,dx, \tag{3.10}$$

for all $0 < \lambda < \lambda(q)$, it implies that $J_\lambda(tw_{e,l}) \to -\infty$ as $t \to \infty$ for all $e \in \mathbb{S}^2$. Thus, there exists $t_1 > 0$ such that, for any $l \ge 0$,

$$J_\lambda(tw_{e,l}) < \alpha^\infty \quad \text{for all } t \ge t_1 \text{ and for all } e \in \mathbb{S}^2. \tag{3.11}$$

Moreover, it is easy to obtain that $J_\lambda(0) = 0 < \alpha^\infty$, $J_\lambda \in C^1(H^1(\mathbb{R}^3),\mathbb{R})$ and $|w_{e,l}|^2 = 2p\alpha^\infty/(p-2)$ for all $l \ge 0$, which implies that there exists $t_2 > 0$ such that, for any $l \ge 0$,

$$J_\lambda(tw_{e,l}) < \alpha^\infty \quad \text{for all } 0 \le t \le t_2 \text{ and for all } e \in \mathbb{S}^2. \tag{3.12}$$

By Brown and Zhang [12] and Willem [27], we also know that

$$J^\infty(tw) = \frac{t^2}{2}|w|^2 - \frac{t^p}{p}\int_{\mathbb{R}^3} w^p\,dx \le \alpha^\infty \quad \text{for all } t > 0. \tag{3.13}$$

Thus, by (3.10),

$$J_\lambda(tw_{e,l}) \le \alpha^\infty + \frac{t^4}{4}\int_{\mathbb{R}^3} K(x)\phi_{w_{e,l}}(x)w_{e,l}^2\,dx - \frac{\lambda t^q}{q}\int_{\mathbb{R}^3} f w_{e,l}^q\,dx \tag{3.14}$$

for all $t > 0$. By (3.11) and (3.12), we only need to show that there exists $\widehat{l}_1 > 0$ such that, for any $l > \widehat{l}_1$,

$$\sup_{t_2 \le t \le t_1} J_\lambda(tw_{e,l}) < \alpha^\infty \quad \text{for all } e \in \mathbb{S}^2.$$

We set $C_0 = \min_{x \in \overline{B^3(0,1)}} w^q(x) > 0$, where $B^3(0,1) = \{x \in \mathbb{R}^3 \mid |x| < 1\}$. Then, by the condition $(\mathcal{D}1)$,

$$\lambda\int_{\mathbb{R}^3} f w_{e,l}^q\,dx \ge \lambda\int_{|x|\ge R_0} f w_{e,l}^q\,dx = \lambda\int_{|x+le|\ge R_0} f(x+le)w^q(x)\,dx \ge \lambda C_0\int_{B^3(0,1)} f(x+le)\,dx \ge \lambda C_0\exp(-r_f l) \tag{3.15}$$

for all $l \geq 2\max\{1, R_0\}$. Moreover, by (3.1) and the condition $(\mathcal{K})$, we have

$$
\begin{aligned}
(3.16)\qquad &\int_{\mathbb{R}^3} K(x)\phi_{w_{e,l}}(x)w_{e,l}^2\,dx \\
&\leq \bigg(\int_{\mathbb{R}^3} K^{6/5}(x)w_{e,l}^{12/5}\,dx\bigg)^{5/6}\bigg(\int_{\mathbb{R}^3}\phi_{w_{e,l}}^6(x)\,dx\bigg)^{1/6} \\
&\leq B_0^2\overline{S}^{-2}S^{-2}|K|_2\frac{2p}{p-2}\alpha^\infty\bigg[C\exp\bigg(-\min\bigg\{\frac{6}{5}r_K,\frac{12}{5}\bigg\}l\bigg)\bigg]^{5/6} \\
&\leq C\exp(-\min\{r_K,2\}l).
\end{aligned}
$$

Since $r_f < \min\{r_K, 2\}$ and $t_2 \leq t \leq t_1$, we can find $\widehat{l}_1 > 2\max\{1, R_0\}$ such that, for any $l > \widehat{l}_1$,

$$
(3.17)\qquad \frac{t^4}{4}\int_{\mathbb{R}^3} K(x)\phi_{w_{e,l}}(x)w_{e,l}^2\,dx < \frac{\lambda t^q}{q}\int_{\mathbb{R}^3} f w_{e,l}^q\,dx
$$

for all $e \in \mathbb{S}^2$ and for all $t \in [t_2, t_1]$. Thus, by (3.11)–(3.14) and (3.17), we obtain that, for any $l > \widehat{l}_1$, $\sup_{t\geq 0} J_\lambda(tw_{e,l}) < \alpha^\infty$ for all $e \in \mathbb{S}^2$. Moreover, by Lemma 2.4, there is a unique $t_\lambda(w_{e,l}) > 0$ such that $t_\lambda(w_{e,l})w_{e,l} \in \mathbf{N}_\lambda$.

(b) When $s = 0$ or 1, by a similar argument in part (a), there exists $\widetilde{t}_1 > 0$ such that

$$
(3.18)\qquad \max\Big\{\sup_{t\geq 0} J_0(tw_{e,l}), \sup_{t\geq 0} J_0(tw_{z_0,l})\Big\} \leq \alpha^\infty + \frac{\widetilde{t}_1 C_0}{q}\exp(-r_f l)
$$

for all $e \in \mathbb{S}^2$, this implies that there exists $\widetilde{l}_1 > 0$ such that, for any $l > \widetilde{l}_1$,

$$
\max\Big\{\sup_{t\geq 0} J_0(tw_{e,l}), \sup_{t\geq 0} J_0(tw_{z_0,l})\Big\} \leq \frac{3}{2}\alpha^\infty \quad \text{for all } e \in \mathbb{S}^2.
$$

Therefore, by $J_0 \in C^2(H^1(\mathbb{R}^3), \mathbb{R})$, there exist positive constants s_0 and $\widetilde{l}$ such that, for any $l > \widetilde{l}$, $\sup_{t\geq 0} J_0(t[sw_{e,l} + (1-s)w_{z_0,l}]) < 2\alpha^\infty$ for all $e \in \mathbb{S}^2$ and for all $\min\{s, 1-s\} \leq s_0$. In the following we always assume that $\min\{s, 1-s\} \geq s_0$. By Lemma 2.4(a) and Lemma 3.1, we may show that there exists $l_1 \geq \widetilde{l}$ such that, for any $l > l_1$,

$$
(3.19)\qquad \sup_{t\leq(\sigma(s_0)/(s^{p-2}+(1-s)^{p-2})^{1/(p-2)})} J_0(t[sw_{e,l} + (1-s)w_{z_0,l}]) < 2\alpha^\infty
$$

for all $e \in \mathbb{S}^2$, where $\sigma(s_0) > 1$ is as in Lemma 3.1. Since

$$
\begin{aligned}
(3.20)\qquad &J_0(t[sw_{e,l} + (1-s)w_{z_0,l}]) \\
&= \frac{t^2}{2}[s^2|w|^2 + (1-s)^2|w|^2 + 2s(1-s)\langle w_{e,l}, w_{z_0,l}\rangle] \\
&\quad + \frac{t^4}{4}\int_{\mathbb{R}^3} K(x)\phi_{sw_{e,l}+(1-s)w_{z_0,l}}(x)(sw_{e,l} + (1-s)w_{z_0,l})^2\,dx \\
&\quad - \frac{t^p}{p}\int_{\mathbb{R}^3}[sw_{e,l} + (1-s)w_{z_0,l}]^p\,dx
\end{aligned}
$$

$$
\begin{aligned}
&\le \frac{t^2}{2}[s^2+2s(1-s)+(1-s)^2]|w|^2 \\
&\quad + \frac{t^4}{4}\overline{S}^{-2}S^{-4}|K|_2^2|sw_{e,l}+(1-s)w_{z_0,l}|^4 \\
&\quad - \frac{t^p}{p}\max\{s^p,(1-s)^p\}\int_{\mathbb{R}^3} w^p\,dx \\
&\le \frac{t^2}{2}|w|^2+\frac{t^4}{4}\overline{S}^{-2}S^{-4}|K|_2^2|w|^4-\frac{t^p}{p2^p}\int_{\mathbb{R}^3} w^p\,dx
\end{aligned}
$$

for all $0 \le s \le 1$ and $e \in \mathbb{S}^2$, there exists $t_1 > 0$ such that, for any $t \ge t_1$,

$$
J_0(t[sw_{e,l}+(1-s)w_{z_0,l}]) < 2\alpha^\infty \quad \text{for all } 0 \le s \le 1 \text{ and for all } e \in \mathbb{S}^2. \tag{3.21}
$$

By (3.19) and (3.21), we only need to show that there exists $l_1 \ge \widetilde{l}$ such that, for any $l > l_1$,

$$
\sup_{(\sigma(s_0)/(s^{p-2}+(1-s)^{p-2}))^{1/(p-2)}\le t\le t_1} J_0(t[sw_{e,l}+(1-s)w_{z_0,l}]) < 2\alpha^\infty \tag{3.22}
$$

for all $e \in \mathbb{S}^2$. By Lemma 2.1 in Bahri-Li [7], there exists $C_p > 0$, such that, for any nonnegative real numbers c, d,

$$
(c+d)^p \ge c^p + d^p + p(c^{p-1}d + cd^{p-1}) - C_p c^{p/2}d^{p/2}.
$$

Then, by (3.13), (3.20) and Lemma 3.1,

$$
\begin{aligned}
&J_0(t[sw_{e,l}+(1-s)w_{z_0,l}]) \\
&\le \frac{t^2}{2}[s^2|w|^2+(1-s)^2|w|^2+2s(1-s)\langle w_{e,l},w_{z_0,l}\rangle] \\
&\quad + \frac{t^4}{4}\int_{\mathbb{R}^3} K(x)\phi_{sw_{e,l}+(1-s)w_{z_0,l}}(x)(sw_{e,l}+(1-s)w_{z_0,l})^2\,dx \\
&\quad - \frac{t^p}{p}\int_{\mathbb{R}^3}(sw_{e,l})^p+[(1-s)w_{z_0,l}]^p+p(sw_{e,l})^{p-1}((1-s)w_{z_0,l}) \\
&\quad + p(sw_{e,l})[(1-s)w_{z_0,l}]^{p-1} - C_p(sw_{e,l})^{p/2}[(1-s)w_{z_0,l}]^{p/2}\,dx \\
&\le 2\alpha^\infty + \frac{t_1^4}{4}\overline{S}^{-2}S^{-2}|K|_2\|w\|^2 \\
&\quad \times \biggl(\int_{\mathbb{R}^3}K^{6/5}(x)(sw_{e,l}+(1-s)w_{z_0,l})^{12/5}\,dx\biggr)^{5/6} \\
&\quad - s(1-s)t^2[t^{p-2}(s^{p-2}+(1-s)^{p-2})-1]\int_{\mathbb{R}^3} w_{e,l}^{p-1}w_{z_0,l}\,dx \\
&\quad + \frac{t_1^pC_p}{p}\int_{\mathbb{R}^3} w_{e,l}^{p/2}w_{z_0,l}^{p/2}\,dx \\
&\le 2\alpha^\infty + \frac{t_1^4}{4}\overline{S}^{-2}S^{-2}|K|_2\|w\|^2
\end{aligned} \tag{3.23}
$$

$$\times\bigg(\int_{\mathbb{R}^3} K^{6/5}(x)(sw_{e,l}+(1-s)w_{z_0,l})^{12/5}\,dx\bigg)^{5/6} - C_0^2[\sigma(s_0)-1]\int_{\mathbb{R}^3} w_{e,l}^{p-1}w_{z_0,l}\,dx + \frac{t_1^pC_p}{p}\int_{\mathbb{R}^3} w_{e,l}^{p/2}w_{z_0,l}^{p/2}\,dx$$

for all $e\in\mathbb{S}^2$, where we have used the result

$$\int_{\mathbb{R}^3} w_{e,l}^{p-1}w_{z_0,l}\,dx = \langle w_{e,l}, w_{z_0,l}\rangle = \int_{\mathbb{R}^3} w_{e,l}w_{z_0,l}^{p-1}\,dx.$$

We first estimate $\int_{\mathbb{R}^3} w_{e,l}^{p-1}w_{z_0,l}\,dx$. Setting $\overline{C}_0=\min_{x\in\overline{B^3(0,1)}} w^{p-1}(x)>0$, by (3.1) and (3.2), for any $\varepsilon>0$, we have

$$\begin{aligned}\int_{\mathbb{R}^3} w_{e,l}^{p-1}w_{z_0,l}\,dx &= \int_{\mathbb{R}^3} w^{p-1}(x)w(x-l(z_0-e))\,dx\\ &\geq \overline{C}_0A_\varepsilon\int_{B^3(0,1)}\exp(-(1+\varepsilon)|x-l(z_0-e)|)\,dx\\ &\geq \overline{C}_0A_\varepsilon\exp(-l(1+\varepsilon)(1+\delta_0)).\end{aligned}\tag{3.24}$$

From (3.2) we have

$$\begin{aligned}\int_{\mathbb{R}^3} w_{e,l}^{p/2}w_{z_0,l}^{p/2}\,dx &\leq B_0^p\int_{|x|<(1+\delta_0)l}\exp\bigg(-\frac{p}{2}(|x|+|x-l(z_0-e)|)\bigg)\,dx\\ &\quad + B_0^p\int_{|x|\geq(1+\delta_0)l}\exp\bigg(-\frac{p}{2}(|x|+|x-l(z_0-e)|)\bigg)\,dx\\ &\leq CB_0^pl^3\int_{|x|<(1+\delta_0)}\exp\bigg(-\frac{pl}{2}|e-z_0|\bigg)\,dx\\ &\quad + CB_0^p\exp\bigg(-\frac{pl}{2}|e-z_0|\bigg)\\ &\leq CB_0^pl^3\exp(-r_f(1-\delta_0)l)\end{aligned}\tag{3.25}$$

for l sufficiently large. By (3.16) and the condition ($\mathcal{K}$), one has

$$\begin{aligned}&\bigg(\int_{\mathbb{R}^3} K^{6/5}(x)(sw_{e,l}+(1-s)w_{z_0,l})^{12/5}\,dx\bigg)^{5/6}\\ &\quad\leq 2^{7/6}\bigg(\int_{\mathbb{R}^3} K^{6/5}(x)w_{e,l}^{12/5}\,dx+\int_{\mathbb{R}^3} K^{6/5}(x)w_{z_0,l}^{12/5}\,dx\bigg)^{5/6}\\ &\quad\leq CB_0^2l^3\exp(-\min\{r_K,2\}l)\leq CB_0^2l^3\exp(-r_f(1-\delta_0)l)\end{aligned}\tag{3.26}$$

for $l\geq 1$. Since

$$1+\delta_0 = 1+\frac{r_f-1}{2(r_f+1)} < r_f\bigg(1-\frac{r_f-1}{2(r_f+1)}\bigg) = r_f(1-\delta_0),$$

we may take $0<\varepsilon<<1$ such that $(1+\varepsilon)(1+\delta_0)<r_f(1-\delta_0)$. Then, by (3.23)–(3.26), there exists $l_1\geq\max\{\widetilde{l},1\}$ such that (3.22) holds. Thus, we can

conclude that, for any $l > l_1$,

$$\sup_{t\geq 0} J_0(t[sw_{e,l} + (1-s)w_{z_0,l}]) < 2\alpha^\infty \quad \text{for all } 0 \leq s \leq 1 \text{ and for all } e \in \mathbb{S}^2.$$

Moreover, by Lemma 2.4(a), there is a unique $t_0(sw_{e,l} + (1-s)w_{z_0,l}) > 0$ such that $t_0(sw_{e,l} + (1-s)w_{z_0,l})\,[sw_{e,l} + (1-s)w_{z_0,l}] \in \mathbf{N}_0$. □

REMARK 3.3. Using (3.15), (3.16) and $r_f < \min\{r_K, 2\}$, it is not difficult to obtain that $\widehat{l_1} \to \infty$ as $\lambda \to 0$.

4. Existence of a positive solution

First, we establish the existence of positive ground state solutions for problem (SP_λ) for $0 < \lambda < \lambda(q)$.

THEOREM 4.1. *For each $0 < \lambda < \lambda(q)$, problem (SP_λ) has a positive ground state solution $(\widehat{u}_0, \phi_{\widehat{u}_0})$.*

PROOF. By analogy with the proof of Ni and Takagi [23], one can show that by the Ekeland variational principle (see [16]), there exists a minimizing sequence $\{u_n\} \subset \mathbf{N}_\lambda$ such that

$$J_\lambda(u_n) = \alpha_\lambda + o(1) \quad \text{and} \quad J'_{\lambda|\mathbf{N}_\lambda}(u_n) = o(1) \quad \text{in } H^{-1}(\mathbb{R}^3).$$

Moreover, by a similar argument to that in the proof of [13, Lemma 4.1],

$$J'_\lambda(u_n) = o(1) \quad \text{in } H^{-1}(\mathbb{R}^3).$$

Since $\alpha_\lambda < \alpha^\infty$ from Proposition 3.2(a), by Lemma 2.2 and Corollary 2.7 there exists a subsequence $\{u_n\}$ and $\widehat{u}_0 \in \mathbf{N}_\lambda$ such that

$$u_n \to \widehat{u}_0 \quad \text{strongly in } H^1(\mathbb{R}^3) \quad \text{and} \quad J_\lambda(\widehat{u}_0) = \alpha_\lambda.$$

Since $J_\lambda(\widehat{u}_0) = J_\lambda(|\widehat{u}_0|)$ and $|\widehat{u}_0| \in \mathbf{N}_\lambda$, by Lemma 2.3, we obtain that $(\widehat{u}_0, \phi_{\widehat{u}_0})$ is a positive solution of problem (SP_λ). □

Set $\alpha_0 = \inf_{u\in\mathbf{N}_0} J_0(u)$. Then by a similar argument to that in the proof of [13, Proposition 6.1], we have

$$\alpha_0 = \inf_{u\in\mathbf{N}_0} J_0(u) = \inf_{u\in\mathbf{N}^\infty} J^\infty(u) = \alpha^\infty, \tag{4.1}$$

and α_0 is not attained. Moreover, we have the following result.

LEMMA 4.2. *Suppose that $\{u_n\}$ is a minimizing sequence for J_0 in $\mathbf{N}_0$. Then*

$$\int_{\mathbb{R}^3} K(x)\phi_{u_n}(x)u_n^2\,dx = o(1).$$

Furthermore, $\{u_n\}$ is a $(\mathrm{PS})_{\alpha^\infty}$-sequence for J^∞ in $H^1(\mathbb{R}^3)$.

PROOF. For each n, there is a unique $t_n > 0$ such that $t_n u_n \in \mathbf{N}^\infty$, that is

$$t_n^2 |u_n|^2 = t_n^p \int_{\mathbb{R}^3} |u_n|^p \, dx.$$

Then, by Lemma 2.4(a),

$$\begin{aligned} J_0(u_n) \geq J_0(t_n u_n) &= J^\infty(t_n u_n) + \frac{t_n^4}{4} \int_{\mathbb{R}^3} K(x)\phi_{u_n}(x) u_n^2 \, dx \\ &\geq \alpha^\infty + \frac{t_n^4}{4} \int_{\mathbb{R}^3} K(x)\phi_{u_n}(x) u_n^2 \, dx. \end{aligned}$$

Since $J_0(u_n) = \alpha^\infty + o(1)$ from (4.1), we have

$$\frac{t_n^4}{4} \int_{\mathbb{R}^3} K(x)\phi_{u_n}(x) u_n^2 \, dx = o(1).$$

We will show that there exists $c_1 > 0$ such that $t_n > c_1$ for all n. Suppose the contrary. Then we may assume $t_n \to 0$ as $n \to \infty$. Since $J_0(u_n) = \alpha^\infty + o(1)$, by Lemma 2.2, we have $|u_n|$ is uniformly bounded and so $|t_n u_n| \to 0$ or $J^\infty(t_n u_n) \to 0$, and this contradicts the fact that $J^\infty(t_n u_n) \geq \alpha^\infty > 0$. Thus,

$$\int_{\mathbb{R}^3} K(x)\phi_{u_n}(x) u_n^2 \, dx = o(1),$$

which implies that

$$|u_n|^2 = \int_{\mathbb{R}^3} |u_n|^p \, dx + o(1) \quad \text{and} \quad J^\infty(u_n) = \alpha^\infty + o(1).$$

Moreover, by Wang and Wu [26, Lemma 7], we have $\{u_n\}$ is a $(\mathrm{PS})_{\alpha^\infty}$-sequence for J^∞ in $H^1(\mathbb{R}^3)$. □

For $u \in H^1(\mathbb{R}^3)$, we define the center mass function from $\mathbf{N}_\lambda$ to the unit ball $B^3(0,1)$ in $\mathbb{R}^3$,

$$m(u) = \frac{1}{|u|_p^p} \int_{\mathbb{R}^3} \frac{x}{|x|} |u(x)|^p \, dx.$$

Clearly, m is continuous from $\mathbf{N}_\lambda$ to $B^3(0,1)$ and $|m(u)| < 1$. Let

$$\theta_\lambda = \inf\{J_\lambda(u) \mid u \in \mathbf{N}_\lambda,\ u \geq 0,\ m(u) = 0\}.$$

Then we have the following result.

LEMMA 4.3. *There exists $\xi_0 > 0$ such that $\alpha^\infty < \xi_0 \leq \theta_0$, where $\theta_0 = \theta_\lambda$ with $\lambda = 0$.*

PROOF. Suppose the contrary. Then there exists a sequence $\{u_n\} \subset \mathbf{N}_0$ and $m(u_n) = 0$ for each n, such that $J_0(u_n) = \alpha^\infty + o(1)$. By Lemma 4.2, $\{u_n\}$ is a $(\mathrm{PS})_{\alpha^\infty}$-sequence in $H^1(\mathbb{R}^3)$ for J^∞. Moreover, by the concentration–compactness principle (see Lions [21], [22]) and the fact that $\alpha^\infty > 0$, there exist a subsequence $\{u_n\}$, a sequence $\{x_n\} \subset \mathbb{R}^3$, and a positive solution $w_0 \in H^1(\mathbb{R}^3)$ of equation (E^∞) such that

$$|u_n(x) - w_0(x - x_n)| \to 0 \quad \text{as } n \to \infty. \tag{4.2}$$

Now we will show that $|x_n| \to \infty$ as $n \to \infty$. Suppose the contrary. Then we may assume that $\{x_n\}$ is bounded and $x_n \to x_0$ for some $x_0 \in \mathbb{R}^3$. Thus, by (4.2),

$$\int_{\mathbb{R}^3} K(x)\phi_{u_n}(x)u_n^2 dx = \int_{\mathbb{R}^3} K(x + x_0)\phi_{w_0}(x + x_0)w_0^2(x)\, dx + o(1),$$

this contradicts the result of Lemma 4.2: $\int_{\mathbb{R}^3} K(x)\phi_{u_n}(x)u_n^2\, dx = o(1)$. Hence we may assume $x_n/|x_n| \to e$ as $n \to \infty$, where $e \in \mathbb{S}^2$. Then, by (4.2) and the Lebesgue dominated convergence theorem, we have

$$0 = m(u_n) = |w_0|_p^{-p} \int_{\mathbb{R}^3} \frac{x + x_n}{|x + x_n|} |w_0(x)|^p\, dx + o(1) = e + o(1) \quad \text{as } n \to \infty,$$

which is a contradiction. Therefore, there exists $\xi_0 > 0$ such that $\alpha^\infty < \xi_0 \leq \theta_0$. □

By Lemma 2.4 and Proposition 3.2, for each $e \in \mathbb{S}^2$ and $l > l_1$ there exists $t_0(w_{e,l}) > 0$ such that $t_0(w_{e,l})w_{e,l} \in \mathbf{N}_0$. Moreover, we have the following result.

Lemma 4.4. *There exists $l_0 \geq l_1$ such that, for any $l \geq l_0$,*

(a) $\alpha^\infty < J_0(t_0(w_{e,l})w_{e,l}) < \xi_0$ *for all* $e \in \mathbb{S}^2$;
(b) $\langle m(t_0(w_{e,l})w_{e,l}), e\rangle > 0$, *for all* $e \in \mathbb{S}^2$.

Proof. (a) Follows from (3.13)–(3.16) and (4.1).

(b) For $x \in \mathbb{R}^3$ with $x + le \neq 0$, we have

$$\begin{aligned}\left(\frac{x + le}{|x + le|}, le\right) &= |x + le| - \frac{1}{|x + le|}(x + le, x)\\ &\geq |x + le| - |x| \geq l|e| - 2|x| = l - 2|x|.\end{aligned}$$

Then

$$\begin{aligned}\langle m(t_0(w_{e,l})w_{e,l}), e\rangle &= \frac{1}{l|w|_p^p} \int_{\mathbb{R}^3} \left(\frac{x + le}{|x + le|}, le\right)|w|^p\, dx\\ &\geq \frac{1}{l|w|_p^p}\left(l \int_{\mathbb{R}^3} |w|^p\, dx - 2\int_{\mathbb{R}^3} |x||w|^p\, dx\right) = 1 - \frac{2c_2}{l},\end{aligned}$$

where $c_2 = |w|_p^{-p} \int_{\mathbb{R}^3} |x||w|^p\, dx$. Thus, there exists $l_0 \geq l_1$ such that, for any $l \geq l_0$,

$$\langle m(t_0(w_{e,l})w_{e,l}), e\rangle \geq 1 - \frac{2c_0}{l} > 0 \quad \text{for all } e \in \mathbb{S}^2. \qquad \square$$

In the following, we will use Bahri–Li's minimax argument [7]. Let

$$\mathbb{B} = \{u \in H^1(\mathbb{R}^3) \setminus \{0\} \mid u \geq 0 \text{ and } |u| = 1\}.$$

We define $I_0(u) = \sup_{t \geq 0} J_0(tu) \colon \mathbb{B} \to \mathbb{R}$. Then, by Lemma 2.4(c), for each $u \in H^1(\mathbb{R}^3) \setminus \{0\}$ there exists

$$t_0(u) = \frac{1}{|u|} t_0\left(\frac{u}{|u|}\right) > 0$$

such that $t_0(u)u \in \mathbf{N}_0$ and

$$I_0\left(\frac{u}{|u|}\right) = J_0(t_0\left(\frac{u}{|u|}\right)\frac{u}{|u|}) = J_0(t_0(u)u). \tag{4.3}$$

Next, we define a map h_0 from $\mathbb{S}^2$ to $\mathbb{B}$ by

$$h_0(e) = \frac{w(x - le)}{|w(x - le)|} = \frac{w_{e,l}}{|w_{e,l}|},$$

where $e \in \mathbb{S}^2$. Then, by (3.18) and (4.3), for $l > l_0$ very large, we have

$$I_0(h_0(e)) = J_0(t_0(w_{e,l})w_{e,l}) < \theta_0 \quad \text{for all } e \in \mathbb{S}^2.$$

We define another map h^* from $\overline{B^3(0,1)}$ to $\mathbb{B}$ by

$$h^*(se + (1 - s)z_0) = \frac{sw_{e,l} + (1 - s)w_{z_0,l}}{|sw_{e,l} + (1 - s)w_{z_0,l}|},$$

where $0 \leq s \leq 1$ and $e \in \mathbb{S}^2$. It is clear that $h^*|_{\mathbb{S}^2} = h_0$. It follows from Proposition 3.2(b) and (4.3) that

$$I_0(h_0(e)) = J_0(t_0(sw_{e,l} + (1 - s)w_{z_0,l})\,[sw_{e,l} + (1 - s)w_{z_0,l}]) < 2\alpha^\infty \tag{4.4}$$

for all $e \in \mathbb{S}^2$. We next define a min-max value. Let

$$\beta_0 = \inf_{\gamma \in \Gamma} \max_{x \in \overline{B^3(0,1)}} I_0(\gamma(x)), \tag{4.5}$$

where

$$\Gamma = \big\{\gamma \in C(\overline{B^3(0,1)}, \mathbb{B}) \mid \gamma|_{\mathbb{S}^2} = h_0\big\}. \tag{4.6}$$

Note that $\mathbb{S}^2 = \partial B^3(0,1)$. Then we have the following result.

Lemma 4.5. *We have* $\alpha^\infty < \xi_0 \leq \theta_0 \leq \beta_0 < 2\alpha^\infty$.

Proof. By Lemmas 4.3 and 4.4, (4.4) and (4.3), we only need to show $\theta_0 \leq \beta_0$. For any $\gamma \in \Gamma$, there exists $t_0(\gamma(x)) > 0$ such that $t_0(\gamma(x))\gamma(x) \in \mathbf{N}_0$ and $t_0(\gamma(x))\gamma(x) = t_0(w_{x,l})w_{x,l}$ for all $x \in \mathbb{S}^2$. Consider the homotopy $H(s,x) \colon [0,1] \times B^3(0,1) \to \mathbb{R}^3$ defined by

$$H(s,x) = (1 - s)m(t_0(\gamma(x))\gamma(x)) + sI(x),$$

where I denotes the identity map. Note that $m(t_0(\gamma(x))\gamma(x)) = m(t_0(w_{x,l})w_{x,l})$ for all $x \in \mathbb{S}^2$. By Lemma 4.4(b), $H(s,x) \neq 0$ for $x \in \mathbb{S}^2$ and $s \in [0,1]$. Therefore,

$$\deg(m(t_0(\gamma)\gamma), B^3(0,1), 0) = \deg(I, B^3(0,1), 0) = 1.$$

There exists $x_0 \in B^3(0,1)$ such that $m(t_0(\gamma(x_0))\gamma(x_0)) = 0$. Hence, for each $\gamma \in \Gamma$, we have

$$\theta_0 = \inf\{J_0(u) \mid u \in \mathbf{N}_0,\ u \geq 0,\ m(u) = 0\} \leq \max_{x \in \overline{B^3(0,1)}} I_0(\gamma(x)).$$

This shows that $\theta_0 \leq \beta_0$. □

Now, we are going to assert that problem (SP_λ) has a positive higher energy solution for $\lambda \leq 0$.

THEOREM 4.6. *Problem* (SP_λ) *with* $\lambda = 0$ *has a positive solution* $(\widetilde{u}_0, \phi_{\widetilde{u}_0})$ *such that* $J_0(\widetilde{u}_0) = \beta_0 > \alpha^\infty$.

PROOF. By Lemma 4.5 and the minimax principle (see Ambrosetti and Rabinowitz [4]), there exists a sequence $\{u_n\} \subset \mathbb{B}$ such that

$$\begin{cases} I_0(u_n) = \beta_0 + o(1), \\ |I_0'(u_n)|_{T^*_{u_n}\mathbb{B}} \equiv \sup\{I_0'(u_n)\omega \mid \omega \in T_{u_n}\mathbb{B}, |\omega| = 1\} = o(1) \quad \text{as } n \to \infty, \end{cases}$$

where $\alpha^\infty < \beta_0 < 2\alpha^\infty$ and $T_{u_n}\mathbb{B} = \{\omega \in H^1(\mathbb{R}^3) \mid \langle \omega, u_n\rangle = 0\}$. By an argument similar to the proof of Proposition 1.7 in Adachi and Tanaka [1], there exists $t_0(u_n) > 0$ such that $t_0(u_n)u_n \in \mathbf{N}_0$ and

$$\begin{cases} J_0(t_0(u_n)u_n) = \beta_0 + o(1), \\ J_0'(t_0(u_n)u_n) = o(1) \quad \text{in } H^{-1}(\mathbb{R}^3) \text{ as } n \to \infty. \end{cases}$$

Thus, by Corollary 2.7, we can conclude that Problem (SP_λ) with $\lambda = 0$ has a positive solution $(\widetilde{u}_0, \phi_{\widetilde{u}_0})$ and such that $J_0(\widetilde{u}_0) = \beta_0$. □

5. Existence of two positive solutions

First, we need the following result.

LEMMA 5.1. *There exists* $d_0 > 0$ *such that if* $u \in \mathbf{N}_0$ *and* $J_0(u) \leq \alpha^\infty + d_0$, *then*

$$\int_{\mathbb{R}^3} \frac{x}{|x|}(|\nabla u|^2 + u^2)dx \neq 0.$$

PROOF. Suppose the contrary. Then there exists a sequence $\{u_n\} \subset \mathbf{N}_0$ such that $J_0(u_n) = \alpha^\infty + o(1)$ and

$$\int_{\mathbb{R}^3} \frac{x}{|x|}(|\nabla u_n|^2 + u_n^2)\,dx = 0.$$

Moreover, by Lemma 4.2, $\{u_n\}$ is a $(\mathrm{PS})_{\alpha^\infty}$-sequence in $H^1(\mathbb{R}^3)$ for J^∞. By the concentration–compactness principle (see Lions [21], [22]) and the fact that $\alpha^\infty > 0$, there exist a subsequence $\{u_n\}$, a sequence $\{x_n\} \subset \mathbb{R}^3$, and a positive solution $w \in H^1(\mathbb{R}^3)$ of equation (E^∞) such that

$$|u_n(x) - w(x - x_n)| \to 0 \quad \text{as } n \to \infty. \tag{5.1}$$

Now we will show that $|x_n| \to \infty$ as $n \to \infty$. Suppose the contrary. Then we may assume that $\{x_n\}$ is bounded and $x_n \to x_0$ for some $x_0 \in \mathbb{R}^3$. Thus, by (5.1),

$$\int_{\mathbb{R}^3} K(x)\phi_{u_n}(x)u_n^2\,dx = \int_{\mathbb{R}^3} K(x+x_0)\phi_w(x+x_0)w^2(x)\,dx + o(1),$$

which contradicts the result of Lemma 4.2: $\int_{\mathbb{R}^3} K(x)\phi_{u_n}(x)u_n^2\,dx = o(1)$. Hence we may assume $x_n/|x_n| \to e_0$ as $n \to \infty$, where $e_0 \in \mathbb{S}^2$. Then, by the Lebesgue dominated convergence theorem, we have

$$\begin{aligned} 0 &= \int_{\mathbb{R}^3} \frac{x}{|x|}(|\nabla u_n|^2 + u_n^2)\,dx \\ &= \int_{\mathbb{R}^3} \frac{x+x_n}{|x+x_n|}(|\nabla w|^2 + w^2)\,dx + o(1) = \frac{2p}{p-2}\alpha^\infty e_0 + o(1), \end{aligned}$$

which is a contradiction. □

For $u \in \mathbf{N}_\lambda$, by Lemma 2.4, there is a unique $t_0(u) > 0$ such that $t_0(u)u \in \mathbf{N}_0$. Moreover, we have the following result.

Lemma 5.2. *There exists a continuous function* $\Lambda\colon [0,\infty) \to [0, S_p^{p/(p-2)})$ *with* $\Lambda(0) = 0$ *such that*

$$t_0(u) \le \big[1 + \lambda|f|_{p/(p-q)}(S_p^{p/(p-2)} - \Lambda(\lambda))^{(q-p)/p}\big]^{1/(p-q_0)}$$

for all $0 < \lambda < \lambda(q)$ *and* $u \in \mathbf{N}_\lambda$, *where* $q_0 = \max\{q, 4\}$.

Proof. Let $u \in \mathbf{N}_\lambda$. Then we have

$$\begin{aligned} S_p\bigg(\int_{\mathbb{R}^3}|u|^p\,dx\bigg)^{2/p} &\le |u|^2 = \int_{\mathbb{R}^3}|u|^p\,dx + \lambda\int_{\mathbb{R}^3} f|u|^q\,dx - \int_{\mathbb{R}^3} K(x)\phi_u(x)u^2\,dx \\ &\le \int_{\mathbb{R}^3}|u|^p\,dx + \lambda\int_{\mathbb{R}^3} f|u|^q\,dx \le \int_{\mathbb{R}^3}|u|^p\,dx + \lambda|f|_{p/(p-q)}\bigg(\int_{\mathbb{R}^3}|u|^p\,dx\bigg)^{q/p}, \end{aligned}$$

which implies that there exists a continuous function $\Lambda\colon [0,\infty) \to [0, S_p^{p/(p-2)})$ with $\Lambda(0) = 0$ such that

$$\int_{\mathbb{R}^3}|u|^p\,dx \ge S_p^{p/(p-2)} - \Lambda(\lambda) > 0. \tag{5.2}$$

We distinguish two cases.

Case 1. $t_0(u) < 1$. Since

$$1 + \lambda|f|_{p/(p-q)}(S_p^{p/(p-2)} - \Lambda(\lambda))^{(q-p)/p} \ge 1$$

for all $\lambda \ge 0$ and $p - q > 0$, we have

$$t_0(u) < 1 \le \big[1 + \lambda|f|_{p/(p-q)}(S_p^{p/(p-2)} - \Lambda(\lambda))^{(q-p)/p}\big]^{1/(p-q_0)}.$$

Case 2. $t_0(u) \geq 1$. Since $t_0(u)u \in \mathbf{N}_0$ for $u \in \mathbf{N}_\lambda$,

$$\begin{aligned}[t_0(u)]^p \int_{\mathbb{R}^3} |u|^p \, dx &= [t_0(u)]^2 |u|^2 + [t_0(u)]^4 \int_{\mathbb{R}^3} K(x)\phi_u(x)u^2 \, dx \\ &\leq [t_0(u)]^{q_0} \bigg(|u|^2 + \int_{\mathbb{R}^3} K(x)\phi_u(x)u^2 \, dx \bigg),\end{aligned}$$

and using (5.2), we have

$$\begin{aligned}[t_0(u)]^{p-q_0} &\leq \frac{|u|^2 + \int_{\mathbb{R}^3} K(x)\phi_u(x)u^2 \, dx}{\int_{\mathbb{R}^3} |u|^p \, dx} = 1 + \lambda \frac{\int_{\mathbb{R}^3} f|u|^q \, dx}{\int_{\mathbb{R}^3} |u|^p \, dx} \\ &\leq 1 + \lambda |f|_{p/(p-q)} \bigg(\int_{\mathbb{R}^3} |u|^p \, dx \bigg)^{(q-p)/p} \\ &\leq 1 + \lambda |f|_{p/(p-q)} (S_p^{p/(p-2)} - \Lambda(\lambda))^{(q-p)/p}.\end{aligned}$$

This completes the proof. □

By the proof of Proposition 3.2, there exist positive numbers $t_\lambda(w_{e,l})$ and $\widehat{l}_1$ such that $t(w_{e,l})w_{e,l} \in \mathbf{N}_\lambda$ and $J_\lambda(t_\lambda(w_{e,l})w_{e,l}) < \alpha^\infty$ for all $l > \widehat{l}_1$.

Let $\Lambda(\lambda)$ be as in Lemma 5.2. Then we have the following result.

LEMMA 5.3. *There exists a positive number $\widetilde{\lambda}(q)$ such that for every $\lambda \in (0, \widetilde{\lambda}(q))$, we have*

$$\int_{\mathbb{R}^3} \frac{x}{|x|} (|\nabla u|^2 + u^2) \, dx \neq 0$$

for all $u \in \mathbf{N}_\lambda$ with $J_\lambda(u) < \alpha^\infty$, where

$$\widetilde{\lambda}(q) = \begin{cases} \lambda_0 & \text{if } 4 \leq q \leq p, \\ \min\{\widehat{\lambda}, \lambda_0\} & \text{if } 2 < q < 4, \end{cases}$$

and $\lambda_0 > 0$ is defined in the proof.

PROOF. For $u \in \mathbf{N}_\lambda$ with $J_\lambda(u) < \alpha^\infty$, by Lemma 2.4, there exists $t_0(u) > 0$ such that $t_0(u)u \in \mathbf{N}_0$. Moreover,

$$J_\lambda(u) = \sup_{t \geq 0} J_\lambda(tu) \geq J_\lambda(t_0(u)u) = J_0(t_0(u)u) - \lambda [t_0(u)]^q \int_{\mathbb{R}^3} f|u|^q \, dx.$$

Thus, by Lemma 5.2 and the Sobolev inequality,

$$\begin{aligned}J_0(t_0(u)u) &\leq J_\lambda(u) + \lambda [t_0(u)]^q \int_{\mathbb{R}^3} f|u|^q \, dx \\ &< \alpha^\infty + \lambda C [1 + \lambda |f|_{p/(p-q)} (S_p^{p/(p-2)} - \Lambda(\lambda))^{(q-p)/p}]^{q/(p-q_0)} |u|^q.\end{aligned} \tag{5.3}$$

Moreover, by (2.1), we obtain there exists $M > 0$ such that

$$|u| \leq M \tag{5.4}$$

for all $u \in \mathbf{N}_\lambda$ with $J_\lambda(u) < \alpha^\infty$. Therefore, by (5.3) and (5.4),

$$J_0(t_0(u)u) < \alpha^\infty + \lambda C M^q [1 + \lambda |f|_{p/(p-q)} (S_p^{p/(p-2)} - \Lambda(\lambda))^{(q-p)/p}]^{q/(p-q_0)}.$$

Let $d_0 > 0$ be as in Lemma 5.1. Then there exists a positive number λ_0 such that for $\lambda \in (0, \lambda_0)$,

$$J_0(t_0(u)u) < \alpha^\infty + d_0. \tag{5.5}$$

Since $t_0(u)u \in \mathbf{N}_0$ and $t_0(u) > 0$, by Lemma 5.1 and (5.5),

$$\int_{\mathbb{R}^3} \frac{x}{|x|} (|\nabla(t_0(u)u)|^2 + (t_0(u)u)^2)\, dx \neq 0,$$

which implies that there exists a positive number $\widetilde{\lambda}(q)$ such that, for every $\lambda \in (0, \widetilde{\lambda}(q))$,

$$\int_{\mathbb{R}^3} \frac{x}{|x|} (|\nabla u|^2 + u^2)\, dx \neq 0$$

for all $u \in \mathbf{N}_\lambda$ with $J_\lambda(u) < \alpha^\infty$. □

In the following, we use an idea of Adachi and Tanaka [1]. For $c \in \mathbb{R}^+$, we define$[J_\lambda \leq c] = \{u \in \mathbf{N}_\lambda \mid u \geq 0, J_{\ \lambda}(u) \leq c\}$. We then try to show that for a sufficiently small $\sigma > 0$,

$$\operatorname{cat}([J_\lambda \leq \alpha^\infty - \sigma]) \geq 2. \tag{5.6}$$

To prove (5.6), we need some preliminaries. Recall the definition of the Lusternik–Schnirelmann category.

Definition 5.4. (a) For a topological space X, we say that a non-empty, closed subset $Y \subset X$ is contractible to a point in X if and only if there exists a continuous mapping $\xi\colon [0,1] \times Y \to X$ such that, for some $x_0 \in X$ and for all $x \in Y$,

$$\xi(0, x) = x \quad \text{and} \quad \xi(1, x) = x_0.$$

(b) We define:

$$\operatorname{cat}(X) = \min \bigg\{ k \in \mathbb{N} \;\bigg|\; \text{there exist closed subsets } Y_1, \ldots, Y_k \subset X \text{ such that } Y_j \text{ is contractible to a point in } X \text{ for all } j \text{ and } \bigcup_{j=1}^k Y_j = X \bigg\}.$$

When there do not exist finitely many closed subsets $Y_1, \ldots, Y_k \subset X$ such that Y_j is contractible to a point in X for all j and $\bigcup_{j=1}^k Y_j = X$, we say that $\operatorname{cat}(X) = \infty$.

We need the following two lemmas.

Lemma 5.5. *Suppose that X is a Hilbert manifold and $F \in C^1(X, \mathbb{R})$. Assume that there exist $\overline{c} \in \mathbb{R}$ and $k \in \mathbb{N}$ such that*:

(a) *$F(x)$ satisfies the Palais–Smale condition for energy levels $c \le \overline{c}$;*
(b) $\operatorname{cat}(\{x \in X \mid F(x) \le \overline{c}\}) \ge k$.

Then $F(x)$ has at least k critical points in $\{x \in X;\ F(x) \le \overline{c}\}$.

PROOF. See Ambrosetti [2, Theorem 2.3]. □

LEMMA 5.6. *Let X be a topological space. Suppose that there are two continuous maps $\Phi\colon \mathbb{S}^2 \to X$ and $\Psi\colon X \to \mathbb{S}^2$ such that $\Psi \circ \Phi$ is homotopic to the identity map of $\mathbb{S}^2$, that is, there exists a continuous map $\zeta\colon [0,1] \times \mathbb{S}^2 \to \mathbb{S}^2$ such that,*

$$\zeta(0,x) = (\Psi \circ \Phi)(x), \quad \zeta(1,x) = x \quad \textit{for each } x \in \mathbb{S}^2.$$

Then $\operatorname{cat}(X) \ge 2$.

PROOF. See Adachi and Tanaka [1, Lemma 2.5]. □

For $l > \widehat{l}_1$, we may define a map $\Phi_{\lambda,l}\colon \mathbb{S}^2 \to H^1(\mathbb{R}^3)$ by

$$\Phi_{\lambda,l}(e)(x) = t_\lambda(w(x-le))w(x-le) \quad \text{for } e \in \mathbb{S}^2,$$

where $t_\lambda(w(x-le))w(x-le)$ is as in the proof of Proposition 3.2. Then we have the following result.

LEMMA 5.7. *There exists a sequence $\{\sigma_l\} \subset \mathbb{R}^+$ with $\sigma_l \to 0$ as $l \to \infty$ such that*

$$\Phi_{\lambda,l}(\mathbb{S}^2) \subset [J_\lambda \le \alpha^\infty - \sigma_l]\,.$$

PROOF. By Proposition 3.2, for each $l > \widehat{l}_1$ we have $t_\lambda(w(x-le))w(x-le) \in \mathbf{N}_\lambda$ and $\sup_{l > \widehat{l}_1} J_\lambda(t_\lambda(w(x-le))w(x-le)) < \alpha^\infty$ for all $e \in \mathbb{S}^2$. Since $\Phi_{\lambda,l}(\mathbb{S}^2)$ is compact, $J_\lambda(t_\lambda(w(x-le))w(x-le)) \le \alpha^\infty - \sigma_l$, so that conclusion holds. □

From Lemma 5.3, we define $\Psi_\lambda\colon [J_\lambda < \alpha^\infty] \to \mathbb{S}^2$ by

$$\Psi_\lambda(u) = \frac{\displaystyle\int_{\mathbb{R}^3} \frac{x}{|x|}(|\nabla u|^2 + u^2)\,dx}{\displaystyle\left|\int_{\mathbb{R}^3} \frac{x}{|x|}(|\nabla u|^2 + u^2)\,dx\right|}.$$

Then we have the following results.

LEMMA 5.8. *There exists $\overline{\lambda} \in (0, \widehat{\lambda}(q)]$ and $\widehat{l}_0 \ge \widehat{l}_1$ such that for $\lambda \in (0, \overline{\lambda})$ and $l > \widehat{l}_0$, the map$\Psi_\lambda \circ \Phi_{\lambda,l}\colon \mathbb{S}^2 \to \mathbb{S}^2$ is homotopic to the identity.*

PROOF. Let

$$\Sigma = \left\{ u \in H^1(\mathbb{R}^3) \setminus \{0\} \;\middle|\; \int_{\mathbb{R}^3} \frac{x}{|x|}(|\nabla u|^2 + u^2)\,dx \ne 0 \right\}.$$

We define $\overline{\Psi}_\lambda\colon \Sigma \to \mathbb{S}^2$ by

$$\overline{\Psi}_\lambda(u) = \frac{\displaystyle\int_{\mathbb{R}^3} \frac{x}{|x|}(|\nabla u|^2 + u^2)\,dx}{\displaystyle\left|\int_{\mathbb{R}^3} \frac{x}{|x|}(|\nabla u|^2 + u^2)\,dx\right|},$$

an extension of Ψ_λ. By Remark 3.3 for $[0,1/2)$,

$$(1-2\theta)\Phi_{\lambda,l}(e) + 2\theta w(x-le) = w(x-le) + o(1) \quad \text{in } H^1(\mathbb{R}^3) \quad \text{as } \lambda \to 0.$$

By an argument similar to that in Lemma 5.1, there exist $\overline{\lambda} \in (0, \widetilde{\lambda}(q)]$ and $\widehat{l}_0 \in [\widehat{l}_1, \infty)$ such that for $\lambda \in (0, \overline{\lambda})$ and $l \in (\widehat{l}_0, \infty)$,

$$(1-2\theta)\Phi_{\lambda,l}(e) + 2\theta w(x-le) \in \Sigma \quad \text{for all } e \in \mathbb{S}^2 \text{ and } \theta \in [1/2, 1)$$

and

$$w\bigg(x - \frac{le}{2(1-\theta)}\bigg) \in \Sigma \quad \text{for all } e \in \mathbb{S}^2 \text{ and } \theta \in [1/2, 1).$$

Now we define $\zeta_l(\theta, e)\colon [0,1] \times \mathbb{S}^2 \to \mathbb{S}^2$ by

$$\zeta_l(\theta,e) = \begin{cases} \overline{\Psi}_\lambda((1-2\theta)\Phi_{\lambda,l}(e) + 2\theta w(x-le)) & \text{for } \theta \in [0,1/2); \\ \overline{\Psi}_\lambda\bigg(w\bigg(x - \dfrac{le}{2(1-\theta)}\bigg)\bigg) & \text{for } \theta \in [1/2,1); \\ e & \text{for } \theta = 1. \end{cases}$$

Then $\zeta_l(0,e) = \overline{\Psi}_\lambda(\Phi_{\lambda,l}(e)) = \Psi_\lambda(\Phi_{\lambda,l}(e))$ and $\zeta_l(1,e) = e$.

First, we claim that $\lim_{theta \to 1^-} \zeta_l(\theta,e) = e$ and $\lim_{\theta \to 1/2^-} \zeta_l(\theta,e) = \overline{\Psi}_\lambda(w(x-le))$.

(a) $\lim_{\theta \to 1^-} \zeta_l(\theta,e) = e$. Since

$$\begin{aligned} &\int_{\mathbb{R}^3} \frac{x}{|x|}\bigg(\bigg|\nabla\bigg[w\bigg(x - \frac{le}{2(1-\theta)}\bigg)\bigg]\bigg|^2 + \bigg[w\bigg(x - \frac{le}{2(1-\theta)}\bigg)\bigg]^2\bigg)\,dx \\ &= \int_{\mathbb{R}^3} \frac{x + le/(2(1-\theta))}{|x + le/(2(1-\theta))|}(|\nabla[w(x)]|^2 + [w(x)]^2)\,dx = \bigg(\frac{2p}{p-2}\bigg)\alpha^\infty e + o(1), \end{aligned}$$

as $\theta \to 1^-$, it follows that $\lim_{\theta \to 1^-} \zeta_l(\theta,e) = e$.

(b) $\lim_{\theta \to 1/2^-} \zeta_l(\theta,e) = \overline{\Psi}_\lambda(w(x-le))$.

Since $\overline{\Psi}_\lambda \in C(\Sigma, \mathbb{S}^2)$, we obtain $\lim_{\theta \to 1/2^-} \zeta_l(\theta,e) = \overline{\Psi}_\lambda(w(x-le))$. Thus, $\zeta_l(\theta,e) \in C([0,1] \times \mathbb{S}^2, \mathbb{S}^2)$ and

$$\begin{aligned} \zeta_l(0,e) &= \Psi_\lambda(\Phi_{\lambda,l}(e)) && \text{for all } e \in \mathbb{S}^2, \\ \zeta_l(1,e) &= e && \text{for all } e \in \mathbb{S}^2, \end{aligned}$$

provided $l > \widehat{l}_0$. $\square$

Theorem 5.9. *For each* $\lambda \in (0, \widetilde{\lambda}(q))$, J_λ *has at least two critical points in* $[J_\lambda < \alpha^\infty]$. *In particular, problem* (SP_λ) *has two positive solutions* $(u_0^{(1)}, \phi_{u_0^{(1)}})$ *and* $(u_0^{(2)}, \phi_{u_0^{(2)}})$ *such that* $u_0^{(i)} \in \mathbf{N}_\lambda$ *for* $i = 1, 2$.

Proof. Applying Lemmas 5.6 and 5.8, we have for $\lambda \in (0, \widetilde{\lambda}(q))$,

$$\mathrm{cat}([J_\lambda \le \alpha^\infty - \sigma_l]) \ge 2.$$

By Corollary 2.7 and Lemma 5.5, $J_\lambda(u)$ has at least two critical points in $[J_\lambda < \alpha^\infty]$. This implies that problem (SP_λ) has two positive solutions $(u_0^{(1)}, \phi_{u_0^{(1)}})$ and $(u_0^{(2)}, \phi_{u_0^{(2)}})$ such that $u_0^{(i)} \in \mathbf{N}_\lambda$ for $i = 1, 2$. □

6. Proof of Theorem 1.2

Given a positive real number $r_0 > q/(p-q)$. Let

$$\Lambda_0 = \min\left\{\left(\frac{r_0 p}{q(r_0+1)} - 1\right), \widetilde{\lambda}(q)\right\} > 0,$$

where $\widetilde{\lambda}(q) > 0$ is as in Lemma 5.3. Then we have the following results.

Lemma 6.1. *We have*

$$\frac{1}{2}(1+\lambda)^{r_0} - \frac{1}{p}(1+\lambda)^{r_0+1} - \frac{p-2}{2p} > 0$$

and

$$\frac{1}{q}(1+\lambda)^{r_0} - \frac{1}{p}(1+\lambda)^{r_0+1} - \frac{p-q}{pq} > 0$$

for all $\lambda \in (0, \Lambda_0)$.

Proof. Let

$$k(\lambda) = \frac{1}{q}(1+\lambda)^{r_0} - \frac{1}{p}(1+\lambda)^{r_0+1} - \frac{p-q}{pq}.$$

Then $k(0) = 0$ and

$$k'(\lambda) = \frac{r_0}{q}(1+\lambda)^{r_0-1} - \frac{r_0+1}{p}(1+\lambda)^{r_0} = (1+\lambda)^{r_0-1}\left(\frac{r_0}{q} - \frac{r_0+1}{p}(1+\lambda)\right) > 0$$

for all $\lambda \in (0, \Lambda_0)$. This implies that $k(\lambda) > 0$ or

$$\frac{1}{2}(1+\lambda)^{r_0} - \frac{1}{p}(1+\lambda)^{r_0+1} - \frac{p-q}{pq} > 0 \quad \text{for all } \lambda \in (0, \Lambda_0).$$

Similar to the argument we also have

$$\frac{1}{2}(1+\lambda)^{r_0} - \frac{1}{p}(1+\lambda)^{r_0+1} - \frac{p-2}{2p} > 0 \quad \text{for all } \lambda \in (0, \Lambda_0).$$

This completes the proof. □

We define $I_\lambda(u) = \sup_{t \ge 0} J_\lambda(tu) \colon \mathbb{B} \to \mathbb{R}$. Then we have the following result.

Lemma 6.2. *For each $\lambda \in (0,\Lambda_0)$ and $u \in \mathbb{B}$ we have*

$$(1+\lambda)^{-r_0} I_0(u) - \frac{\lambda(p-q)}{pq}\,|f|_{p/(p-q)} \le I_\lambda(u) \le I_0(u).$$

Proof. Let $u \in \mathbb{B}$. Using Lemma A.2 in Appendix, we obtain

$$\frac{\lambda}{q}\bigg(\int_{\mathbb{R}^3} |f|^{p/(p-q)}\,dx\bigg)^{1-q/p}\bigg(\int_{\mathbb{R}^3} |t_0(u)u|^p\bigg)^{q/p} \le \frac{\lambda(p-q)}{pq}|f|_{p/(p-q)} + \frac{\lambda t_0^p(u)}{p}\int_{\mathbb{R}^3}|u|^p\,dx. \tag{6.1}$$

Then, by Lemma 2.4, the Hölder inequality, Lemma 6.1 and (6.1), (4.3), we have

$$\begin{aligned}
I_\lambda(u) &= \sup_{t\ge 0} J_\lambda(tu) \ge J_\lambda(t_0(u)u)\\
&= \frac{t_0^2(u)}{2}\int_{\mathbb{R}^3}(|\nabla u|^2+u^2)dx + \frac{t_0^4(u)}{4}\int_{\mathbb{R}^3}K(x)\phi_u(x)u^2\,dx\\
&\quad - \frac{\lambda t_0^q(u)}{q}\int_{\mathbb{R}^3} f|u|^q dx - \frac{t_0^p(u)}{p}\int_{\mathbb{R}^3}|u|^p\,dx\\
&\ge \frac{t_0^2(u)}{2}\int_{\mathbb{R}^3}(|\nabla u|^2+u^2)\,dx + \frac{t_0^4(u)}{4}\int_{\mathbb{R}^3}K(x)\phi_u(x)u^2\,dx\\
&\quad - \frac{\lambda}{q}\bigg(\int_{\mathbb{R}^3}|f|^{p/(p-q)}\,dx\bigg)^{1-q/p}\bigg(\int_{\mathbb{R}^3}|t_0(u)u|^p\bigg)^{q/p} - \frac{t_0^p(u)}{p}\int_{\mathbb{R}^3}|u|^p\,dx\\
&\ge \frac{t_0^2(u)}{2}\int_{\mathbb{R}^3}(|\nabla u|^2+u^2)\,dx + \frac{t_0^4(u)}{4}\int_{\mathbb{R}^3}K(x)\phi_u(x)u^2\,dx\\
&\quad - \frac{\lambda t_0^p(u)}{p}\int_{\mathbb{R}^3}|u|^p\,dx - \frac{\lambda(p-q)}{pq}\,|f|_{p/(p-q)} - \frac{t_0^p(u)}{p}\int_{\mathbb{R}^3}|u|^p\,dx\\
&= \frac{t_0^2(u)}{2}\int_{\mathbb{R}^3}(|\nabla u|^2+u^2)\,dx + \frac{t_0^4(u)}{4}\int_{\mathbb{R}^3}K(x)\phi_u(x)u^2\,dx\\
&\quad - \frac{(1+\lambda)t_0^p(u)}{p}\int_{\mathbb{R}^3}|u|^p\,dx - \frac{\lambda(p-q)}{pq}|f|_{p/(p-q)}\\
&= \frac{t_0^2(u)}{2}\int_{\mathbb{R}^3}(|\nabla u|^2+u^2)\,dx\\
&\quad + \frac{t_0^4(u)}{4}\int_{\mathbb{R}^3}K(x)\phi_u(x)u^2\,dx - \frac{\lambda(p-q)}{pq}|f|_{p/(p-q)}\\
&\quad - \frac{1+\lambda}{p}\bigg[t_0^2(u)\int_{\mathbb{R}^3}(|\nabla u|^2+u^2)\,dx + t_0^4(u)\int_{\mathbb{R}^3}K(x)\phi_u(x)u^2\,dx\bigg]\\
&= \bigg(\frac{1}{2}-\frac{1+\lambda}{p}\bigg)t_0^2(u)\int_{\mathbb{R}^3}(|\nabla u|^2+u^2)\,dx\\
&\quad + \bigg(\frac{1}{q}-\frac{1+\lambda}{p}\bigg)t_0^q(u)\int_{\mathbb{R}^3}K(x)\phi_u(x)u^2\,dx - \frac{\lambda(p-q)}{pq}|f|_{p/(p-q)}
\end{aligned}$$

$$\geq (1+\lambda)^{-r_0} J_0(t_0(u)u) - \frac{\lambda(p-q)}{pq} |f|_{p/(p-q)}$$
$$= (1+\lambda)^{-r_0} I_0(u) - \frac{\lambda(p-q)}{pq} |f|_{p/(p-q)}.$$

Moreover, $J_\lambda(tu) \leq J_0(tu) \leq I_0(u)$ for all $t > 0$. Then $I_\lambda(u) \leq I_0(u)$. □

We observe that if λ is sufficiently small, the minimax argument in Section 4 also works for J_λ. Let $l > \max\{l_0, \widehat{l_0}\}$ be very large and let

$$\beta_\lambda = \inf_{\gamma\in\Gamma} \max_{x\in \overline{B^3(0,1)}} I_\lambda(\gamma(x)),$$

where Γ is as in (4.6). Then by (4.5) and Lemma 6.2, for $\lambda \in (0, \Lambda_0)$, we have

$$(1+\lambda)^{-r_0}\beta_0 - \frac{\lambda(p-q)}{pq}|f|_{p/(p-q)} \leq \beta_\lambda \leq \beta_0. \tag{6.2}$$

Moreover, we have the following result.

Theorem 6.3. *There exists a positive number $\Lambda_* \leq \Lambda_0$ such that*

$$\alpha^\infty < \beta_\lambda < 2\alpha^\infty \quad \text{for } \lambda \in (0, \Lambda_*).$$

Furthermore, problem (SP$_\lambda$) *has a positive solution $(u_0^{(3)}, \phi_{u_0^{(3)}})$ such that*

$$J_\lambda(u_0^{(3)}) = \beta_\lambda.$$

Proof. By (4.1), Theorem 4.1 and Lemma 6.2, we have

$$(1+\lambda)^{-r_0}\alpha^\infty - \frac{\lambda(p-q)}{pq}|f|_{p/(p-q)} \leq \alpha_\lambda < \alpha^\infty.$$

For any $\varepsilon > 0$, there exists a positive number $\overline{\lambda}_1 \leq \Lambda_0$ such that, for $\lambda \in (0, \overline{\lambda}_1)$, $\alpha^\infty - \varepsilon < \alpha_\lambda < \alpha^\infty$. Thus, $2\alpha^\infty - \varepsilon < \alpha^\infty + \alpha_\lambda < 2\alpha^\infty$.

Applying (6.2) for any $\delta > 0$ there exists a positive number $\overline{\lambda}_2 \leq \Lambda_0$ such that, for $\lambda \in (0, \overline{\lambda}_2)$, $\beta_0 - \delta < \beta_\lambda \leq \beta_0$. Moreover, by Theorem 4.6, $\alpha^\infty < \beta_0 < 2\alpha^\infty$.

Fix a small $0 < \varepsilon < 2\alpha^\infty - \beta_0$, choosing a $\delta > 0$ such that, for $\lambda \in (0, \Lambda_*)$, we get

$$\alpha^\infty < \beta_\lambda < 2\alpha^\infty - \varepsilon < \alpha^\infty + \alpha_\lambda < 2\alpha^\infty,$$

where $\Lambda_* = \min\{\overline{\lambda}_1, \overline{\lambda}_2\}$. Similar to the argument in the proof of Theorem 4.6, we can conclude that the problem (SP$_\lambda$) has a positive solution $(u_0^{(3)}, \phi_{u_0^{(3)}})$ such that $J_\lambda(u_0^{(3)}) = \beta_\lambda$. This completes the proof. □

We can now complete the proof of Theorem 1.2.

(a) and (b). By Theorems 4.1 and 4.6, the results (a) and (b) hold.

(c) By Theorems 5.9 and 6.3, there exists a positive number Λ_* such that for $\lambda \in (0, \Lambda_*)$, problem (SP$_\lambda$) has three positive solutions $(u_0^{(1)}, \phi_{u_0^{(1)}})$, $(u_0^{(2)}, \phi_{u_0^{(2)}})$ and $(u_0^{(3)}, \phi_{u_0^{(3)}})$ with

$$0 < J_\lambda(u_0^{(i)}) < \alpha^\infty < J_\lambda(u_0^{(3)}) < 2\alpha^\infty \quad \text{for } i = 1, 2.$$

This completes the proof of Theorem 1.2. □

Appendix A

Lemma A.1. *Suppose that a, b, c are positive constants and $2 < q \le 4 < p < 6$. Then there exists a positive number*

$$\overline{\lambda} = \frac{a}{c}\left[\frac{a(q-2)}{b(p-q)}\right]^{(2-q)/(p-2)} + \frac{b}{c}\left[\frac{a(q-2)}{b(p-q)}\right]^{(p-q)/(p-2)}$$

such that for any $\lambda < \overline{\lambda}$, the function

$$y(x) = -ax^2 - bx^p + \lambda c x^q < 0 \quad \textit{for all } x > 0.$$

Proof. Let

$$Y(x) = \frac{a}{c}\,x^{2-q} + \frac{b}{c}\,x^{p-q} \quad \text{for } x > 0.$$

Clearly, $Y(x) \to +\infty$ as $x \to 0^+$ and $Y(x) \to +\infty$ as $x \to +\infty$, and

$$Y'(x) = \frac{a}{c}(2-q)\,x^{1-q} + \frac{b}{c}(p-q)x^{p-q-1}.$$

Thus, Y has an absolute minimum at point $x_0 = [a(q-2)/(b(p-q))]^{1/(p-2)}$ and $Y'(x_0) = 0$. Take

$$\overline{\lambda} = Y(x_0) = \frac{a}{c}\left[\frac{a(q-2)}{b(p-q)}\right]^{(2-q)/(p-2)} + \frac{b}{c}\left[\frac{a(q-2)}{b(p-q)}\right]^{(p-q)/(p-2)} > 0,$$

then for any $\lambda < \overline{\lambda}$, we obtain

$$y(x) = -cx^q(Y(x) - \lambda) < 0 \quad \text{for all } x > 0.$$ □

Lemma A.2. *Suppose that a, b are positive constants. Then the function*

$$y(x) = a^{1-x}b^x - (1-x)a - bx \le 0 \quad \textit{for all } x \in [0,1].$$

Proof. Clearly, $y(x)$ is a differentiable function on $x \in [0,1]$, and $y(0) = y(1) = 0$. It is easy to obtain that

$$y'(x) = \left(\frac{b}{a}\right)^x a \ln\frac{b}{a} + a - b.$$

Thus, y has an absolute minimum at point $x_0 = \log_{b/a}^{(b-a)/(a\ln(b-a))} \in (0,1)$ satisfying $y'(x_0) = 0$ and $y''(x_0) > 0$, which implies that

$$y(x) \le 0 \quad \text{for all } x \in [0,1].$$ □

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