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EXISTENCE AND NONEXISTENCE OF LEAST ENERGY NODAL SOLUTIONS FOR A CLASS OF ELLIPTIC PROBLEM IN $\mathbb{R}^2$

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Abstract. In this work, we prove the existence of least energy nodal solutions for a class of elliptic problem in both cases, bounded and unbounded domain, when the nonlinearity has exponential critical growth in $\mathbb{R}^2$. Moreover, we also prove a nonexistence result of least energy nodal solution for the autonomous case in whole $\mathbb{R}^2$.

1. Introduction

This paper concerns with the existence of least energy nodal solutions for the following class of elliptic problem

$$\text{(P)} \qquad \begin{cases} -\Delta u + V(x)u = f(u) & \text{in } \Omega, \\ u \in H_0^1(\Omega), \end{cases}$$

where $\Omega \subset \mathbb{R}^2$ is a smooth bounded domain or $\Omega = \mathbb{R}^2$, $V : \overline{\Omega} \to \mathbb{R}$ is a continuous function verifying some hypotheses which will be fix later on. Concerning the nonlinearity, we assume that $f : \mathbb{R} \to \mathbb{R}$ is a C^1-function, which can have an exponential critical growth at both $+\infty$ and $-\infty$, that is, it behaves like $e^{\alpha_0 s^2}$,

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as $|s| \to \infty$, for some $\alpha_0 > 0$. More precisely,

$$\begin{aligned} &\lim_{|s|\to\infty} \frac{f(s)}{e^{\alpha|s|^2}} = 0 \quad \text{for all } \alpha > \alpha_0, \\ &\lim_{|s|\to\infty} \frac{f(s)}{e^{\alpha|s|^2}} = \infty \quad \text{for all } \alpha < \alpha_0, \end{aligned} \tag{1.1}$$

(see [20]). In the last years, we have observed that the existence of nodal solution has received a special attention of a lot of researches. In Cerami, Solimini and Struwe [19], the authors showed the existence of multiple nodal solutions for the following class of elliptic problem with critical growth

$$\begin{cases} -\Delta u - \lambda u = |u|^{2^*-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{P$_1$}$$

where $\Omega = B_R(0) \subset \mathbb{R}^N$, $N \geq 7$, $2^* = 2N/(N-2)$ and $\lambda \in [0, \lambda_1]$, with λ_1 being the first eigenvalue of $(-\Delta, H_0^1(\Omega))$. In Bartsch and Willem [12], the existence of infinitely many radial nodal solutions was proved for the problem

$$\begin{cases} -\Delta u + u = f(|x|, u) & \text{in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{cases} \tag{P$_2$}$$

where f is a continuous function with subcritical growth and verifying some hypotheses. In Cao and Noussair [17], the authors studied the existence and multiplicity of positive and nodal solutions for the following class of problems

$$\begin{cases} -\Delta u + u = Q(x)|u|^{p-2}u & \text{in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N) \end{cases} \tag{P$_2$}$$

by supposing $2 < p < (N+2)/(N-2)$, $N \geq 3$ and some technical conditions on Q. In that paper, the main result connects the number of positive and nodal solutions with the number of maximum points of function Q.

In Castro, Cossio and Neuberger [18] and Bartsch and Wang [13], the authors studied the existence of nodal solution for a problem like

$$\begin{cases} -\Delta u = f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{P$_3$}$$

where Ω is a smooth bounded domain and f verifies some hypotheses. In [18], it was assumed that f is superlinear, while in [10] that f is asymptotically linear at infinity. In Bartsch and Weth [10], existence of multiple nodal solutions was also considered for problem (P$_3$).

In Noussai and Wei [23], [24], existence and concentration of nodal solutions were proved for the problem

$$(\mathrm{P}_4) \qquad \begin{cases} -\varepsilon^2\Delta u + u = f(u) & \text{in } \Omega, \\ Bu = 0 & \text{on } \partial\Omega, \end{cases}$$

when $\varepsilon \to 0$, where Ω is smooth bounded domain, $Bu = u$ in [23] and $Bu = \partial u/\partial\eta$ in [24].

In Bartsch and Wang [14], the authors have considered the existence and concentration of nodal solutions for the following class of problem

$$(\mathrm{P}_5) \qquad \begin{cases} -\Delta u + (\lambda a(x) + 1)u = f(u) & \text{in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{cases}$$

when $\lambda \to +\infty$, by supposing that f has a subcritical growth and $a\colon \mathbb{R}^N \to \mathbb{R}$ is a nonnegative continuous function with $a^{-1}(\{0\})$ being nonempty and verifying

$$\mu(\{x \in \mathbb{R}^N : a(x) \leq M_0\}) < +\infty \quad \text{for some } M_0 > 0,$$

where μ denotes the Lebesgue measure.

In [8], Bartsch, Liu and Weth have showed the existence of nodal solutions with exactly two nodal regions for the problem

$$(\mathrm{P}_6) \qquad \begin{cases} -\Delta u + a(x)u = f(u) & \text{in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{cases}$$

where a is a nonnegative function verifying some conditions, among which we highlight

$$\mu(\{x \in B_r(y) : a(x) \leq M\}) \to 0 \quad \text{as } |y| \to +\infty \text{ for any } M, r > 0.$$

The reader can found more results involving nodal solutions in the papers of Bartsch, Weth and Willem [11], Alves and Soares [5], Bartsch, Clapp and Weth [7], Zou [27] and their references.

After a literature review, we have observed that there are few papers in the literature where existence of nodal solution has been considered for the case where the nonlinearity has an exponential critical growth. The authors know only the references Adimurthi and Yadava [1], Alves [2] and Alves and Soares [6]. In [1], the authors have proved the existence of infinitely many radial solutions for problem (P_3) when $\Omega = B_R(0) \subset \mathbb{R}^2$. In [2], the author has proved the existence of a nodal solution for a class of problems in exterior domains with Neumann boundary conditions, and in [6], the existence of a nodal solution has been established for a problem like

$$\begin{cases} -\varepsilon^2\Delta u + V(x)u = f(u) & \text{in } \mathbb{R}^2, \\ u \in H^1(\mathbb{R}^2), \end{cases}$$

for ε small enough and V verifying some technical conditions.

Motivated by this fact, our goal in the present paper is proving the existence of least energy nodal solution for problem (P) when Ω is a smooth bounded domain or $\Omega = \mathbb{R}^2$. Here, we also show a nonexistence result of least energy solution for (P) when the potential V is constant. Since we will work with exponential critical growth in whole $\mathbb{R}^2$, a key inequality in our arguments is the Trudinger–Moser inequality for bounded domains, see [22] and [25], which claims that for any $u \in H_0^1(\Omega)$,

$$\int_\Omega e^{\alpha u^2}\,dx < +\infty, \quad \text{for every } \alpha > 0. \tag{1.2}$$

Moreover, there exists a positive constant $C = C(\alpha, |\Omega|)$ such that

$$\sup_{||u||_{H_0^1(\Omega)} \le 1} \int_\Omega e^{\alpha u^2}\,dx \le C, \quad \text{for all } \alpha \le 4\pi. \tag{1.3}$$

A version of the above inequality in whole space $\mathbb{R}^2$ has been proved by Cao [16] and has the following statement:

$$\int_{\mathbb{R}^2} (e^{\alpha u^2} - 1)\,dx < +\infty, \quad \text{for all } u \in H^1(\mathbb{R}^2) \text{ and } \alpha > 0. \tag{1.4}$$

Furthermore, if $\alpha \le 4\pi$ and $|u|_{L^2(\mathbb{R}^2)} \le M$, there exists a constant $C_1 = C_1(M, \alpha)$ such that

$$\sup_{|\nabla u|_{L^2(\mathbb{R}^2)} \le 1} \int_{\mathbb{R}^2} (e^{\alpha u^2} - 1)\,dx \le C_1. \tag{1.5}$$

Hereafter, the function f satisfies the ensuing assumptions:

(f_1) There is $C > 0$ such that
$$|f(s)| \le Ce^{4\pi|s|^2} \quad \text{for all } s \in \mathbb{R};$$
(f_2) $\lim_{s\to 0} f(s)/s = 0$;
(f_3) There is $\theta > 2$ such that
$$0 < \theta F(s) \le sf(s), \quad \text{for all } s \in \mathbb{R} \setminus \{0\}.$$
(f_4) $f(s)/|s|$ is a strictly increasing function of s on $\mathbb{R} \setminus \{0\}$.
(f_5) There exist constants $p > 2$ and $C_p > 0$ such that
$$\mathrm{sign}(s)f(s) \ge C_p|s|^{p-1} \quad \text{for all } s \in \mathbb{R} \setminus \{0\},$$
where
$$sign(s) = \begin{cases} 1 & \text{if } s > 0, \\ -1 & \text{if } s < 0. \end{cases}$$

Here $F(s) := \int_0^s f(t)\,dt$ is a primitive of f. Clearly these assumptions hold for

$$f(s) = 2\alpha C_p |s|^{p-2} s e^{\alpha s^2}, \quad 0 < \alpha < 4\pi.$$

Our main result related to the case where Ω is a bounded domain is the following:

THEOREM 1.1. *Let Ω be a bounded domain and $V\colon \overline{\Omega} \to \mathbb{R}$ be a nonnegative continuous function. If* (f_1)–(f_5) *occur, then problem* (P) *possesses a least energy nodal solution, provided that*

$$C_p > \left[\beta_p \frac{2\theta}{\theta - 2}\right]^{(p-2)/2}, \tag{1.6}$$

where $\beta_p = \inf_{\mathcal{M}_\Omega^p} I_p$, $\mathcal{M}_\Omega^p = \{u \in H_0^1(\Omega) : u^\pm \neq 0$ *and* $I_p'(u^\pm)u^\pm = 0\}$ *and*

$$I_p(u) = \frac{1}{2}\int_\Omega (|\nabla u|^2 + V(x)|u|^2)\,dx - \frac{1}{p}\int_\Omega |u|^p\,dx.$$

For the case where $\Omega = \mathbb{R}^2$, we have two results. The first one is a nonexistence result of least energy nodal solution whose statement is the following:

THEOREM 1.2. *Suppose that $V(x) = V_0 > 0$ for all $x \in \mathbb{R}^2$ and f satisfies* (f_1)–(f_5). *Then, the autonomous problem*

$$\begin{cases} -\Delta u + V_0 u = f(u) & \textit{in } \mathbb{R}^2, \\ u \in H^1(\mathbb{R}^2), \end{cases} \tag{P}$$

does not have a least energy nodal solution, provided that

$$C_p > \left[\chi_p \frac{2\theta}{\theta - 2}\right]^{(p-2)/2}, \tag{1.7}$$

where $\chi_p = \inf_{\mathcal{M}_{B_1(0)}} J_p$, $\mathcal{M}_{B_1(0)} = \{u \in H_0^1(B_1(0)) : u^\pm \neq 0$ *and* $J_p'(u^\pm)u^\pm = 0\}$ *and*

$$J_p(u) = \frac{1}{2}\int_{B_1(0)} (|\nabla u|^2 + V_0|u|^2)\,dx - \frac{1}{p}\int_{B_1(0)} |u|^p\,dx.$$

Our second result is related to the existence of least energy nodal solution for a non-autonomous problem. For this case, we will assume that f is an odd function and the ensuing hypotheses on function V:

- (V_1) There exists a constant $V_0 > 0$ such that $V_0 \leq V(x)$ for all $x \in \mathbb{R}^2$;
- (V_2) There exists a continuous $\mathbb{Z}^2$-periodic function $V_\infty\colon \mathbb{R}^2 \to \mathbb{R}$ satisfying

$$V(x) \leq V_\infty(x) \quad \text{for all } x \in \mathbb{R}^2 \quad \text{and} \quad \lim_{|x|\to\infty} |V(x) - V_\infty(x)| = 0.$$

We recall that a function $h\colon \mathbb{R}^2 \to \mathbb{R}$ is $\mathbb{Z}^2$-periodic when

$$h(x) = h(x + y), \quad \text{for all } x \in \mathbb{R}^2 \text{ and } y \in \mathbb{Z}^2.$$

(V_3) There exist $\mu < 1/2$ and $C > 0$ such that

$$V(x) \leq V_\infty(x) - Ce^{-\mu|x|}, \quad \text{for all } x \in \mathbb{R}^2.$$

Our main result involving the above hypotheses is the following:

THEOREM 1.3. *Suppose that* (V_1)–(V_3) *and* (f_1)–(f_5) *hold with f being an odd function. If* (1.7) *occurs replacing V_0 by V, the elliptic problem*

$$\text{(P)} \qquad \begin{cases} -\Delta u + V(x)u = f(u) & \text{in } \mathbb{R}^2, \\ u \in H^1(\mathbb{R}^2), \end{cases}$$

possesses a least energy nodal solution with exactly two nodal domains.

We conclude this section by giving a sketch of the proofs. The basic idea goes as follows. To prove Theorem 1.1 we will use the Nehari method and the deformation lemma. Our inspiration comes from [19], however in that paper the authors used a deformation lemma in cones together with the fact that the nonlinearity is odd. Here, we develop a new approach to get a Palais–Smale sequence of nodal functions associated to the least energy nodal level, for details see Section 2. In order to prove Theorem 1.3, we invoke Theorem 1.1 to obtain a sequence (u_n) of least energy nodal solutions to problem (P) when $\Omega = B_n(0)$. Then, we prove that (u_n) is weakly convergent in $H^1(\mathbb{R}^2)$, and its weak limit is a least energy nodal solution of the problem (P).

2. Bounded domain

In this section, we consider the existence of least energy nodal solution for problem (P) when Ω is a smooth bounded domain. Let us denote by E the Sobolev space $H_0^1(\Omega)$ endowed with the norm

$$\|u\|^2 = \int_\Omega (|\nabla u|^2 + V(x)|u|^2)\, dx.$$

It is easy to check that the above norm is equivalent to usual norm in $H_0^1(\Omega)$. Hereafter, we will be denoted by $\|\cdot\|_*$ the usual norm in $H_0^1(\Omega)$.

From assumptions (f_1) and (f_2), given $\varepsilon > 0$, $q \geq 1$ and $\alpha > 4$, there exists a positive constant $C = C(\varepsilon, q, \alpha)$ such that

$$\text{(2.1)} \qquad |sf(s)|, \quad |F(s)| \leq \varepsilon\frac{s^2}{2} + C|s|^q e^{\alpha\pi s^2}, \quad \text{for all } s \in \mathbb{R}.$$

Thus, by Trudinger–Moser inequality (1.2), $F(u) \in L^1(\mathbb{R}^2)$ for all $u \in E$, from where it follows that Euler–Lagrange functional associated with (P) $I\colon E \to \mathbb{R}$ given by

$$I(u) = \frac{1}{2}\|u\|^2 - \int_\Omega F(u)\, dx$$

is well defined. Furthermore, using standard arguments, we see that I is a C^1 functional on E with

$$I'(u)v = \int_\Omega [\nabla u \nabla v + V(x)uv]\,dx - \int_\Omega f(u)v\,dx, \quad \text{for all } v \in E.$$

Consequently, critical points of I are precisely the weak solutions of problem (P). We know that every nontrivial critical point of I is contained in the Nehari manifold $\mathcal{N}_\Omega = \{u \in E \setminus \{0\} : I'(u)u = 0\}$. Since we are interested in least energy nodal solution, we define the nodal Nehari set $\mathcal{M}_\Omega = \{u \in E : u^\pm \neq 0,$ $I'(u^\pm)u^\pm = 0\}$ and $c^*_\Omega = \inf_{u \in \mathcal{M}_\Omega} I(u)$.

By a least energy nodal solution, we understand as being a function $u \in \mathcal{M}_\Omega$ such that

$$I(u) = c^*_\Omega \quad \text{and} \quad I'(u) = 0.$$

Next, we state some necessary results to prove Theorem 1.1. The proofs of some of them are in Section 5.

LEMMA 2.1. *There exists $A > 0$ such that*

$$c^*_\Omega \leq A < \left(\frac{1}{2} - \frac{1}{\theta}\right).$$

PROOF. Let $\widetilde{u} \in \mathcal{M}^p_\Omega \subset H^1_0(\Omega)$ verifying

$$I_p(\widetilde{u}) = \beta_p \quad \text{and} \quad I'_p(\widetilde{u}) = 0. \tag{2.2}$$

The reader can find the proof of the above claim in Bartsch and Weth [9]. Once $\widetilde{u}^\pm \neq 0$ there exist $0 < s, t$ such that $s\widetilde{u}^+, t\widetilde{u}^- \in \mathcal{N}_\Omega$ and $s\widetilde{u}^+ + t\widetilde{u}^- \in \mathcal{M}_\Omega$. Then,

$$c^*_\Omega \leq I(s\widetilde{u}^+ + t\widetilde{u}^-) = I(s\widetilde{u}^+) + I(t\widetilde{u}^-),$$

leading to

$$\begin{aligned} c^*_\Omega \leq \frac{s^2}{2} &\int_\Omega (|\nabla \widetilde{u}^+|^2 + V(x)|\widetilde{u}^+|^2)\,dx - \int_\Omega F(s\widetilde{u}^+)\,dx \\ + \frac{t^2}{2} &\int_\Omega (|\nabla \widetilde{u}^-|^2 + V(x)|\widetilde{u}^-|^2)\,dx - \int_\Omega F(t\widetilde{u}^-)\,dx. \end{aligned}$$

By (f_5),

$$c^*_\Omega \leq \left(\frac{s^2}{2} - \frac{C_p s^p}{p}\right) \int_\Omega |\widetilde{u}^+|^p\,dx + \left(\frac{t^2}{2} - \frac{C_p t^p}{p}\right) \int_\Omega |\widetilde{u}^-|^p\,dx,$$

and so,

$$c^*_\Omega \leq \max_{r \geq 0} \left\{\frac{r^2}{2} - \frac{C_p r^p}{p}\right\} \int_\Omega |\widetilde{u}|^p\,dx.$$

A direct computation gives

$$\max_{r \geq 0} \left\{\frac{r^2}{2} - \frac{C_p r^p}{p}\right\} = C_p^{2/(2-p)} \left(\frac{1}{2} - \frac{1}{p}\right),$$

then

$$c_\Omega^* \le C_p^{2/(2-p)}\left(\frac{1}{2}-\frac{1}{p}\right)\int_\Omega |\widetilde{u}|^p\,dx.$$

Using (2.2) in the above inequality, we get

$$c_\Omega^* \le C_p^{2/(2-p)}\beta_p := A. \tag{2.3}$$

From (1.6), $A < (1/2 - 1/\theta)$, finishing the proof. □

The next lemma shows two important limits involving the function.

Lemma 2.2. *Let (u_n) be a sequence in E satisfying*:

(a) $b := \sup_{n\in\mathbb{N}} \|u_n\|^2 < 1$;
(b) $u_n \rightharpoonup u$ *in* $H_0^1(\Omega)$, *and*
(c) $u_n(x) \to u(x)$ *almost everywhere in* Ω.

Then,

$$\lim_n \int_\Omega f(u_n)u_n\,dx = \int_\Omega f(u)u\,dx, \tag{2.4}$$

$$\lim_n \int_\Omega f(u_n)v\,dx = \int_\Omega f(u)v\,dx, \tag{2.5}$$

for any $v \in E$.

Proof. See Section 5. □

The result below is well known for problems in $\mathbb{R}^N$ with $N \ge 3$. Here, we decide to write its proof, because we are working with exponential critical growth.

Lemma 2.3. *There exists $m_0 > 0$ such that $0 < m_0 \le \|u\|^2$ for all $u \in \mathcal{N}_\Omega$.*

Proof. We start by fixing $q > 2$ in (2.1). Suppose by contradiction that above inequality is false. Then, there exists a sequence $(u_n) \subset \mathcal{N}_\Omega$ such that $\|u_n\|^2 \to 0$, as $n \to \infty$. Since $u_n \in \mathcal{N}_\Omega$,

$$\|u_n\|^2 = \int_\Omega f(u_n)u_n\,dx.$$

Then, from (2.1),

$$\|u_n\|^2 \le \varepsilon|u_n|_2^2 + C\int_\Omega |u_n|^q e^{\alpha|u_n|^2},dx, \quad \text{for some } \alpha > 4\pi.$$

By Sobolev imbedding and Hölder inequality,

$$\|u_n\|^2 \le C_1\varepsilon\|u_n\|^2 + C|u_n|_{2q}^q\left(\int_\Omega e^{2\alpha|u_n|^2}\,dx\right)^{1/2}.$$

Using again Sobolev imbedding,

$$(1-C_1\varepsilon)\|u_n\|^2 \le C_2\|u_n\|^q\left(\int_\Omega e^{2\alpha|u_n|^2}\,dx\right)^{1/2}.$$

Choosing $\varepsilon > 0$ sufficiently small such that $C_3 := (1 - C_1\varepsilon)/C_2 > 0$, we find that

$$0 < C_3 \leq \|u_n\|^{q-2}\bigg(\int_\Omega e^{2\alpha|u_n|^2}\,dx\bigg)^{1/2}. \tag{2.6}$$

Since $\|u_n\|^2 \to 0$, as $n \to \infty$, there is $n_0 \in \mathbb{N}$ such that $2\alpha\|u_n\|^2 \leq 4\pi$, for all $n \geq n_0$. From Trudinger–Moser inequality (1.3), it follows that

$$\int_\Omega e^{2\alpha|u_n|^2}\,dx = \int_\Omega e^{2\alpha\|u_n\|_*^2(|u_n|/\|u_n\|_*)^2}\,dx \leq \int_\Omega e^{4\pi(|u_n|/\|u_n\|_*)^2}\,dx \leq C$$

for all $n \geq n_0$. Here, we have used that $\|u\|_* \leq \|u\|$ for all $u \in H_0^1(\Omega)$. Thereby, from (2.6),

$$0 < \bigg(\frac{C_3}{\sqrt{C}}\bigg)^{1/(q-2)} \leq \|u_n\|, \quad \text{for all } n \geq n_0,$$

which contradicts the fact that $\|u_n\| \to 0$, as $n \to \infty$. $\square$

COROLLARY 2.4. *For all $u \in \mathcal{M}_\Omega$, $0 < m_0 \leq \|u^\pm\|^2 \leq \|u\|^2$.*

COROLLARY 2.5. *There exists $\delta_2 > 0$ such that $I(u^\pm)$, $I(u) \geq 2\delta_2$ for all $u \in \mathcal{M}_\Omega$.*

PROOF. Firstly, observe that if $u \in \mathcal{N}_\Omega$,

$$I(u) = I(u) - \frac{1}{\theta}I'(u)u = \bigg(\frac{1}{2} - \frac{1}{\theta}\bigg)\|u\|^2 - \int_\Omega \bigg(F(u) - \frac{1}{\theta}f(u)u\bigg)\,dx.$$

Then, from (f_3) and Lemma 2.3,

$$I(u) \geq \bigg(\frac{1}{2} - \frac{1}{\theta}\bigg)\|u\|^2 \geq \bigg(\frac{1}{2} - \frac{1}{\theta}\bigg)m_0 := 2\delta_2 \quad \text{for all } u \in \mathcal{N}_\Omega.$$

Now, the result follows by observing that $u \in \mathcal{M}_\Omega$ implies that $u, u^\pm \in \mathcal{N}_\Omega$. $\square$

Now, we prove some results related to the following set

$$\widetilde{S}_\lambda := \{u \in \mathcal{M}_\Omega : I(u) < c_\Omega^* + \lambda\}.$$

The above set will be crucial to show the existence of a (PS) sequence of nodal functions associated with c_Ω^*.

LEMMA 2.6. *For all $u \in \widetilde{S}_\lambda$, we have $0 < m_0 \leq \|u^\pm\|^2 \leq \|u\|^2 \leq m_\lambda < 1$, for $\lambda > 0$ sufficiently small.*

PROOF. See Section 5. $\square$

LEMMA 2.7. *For each $q > 1$, there exists $\delta_q > 0$ such that*

$$0 < \delta_q \leq \int_\Omega |u^\pm|^q\,dx \leq \int_\Omega |u|^q\,dx, \quad \textit{for all } u \in \widetilde{S}_\lambda.$$

PROOF. See Section 5. $\square$

Lemma 2.8. *There exists $R > 0$ such that*

$$I\bigg(\frac{1}{R}u^{\pm}\bigg), \quad I(Ru^{\pm}) < \frac{1}{2}I(u^{\pm}), \quad \textit{for all } u \in \widetilde{S}_\lambda.$$

Proof. Let $u \in \widetilde{S}_\lambda$ and $R > 0$. By definition of I and (f_3),

$$I\bigg(\frac{1}{R}u^{\pm}\bigg) = \frac{1}{2R^2}\|u^{\pm}\|^2 - \int_\Omega F\bigg(\frac{1}{R}u^{\pm}\bigg)\,dx \le \frac{1}{2R^2}\|u^{\pm}\|^2.$$

Hence, by Lemma 2.6

$$I\bigg(\frac{1}{R}u^{\pm}\bigg) \le \frac{m_\lambda}{2R^2}.$$

From this, we can fix $R > 0$ large enough such that $m_\lambda/(2R^2) < \delta_2$, which implies, by Corollary 2.5,

$$I\bigg(\frac{1}{R}u^{\pm}\bigg) < \delta_2 \le \frac{1}{2}I(u^{\pm}), \quad \text{for all } u \in \widetilde{S}_\lambda.$$

By (f_3), there are constants $b_1, b_2 > 0$ verifying $F(t) \ge b_1|t|^\theta - b_2$, for all $t \in \mathbb{R}$. Then,

$$I(Ru^{\pm}) = \frac{R^2}{2}\|u^{\pm}\|^2 - \int_\Omega F(Ru^{\pm})\,dx \le \frac{R^2 m_\lambda}{2} - b_1 R^\theta \int_\Omega |u^{\pm}|^\theta\,dx + b_2|\Omega|.$$

By Lemma 2.7, there is $\delta_\theta > 0$ such that

$$\int_\Omega |u^{\pm}|^\theta\,dx \ge \delta_\theta.$$

Thus,

$$I(Ru^{\pm}) = \frac{R^2}{2}\|u^{\pm}\|^2 - \int_\Omega F(Ru^{\pm})\,dx \le \frac{R^2 m_\lambda}{2} - b_1 R^\theta \delta_\theta + b_2|\Omega|.$$

Since $\theta > 2$, we conclude that $I(Ru^{\pm}) < 0 < \delta_2 \le I(u^{\pm})/2$, for all $u \in \widetilde{S}_\lambda$, for $R > 0$ large enough. □

From now on, we consider the following sets

$$\begin{aligned} S &= \{sRu^+ + tRu^- : u \in \widetilde{S}_\lambda \text{ and } s,t \in [1/R^2, 1]\}, \\ P &= \{u \in E : u \ge 0 \text{ a.e. in } \Omega\} \end{aligned}$$

and $\Lambda = P \cup (-P)$.

In what follows, for a subset $\Theta \subset H_0^1(\Omega)$ and $r > 0$ we denote by Θ_r the set $\Theta_r = \{u \in H_0^1(\Omega) : \operatorname{dist}(u, \Theta) \le r\}$, and by $\operatorname{dist}(A, B)$ the distance between sets of $H_0^1(\Omega)$.

Lemma 2.9. $d_0 := \operatorname{dist}(S, \Lambda) > 0$.

Proof. The lemma follows by using contradiction argument combined with Rellich Imbedding and Lemma 2.7. □

PROPOSITION 2.10 (Main Proposition). *Given* $\varepsilon, \delta > 0$, *there exists* $u \in I^{-1}([c_\Omega^* - 2\varepsilon, c_\Omega^* + 2\varepsilon]) \cap S_{2\delta}$ *verifying* $\|I'(u)\| < 4\varepsilon/\delta$.

PROOF. In fact, otherwise, there exist $\varepsilon_o, \delta_o > 0$ such that

$$\|I'(u)\| \geq \frac{4\varepsilon_o}{\delta_o}, \quad \text{for all } u \in I^{-1}([c_\Omega^* - 2\varepsilon_o, c_\Omega^* + 2\varepsilon_o]) \cap S_{2\delta_o}.$$

Thus, for each $n \in \mathbb{N}^*$,

$$\|I'(u)\| \geq \frac{4\varepsilon_o/n}{\delta_o/n}, \quad \text{for all } u \in I^{-1}([c_\Omega^* - 2\varepsilon_o, c_\Omega^* + 2\varepsilon_o]) \cap S_{2\delta_o}.$$

Since

$$I^{-1}([c_\Omega^* - 2\varepsilon_o/n, c_\Omega^* + 2\varepsilon_o/n]) \cap S_{2\delta_o/n} \subset I^{-1}([c_\Omega^* - 2\varepsilon_o, c_\Omega^* + 2\varepsilon_o]) \cap S_{2\delta_o},$$

we get

$$\|I'(u)\| \geq \frac{4\varepsilon_o/n}{\delta_o/n}, \quad \text{for all } u \in I^{-1}([c_\Omega^* - 2\varepsilon_o/n, c_\Omega^* + 2\varepsilon_o/n]) \cap S_{2\delta_o/n}.$$

Then, we can fix $n \in \mathbb{N}$ large enough such that

$$\overline{\varepsilon} := \frac{\varepsilon_o}{n} < \min\left\{\frac{2\delta_2}{5}, \lambda\right\}, \qquad \overline{\delta} := \frac{\delta_o}{n} < \frac{d_0}{2} \tag{2.7}$$

and

$$\|I'(u)\| \geq \frac{4\overline{\varepsilon}}{\overline{\delta}}, \quad \text{for all } u \in I^{-1}([c_\Omega^* - 2\overline{\varepsilon}, c_\Omega^* + 2\overline{\varepsilon}]) \cap S_{2\overline{\delta}}.$$

In view of the above hypotheses, [26, Lemma 2.3] yields a continuous map $\eta\colon E \to E$ satisfying:

(1) $\eta(u) = u$, for all $u \notin I^{-1}([c_\Omega^* - 2\overline{\varepsilon}, c_\Omega^* + 2\overline{\varepsilon}]) \cap S_{2\overline{\delta}}$;
(2) $\|\eta(u) - u\| \leq \overline{\delta}$ for all $u \in E$;
(3) $\eta(I^{c_\Omega^* + \overline{\varepsilon}} \cap S) \subset I^{c_\Omega^* - \overline{\varepsilon}} \cap S_{\overline{\delta}}$;
(4) η is a homeomorphism.

From the definition of c_Ω^*, for such $\overline{\varepsilon} > 0$, there exists $u_* \in \mathcal{M}_\Omega$ such that

$$I(u_*) < c_\Omega^* + \overline{\varepsilon}/2. \tag{2.8}$$

Now, consider $\gamma\colon [1/R^2, 1]^2 \to E$ given by

$$\gamma(s,t) = \eta(sRu_*^+ + tRu_*^-).$$

Once $u_*^\pm \in \mathcal{N}_\Omega$, in view of ($f_4$) we have the following classical result

$$I(u_*^\pm) = \max_{s>0} I(su_*^\pm),$$

see e.g. [26, Lemma 4.1]. Thus

$$I(sRu_*^+ + tRu_*^-) = I(sRu_*^+) + I(tRu_*^-) \leq I(u_*^+) + I(u_*^-) = I(u_*).$$

Thereby, (2.7) and (2.8) give

$$I(sRu_*^+ + tRu_*^-) \leq I(u_*) < c_\Omega^* + \frac{\overline{\varepsilon}}{2} < c_\Omega^* + \overline{\varepsilon} < c_\Omega^* + \lambda,$$

for all $s,t \in [1/R^2,1]$. Then, $u_* \in \widetilde{S}_\lambda$ and $sRu_*^+ + tRu_*^- \in I^{c^*_\Omega+\overline{\varepsilon}} \cap S$, which implies, by item (3),

$$I(\gamma(s,t)) = I(\eta(sR{u_*}^+ + tR{u_*}^-)) < c^*_\Omega - \overline{\varepsilon}, \quad \text{for all } (s,t) \in [1/R^2,1]^2. \tag{2.9}$$

From item (2), $\|\gamma(s,t) - (sRu_*^+ + tRu_*^-)\| \le \overline{\delta}$, then by the choice of $\overline{\delta}$ made in (2.7), for $v \in \Lambda$, we have

$$\begin{aligned}\|\gamma(s,t) - v\| &= \|\gamma(s,t) - (sRu_*^+ + tRu_*^-) + (sRu_*^+ + tRu_*^-) - v\| \\ &\ge \|(sRu_*^+ + tRu_*^-) - v\| - \|\gamma(s,t) - (sRu_*^+ + tRu_*^-)\| \\ &\ge d_0 - \overline{\delta} > d_0 - \frac{d_0}{2} = \frac{d_0}{2} > 0,\end{aligned}$$

for all $s,t \in [1/R^2,1]$. Therefore,

$$\gamma(s,t)^\pm \ne 0, \quad \text{for all } (s,t) \in [1/R^2,1]^2. \tag{2.10}$$

Claim 2.11. *There exists $(s_0,t_0) \in [1/R^2,1]^2$ such that*

$$I'(\gamma(s_0,t_0)^\pm)(\gamma(s_0,t_0)^\pm) = 0.$$

Suppose, for a moment, that this claim is true. From (2.10), $\gamma(s_0,t_0) \in \mathcal{M}_\Omega$, and so, $I(\gamma(s_0,t_0)) \ge c^*_\Omega$, which contradicts (2.9), proving the proposition. $\square$

Proof of Claim 2.11. Let us define $Q := [1/R^2,1]^2$ and the functions $H,G\colon Q \to \mathbb{R}^2$ by

$$\begin{aligned}H(s,t) &:= (I'(\gamma(s,t)^+)(\gamma(s,t)^+), I'(\gamma(s,t)^-)(\gamma(s,t)^-)), \\ G(s,t) &:= (I'(sR{u_*}^+)(sRu_*^+), I'(tRu_*^-)(tRu_*^-)).\end{aligned}$$

Since

$$\gamma(s,t) = \eta(sRu_*^+ + tRu_*^-) = sRu_*^+ + tRu_*^-, \quad \text{for all } (s,t) \in \partial Q, \tag{2.11}$$

we have $\gamma(s,t)^+ = sRu_*^+$ and $\gamma(s,t)^- = tRu_*^-$,for all $(s,t) \in \partial Q$, and $H \equiv G$ on ∂Q.

To see (2.11), let $s = 1/R^2$ and $t \in [1/R^2,1]$. By Lemma 2.8,

$$\begin{aligned}I(sR{u_*}^+ + tR{u_*}^-) &= I\bigg(\frac{1}{R}{u_*}^+\bigg) + I(tR{u_*}^-) \\ &< \frac{I({u_*}^+)}{2} + I({u_*}^-) = I(u_*) - \frac{I({u_*}^+)}{2}.\end{aligned}$$

From (2.8), Corollary 2.5 and the choice of $\overline{\varepsilon} > 0$ made in (2.7), we obtain

$$I(sR{u_*}^+ + tR{u_*}^-) < c^*_\Omega + \frac{\overline{\varepsilon}}{2} - \delta_2 < c^*_\Omega - 2\overline{\varepsilon},$$

i.e.

$$\frac{1}{R}{u_*}^+ + tR{u_*}^- \notin I^{-1}([c^*_\Omega - 2\overline{\varepsilon}, c^*_\Omega + 2\overline{\varepsilon}]) \cap S_{2\overline{\delta}},$$

for all $t \in [1/R^2, 1]$. From this, item (1) yields

$$\gamma\bigg(\frac{1}{R^2}, t\bigg) = \eta\bigg(\frac{1}{R}{u_*}^+ + tR{u_*}^-\bigg) = \frac{1}{R}{u_*}^+ + tR{u_*}^-.$$

The other cases are similar. Then, $d(H, \dot{Q}, (0,0)) = d(G, \dot{Q}, (0,0))$ and in view of (f_4), we have $d(G, \dot{Q}, (0,0)) = 1 \neq 0$. From Brouwer's degree property, there exists $(s_0, t_0) \in Q$ such that $H(s_0, t_0) = 0$, i.e. $I'(\gamma(s_0, t_0)^\pm)(\gamma(s_0, t_0)^\pm) = 0$, and the proof is complete. □

PROOF OF THEOREM 1.1. For each $n \in \mathbb{N}$, consider $\varepsilon = 1/(4n)$ and $\delta = 1/\sqrt{n}$. From Proposition 2.10, there exists $u_n \in S_{2/\sqrt{n}}$ with

$$u_n \in I^{-1}\bigg(\bigg[c_\Omega^* - \frac{1}{2n}, c_\Omega^* + \frac{1}{2n}\bigg]\bigg) \quad \text{and} \quad \|I'(u_n)\| \leq \frac{1}{\sqrt{n}}.$$

Thus, there is $(v_n) \subset S$ satisfying $I(v_n) \to c_\Omega^*$ and $I'(v_n) \to 0$, in other words, (v_n) is a $(\mathrm{PS})_{c_\Omega^*}$ of nodal functions for I.

CLAIM 2.12. *The sequence (v_n) is bounded in E and for a subsequence of (v_n), still denoted by (v_n),*

$$\limsup_{n \in \mathbb{N}} \|v_n\|^2 < 1.$$

Indeed, since $(v_n) \subset S$, it is easy to see that (v_n) is bounded in E. Thus, $I'(v_n)v_n = o_n(1)$ and

$$c_\Omega^* + o_n(1) = I(v_n) - \frac{1}{\theta}I'(v_n)v_n = \bigg(\frac{1}{2} - \frac{1}{\theta}\bigg)\|v_n\|^2 - \int_\Omega [F(v_n) - \frac{1}{\theta}f(v_n)v_n]\,dx$$

The above equality together with (f_3) and Lemma 2.1 gives

$$\limsup_n \|v_n\| \leq \frac{c_\Omega^*}{(1/2 - 1/\theta)} < 1.$$

Now, let $v_0 \in E$ the weak limit of (v_n). Combining Claim 2.12 with Lemma 2.2, we deduce that v_0 is a weak solution to problem (P). Finally, to conclude the proof, we must prove that $v_0^\pm \neq 0$. We know that $v_n \rightharpoonup v_0$ in $H_0^1(\Omega)$; $v_n(x) \to v_0(x)$ almost everywhere in Ω and $v_n \to v_0$ in $L^q(\Omega)$, for all $q \geq 1$.

On the other hand, using that $v_n \in S$, there are $s_n, t_n \in [1/R^2, 1]$ and $u_n \in \widetilde{S}_\lambda$, such that

$$\begin{aligned} v_n &= s_nRu_n^+ + t_nRu_n^- \rightharpoonup s_0Ru_0^+ + t_0Ru_0^- && \text{in } E, \\ v_n(x) &= s_nRu_n^+(x) + t_nRu_n^-(x) \to s_0Ru_0^+(x) + t_0Ru_0^-(x) && \text{a.e. in } \Omega, \end{aligned}$$

for some $s_0, t_0 \in [1/R^2, 1]$, where $u_0 \in E$ is the weak limit of the sequence $(u_n) \subset \mathcal{M}_\Omega$. By uniqueness of limit, we have $v_0 = s_0Ru_0^+ + t_0Ru_0^-$. From Lemma 2.7, we obtain $u_0^\pm \neq 0$, which implies that $v_0^+ = s_0Ru_0^+ \neq 0$ and $v_0^- = s_0Ru_0^- \neq 0$ and the proof of Theorem 1.1 is complete. □

3. Unbounded domain

From now on, we consider the problem (P) with $\Omega = \mathbb{R}^2$. From (V_1), it is possible to show that

$$\|u\| = \bigg(\int_{\mathbb{R}^2} (|\nabla u|^2 + V(x)|u|^2)\,dx\bigg)^{1/2}$$

is a norm on $H^1(\mathbb{R}^2)$, which is equivalent to the usual norm in $H^1(\mathbb{R}^2)$. Hereafter, E denotes $H^1(\mathbb{R}^2)$ endowed with the above norm.

From assumptions (f_1) and (f_2), given $\varepsilon > 0$, $q \geq 1$ and $\beta > 4$, there exists a positive constant $C = C(\varepsilon, q, \beta)$ such that

$$|sf(s)|,\ |F(s)| \leq \varepsilon\frac{s^2}{2} + C|s|^q(e^{\beta\pi s^2} - 1), \quad \text{for all } s \in \mathbb{R}.$$

Thus, by the Trudinger–Moser inequality (1.4), we have $F(u) \in L^1(\mathbb{R}^2)$ for all $u \in H^1(\mathbb{R}^2)$. Therefore, the Euler–Lagrange functional associated with (P) given by

$$I(u) = \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^2} F(u)\,dx, \quad u \in E$$

is well defined. Furthermore, using standard arguments, we see that I is a C^1 functional on E with

$$I'(u)v = \int_{\mathbb{R}^2} \bigg[\nabla u \nabla v + V(x)uv\bigg]\,dx - \int_{\mathbb{R}^2} f(u)v\,dx, \quad \text{for all } v \in E.$$

Consequently, critical points of I are precisely the weak solutions of problem (P). Every nontrivial critical point of I is contained in the Nehari manifold

$$\mathcal{N} = \{u \in E \setminus \{0\} : I'(u)u = 0\}.$$

A critical point $u \neq 0$ of I is a ground state if $I(u) = c_1$, where $c_1 = \inf_{u\in\mathcal{N}} I(u)$.

Since we are interested in least energy nodal solution, we define the nodal Nehari set $\mathcal{M} = \{u \in E : u^{\pm} \neq 0,\ I'(u^{\pm})u^{\pm} = 0\}$ and $c^* = \inf\limits_{u\in\mathcal{M}} I(u)$. Here, it is important to observe that every nodal solution of (P) lies in $\mathcal{M}$.

Next, we state some necessary results to prove Theorem 1.3. The proofs of some of them are in Section 5. The first one can be found in Alves, Carrião and Medeiros [3].

Lemma 3.1. *Let $F \in C^2(\mathbb{R}, \mathbb{R}_+)$ be a convex and even function such that $F(0) = 0$ and $f(s) = F'(s) \geq 0$, for all $s \in [0, +\infty)$. Then, for all $t, s \geq 0$,*

$$|F(t - s) - F(t) - F(s)| \leq 2(f(t)s + f(s)t).$$

Remark 3.2. Notice that, if f is an odd function satisfying the hypotheses (f_1)–(f_4), then the primitive F of f verifies the hypotheses from Lemma 3.1.

The following two results are essentially due to Alves, do Ó and Miyagaki and its proof can be found in [4].

THEOREM 3.3. *Suppose that* (V_1)–(V_2) *and* (f_1)–(f_5) *hold. Then*

$$(\mathrm{P}_\infty) \qquad \begin{cases} -\Delta u + V_\infty(x)u = f(u) & \text{in } \mathbb{R}^2, \\ u \in H^1(\mathbb{R}^2), \end{cases}$$

possesses a positive ground state solution, i.e. there exists $\overline{u} \in H^1(\mathbb{R}^2)$ *such that* $\overline{u} > 0$, $I_\infty(\overline{u}) = c_\infty$ *and* $I'_\infty(\overline{u}) = 0$, *where*

$$I_\infty(u) = \frac{1}{2}\int_{\mathbb{R}^2} \left(|\nabla u|^2 + V_\infty(x)u^2\right) dx - \int_{\mathbb{R}^2} F(u)\, dx, \quad u \in H^1(\mathbb{R}^2),$$

$c_\infty = \inf\limits_{u \in \mathcal{N}_\infty} I_\infty(u)$ *and* $\mathcal{N}_\infty$ *denotes the Nehari manifold*

$$\mathcal{N}_\infty = \{u \in H^1(\mathbb{R}^2) \setminus \{0\} : I'_\infty(u)u = 0\}.$$

The second result deal with the asymptotically periodic case.

THEOREM 3.4. *Suppose that* (V_1)–(V_2) *and* (f_1)–(f_5) *hold. Then, problem* (P) *possesses a positive ground state solution, i.e. there exists* $u_1 \in H^1(\mathbb{R}^2)$ *such that* $u_1 > 0$, $I(u_1) = c_1$ *and* $I'(u_1) = 0$.

Employing the same arguments explored by Alves [2], it is possible to prove the following result:

THEOREM 3.5. *Assume that* (f_1) *and* (f_2) *hold. Then, any positive solution* $\overline{u}$ *of problem rom(*P_∞*) with* $\|\overline{u}\|_{H^1(\mathbb{R}^2)} < 1$ *satifies:*

(a) $\lim\limits_{|x|\to\infty} \overline{u}(x) = 0$, *and*

(b) $C_1 e^{-a|x|} \le \overline{u} \le C_2 e^{-b|x|}$ *in* $\mathbb{R}^2$, *where* C_1 *and* C_2 *are positive constants and* $0 < b < 1 < a$. *Moreover, we choose* $a = 1 + \delta$, $b = 1 - \delta$ *for* $\delta > 0$.

The same result hold for $u_1 > 0$ *given in Theorem* 3.4.

The next proposition is a key point in our arguments to get nodal solutions, because it gives an estimate from above of c^*.

PROPOSITION 3.6. *Suppose that* (V_1)–(V_3), (f_1)–(f_5) *and* (1.7) *hold. Then* $c^* < c_1 + c_\infty$.

PROOF. See Section 5. □

The below lemma establishes a condition to conclude when the weak limit of a (PS) sequence is nontrivial.

LEMMA 3.7. *Assume that* (V_1)–(V_3) *and* (f_1)–(f_5) *hold. If* $(u_n) \subset E$ *is such that* $I(u_n) \to \sigma$, $u_n \rightharpoonup u$, $I'(u_n)u_n \to 0$ *and*

$$\liminf_{n\to\infty} \int_{\mathbb{R}^2} f(u_n)u_n\, dx > 0,$$

then $u \neq 0$, *provided that* $0 < \sigma < c_\infty$.

Proof. See Section 5. □

Proof of Theorem 1.3. Applying Theorem 1.1 with $\Omega = B_n(0)$ and $n \in \mathbb{N}$, there is a nodal solution $u_n \in H_0^1(B_n(0))$ for (P) satisfying

$$I(u_n) = c_n^* \quad \text{and} \quad I'(u_n) = 0, \quad \text{where } c_n^* = c_{B_n(0)}^*.$$

Here, we also denote by I the functional associated with (P), because its restriction to $H_0^1(B_n)$ coincides with the functional associated with (P).

Claim 3.8. *The limit* $\lim_{n\to\infty} c_n^* = c^*$ *holds.*

Indeed, we begin recalling that (c_n^*) is a non-increasing sequence and bounded from bellow by c^*. If $\lim c_n^* = \widehat{c} > c^*$, then there exists $\phi \in \mathcal{M}$ such that $I(\phi) < \widehat{c}$. Take $(\omega_n) \subset C_0^\infty(\mathbb{R}^2)$ and $t_n^\pm > 0$ such that $\omega_n^\pm \neq 0$, $\omega_n \to \phi$ in $H^1(\mathbb{R}^2)$ and $t_n^\pm \omega_n^\pm \in \mathcal{N}$. Thereby, $I(\omega_n) = I(\omega_n^+) + I(\omega_n^-) \to I(\phi) \geq c^* > 0$, $I(\omega_n^\pm) \to I(\phi^\pm)$ and $I'(\omega_n^\pm)\omega_n^\pm \to I'(\phi^\pm)\phi^\pm = 0$.

Then, if we define $\phi_n := t_n^+\omega_n^+ + t_n^-\omega_n^- \in \mathcal{M}$, by using similar arguments as in the proof of Lemma 3.7, it is possible to prove that $t_n^\pm \to 1$ and $I(t_n^\pm \omega_n^\pm) \to I(\phi^\pm)$, leading to, $I(\phi_n) \to I(\phi)$. Therefore, we can fix $n_0 \in \mathbb{N}$ such that $I(\phi_{n_0}) < \widehat{c}$, for all $n \geq n_0$. On the other hand, fixing $n_1 \in \mathbb{N}$ such that $\phi_{n_0} \in \mathcal{M}_{B_{n_1}}$, it follows that $c_{n_1} \leq I(\phi_{n_0}) < \widehat{c}$, which contradicts the definition of $\widehat{c}$.

From (f_3), we know that (u_n) is a bounded sequence in E. Thus, we can assume that (u_n) is weakly convergent to u, for some $u \in E$. Once

$$c^* = \lim_n c_n^* = \lim_n I(u_n) \quad \text{and} \quad I'(u_n)v = 0, \quad \text{for all } v \in H_0^1(B_n),$$

a direct computation gives that u is a weak solution for (P).

Now, our goal is proving that $u \in \mathcal{M}$ and $I(u) = c^*$. In fact, taking a subsequence if necessary, we can assume that $I(u_n^\pm) \to \sigma^\pm$, where $c^* = \sigma^+ + \sigma^-$.

Using that $u_n^+, u_n^- \in \mathcal{N}$, we derive $\sigma^\pm \geq c_1 > 0$. From Proposition 3.6, it follows that $\sigma^\pm < c_\infty$. Since

$$\liminf_{n\to\infty} \int_{\mathbb{R}^2} f(u_n^\pm)u_n^\pm > 0,$$

Lemma 3.7 yields $u^\pm \neq 0$. Therefore, $u \in \mathcal{M}$ and $I(u) \geq c^*$. To complete the proof, by Fatou's Lemma, we see that

$$\begin{aligned} 2c^* &= \liminf_{n\to\infty} [2I(u_n) - I'(u_n)u_n] = \liminf_{n\to\infty} \int_{\mathbb{R}^2} (f(u_n)u_n - 2F(u_n))\,dx \\ &\geq \int_{\mathbb{R}^2} (f(u)u - 2F(u))\,dx = 2I(u) - I'(u)u = 2I(u) \geq 2c^*. \end{aligned}$$

Hence, $I(u) = c^*$, which proves that (P) has a nodal solution. In order to establish that the nodal solution has exactly two nodal domains, we refer the reader to [9, Theorem 2.3]. □

4. Nonexistence result

In this section, we prove a nonexistence result of least energy nodal solution for the following autonomous problem

$$\text{(Q)} \qquad \begin{cases} -\Delta u + V_0 u = f(u) & \text{in } \mathbb{R}^2, \\ u \in H^1(\mathbb{R}^2), \end{cases}$$

that is, we prove that $\widehat{c} := \inf_{\mathcal{M}} J$ is not attained, where J is the energy functional defined on $H^1(\mathbb{R}^2)$ associated with (Q) and $\mathcal{M}$ is the nodal Nehari set

$$\mathcal{M} := \{u \in H^1(\mathbb{R}^2) : u^\pm \neq 0 \text{ and } J'(u^\pm)u^\pm = 0\}.$$

For this, we define

$$f_+(t) = \begin{cases} f(t) & \text{for } t \geq 0, \\ 0 & \text{for } t \leq 0, \end{cases}$$

and the functional J_+ defined on $H^1(\mathbb{R}^2)$ by

$$J_+(u) := \int_{\mathbb{R}^2} (|\nabla u|^2 + V_0 |u|^2)\, dx - \int_{\mathbb{R}^2} F_+(u)\, dx,$$

where F_+ is the primitive of f_+ with $F_+(0) = 0$. From [4, Theorem 1.1], the number $c_+ = \inf_{\mathcal{N}_+} J_+$ where $\mathcal{N}_+ := \{u \in H^1(\mathbb{R}^2) \setminus \{0\} : J_+'(u)u = 0\}$, is a critical value of J_+. Let v be the corresponding critical point. It is easy to see that $v^- = 0$. Thus, v is a nonnegative function and by the maximum principle, $v > 0$ on $\mathbb{R}^2$. In particular, v is a positive critical point of J.

Analogously, if we define

$$f_-(t) = \begin{cases} 0 & \text{for } t \geq 0, \\ f(t) & \text{for } t < 0, \end{cases}$$

and denote by J_- the corresponding functional and by $\mathcal{N}_-$ the Nehari manifold, then $c_- := \inf\limits_{\mathcal{N}_-} J_-$ is a critical value of J_-.

The next proposition is a key point in our argument to prove the nonexistence result, because it gives an exact estimate of $\widehat{c}$.

PROPOSITION 4.1. *Under assumptions* (f_1)–(f_5), *we have* $\widehat{c} = c_+ + c_-$.

PROOF. Let $v, w \in H^1(\mathbb{R}^2)$ verifying

$$\begin{aligned} J_+(v) &= c_+, \quad J_+'(v) = 0, \quad v(x) > 0, \quad \text{for all } x \in \mathbb{R}^2, \\ J_-(w) &= c_-, \quad J_-'(w) = 0, \quad w(x) < 0, \quad \text{for all } x \in \mathbb{R}^2, \end{aligned}$$

and consider $\varphi \in C_0^\infty(\mathbb{R}^2)$ be a cut-off function satisfying

$$\operatorname{supp} \varphi \subset B_2(0), \quad 0 \leq \varphi \leq 1, \quad \varphi = 1 \quad \text{on } B_1(0) \quad \text{and} \quad |\nabla \varphi|_\infty \leq C.$$

Using the above function, for each $R>0$ fixed, we set $\varphi_R(x)=\varphi(x/R)$. Then,

$$\operatorname{supp}\varphi_R\subset B_{2R}(0),\quad 0\le\varphi_R\le 1,\quad \varphi_R=1\quad\text{on } B_R(0)\quad\text{and}\quad |\nabla\varphi_R|_\infty\le C/R.$$

Now, we consider the functions:

$$v_R(x):=\varphi_R(x)v(x),\quad w_R(x):=\varphi_R(x)w(x)\quad\text{and}\quad w_{R,n}:=w_R(x-x_n),$$

where $x_n=(n,0)$. Clearly, for n large enough, $\operatorname{supp}v_R\cap\operatorname{supp}w_{R,n}=\emptyset$.

Let t_R, s_R be the positive real numbers such that

$$J'(t_Rv_R)t_Rv_R=0\quad\text{and}\quad J'(s_Rw_{R,n})s_Rw_{R,n}=0.$$

Since

$$t_R^2\int_{\mathbb{R}^2}(|\nabla v_R|^2+V_0|v_R|^2)\,dx=\int_{\mathbb{R}^2}f_+(t_Rv_R)t_Rv_R$$

and $v_R\to v$ in $H^1(\mathbb{R}^2)$ as $R\to+\infty$, it is possible to show, by using similar arguments given in the proof of Lemma 3.7, that $t_R\to 1$, as $R\to+\infty$. Similarly,

$$s_R^2\int_{\mathbb{R}^2}(|\nabla w_{R,n}|^2+V_0|w_{R,n}|^2)\,dx=\int_{\mathbb{R}^2}f_+(s_Rw_{R,n})s_Rw_{R,n}.$$

Since $w_R\to w$ in $H^1(\mathbb{R}^2)$ as $R\to+\infty$, we derive that $s_R\to 1$, as $R\to+\infty$. Now, note that $u_R:=t_Rv_R+s_Rw_{R,n}\in\mathcal{M}$ with $u_R^+=t_Rv_R$ and $u_R^-=s_Rw_{R,n}$ for $n\in\mathbb{N}$ large enough. Then,

$$\widehat{c}\le J(t_Rv_R+s_Rw_{R,n})=J(t_Rv_R)+J(s_Rw_{R,n})$$

Using the invariance of $\mathbb{R}^2$ under translations, by taking $R\to+\infty$, we obtain $\widehat{c}\le J(v)+J(w)$. Since $J(v)=J_+(v)=c_+$ and $J(w)=J_-(w)=c_-$, it follows that $\widehat{c}\le c_++c_-$.

On the other hand, it is obvious that $\widehat{c}\ge c_++c_-$. Therefore, we can conclude that $\widehat{c}=c_++c_-$. □

Proof of Theorem 1.2. Suppose by contradiction that there exists $u\in\mathcal{M}$ such that $J(u)=\widehat{c}$. Thus, $u^+\in\mathcal{N}_+$ and $u^-\in\mathcal{N}_-$, from where it follows that

$$c_++c_-\le J_+(u^+)+J_-(u^-)=J(u)=\widehat{c}=c_++c_-,$$

and so, $J_+(u^+)=c_+$ and $J_-(u^-)=c_-$. Thereby, u^+ is a critical point of J_+ and u^- is a critical point of J_-.Then, by maximum principle, we must have

$$u^+(x)>0,\quad\text{for all } x\in\mathbb{R}^2\quad\text{and}\quad u^-(x)<0,\quad\text{for all } x\in\mathbb{R}^2,$$

which is impossible. □

Remark 4.2. A version of Theorem 1.2 can be made for $N\ge 3$, by supposing that f has a subcritical growth.

REMARK 4.3. We can define $H_r^1(\mathbb{R}^2) := \{u \in H : u \text{ is a radial function}\}$, $\mathcal{M}_r := \mathcal{M} \cap H_r^1(\mathbb{R}^2)$ and $c_r^* = \inf_{\mathcal{M}_r} J$. Under the assumptions of Theorem 1.2, there exist a minimizer $u \in \mathcal{M}_r$ which is a critical point of I on $H^1(\mathbb{R}^2)$. To prove this, we combine the symmetric criticality principle with arguments as in the proof of Theorem 1.1. It is clear that $c^* \le c_r^*$, and so, as a consequence of our nonexistence result, we have $c^* < c_r^*$. A similar inequality in bounded domain like annulus for $N \ge 3$ was proved in [11] .

5. Proofs of lemmas and propositions

PROOF OF LEMMA 2.2. From (f_1), $|f(u_n)u_n| \le C|u_n|e^{4\pi|u_n|^2}$, for all $n \in \mathbb{N}$. We claim that

$$\int_\Omega |u_n|e^{4\pi|u_n|^2}\,dx \to \int_\Omega |u|e^{4\pi|u|^2}\,dx, \quad \text{as } n \to \infty.$$

Effectively, consider $t > 1$ with $t \approx 1$. Note that

$$\int_\Omega (e^{4\pi|u_n|^2})^t\,dx = \int_\Omega e^{4\pi t\|u_n\|^2(|u_n|/\|u_n\|)^2}\,dx \le \int_\Omega e^{4\pi tb(|u_n|/\|u_n\|)^2}\,dx.$$

Now, since $b < 1$, we can fix $t > 1$ with $t \approx 1$, such that $tb < 1$. Consequently, by Trudinger–Moser inequality,

$$\sup_n \int_\Omega (e^{4\pi|u_n|^2})^t\,dx \le \sup_{\|v\|\le 1} \int_\Omega e^{4\pi tb|v|^2}\,dx < \infty.$$

Thus, the sequence $(e^{4\pi|u_n|^2})$ is bounded in $L^t(\Omega)$ and $e^{4\pi|u_n(x)|^2} \to e^{4\pi|u(x)|^2}$ almost everywhere in Ω. This implies that,

$$e^{4\pi|u_n|^2} \rightharpoonup e^{4\pi|u|^2} \quad \text{in } L^t(\Omega). \tag{5.1}$$

On the other hand,

$$|u_n| \to |u| \quad \text{in } L^{t'}(\Omega) \tag{5.2}$$

where $1/t + 1/t' = 1$. Now, (5.1) combined with (5.2) gives

$$\int_\Omega |u_n|e^{4\pi|u_n|^2}\,dx \to \int_\Omega |u|e^{4\pi|u|^2}\,dx.$$

Hence, $|u_n|e^{4\pi|u_n|^2} \to |u|e^{4\pi|u|^2}$ in $L^1(\Omega)$. Thus, from [15, Theorem IV.9], there are a subsequence of (u_n) and $h \in L^1(\Omega)$ such that $|u_n|e^{4\pi|u_n|^2} \le h$ almost everywhere in Ω. Thereby, $|f(u_n)u_n| \le h$ almost everywhere in Ω. By Lebesgue's Theorem, it follows that

$$\int_\Omega f(u_n)u_n\,dx \to \int_\Omega f(u)u\,dx.$$

The proof of (2.5) follows by using the same type of arguments. $\square$

PROOF OF LEMMA 2.6. Since $\widetilde{S}_\lambda \subset \mathcal{M}_\Omega$, in view of Corollary 2.4, we only need to prove that there exist $m_\lambda > 0$ such that $\|u\|^2 \leq m_\lambda < 1$ for all $u \in \widetilde{S}_\lambda$. For each $u \in \widetilde{S}_\lambda$, we have

$$c_\Omega^* + \lambda \geq I(u) = I(u) - \frac{1}{\theta} I'(u)u = \left(\frac{1}{2} - \frac{1}{\theta}\right)\|u\|^2 - \int_\Omega \left(F(u) - \frac{1}{\theta} f(u)u\right) dx.$$

From Ambrosetti–Rabinowitz condition (f_3),

$$c_\Omega^* + \lambda \geq \left(\frac{1}{2} - \frac{1}{\theta}\right)\|u\|^2.$$

On the other hand, by Lemma 2.1, we can fix $\lambda > 0$ sufficiently small such that

$$A + \lambda < \left(\frac{1}{2} - \frac{1}{\theta}\right),$$

where A was given in (2.3). Therefore

$$\|u\|^2 \leq \frac{c_\Omega^* + \lambda}{\left(\dfrac{1}{2} - \dfrac{1}{\theta}\right)} \leq m_\lambda < 1, \quad \text{where } m_\lambda := \frac{A + \lambda}{\left(\dfrac{1}{2} - \dfrac{1}{\theta}\right)}. \qquad \square$$

PROOF OF LEMMA 2.7. Since $u \in \widetilde{S}_\lambda \subset \mathcal{M}_\Omega$,

$$\|u^\pm\|^2 = \int_\Omega f(u^\pm)u^\pm \, dx.$$

Then, from (f_1),

$$\|u^\pm\|^2 \leq C \int_\Omega |u^\pm| e^{4\pi |u^\pm|^2} \, dx.$$

Using Sobolev imbedding and Höder inequality, for $1 < t_1$ and $1 < t_2 \approx 1$ such that $1/t_1 + 1/t_2 = 1$, we obtain

$$\|u^\pm\|^2 \leq |u^\pm|_{L^{t_1}} \left(\int_\Omega e^{4\pi t_2 |u^\pm|^2} \, dx\right)^{1/t_2}.$$

From Corollary 2.4,

$$m_0 \leq |u^\pm|_{L^{t_1}} \left(\int_\Omega e^{4\pi t_2 \|u^\pm\|^2 (|u^\pm|/\|u^\pm\|)^2} \, dx\right)^{1/t_2},$$

and, by Lemma 2.6, it follows that

$$m_0 \leq |u^\pm|_{L^{t_1}} \bigg(\int_\Omega e^{4\pi t_2 m_\lambda (|u^\pm|/\|u^\pm\|)^2} \, dx)^{1/t_2}.$$

Since $m_\lambda < 1$, we can fix $1 < t_2$ near 1 such that $t_2 m_\lambda < 1$. From Trudinger–Moser inequality (1.3), there exists a constant $C > 0$ such that

$$\int_\Omega e^{4\pi t_2 m_\lambda (|u^\pm|/\|u^\pm\|)^2} \, dx \leq C \quad \text{for all } u \in \widetilde{S}_\lambda.$$

Thereby, for some $C_1 > 0$, $C_1 \leq |u^\pm|_{L^{t_1}}$ for all $u \in \widetilde{S}_\lambda$. Now, the lemma follows applying interpolation. $\square$

PROOF OF PROPOSITION 3.6. Let $\overline{u}$ be a positive ground state solution of (P_∞) and u_1 is a positive ground state of (P) given by Theorems 3.3 and 3.4, respectively. Let us define $\overline{u}_n(x) = \overline{u}(x - x_n)$, where $x_n = (0, n)$ and for $\alpha, \beta > 0$

$$
\begin{aligned}
h^{\pm}(\alpha,\beta,n) = \int_{\mathbb{R}^2} \Big(|\nabla(\alpha u_1 - \beta\overline{u}_n)^{\pm}|^2 + V(x)|(\alpha u_1 - \beta\overline{u}_n)^{\pm}|^2\Big)\,dx \\
- \int_{\mathbb{R}^2} f((\alpha u_1 - \beta\overline{u}_n)^{\pm})(\alpha u_1 - \beta\overline{u}_n)^{\pm}\,dx.
\end{aligned} \tag{5.3}
$$

Recalling that $I'(u_1)u_1 = 0$ and using (f_4), we get

$$
\begin{aligned}
\int_{\mathbb{R}^2} (|\nabla(u_1/2)|^2 + V(x)(u_1/2)^2)\,dx - \int_{\mathbb{R}^2} f(u_1/2)(u_1/2) \\
= \int_{\mathbb{R}^2} \left(\frac{f(u_1)}{u_1} - \frac{f(u_1/2)}{(u_1/2)}\right)\left(\frac{u_1}{2}\right)^2 dx > 0.
\end{aligned} \tag{5.4}
$$

and

$$
\begin{aligned}
\int_{\mathbb{R}^2} (|\nabla(2u_1)|^2 + V(x)|2u_1|^2)\,dx - \int_{\mathbb{R}^2} f(2u_1)(2u_1) \\
= \int_{\mathbb{R}^2} \left(\frac{f(u_1)}{u_1} - \frac{f(2u_1)}{2u_1}\right)(2u_1)^2\,dx < 0.
\end{aligned} \tag{5.5}
$$

By (V_2), for n large enough, there holds

$$
\int_{\mathbb{R}^2} (|\nabla(\overline{u}_n/2)|^2 + V(x)(\overline{u}_n/2)^2)\,dx - \int_{\mathbb{R}^2} f(\overline{u}_n/2)(\overline{u}_n/2) > 0 \tag{5.6}
$$

and

$$
\int_{\mathbb{R}^2} (|\nabla(2\overline{u}_n)|^2 + V(x)(2\overline{u}_n)^2)\,dx - \int_{\mathbb{R}^2} f(2\overline{u}_n)(2\overline{u}_n) < 0. \tag{5.7}
$$

Hence, from (5.4)–(5.7), there exists $n_0 > 0$ such that

$$
\begin{cases} h^+(1/2,\beta,n) > 0, \\ h^+(2,\beta,n) < 0 \end{cases} \tag{5.8}
$$

for $n \geq n_0$ and $\beta \in [1/2, 2]$. Now, for all $\alpha \in [1/2, 2]$ we have

$$
\begin{cases} h^-(\alpha,1/2,n) > 0, \\ h^-(\alpha,2,n) < 0. \end{cases} \tag{5.9}
$$

From this, we can apply a variant of the Mean Value Theorem due to Miranda [21], to obtain $\alpha^*, \beta^* \in [1/2, 2]$, which depend on n and verify $h^{\pm}(\alpha^*, \beta^*, n) = 0$, for any $n \geq n_0$. Thus, $\alpha^* u_1 - \beta^* \overline{u}_n \in \mathcal{M}$, for $n \geq n_0$.

In view of the definition of c^*, it suffices to show that

$$
\sup_{1/2 \leq \alpha,\beta \leq 2} I(\alpha u_1 - \beta\overline{u}_n) < c_1 + c_\infty \quad \text{for } n \geq n_0.
$$

In order to do this, first we use Lemma 3.1 to get the ensuing estimate

$$I(\alpha u_1 - \beta \overline{u}_n) \leq \frac{1}{2}\int_{\mathbb{R}^2} (|\nabla(\alpha u_1)|^2 + |\nabla(\beta\overline{u}_n)|^2)\,dx$$
$$+ \frac{1}{2}\int_{\mathbb{R}^2} V(x)(|\alpha u_1|^2 + |\beta\overline{u}_n|^2)\,dx - \alpha\beta\int_{\mathbb{R}^2} (\nabla u_1 \nabla\overline{u}_n + V(x)u_1\overline{u}_n)dx - A_1,$$

where

$$A_1 = \int_{\mathbb{R}^2} F(\alpha u_1)\,dx + \int_{\mathbb{R}^2} F(\beta\overline{u}_n)\,dx - 2\int_{\mathbb{R}^2} [f(\alpha u_1)\beta\overline{u}_n + f(\beta\overline{u}_n)\alpha u_1]\,dx$$

Since u_1 is a positive solution of (P), we know that

$$\int_{\mathbb{R}^2} (\nabla u_1 \nabla \overline{u}_n + V(x)u_1\overline{u}_n)\,dx \geq 0.$$

Therefore

$$I(\alpha u_1 - \beta\overline{u}_n) \leq I(\alpha u_1) + I_\infty(\beta\overline{u}_n) + 2\alpha\int_{\mathbb{R}^2} f(\beta\overline{u}_n)u_1\,dx + 2\beta\int_{\mathbb{R}^2} f(\alpha u_1)\overline{u}_n\,dx + \frac{\beta^2}{2}\int_{\mathbb{R}^2} (V(x) - V_\infty(x))\overline{u}_n^2\,dx. \tag{5.10}$$

From (V_3),

$$\int_{\mathbb{R}^2} (V(x) - V_\infty(x))\overline{u}_n^2\,dx \leq -Ce^{-\mu n}$$

and, by (f_1)–(f_2),

$$\int_{\mathbb{R}^2} f(\alpha u_1)\overline{u}_n\,dx \leq \varepsilon\alpha\int_{\mathbb{R}^2} u_1\overline{u}_n\,dx + C_\alpha\int_{\mathbb{R}^2} (e^{\tau\alpha^2 u_1^2} - 1)u_1\overline{u}_n\,dx, \quad \text{for } \tau > 4\pi.$$

Notice that from Theorem 3.5,

$$\int_{B_{n/2}} u_1\overline{u}_n\,dx \leq C_2\int_{B_{n/2}} u_1 e^{-b|x-x_n|}\,dx.$$

Once $|x - x_n| \geq |x_n| - |x| = n - |x|$ and $|x| \leq n/2$, we find that $|x - x_n| \geq n/2$, from where it follows that

$$\int_{B_{n/2}} u_1\overline{u}_n\,dx \leq C_2\int_{B_{n/2}} u_1 e^{-bn/2}\,dx = Ce^{-bn/2}$$

and

$$\int_{\mathbb{R}^2\setminus B_{n/2}} u_1\overline{u}_n\,dx \leq C_2\int_{\mathbb{R}^2\setminus B_{n/2}} e^{-b|x|}\overline{u}_n\,dx$$
$$\leq C_2 e^{-bn/2}\int_{\mathbb{R}^2} \overline{u}_n\,dx = C_2 e^{-bn/2}\int_{\mathbb{R}^2} \overline{u}\,dx.$$

Therefore

$$\int_{\mathbb{R}^2} u_1\overline{u}_n\,dx \leq Ce^{-bn/2}.$$

Moreover, since $u_1 \in L^\infty(\mathbb{R}^2)$,

$$\int_{\mathbb{R}^2} (e^{\tau\alpha^2 u_1^2} - 1)u_1\overline{u}_n\,dx \leq C\int_{\mathbb{R}^2} u_1\overline{u}_n \leq Ce^{-bn/2}.$$

Therefore

$$\int_{\mathbb{R}^2} f(\alpha u_1)\overline{u}_n\,dx \le Ce^{-bn/2} \quad \text{and} \quad \int_{\mathbb{R}^2} f(\beta\overline{u}_n)u_1\,dx \le Ce^{-bn/2}.$$

Then, from (5.10),

$$I(\alpha u_1 - \beta\overline{u}_n) \le \sup_{\alpha\ge 0} I(\alpha u_1) + \sup_{\beta\ge 0} I_\infty(\beta\overline{u}_n) + C(e^{-bn/2} - e^{-\mu n}).$$

Since $\mu < 1/2$, for n large enough, we know that $e^{-bn/2} - e^{-\mu n} < 0$, from where it follows that $\sup\limits_{1/2\le\alpha,\beta\le 2} I(\alpha u_1 - \beta\overline{u}_n) < c_1 + c_\infty$. Consequently, $c^* < c_1 + c_\infty$, finishing the proof of the proposition. □

PROOF OF LEMMA 3.7. Suppose, by contradiction that $u \equiv 0$. From (V_2), given $\varepsilon > 0$ there exists $R = R(\varepsilon) > 0$ such that $|V(x) - V_\infty(x)| < \varepsilon$, for $|x| \ge R$. As a consequence of $u \equiv 0$, we get

$$\int_{B_R} |V(x) - V_\infty(x)||u_n|^2\,dx \to 0.$$

The inequality

$$\int_{\mathbb{R}^2} |V(x) - V_\infty(x)||u_n|^2\,dx \le \int_{B_R} |V(x) - V_\infty(x)||u_n|^2\,dx + \varepsilon\int_{\mathbb{R}^2\setminus B_R} |u_n|^2\,dx,$$

together with the boundedness of (u_n) in $H^1(\mathbb{R}^2)$ yields $|I(u_n) - I_\infty(u_n)| \to 0$ as $n \to \infty$.

A similar argument shows that $|I'(u_n)u_n - I'_\infty(u_n)u_n| \to 0$ as $n \to \infty$. Consequently,

$$I_\infty(u_n) = \sigma + o_n(1) \quad \text{and} \quad I'_\infty(u_n)u_n = o_n(1). \tag{5.11}$$

In what follows, we fix $s_n > 0$ verifying $s_n u_n \in \mathcal{N}_\infty$. We claim that (s_n) converges to 1 as $n \to \infty$. Effectivelly, we start proving that

$$\limsup s_n \le 1. \tag{5.12}$$

Suppose by contradiction that there exists a subsequence of (s_n), still denoted by (s_n), such that $s_n \ge 1 + \delta$ for all $n \in \mathcal{N}$, for some $\delta > 0$. From (5.11),

$$\int_{\mathbb{R}^2} (|\nabla u_n|^2 + V_\infty(x)|u_n|^2)\,dx = \int_{\mathbb{R}^2} f(u_n)u_n\,dx + o_n(1) \tag{5.13}$$

On the other hand, since $s_n u_n \in \mathcal{N}_\infty$,

$$s_n \int_{\mathbb{R}^2} (|\nabla u_n|^2 + V_\infty(x)|u_n|^2)\,dx = \int_{\mathbb{R}^2} f(s_n u_n)u_n\,dx. \tag{5.14}$$

Consequently

$$\int_{\mathbb{R}^2} \left(\frac{f(s_n u_n)}{s_n u_n} - \frac{f(u_n)}{u_n}\right)|u_n|^2\,dx = o_n(1). \tag{5.15}$$

We claim that there exist $(y_n) \subset \mathbb{Z}^2$ with $|y_n| \to \infty$, $r > 0$ and $\beta > 0$ such that

$$\int_{B_r(y_n)} u_n^2\,dx \geq \beta > 0.$$

Indeed, in the contrary case, using a version of Lions' results for critical growth in $\mathbb{R}^2$ due to Alves, do Ó and Miyagaki [4], we derive

$$\lim_{n\to+\infty} \int_{\mathbb{R}^2} f(u_n)u_n\,dx = 0,$$

which is contrary to our assumption.

Now, let $v_n(x) := u_n(x+y_n)$. Once that (u_n) is bounded in $H^1(\mathbb{R}^2)$, it is easy to show that (v_n) is also bounded in $H^1(\mathbb{R}^2)$. Therefore, for some subsequence, we can assume that (v_n) is weakly convergent, and we will denote by $\widetilde{v}$ its weak limit in $H^1(\mathbb{R}^2)$. Observing that

$$\int_{B_r(0)} |v_n|^2\,dx = \int_{B_r(y_n)} |u_n|^2\,dx \geq \beta > 0,$$

we deduce that $\widetilde{v} \neq 0$ in $H^1(\mathbb{R}^2)$. Now, (5.15), (f_4) and Fatou's Lemma lead to

$$0 < \int_{\mathbb{R}^2} \bigg(\frac{f((1+\delta)\widetilde{v})}{(1+\delta)\widetilde{v}} - \frac{f(\widetilde{v})}{\widetilde{v}} \bigg) \widetilde{v}^2\,dx \leq 0,$$

which is impossible. Hence $\limsup\limits_{n\to\infty} s_n \leq 1$.

If $s_0 = \limsup\limits_{n\to\infty} s_n < 1$, we can assume that $s_n < 1$ for n large enough. Then, by Fatou's Lemma

$$\begin{aligned} &0 < \int_{\mathbb{R}^2} \bigg(\frac{f(\widetilde{v})}{\widetilde{v}} - \frac{f(s_o\widetilde{v})}{s_o\widetilde{v}} \bigg) \widetilde{v}^2\,dx \leq 0 && \text{if } s_o > 0, \\ &0 < \int_{\mathbb{R}^2} f(\widetilde{v})\widetilde{v} \leq 0 && \text{if } s_o = 0, \end{aligned}$$

which are impossible. Hence, $\limsup\limits_{n\to\infty} s_n = 1$, and so, for some subsequence,

$$\lim_{n\to\infty} s_n = 1. \tag{5.16}$$

As a consequence of (5.13), (5.14), (5.16) and the fact that f is odd, we have

$$\int_{\mathbb{R}^2} F(s_n u_n)\,dx - \int_{\mathbb{R}^2} F(u_n)\,dx = o_n(1)$$

and

$$(s_n^2 - 1) \int_{\mathbb{R}^2} (|\nabla u_n|^2 + V_\infty(x)|u_n|^2)\,dx = o_n(1),$$

from where it follows that $I_\infty(s_n u_n) = I_\infty(u_n) + o_n(1)$. Then

$$c_\infty \leq I_\infty(s_n u_n) = \sigma + o_n(1).$$

Taking $n \to +\infty$, we find $c_\infty \leq \sigma$, which is impossible because $\sigma < c_\infty$. This contradiction comes from the assumption that $u \equiv 0$. $\square$

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