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TOPOLOGICAL AND MEASURE PROPERTIES OF SOME SELF-SIMILAR SETS

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Abstract. Given a finite subset $\Sigma \subset \mathbb{R}$ and a positive real number $q < 1$ we study topological and measure-theoretic properties of the self-similar set $K(\Sigma; q) = \Big\{ \sum_{n=0}^{\infty} a_n q^n : (a_n)_{n\in\omega} \in \Sigma^\omega \Big\}$, which is the unique compact solution of the equation $K = \Sigma + qK$. The obtained results are applied to studying partial sumsets $E(x) = \Big\{ \sum_{n=0}^{\infty} x_n \varepsilon_n : (\varepsilon_n)_{n\in\omega} \in \{0,1\}^\omega \Big\}$ of multigeometric sequences $x = (x_n)_{n\in\omega}$. Such sets were investigated by Ferens, Morán, Jones and others. The aim of the paper is to unify and deepen existing piecemeal results.

1. Introduction

Suppose that $x = (x_n)_{n=1}^{\infty}$ belongs to $l_1 \setminus c_{00}$ which means that x is an absolutely summable sequence with infinitely many nonzero terms. Let

$$E(x) = \left\{ \sum_{n=1}^{\infty} \varepsilon_n x_n : (\varepsilon_n)_{n=1}^{\infty} \in \{0,1\}^{\mathbb{N}} \right\}$$

denotes the set of all subsums of the series $\sum_{n=1}^{\infty} x_n$, called *the achievement set* (or *a partial sumset*) of x. The investigation of topological properties of achievement

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sets was initiated almost one hundred years ago. In 1914 Soichi Kakeya [8] presented the following result:

Theorem 1.1 (Kakeya). *For any sequence $x \in l_1 \setminus c_{00}$*

(a) *$E(x)$ is a perfect compact set.*
(b) *If $|x_n| > \sum_{i>n} |x_i|$ for almost all n, then $E(x)$ is homeomorphic to the ternary Cantor set.*
(c) *If $|x_n| \le \sum_{i>n} |x_i|$ for almost all n, then $E(x)$ is a finite union of closed intervals. In the case of non-increasing sequence x, the last inequality is also necessary for $E(x)$ to be a finite union of intervals.*

Moreover, Kakeya conjectured that $E(x)$ is either nowhere dense or a finite union of intervals. Probably, the first counterexample to this conjecture was given by Weinstein and Shapiro ([16]) and, independently, by Ferens ([5]). The simplest example was presented by Guthrie and Nymann [6]: for the sequence $c = ((5 + (-1)^n)/4^n)_{n=1}^{\infty}$, the set $T = E(c)$ contains an interval but is not a finite union of intervals. In the same paper they formulated the following theorem, finally proved in [12]:

Theorem 1.2. *For any sequence $x \in l_1 \setminus c_{00}$, $E(x)$ is one of the following sets*:

(a) *a finite union of closed intervals*;
(b) *homeomorphic to the Cantor set*;
(c) *homeomorphic to the set T.*

Note that the set $T = E(c)$ is homeomorphic to $C \cup \bigcup_{n=1}^{\infty} S_{2n-1}$, where S_n denotes the union of the 2^{n-1} open middle thirds which are removed from $[0,1]$ at the n-th step in the construction of the Cantor ternary set C. Such sets are called Cantorvals (to emphasize their similarity to unions of intervals and to the Cantor set simultaneously). Formally, a *Cantorval* (more precisely, an $\mathcal{M}$-Cantorval, see [9]) is a non-empty compact subset S of the real line such that S is the closure of its interior, and both endpoints of any component with non-empty interior are accumulation points of one-point components of S. A non-empty subset C of the real line $\mathbb{R}$ will be called a *Cantor set* if it is compact, zero-dimensional, and has no isolated points.

Let us observe that Theorem 1.2 says that l_1 can be divided into 4 sets: c_{00} and the sets connected with cases (a), (b) and (c). Some algebraic and topological properties of these sets have been recently considered in [1].

We will describe sequences constructed by Weinstein and Shapiro, Ferens and Guthrie and Nymann using the notion of multigeometric sequence. We call a sequence *multigeometric* if it is of the form

$$(k_0, \dots, k_m, k_0 q, \dots, k_m q, k_0 q^2, \dots, k_m q^2, k_0 q^3 \dots)$$

for some positive numbers $k_0, \dots, k_m$ and $q \in (0, 1)$. We will denote such a sequence by $(k_0, \dots, k_m; q)$. Keeping in mind that the type of $E(x)$ (in the sense of the classification given in Theorem 1.2) is the same as $E(\alpha x)$ for any $\alpha > 0$, we can describe the Weinstein–Shapiro sequence as $a = (8, 7, 6, 5, 4; 1/10)$, the Ferens sequence as $b = (7, 6, 5, 4, 3; 2/27)$ and the Guthrie–Nymann sequence as $c = (3, 2; 1/4)$.

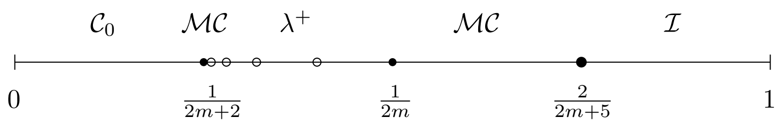
Another interesting example of a sequence d with $E(d)$ being Cantorval was presented by R. Jones in ([7]). The sequence is of the form$d = (3, 2, 2, 2; 19/109)$.

In fact, Jones constructed continuum many sequences generating Cantorvals, indexed by a parameter q, by proving that, for any positive number q with

$$\frac{1}{5} \le \sum_{n=1}^{\infty} q^n < \frac{2}{9}$$

(i.e. $1/6 \le q < 2/11$) the achievement set of the sequence $(3, 2, 2, 2; q)$ is a Cantorval.

The structure of the achievement sets $E(x)$ for multigeometric sequences x was studied in the paper [3], which contains a necessary condition for $E(x)$ to be an interval and sufficient conditions for $E(x)$ to contain an interval or have Lebesgue measure zero. In the case of a Guthrie–Nymann–Jones sequence $x_q = (3, 2, \dots, 2; q)$, of rank m (i.e. with m repeated 2's), the set $E(x_q)$ is an interval if and only if $q \ge 2/(2m+5)$, $E(x_q)$ is a Cantor set of measure zero if $q < 1/(2m+2)$, and $E(x_q)$ is a Cantorval if $q \in \{1/(2m+2)\} \cup [1/(2m), 2/(2m+5))$. In this paper we reveal some structural properties of the sets $E(x_q)$ for q belonging to the "mysterious" interval $(1/(2m+2), 1/(2m))$. In particular, we shall show that for almost all q in this interval the set $E(x_q)$ has positive Lebesgue measure and there is a decreasing sequence (q_n) convergent to $1/(2m+2)$ for which $E(x_{q_n})$ is a Cantor set of zero Lebesgue measure. The above description of the structure of $E(x_q)$ can be presented as follows:



where $\mathcal{C}_0$ (resp. $\mathcal{MC}$, $\mathcal{I}$) indicates sets of numbers q for which the set $E(x_q)$ is a Cantor set of zero Lebesgue measure (resp. a Cantorval, an interval). The symbol λ^+ indicates that for almost all q in a given interval the sets $E(x_q)$ have positive Lebesgue measure, which means that the set $Z = \{q \in (1/(2m+2), 1/(2m)) :$

$\lambda(E(x_q)) = 0\}$ has Lebesgue measure $\lambda(Z) = 0$. Similar diagrams we use later in this paper.

The achievement sets of multigeometric sequences are special cases of self-similar sets of the form

$$K(\Sigma; q) = \bigg\{ \sum_{n=0}^{\infty} a_n q^n : (a_n)_{n=0}^{\infty} \in \Sigma^\omega \bigg\}$$

where $\Sigma \subset \mathbb{R}$ is a set of real numbers and $q \in (0,1)$. The set $K(\Sigma; q)$ is self-similar in the sense that $K(\Sigma; q) = \Sigma + q \cdot K(\Sigma; q)$. Moreover, the set $K(\Sigma; q)$ can be found as the unique compact solution $K \subset \mathbb{R}$ of the equation $K = \Sigma + qK$. This follows from the Banach Fixed Point Theorem applied to the contraction mapping $K \mapsto \Sigma + qK$ on the hyperspace $H(\mathbb{R})$ of compact sets of real numbers.

It is easily seen that for a multigeometric sequence $x_q = (k_0, \ldots, k_m; q)$ the achievement set $E(x)$ coincides with the self-similar set $K(\Sigma; q)$ for the set

$$\Sigma = \bigg\{ \sum_{n=0}^{m} k_n \varepsilon_n : (\varepsilon_n)_{n=0}^{m} \in \{0,1\}^{m+1} \bigg\}$$

of all possible sums of the numbers $k_0, \ldots, k_m$. This makes possible to apply for studying the achievement sets $E(x_q)$ the theory of self-similar sets developed in [14] and [13].

In this paper we shall describe some topological and measure properties of the self-similar sets $K(\Sigma; q)$ depending on the value of the similarity ratio $q \in (0,1)$, and shall apply the obtained result to establishing topological and measure properties of achievement sets of multigeometric progressions. To formulate the principal results we need to introduce some numerical characteristics of compact subsets $A \subset \mathbb{R}$.

Given a compact subset $A \subset \mathbb{R}$ containing more than one point let

$$\operatorname{diam} A = \sup\{|a - b| : a, b \in A\}$$

be the diameter of A and $\delta(A) = \inf\{|a - b| : a, b \in A,\ a \neq b\}$ and $\Delta(A) = \sup\{|a - b| : a, b \in A,\ (a, b) \cap A = \emptyset\}$ be the smallest and largest gaps in A, respectively. Observe that A is an interval (equal to $[\min A, \max A]$) if and only if $\Delta(A) = 0$. Also put

$$I(A) = \frac{\Delta(A)}{\Delta(A) + \operatorname{diam} A} \quad \text{and} \quad i(A) = \inf\{I(B) : B \subset A,\ 2 \leq |B| < \omega\}.$$

In particular, given a finite subset $\Sigma \subset \mathbb{R}$ of cardinality $|\Sigma| \geq 2$, we will write it as $\Sigma = \{\sigma_1, \ldots, \sigma_s\}$ for real numbers $\sigma_1 < \ldots < \sigma_s$. Then we have

$$\operatorname{diam}(\Sigma) = \sigma_s - \sigma_1, \quad \delta(\Sigma) = \min_{i<s}(\sigma_{i+1} - \sigma_i), \quad \text{and} \quad \Delta(\Sigma) = \max_{i<s}(\sigma_{i+1} - \sigma_i).$$

Theorem 1.3. *Let $\Sigma = \{\sigma_1, \ldots, \sigma_s\}$ for some real numbers $\sigma_1 < \ldots < \sigma_s$. The self-similar sets $K(\Sigma; q)$ where $q \in (0,1)$ have the following properties*:

(a) $K(\Sigma;q)$ *is an interval if and only if* $q \ge I(\Sigma)$;

(b) $K(\Sigma;q)$ *is not a finite union of intervals if* $q < I(\Sigma)$ *and* $\Delta(\Sigma) \in \{\sigma_2 - \sigma_1, \sigma_s - \sigma_{s-1}\}$;

(c) $K(\Sigma;q)$ *contains an interval if* $q \ge i(\Sigma)$;

(d) *if*

$$d = \frac{\delta(\Sigma)}{\operatorname{diam}(\Sigma)} < \frac{1}{3+2\sqrt{2}} \quad \text{and} \quad \frac{1}{|\Sigma|} < \frac{\sqrt{d}}{1+\sqrt{d}},$$

then for almost all $q \in (1/|\Sigma|, \sqrt{d}/(1+\sqrt{d}))$ *the set* $K(\Sigma;q)$ *has positive Lebesgue measure and the set* $K(\Sigma;\sqrt{q})$ *contains an interval*;

(e) $K(\Sigma;q)$ *is a Cantor set of zero Lebesgue measure if* $q < 1/|\Sigma|$ *or, more generally, if* $q^n < 1/|\Sigma_n|$ *for some* $n \in \mathbb{N}$ *where*

$$\Sigma_n = \left\{ \sum_{k=0}^{n-1} a_k q^k : (a_k)_{k=0}^{n-1} \in \Sigma^n \right\};$$

(f) *if* $\Sigma \supset \{a, a+1, b+1, c+1, b+|\Sigma|, c+|\Sigma|\}$ *for some real numbers* $a,b,c \in \mathbb{R}$ *with* $b \ne c$, *then there is a strictly decreasing sequence* $(q_n)_{n\in\omega}$ *with* $\lim\limits_{n\to\infty} q_n = 1/|\Sigma|$ *such that the sets* $K(\Sigma;q_n)$ *has Lebesgue mesure zero.*

The statements (a)–(c) from this theorem will be proved in Section 2, the statement (d) in Section 3 and (e), (f) in Section 4. Writing that for almost all q in an interval (a,b) some property $\mathcal{P}(q)$ holds we have in mind that the set $Z = \{q \in (a,b) : \mathcal{P}(q) \text{ does not hold}\}$ has Lebesgue measure $\lambda(Z) = 0$.

2. Intervals and Cantorvals

In this section we generalize results of [3] detecting the self-similar sets $K(\Sigma;q)$ which are intervals or Cantorvals. In the following theorem we prove the statements (a)–(c) of Theorem 1.3.

Theorem 2.1. *Let* $q \in (0,1)$ *and* $\Sigma = \{\sigma_1, \dots, \sigma_s\} \subset \mathbb{R}$ *be a finite set with* $\sigma_1 < \dots < \sigma_s$. *The self-similar set* $K(\Sigma;q) = \left\{ \sum\limits_{i=0}^{\infty} a_i q^i : (a_i)_{i\in\omega} \in \Sigma^\omega \right\}$

(a) *is an interval if and only if* $q \ge I(\Sigma)$;

(b) *contains an interval if* $q \ge i(\Sigma)$;

(c) *is not a finite union of intervals if* $q < I(\Sigma)$ *and* $\Delta(\Sigma) \in \{\sigma_2 - \sigma_1, \sigma_s - \sigma_{s-1}\}$.

Proof. (a) Observe that $\operatorname{diam} K(\Sigma;q) = \operatorname{diam}(\Sigma)/(1-q)$. Assuming that $q \ge I(\Sigma) = \Delta(\Sigma)/(\Delta(\Sigma) + \operatorname{diam}\Sigma)$, we conclude that $\Delta(\Sigma) \le q \cdot \operatorname{diam}(\Sigma)/(1-q) = q \cdot \operatorname{diam} K(\Sigma;q)$, which implies that

$$\Delta(K(\Sigma;q)) = \Delta(\Sigma + q \cdot K(\Sigma;q)) \le \Delta(q \cdot K(\Sigma;q)) = q \cdot \Delta(K(\Sigma;q)).$$

Since $q < 1$ this inequality is possible only in case $\Delta(K(\Sigma; q)) = 0$, which means that $K(\Sigma; q)$ is an interval.

If $q < \Delta(\Sigma)/(\Delta(\Sigma) + \mathrm{diam}\Sigma)$, then $\Delta(\Sigma) > q \cdot \mathrm{diam}(\Sigma)/(1 - q) = q \cdot \mathrm{diam}(K(\Sigma; q))$ and we can find two consequtive points $a < b$ in Σ with $b = a + \Delta(\Sigma) > a + \mathrm{diam}(qK(\Sigma; q))$ and conclude that $[a, b] \cap K(\Sigma; q) = [a, b] \cap (\Sigma + qK(\Sigma; q)) \subset [a, a + \mathrm{diam}(q\, K(\Sigma; q))] \neq [a, b]$, so $K(\Sigma; q)$ is not an interval.

(b) Now assume that $q \geq i(\Sigma)$ and find a subset $B \subset \Sigma$ such that $I(B) = i(\Sigma) < q$. By the preceding item, the self-similar set $K(B; q) = B + qK(B; q)$ is an interval. Consequently, $K(\Sigma; q)$ contains the interval $K(B; q)$.

(c) Finally assume that $\Delta(\Sigma) = \sigma_2 - \sigma_1$ and $q < I(\Sigma)$. Since for every $a \in \Sigma$ we get $K(\Sigma - a; q) = K(\Sigma; q) - a/(1 - q)$, we can replace Σ by its shift and assume that $\sigma_1 = 0$ and hence $\Delta(\Sigma) = \sigma_2 - \sigma_1 = \sigma_2$. It follows from $q < I(\Sigma) = \sigma_2/(\sigma_2 + \mathrm{diam}\Sigma)$ that for any $j \in \mathbb{N}$, the interval $\Big(\sum_{n=j+1}^{\infty} q^n \sigma_s, q^j \sigma_2\Big)$ is nonempty and disjoint from $K(\Sigma; q)$. Hence, no interval of the form $[0, \varepsilon]$ is included in $K(\Sigma; q)$. But $0 \in K(\Sigma; q)$, so $K(\Sigma; q)$ is not a finite union of closed intervals. By analogy we can consider the case $\Delta(\Sigma) = \sigma_s - \sigma_{s-1}$. $\square$

In particular, Theorem 2.1 implies:

Corollary 2.2. *For $\Sigma = \{0, \dots, s-1\}$ the set $K(\Sigma; q)$ is an interval if and only if $q \geq I(\Sigma) = 1/|\Sigma|$.*

Corollary 2.3. *If $\{k, \dots, k + n - 1\} \subset \Sigma$, then $i(\Sigma) \leq 1/n$ and for every $q \geq 1/n$ the set $K(\Sigma; q)$ contains an interval.*

In particular, for the Guthrie–Nymann–Jones multigeometric sequence $x_q = (3, 2, \dots, 2; q)$ of rank m the sumset $\Sigma = \{0, 2, \dots, 2m + 1, 2m + 3\}$ has cardinality $|\Sigma| = 2m + 2$, $I(\Sigma) = \Delta(\Sigma)/(\Delta(\Sigma) + \mathrm{diam}\,\Sigma) = 2/(2m + 5)$, $i(\Sigma) = \min\{1/(2m), 2/(2m + 5)\}$, and $d = \delta(\Sigma)/\mathrm{diam}(\Sigma) = 1/(2m + 3)$. So, for $q \in [2/(2m + 5), 1)$ the set $E(x_q) = K(\Sigma; q)$ is an interval.

For $q \in [1/(2m), 2/(2m + 5))$ the set $E(x_q)$ has a nonempty interior but it is not an interval. Hence, by Theorem 1.2, it is a Cantorval.

3. Sets of positive measure

In this section we shall prove the statement (d) of Theorem 1.3 detecting numbers q for which the self-similar set $K(\Sigma; q)$ has positive Lebesgue measure $\lambda(K(\Sigma; q))$. For this we shall apply the deep results of Boris Solomyak [14] related to the distribution of the random series $\sum_{n=0}^{\infty} a_n \lambda^n$, where the coefficients $a_n \in \Sigma$ are chosen independently with probability $1/|\Sigma|$ each.

Given a finite subset $\Sigma \subset \mathbb{R}$ consider the number

$$\alpha(\Sigma) = \inf \Big\{ x \in (0,1) : \exists (a_n)_{n\in\omega} \in (\Sigma - \Sigma)^\omega \setminus \{0\}^\omega \\ \text{such that } \sum_{n=0}^{\infty} a_n x^n = 0 \text{ and } \sum_{n=1}^{\infty} n a_n x^{n-1} = 0 \Big\}.$$

The first part of the following theorem was proved by Solomyak in [14, 1.2]:

Theorem 3.1. *Let $\Sigma \subset \mathbb{R}$ be a finite subset. If $1/|\Sigma| < \alpha(\Sigma)$, then for almost all q in the interval $(1/|\Sigma|, \alpha(\Sigma))$ the self-similar set $K(\Sigma; q)$ has positive Lebesgue measure and the set $K(\Sigma; \sqrt{q})$ contains an interval.*

Proof. By Theorem 1.2 of [14], for almost all $q \in (1/|\Sigma|, \alpha(\Sigma))$ the self-similar set $K(\Sigma; q)$ has positive Lebesgue measure. Since $K(\Sigma; \sqrt{q}) = K(\Sigma; q) + \sqrt{q} \cdot K(\Sigma; q)$, the set $K(\Sigma; \sqrt{q})$ contains an interval, being the sum of two sets of positive Lebesgue measure (according to the famous Steinhaus Theorem [15]).□

The definition of Solomyak's constant $\alpha(\Sigma)$ does not suggest any efficient way of its calculation. In [14] Solomyak found an efficient lower bound on $\alpha(\Sigma)$ based on the notion of a $(*)$-function, i.e., a function of the form

$$g(x) = -\sum_{k=1}^{n-1} x^k + \gamma x^n + \sum_{k=n+1}^{\infty} x^k$$

for some $n \in \mathbb{N}$ and $\gamma \in [-1, 1]$. In Lemma 3.1 [14] Solomyak proved that every $(*)$-function $g(x)$ has a unique critical point on $[0, 1)$ at which g takes its minimal value. Moreover, for every $d > 0$ there is a unique $(*)$-function $g_d(x)$ such that $\min_{[0,1)} g_d = -d$. The unique critical point $x_d \in g_d^{-1}(-d) \in [0, 1)$ of g_d will be denoted by $\underline{\alpha}(d)$. The following lower bound on the number $\alpha(\Sigma)$ follows from Proposition 3.2 and inequality (15) in [14].

Lemma 3.2. *For every finite set $\Sigma \subset \mathbb{R}$ of cardinality $|\Sigma| \geq 2$ we get*

$$\alpha(\Sigma) \geq \underline{\alpha}(d) \quad \textit{where } d = \frac{\delta(\Sigma)}{\operatorname{diam}(\Sigma)}.$$

The function $\underline{\alpha}(d)$ can be calculated effectively (at least for $d \leq 1/2$).

Lemma 3.3. *If $0 < d \leq 1/(3 + 2\sqrt{2})$, then*

$$\underline{\alpha}(d) = \frac{\sqrt{d}}{1 + \sqrt{d}}.$$

Proof. Observe that the minimal value of the $(*)$-function

$$g(x) = -x + \sum_{k=2}^{\infty} x^k = -x + \frac{x^2}{1-x}$$

is equal to $-1/(3+2\sqrt{2})$, which implies that for $d \in (0, 1/(3+2\sqrt{2})]$ the number $\underline{\alpha}(d)$ is equal to the critical point of the unique $(*)$-function

$$g(x) = \gamma x + \sum_{k=2}^{\infty} x^k = -1 + (\gamma - 1)x + \frac{1}{1-x}$$

with $\min_{[0,1)} g = -d$. This $(*)$-function has derivative $g'(x) = (\gamma - 1) + 1/(1-x)^2$. If x is the critical point of g, then $1 - \gamma = 1/(1-x)^2$ and the equality

$$d = -1 + (\gamma - 1)x + \frac{1}{1-x} = -1 - \frac{x}{(1-x)^2} + \frac{1}{1-x}$$

has the solution

$$x = 1 - \frac{1}{1+\sqrt{d}} = \frac{\sqrt{d}}{1+\sqrt{d}}$$

which is equal to $\underline{\alpha}(d)$. □

For $d > 1/(3+2\sqrt{2})$ the formula for $\underline{\alpha}(d)$ is more complex.

Lemma 3.4. *If $1/(3+2\sqrt{2}) \le d \le 1/2$, then the value*

$$\underline{\alpha}(d) = \frac{1+d}{3} + \frac{\sqrt[3]{2} \cdot R}{6} + \frac{2d^2 - 8d - 1}{3\sqrt[3]{2} \cdot R}$$

where

$$R = \sqrt[3]{4d^3 - 24d^2 + 21d - 5 + 3\sqrt{3}\sqrt{1 - 8d^3 + 39d^2 - 6d}}$$

can be found as the unique real solution of the qubic equation

$$2(x-1)^3 + (4-2d)(x-1)^2 + 3(x-1) + 1 = 0.$$

Proof. Since the minimal values of the $(*)$-functions

$$g_1(x) = -x + \sum_{k=2}^{\infty} x^k \quad \text{and} \quad g(x) = -x - x^2 + \sum_{k=3}^{\infty} x^k$$

are equal to $-1/(3+2\sqrt{2})$ and $-1/2$, respectively, for $d \in [1/(3+2\sqrt{2}), 1/2]$ the number $\underline{\alpha}(d)$ is equal to the critical point of a unique $(*)$-function

$$g(x) = -x + \gamma x^2 + \sum_{k=3}^{\infty} x^k = -1 - 2x + (\gamma - 1)x^2 + \frac{1}{1-x}$$

with $\min_{[0,1)} g = -d$. At the critical point x the derivative of g equals zero:

$$0 = g'(x) = -2 + 2(\gamma - 1)x + \frac{1}{(1-x)^2}$$

which implies that

$$\gamma - 1 = \frac{1}{2x}\Big(2 - \frac{1}{(1-x)^2}\Big) = \frac{2x^2 - 4x + 1}{2x(1-x)^2}.$$

After substitution of $\gamma-1$ to the formula of the function $g(x)$, we get

$$-d=-1-2x-\frac{2x^3-4x^2+x}{2(1-x)^2}+\frac{1}{1-x}.$$

This equation is equivalent to the qubic equation

$$2(x-1)^3+(4-2d)(x-1)^2+3(x-1)+1=0.$$

Solving this equation with the Cardano formulas we can get the solution $\underline{\alpha}(d)$ written in the lemma. $\square$

Remark 3.5. Calculating the value $\underline{\alpha}(d)$ for some concrete numbers d, we get

$$\underline{\alpha}\left(\frac{1}{5}\right)\approx 0.32482,\quad \underline{\alpha}\left(\frac{1}{4}\right)\approx 0.37097,\quad \underline{\alpha}\left(\frac{1}{3}\right)\approx 0.42773,\quad \underline{\alpha}\left(\frac{1}{2}\right)=0.5.$$

Theorem 3.1 and Lemma 3.3 imply:

Corollary 3.6. *Let $\Sigma\subset\mathbb{R}$ be a finite subset containing more than three points and $d=\delta(\Sigma)/\mathrm{diam}(\Sigma)$. If $d\le 1/(3+2\sqrt{2})$ and $\sqrt{d}/(1+\sqrt{d})>1/|\Sigma|$, then for almost all q in the interval $(1/|\Sigma|,\sqrt{d}/(1+\sqrt{d}))$ the self-similar set $K(\Sigma;q)$ has positive Lebesgue measure and the set $K(\Sigma;\sqrt{q})$ contains an interval.*

Remark 3.7. Theorem 2.1 says that for $q\in[i(\Sigma),1)$ the set $K(\Sigma;q)$ contains an interval. By Theorem 3.1 under certain conditions the same is true for almost all $q\in[1/\sqrt{|\Sigma|},\sqrt{\alpha(\Sigma)})$. Let us remark that the numbers $i(\Sigma)$ and $1/\sqrt{|\Sigma|}$ are incomparable in general. Indeed, for the multigeometric sequence $(1,\dots,1;q)$ containing $k>1$ units the set $\Sigma=\{0,\dots,k\}$ has

$$i(\Sigma)=I(\Sigma)=\frac{1}{k+1}=\frac{1}{|\Sigma|}<\frac{1}{\sqrt{|\Sigma|}}.$$

On the other hand, for the multigeometric sequence $(3^{k-1},\dots,3,1;q)$ the set $\Sigma=\left\{\sum_{n=0}^{k-1}3^n\varepsilon_n : (\varepsilon_n)_{n<k}\in\{0,1\}^k\right\}$ has cardinality $|\Sigma|=2^k$, diameter $\mathrm{diam}(\Sigma)=(3^k-1)/2$, $d=\delta(\Sigma)/\mathrm{diam}(\Sigma)=2/(3^k-1)$ and

$$i(\Sigma)=I(\Sigma)=\frac{1}{4}+\frac{1}{4\cdot 3^{k-1}}>\frac{1}{\sqrt{2}^k}=\frac{1}{\sqrt{|\Sigma|}}.$$

Corollary 3.6 guarantees that for almost all $q\in(1/\sqrt{2}^k,\sqrt[4]{d}/\sqrt{1+\sqrt{d}})$ the set $K(\Sigma;q)$ contains an interval.

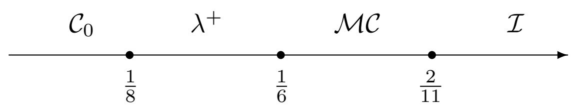
Multigeometric sequences of the form $(k+m,\dots,k+1,k;q)$ with $m\ge k$ we will call, after [2], *Ferens-like sequences.* The achievement set $E(x)$ for a Ferens-like sequence coincides with the self-similar set $K(\Sigma;q)$ for the set $\Sigma=\{0,k,k+1,\dots,n-k,n\}$, where $n=(m+1)(2k+m)/2$. Sets $K(\Sigma;q)$ with Σ of this form will be called *Ferens-like fractals.*

Note that Guthrie–Nymann–Jones sequence of rank m generates a Ferens-like fractal (with $\Sigma = \{0, 2, 3, \dots, 2m+1, 2m+3\}$. There are also Ferens-like fractals which are not originated by any multigeometric sequence (for example $K(\Sigma; q)$ with $\Sigma = \{0, 4, 5, 6, 7, 11\}$). However, as an easy consequence of the main theorem of [11], we obtain for Ferens-like fractals a "richotomy" analogous to that formulated in Theorem 1.2. Moreover, some theorems formulated for multigeometric sequences are in fact proved for $K(\Sigma; q)$ (see for example Theorem 2 in [3]).

Example 3.8. For the Ferens-like sequence $x_q = (4, 3, 2; q)$ we get $\Sigma = \{0, 2, 3, 4, 5, 6, 7, 9\}$,

$$d = \frac{\delta(\Sigma)}{\operatorname{diam}(\Sigma)} = \frac{1}{9} < \frac{1}{3+2\sqrt{2}} \quad \text{and} \quad \frac{\sqrt{d}}{1+\sqrt{d}} = \frac{1}{4} > \frac{1}{6} = i(\Sigma).$$

By Corollary 3.6 (and Theorem 2.1), for almost all numbers $q \in (1/8, 1)$ the achievement set $E(x_q) = K(\Sigma; q)$ has positive Lebesgue measure (for $q < 2/11 = I(\Sigma)$ it is not a finite union of intervals). By Theorem 2.1, for any $q \in [i(\Sigma), I(\Sigma)) = [1/6; 2/11)$ the set $K(\Sigma; q)$ is a Cantorval. The structure of the sets $E(x_q) = K(\Sigma; q)$ is described in the diagram:



More generally, for any Ferens-like fractal, $|\Sigma| = n-2k+3$, $\Delta(\Sigma) = k$, $\delta(\Sigma) = 1$, $I(\Sigma) = k/(n+k)$, $i(\Sigma) = \min(1/(|\Sigma|-2), I(\Sigma))$ and $d = 1/n$. Moreover, if $n \geq 7$ then $\underline{\alpha}(d) = 1/(\sqrt{n}+1)$. Therefore, one can check that for any Ferens-like sequence we have $\underline{\alpha}(d) > i(\Sigma)$, and we can draw an analogous diagram. The same result we can obtain for any Ferens-like fractal with $k = 2$ (even if it is not originated by any Ferens-like sequence). However, there are Ferens-like fractals with $\underline{\alpha}(d) < i(\Sigma)$ (for example $K(\Sigma; q)$ with $\Sigma = \{0, 3, 4, 7\}$ or $\Sigma = \{0, 4, 5, 6, 7, 11\}$).

Example 3.9. For the Guthrie–Nymann–Jones sequence $x_q = (3, 2, \dots, 2; q)$ of rank $m \geq 2$ we get $\Sigma = \{0, 2, 3, \dots, 2m+1, 2m+3\}$, $|\Sigma| = 2m+2$, $I(\Sigma) = 2/(2m+5)$, $i(\Sigma) = \min\{1/(2m), 2/(2m+5)\}$, $d = 1/(2m+3)$ and $\underline{\alpha}(d) = 1/(1+\sqrt{2m+3})$. Moreover, we have $d < 1/(3+2\sqrt{2})$ and $\underline{\alpha}(d) \geq i(\Sigma) > 1/(2m+2) = 1/|\Sigma|$. So, we can apply Corollary 3.6 and conclude that for almost all numbers $q \in (1/(2m+2), 1/(2m))$ the self-similar set $K(\Sigma; q)$ has positive Lebesgue measure. By Theorem 2.1, for any $q \in [i(\Sigma), 2/(2m+5))$ the set $K(\Sigma; q)$ is a Cantorval and for all $q \in [2/(2m+5), 1)$ it is an interval.

For $m = 1$ we obtain $\underline{\alpha}(d) = \underline{\alpha}(1/5) > 2/7$. Therefore, for almost all numbers $q \in (1/4, 2/7)$ the set $K(\Sigma; q)$ has positive Lebesgue measure.

4. Self-similar sets of zero Lebesgue measure

The results of the preceding section yield conditions under which for almost all q in an interval $[1/|\Sigma|, \alpha(\Sigma))$ the set $K(\Sigma; q)$ has positive Lebesgue measure. In this section we shall show that this interval can contain infinitely many numbers q with $\lambda(K(\Sigma; q)) = 0$ thus proving the statements (e) and (f) of Theorem 1.3.

Theorem 4.1. *If there exists $n \in \mathbb{N}$ such that*

$$\left|\sum_{i=0}^{n-1} q^i \Sigma\right| \cdot q^n < 1$$

then the set $K(\Sigma, q)$ has measure zero.

Proof. Denote $K := K(\Sigma, q)$ and assume that $\lambda(K) > 0$. From the equality $K = \Sigma + qK$ we obtain, by induction, that

$$K = \sum_{i=0}^{n-1} q^i \Sigma + q^n K.$$

Let $\Sigma_n = \sum\limits_{i=0}^{n-1} q^i \Sigma$. If $|\Sigma_n| \cdot q^n < 1$, then $\lambda(K) \leq |\Sigma_n| \cdot q^n \cdot \lambda(K) < 1 \cdot \lambda(K)$ which is impossible. □

Remark 4.2. M. Morán in [10] considered achivement sets of multigeometric sequences with complex values. He proved (in our notation) that for $q < \delta(\Sigma)/(\delta(\Sigma) + \operatorname{diam}\Sigma)$ the set $E(x)$ has measure zero. Observe that

$$\frac{\delta(\Sigma)}{\delta(\Sigma) + \operatorname{diam}\Sigma} \leq \frac{1}{|\Sigma|},$$

hence the Morán condition follows from Theorem 4.1 (for $n = 1$). Equality in the previous formula holds if and only if all differences $\sigma_{i+1} - \sigma_i$ are equal.

To use Theorem 4.1 we need a technical lemma:

Lemma 4.3. *For any integer numbers $s > 1$ and $n > 1$ the unique positive solution q of the equation*

$$x + x^2 + \ldots + x^{n-1} = \frac{1}{s-1} \tag{4.1}$$

is greater than $1/s$. Moreover, there is $n_0 \in \mathbb{N}$ such that, for any $n > n_0$,

$$\left(s^n - 2^{n-1}\right) \cdot q^n < 1. \tag{4.2}$$

Proof. Clearly

$$\sum_{i=1}^{n-1}\left(\frac{1}{s}\right)^i=\frac{1}{s-1}\cdot\left(1-\frac{1}{s^{n-1}}\right)<\frac{1}{s-1},$$

so $q>1/s$. From the equality

$$\frac{1}{s-1}=\sum_{i=1}^{n-2}\left(\frac{1}{s}\right)^i+\frac{1}{(s-1)s^{n-2}}$$

we obtain

$$q^{n-1}=\frac{1}{s-1}-\sum_{i=1}^{n-2}q^i<\frac{1}{s-1}-\sum_{i=1}^{n-2}\left(\frac{1}{s}\right)^i=\frac{1}{(s-1)s^{n-2}}.$$

Using the latter inequality and the equality $1/(s-1)=(q-q^n)/(1-q)$ we have

$$\frac{1-q}{s-1}=q\left(1-q^{n-1}\right)>q\left(1-\frac{1}{(s-1)\,s^{n-2}}\right).$$

Therefore, $1-q>(s-1)q-q/s^{n-2}$ (which means that $sq-q/s^{n-2}<1$) and finally

$$q<\frac{1}{s(1-1/s^{n-1})}. \tag{4.3}$$

From Bernoulli's inequality it follows that

$$\left(1-\frac{1}{s^{n-1}}\right)^n\geq 1-\frac{n}{s^{n-1}}$$

and, by (4.3), we have

$$q^n<\frac{1}{s^n\cdot(1-n/s^{n-1})}.$$

Consequently,

$$(s^n-2^{n-1})\cdot q^n<\frac{s^n\cdot(1-2^{n-1}/s^n)}{s^n\cdot(1-n/s^{n-1})}.$$

Obviously, for n greater then some n_0, $2^{n-1}/s>n$ and hence $2^{n-1}/s^n>n/s^{n-1}$ which proves (4.2). $\square$

Theorem 4.4. *If a finite subset $\Sigma\subset\mathbb{R}$ contains the set $\{a,a+1,b+1,c+1,b+|\Sigma|,c+|\Sigma|\}$ for some real numbers a, b, c with $b\neq c$, then there is a decreasing sequence $(q_n)_{n=1}^\infty$ tending to $1/|\Sigma|$ such that, for any $n\in\mathbb{N}$, the self-similar set $K(\Sigma,q_n)$ has Lebesgue measure zero.*

Proof. Let $s=|\Sigma|$ and for every n denote by q_n the unique positive solution of the equation (4.1) from Lemma 4.3. Let n_0 be a natural number such that

$$(s^n-2^{n-1})\cdot(q_n)^n<1\quad\text{for any } n>n_0.$$

Clearly $(q_n)_{n=n_0}^\infty$ is a decreasing sequence and $\lim_{n\to\infty}q_n=1/s$. It suffices to show that $K(\Sigma,q)$ has measure zero for $n>n_0$.

Taking into account that each q_n is a solution of (4.1), we conclude that

$$a + \sum_{i=1}^{n-1} (s-1+\varepsilon_i)(q_n)^i = (a+1) + \sum_{i=1}^{n-1} \varepsilon_i (q_n)^i$$

for any $\varepsilon_i \in \{b+1, c+1\} \subset \Sigma$. Therefore

$$\left| \sum_{i=1}^{n-1} (q_n)^i \Sigma \right| \leq s^n - 2^{n-1}.$$

Hence, by Lemma 4.3,

$$\left| \sum_{i=1}^{n-1} (q_n)^i \Sigma \right| \cdot (q_n)^n < 1.$$

and we can apply Theorem 4.1 to conclude that $K(\Sigma, q)$ has Lebesgue measure zero. □

The condition

$$(*) \qquad \{a, a+1, b+1, c+1, b+|\Sigma|, c+|\Sigma|\} \subset \Sigma$$

looks a bit artificial but it can be easily verified for many sumsets Σ of multigeometric sequences.

In particular, for the Guthrie–Nymann–Jones sequence of rank $m \geq 1$ $x_q = (3, 2, \ldots, 2; q)$, the sumset $\Sigma = \{0, 2, 3, \ldots, 2m+1, 2m+3\}$ has cardinality $|\Sigma| = 2m+2$. Observe that for the set Σ the condition $(*)$ holds for $a = 2$, $b = 1$ and $c = -1$. Because of that, Theorem 4.4 yields a sequence $(q_n)_{n=1}^{\infty} \searrow 1/(2m+2)$ such that for every $n \in \mathbb{N}$ the self-similar set $E(x_{q_n})$ is a Cantor sets of zero Lebesgue measure.

By [3], for $q = 1/(2m+2)$ the achievement set $E(x_q)$ is a Cantorval. Therefore, if $m > 2$, there are three ratios $p < q < r$ such that $E(x_p)$ and $E(x_r)$ are Cantor sets while $E(x_q)$ is a Cantorval. To the authors' best knowledge, this is the first result of this type for multigeometric sequences.

Now we will focus on Ferens-like sequences $x_q = (m+k, \ldots, k; q)$ where $m \geq k$.

For $k = 1$ the Ferens-like sequence $x_q = (m+1, \ldots, 2, 1; q)$ has

$$\Sigma = \{0, 1, 2, \ldots, (m+2)(m+1)/2\}.$$

The set $E(x_q)$ is a Cantor set (for $q < 1/|\Sigma|$) or an interval (for $q \geq 1/|\Sigma|$); see Theorem 7 in [3]), Theorem 1.1 or Theorem 2.1.

For $k = 2$, the "shortest" Ferens-like sequence is $x_q = (4, 3, 2; q)$. For this sequence

$$\Sigma = \{0, 2, 3, 4, 5, 6, 7, 9\}.$$

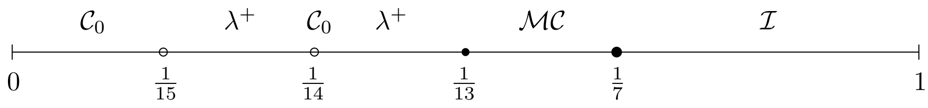
Note that the Guthrie–Nymann–Jones sequence $(3, 2, 2, 2; q)$ has the same Σ (see Example 3.9). It follows that $E(x_q)$ is a Cantor set for $q \in (0, 1/8)$ and $E(x_q)$ is a Cantorval for $q = 1/8$. By Theorem 2.1, $K(\Sigma; q)$ is an interval for

$q \geq I(\Sigma) = 2/11$ and a Cantorval for $q \in (1/6, 2/11)$. As shown in Example 3.9, for almost all $q \in (1/8, 1/6)$ the set $K(\Sigma; q)$ has positive Lebesgue measure. Using Theorem 4.4, we can find a decreasing sequence (q_n) tending to $1/8$ for which the sets $K(\Sigma; q_n)$ have zero Lebesgue measure.

For $k = 3$ the "shortest" Ferens-like sequence is $x_q = (6, 5, 4, 3; q)$. For this sequence $\Sigma = \{0, 3, \ldots, 15, 18\}$ and $|\Sigma| = 15$. Since $1 \in \Sigma/15$ the set $\Sigma_2 = \Sigma + \Sigma/15$ has less than $|15|^2$ elements (for example 4 can be presented as $4 + 0$ or as $3 + 1$). Therefore $|\Sigma_2|/15^2 < 1$ and for $q = 1/15$ the set $E(x_q)$ is a Cantor set according to Theorem 4.1. Moreover, calculating for $q = 1/14 > 1/15$ the cardinality

$$|\Sigma_3| = |\Sigma + q\Sigma + q^2\Sigma| = 2655 < 14^3$$

and applying Theorem 4.1, we conclude that the achievement set $E(x_q)$ is a Cantor set of zero Lebesgue measure for $q = 1/14$. On the other hand, Corollary 3.6 implies that for almost all $q \in (1/15, 1/(1 + \sqrt{18}))$ the achievement set $E(x_q)$ has positive Lebesgue measure. The set Σ has $i(\Sigma) = 1/13$ and $I(\Sigma) = 3/21 = 1/7$. So, in this case we have the diagram:



As in the previous case, we can use Theorem 4.4 (taking $a = b = 3$ and $c = -1$) and find a decreasing sequence (q_n) tending to $1/15$ such that all $E(x_{q_n})$ have zero Lebesgue measure.

Suppose now that $k > 3$. For the Ferens-like sequence $x_q = (k + m, \ldots, k + 1, k; q)$ its sumset Σ contains the number $|\Sigma|$, which implies that $|\Sigma + q\Sigma| < |\Sigma|^2$ for $q = \frac{1}{|\Sigma|}$ and therefore $E(x_q)$ is a Cantor set of zero measure according to Theorem 4.1.

5. Rational ratios

For a contraction ratio $q \in \{1/(n + 1) : n \in \mathbb{N}\}$ self-similar sets of positive Lebesgue measure can be characterized as follows:

Theorem 5.1. *Let $\Sigma \subset \mathbb{Z}$ be a finite set, $q \in \{1/(n + 1) : n \in \mathbb{N}\}$ and $\Sigma_n = \sum_{i=0}^{n-1} q^i \Sigma$ for $n \in \mathbb{N}$. For the compact set $K = K(\Sigma; q)$ the following conditions are equivalent*:

(a) $|\Sigma_n| \cdot q^n \geq 1$ *for all* $n \in \mathbb{N}$,
(b) $\inf_{n \in \mathbb{N}} |\Sigma_n| \cdot q^n > 0$,
(c) $\lambda(K) > 0$.

Proof. The implication (c)⇒(a) follows from Theorem 4.1 while (a)⇒(b) is trivial. It remains to prove (b)⇒(c). Suppose that $\lambda(K) = 0$. Given any $r > 0$ consider the r-neighbourhood $H(K, r) = \{h \in \mathbb{R} : \mathrm{dist}(h, K) < r\}$ of the set $K = K(\Sigma; q)$. Take any point $z \in \left\{ \sum_{i=n}^{\infty} x_i q^i : x_i \in \Sigma \text{ for all } i \geq n \right\}$ and observe that $\Sigma_n + z \subset K = \left\{ \sum_{i=0}^{\infty} x_i q^i : (x_i)_{i\in\omega} \in \Sigma^\omega \right\}$, which implies that $H(\Sigma_n + z, r) \subset H(K, r)$ for all $r > 0$. The continuity of the Lebesgue measure implies that $\lambda(H(K, r)) \to 0$ when r tends to zero. It follows from $\Sigma \subset \mathbb{Z}$ and $1/q \in \mathbb{N}$ that $\Sigma_n \subset q^{n-1} \cdot \mathbb{Z}$. Hence, for any two different points x and y from Σ_n, the distance between x and y is no less then $q^{n-1} > q^n$. Therefore, for any $n \in \mathbb{N}$,

$$|\Sigma_n| \cdot q^n = \lambda\big(H\big(\Sigma_n, q^n/2\big)\big) = \lambda\big(H\big(\Sigma_n + z, q^n/2\big)\big) \leq \lambda\big(H\big(K, q^n/2\big)\big)$$

which means that $\lim_{n\to\infty} |\Sigma_n| \cdot q^n = 0$. □

Theorem 5.1 combined with Corollary 2.3 of [13] implies the following corollary.

Corollary 5.2. *For a finite subset $\Sigma \subset \mathbb{Z}$ and the number $q = 1/|\Sigma| < 1$ the following conditions are equivalent*:

(a) $K(\Sigma; q)$ *has positive Lebesgue measure,*
(b) $K(\Sigma; q)$ *contains an interval,*
(c) *for every $n \in \mathbb{N}$ the set $\Sigma_n = \sum_{k=0}^{n-1} q^k \Sigma$ has cardinality $|\Sigma_n| = |\Sigma|^n$.*

Problem 5.3. Is it true that for a finite set $\Sigma \subset \mathbb{Z}$ and any (rational) $q \in (0, 1)$ the self-similar set $K(\Sigma; q)$ has positive Lebesgue measure if and only if it contains an interval?

Remark 5.4. According to [4], there exists a 10-element set Σ on the complex plane $\mathbb{C}$ such that for $q = 1/3$ the self-similar compact set $K(\Sigma; q) = \Sigma + qK(\Sigma; q) \subset \mathbb{C}$ has positive Lebesgue measure and empty interior in $\mathbb{C}$.

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