Topological **M**ethods in **N**onlinear **A**nalysis
Volume 46, No. 2, 2015, 835–865
DOI: 10.12775/TMNA.2015.070

ON NONHOMOGENEOUS BOUNDARY VALUE PROBLEM FOR THE STEADY NAVIER–STOKES SYSTEM IN DOMAIN WITH PARABOLOIDAL AND LAYER TYPE OUTLETS TO INFINITY

Kristina Kaulakytė

Abstract. The nonhomogeneous boundary value problem for the steady Navier–Stokes system is studied in a domain Ω with two layer type and one paraboloidal outlets to infinity. The boundary $\partial\Omega$ is multiply connected and consists of the outer boundary S and the inner boundary Γ. The boundary value $\mathbf{a}$ is assumed to have a compact support. The flux of $\mathbf{a}$ over the inner boundary Γ is supposed to be sufficiently small. We do not impose any restrictions on fluxes of $\mathbf{a}$ over the unbounded components of the outer boundary S. The existence of at least one weak solution is proved.

1. Introduction

In this paper we study the nonhomogeneous boundary value problem for the steady Navier–Stokes equations

$$\begin{cases} -\nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = 0 & \text{in } \Omega, \\ \operatorname{div}\,\mathbf{u} = 0 & \text{in } \Omega, \\ \mathbf{u} = \mathbf{a} & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

2010 *Mathematics Subject Classification.* 35Q30, 35J65, 76D03, 76D05.
Key words and phrases. Navier–Stokes equations, nonhomogeneous boundary value problem, layer type outlet, nonzero flux.
The research leading to these results has received funding from Lithuanian-Swiss cooperation programme to reduce economic and social disparities within the enlarged European Union under project agreement No. CH-3-SMM-01/01.

in a domain $\Omega \subset \mathbb{R}^3$ having layer type and paraboloidal outlets to infinity. Here the vector-valued function $\mathbf{u} = \mathbf{u}(x) = \big(u_1(x), u_2(x), u_3(x)\big)$ and the scalar function $p = p(x)$ are the unknown velocity field and the pressure of the fluid, while $\mathbf{a}(x) = (a_1(x), a_2(x), a_3(x))$ is the given boundary value; $\nu > 0$ is the constant coefficient of the viscosity.

In bounded domains Ω with the multiply connected boundaries $\partial\Omega$, consisting of N disjoint components Γ_j, problem (1.1) was studied first by J. Leray in 1933 (see [24]), and thereafter by many authors (see [1]–[9], [12], [15]–[19], [26], [25], [27]–[29], [44], [45], etc.). In case of a bounded domain Ω continuity equation (1.1_2) implies the necessary compatibility condition for the solvability of problem (1.1):

$$\int_{\partial\Omega} \mathbf{a}\cdot\mathbf{n}\,dS = \sum_{j=1}^{N}\int_{\Gamma_j} \mathbf{a}\cdot\mathbf{n}\,dS = 0, \tag{1.2}$$

where $\mathbf{n}$ is a unit vector of the outward (with respect to Ω) normal to $\partial\Omega$. However, for a long time the existence of a weak solution $\mathbf{u} \in W^{1,2}(\Omega)$ to problem (1.1) was proved either under the condition of zero fluxes

$$\mathcal{F}_j = \int_{\Gamma_j} \mathbf{a}\cdot\mathbf{n}\,dS = 0, \quad j = 1, \dots, N, \tag{1.3}$$

(e.g. [24], [19], [20], [45]), or assuming the fluxes $\mathcal{F}_j$ to be sufficiently small (e.g. [2], [5], [6], [9], [18]), or under the certain symmetry assumptions on the domain Ω and the boundary value $\mathbf{a}$ (e.g. [1], [7], [8], [26], [34]–[36], [16]), or assuming that the arbitrary large flux $\mathcal{F}$ has the “correct” sign (see [15]). Condition (1.3) requires the fluxes $\mathcal{F}_j$ of the boundary value $\mathbf{a}$ to be zero separately on each connected component Γ_j of the boundary $\partial\Omega$, while the compatibility condition (1.2) means only that the total flux is equal to zero. Obviously, condition (1.3) is stronger than (1.2), and (1.3) does not allow the presence of sinks and sources. In [24] J. Leray formulated a question whether problem (1.1) is solvable only under the necessary compatibility condition (1.2). In general case this so called *Leray's problem* was an open problem for 80 years. Fortunately, recently Leray's problem was solved for a 2-dimensional bounded multiply connected domain (see [17]).

In domains with noncompact boundaries problem (1.1) with the homogeneous boundary conditions was exhaustively studied during the last 35 years (e.g. [10], [13], [21]–[23], [33], [37]–[42]). However, not much is known about the nonhomogeneous boundary value problem (1.1) in the domain with noncompact boundaries. To the best of our knowledge problem (1.1) with nonhomogeneous boundary condition for the first time was solved without prescribing a “smallness condition” in 1999 in [30]. Later H. Fujita and H. Morimoto [25]–[27] studied problem (1.1) in the symmetric two-dimensional multiply connected domains Ω

with channel-like outlets to infinity containing a finite number of "holes". Assuming that the boundary value $\mathbf{a}$ is zero on the "outer" boundary and that $\mathbf{a}$ satisfies the symmetry assumptions on the bounded connected components of $\partial\Omega$, it is proved in [25]–[27] that problem (1.1) admits at least one solution which tends in every channel to a corresponding Poiseuille flow. Notice that the fluxes of Poiseuille flows are assumed to be sufficiently small. In 2010 J. Neustupa [31], [32] studied problem (1.1) in unbounded domains Ω with the multiply connected boundaries. He supposed that the fluxes of $\mathbf{a}$ over bounded components of the boundary are "small", but did not impose any conditions on the fluxes over the unbounded parts of $\partial\Omega$ (of course, the total flux is supposed to be equal to zero). In [32] the existence of at least one solution to (1.1) is proved assuming that the boundary value $\mathbf{a}$ admits a solenoidal extension $\mathbf{A}$ with $\mathbf{A} \in L^3(\Omega)$, $\nabla\mathbf{A} \in L^2(\Omega)$, and the found solutions have finite Dirichlet integrals. This imposes a restriction on the domain Ω: it should expand at infinity sufficiently rapidly, in order to have enough place to "transfer" a flux of the fluid from a bounded part of Ω to infinity.

Recently problem (1.1) was studied in a class of domains $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, having paraboloidal outlets to infinity (see [14]). Assuming, as in [32], that the fluxes of $\mathbf{a}$ over the bounded connected components of the inner boundary are sufficiently small, we do not impose any restrictions on the fluxes of the boundary value $\mathbf{a}$ over the noncompact connected components of the outer boundary. Under these conditions in [14] the existence of at least one weak solution to problem (1.1) which has, additionally, the prescribed fluxes over the cross-sections of outlets to infinity, was proved. This solution can have finite or infinite Dirichlet integral depending on the geometrical properties of the outlets. The proofs in [14] are based on a special construction of the extension $\mathbf{A}$ of the boundary value $\mathbf{a}$ which satisfies the Leray–Hopf inequality and allows to get the effective estimates of the solution.

In this paper the results obtained in [14] are extended to a class of the noncompact domains $\Omega \subset \mathbb{R}^3$ having paraboloidal and layer type outlets to infinity (see Subsection 2.2 for the exact definitions). Under the same assumptions as in [14] it is proved the existence of at least one weak solution of problem (1.1).

2. Preliminaries

2.1. Notation and function spaces. Let V be a Banach space. The norm of an element u in the function space V is denoted by $\|u\|_V$. Vector-valued functions are denoted by bold letters; spaces of scalar and vector-valued functions are not distinguished in notation. The vector-valued function $\mathbf{u} = (u_1, \dots, u_n)$ belongs to the space V, if $u_i \in V, i = 1, \dots, n$, and $\|\mathbf{u}\|_V = \sum\limits_{i=1}^{n} \|u_i\|_V$.

Let Ω be a domain in $\mathbb{R}^n$. As usual, denote by $C^\infty(\Omega)$ the set of all infinitely differentiable functions defined on Ω and let $C_0^\infty(\Omega)$ be the subset of all functions from $C^\infty(\Omega)$ with compact support in Ω. For a given nonnegative integer k and $q > 1$, $L^q(\Omega)$ and $W^{k,q}(\Omega)$ indicate the usual Lebesgue and Sobolev spaces; $W^{k-1/q,q}(\partial\Omega)$ is the trace space on $\partial\Omega$ of functions from $W^{k,q}(\Omega)$; $\mathring{W}^{k,q}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in the norm of $W^{k,q}(\Omega)$; we shall write $u \in W^{k,q}_{\mathrm{loc}}(\Omega)$ if $u \in W^{k,q}(\Omega')$ for any bounded subdomain Ω' with $\overline{\Omega'} \subset \overline{\Omega}$.

Let $D(\Omega)$ be the Hilbert space of vector functions formed as the closure of $C_0^\infty(\Omega)$ in the Dirichlet norm $\|\mathbf{u}\|_{D(\Omega)} = \|\nabla\mathbf{u}\|_{L^2(\Omega)}$ generated by the scalar product

$$(\mathbf{u}, \mathbf{v}) = \int_\Omega \nabla\mathbf{u} : \nabla\mathbf{v}\, dx,$$

where

$$\nabla\mathbf{u} : \nabla\mathbf{v} = \sum_{j=1}^n \nabla u_j \cdot \nabla v_j = \sum_{j=1}^n \sum_{k=1}^n \frac{\partial u_j}{\partial x_k}\frac{\partial v_j}{\partial x_k}.$$

Denote by $J_0^\infty(\Omega)$ the set of all solenoidal ($\operatorname{div}\mathbf{u} = 0$) vector fields $\mathbf{u}$ from $C_0^\infty(\Omega)$. By $\widehat{H}(\Omega)$ we indicate the subspace of $D(\Omega)$ consisting of solenoidal vector fields, and by $H(\Omega)$ – the space formed as the closure of $J_0^\infty(\Omega)$ in the Dirichlet norm. Obviously, $H(\Omega) \subset \widehat{H}(\Omega)$. In general, the spaces $\widehat{H}(\Omega)$ and $H(\Omega)$ do not coincide (see, for example, [10], [21], [42], [13], [38]). However, if Ω is a bounded domain with Lipschitz boundary, then $H(\Omega) = \widehat{H}(\Omega)$ (see [21]).

Let $\mathcal{M}$ be a closed set in $\mathbb{R}^n, n = 2, 3$. Denote by $\Delta_{\mathcal{M}}(x)$ a regularized distance from the point x to a set $\mathcal{M}$. Notice that $\Delta_{\mathcal{M}}(x)$ is an infinitely differentiable function in $\mathbb{R}^n \setminus \mathcal{M}$ and the following inequalities

$$a_1 d_{\mathcal{M}}(x) \le \Delta_{\mathcal{M}}(x) \le a_2 d_{\mathcal{M}}(x), \qquad |D^\alpha \Delta_{\mathcal{M}}(x)| \le a_3 d_{\mathcal{M}}^{1-|\alpha|}(x) \tag{2.1}$$

hold. Here $d_{\mathcal{M}} = dist(x, \mathcal{M})$ is the real distance from x to $\mathcal{M}$, the positive constants a_1, a_2 depend only on the dimension n, while a_3 depends on n and on the order of differentiation $|\alpha|$ (see [43]).

2.2. Domains with outlets to infinity. Let $\Omega \subset \mathbb{R}^3$ be an unbounded domain which splits outside the ball $B_{R_0}(0) = \{x \in \mathbb{R}^3 : |x| < R_0\}$ into three noncompact disjoint components ([1]), i.e.

$$\Omega = \Omega_0 \cup D_1 \cup D_2 \cup D_3,$$

where $\Omega_0 = \Omega \cap B_{R_0}(0)$ and the unbounded components D_1, D_2 are layer type outlets to infinity, while D_3 is a paraboloidal outlet.

([1]) In order not to loose the main idea in the technical details, we take a domain with three outlets to infinity. In general we can take a finite number of outlets to infinity.

We assume that the outlet D_3 is connected to the layer D_2. Layers D_1, D_2 are connected to each other by a finite cylinder which we denote $H \subset \Omega_0$ (see Figure 1). Outlets D_1, D_2, D_3 have the forms (^{2}):

$$\begin{aligned} D_i &= \{z^{(i)} \in \mathbb{R}^3 : 0 < z_3^{(i)} < g_i(|z^{(i)\prime}|)\ |z^{(i)\prime}| > 1\}, \quad i = 1,2, \\ D_3 &= \{z^{(3)} \in \mathbb{R}^3 : |z^{(3)\prime}| < g_3(z_3^{(3)}),\ z_3^{(3)} > 1\}, \end{aligned}$$

where the functions $g_i(t)$, $i = 1,2$, possess the following properties:

$$\mu_1 g_i(t) \le \max_{t \le t_1 \le 2t} g_i(t_1) \le \mu_2 g_i(t), \quad g_i(t) \ge 1, \text{ for all } t,$$

with the certain positive constants μ_1, μ_2,

$$|g_i(t_1) - g_i(t_2)| \le L_i(t)|t_1 - t_2|, \quad t_1, t_2 \in [t, 2t],$$

and for $L_i(t)$ holds the inequality

$$\frac{L_i(t) \cdot t}{g_i(t)} \le \text{const}, \quad L_i(t) \le \text{const} \quad \text{for all } t,\ i = 1,2;$$

the function $g_3(t)$ satisfies the Lipschitz condition

$$|g_3(t_1) - g_3(t_2)| \le L_3|t_1 - t_2|, \quad t_1, t_2 \ge 1, \quad g_3(t) \ge 1 \quad \text{for all } t.$$

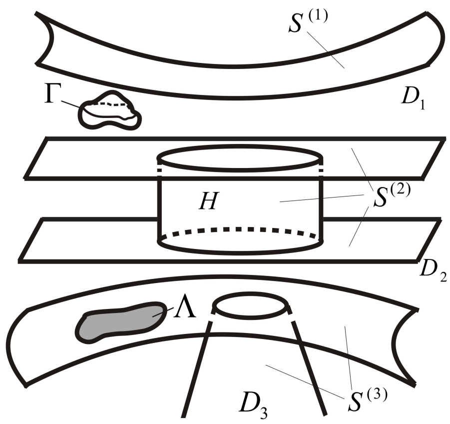

FIGURE 1. Domain Ω.

We assume that:

(i) The boundary $\partial\Omega$ is Lipschitz.
(ii) The bounded domain Ω_0 has the form $\Omega_0 = G_0 \setminus G$, where G_0 and G are bounded simply connected domains such that $\overline{G} \subset G_0$.

(^{2}) Note that $z^{(i)}$ means the local coordinate system in the outlet D_i, while x is the global system.

(iii) The outer boundary $S = \partial\Omega \setminus \Gamma$, $\Gamma = \partial G$, consists of three disjoint unbounded connected components:

$$\begin{aligned} S^{(1)} &= \big\{z^{(1)} \in \partial\Omega : z_3^{(1)} = g_1(|z^{(1)\prime}|)\big\},\\ S^{(2)} &= \big\{z^{(1)} \in \partial\Omega : z_3^{(1)} = 0,\ |z^{(1)\prime}| > 1\big\} \cup \partial H^* \cup \Upsilon^{(1)}\\ &\quad \cup \big\{z^{(2)} \in \partial\Omega : z_3^{(2)} = 0,\ |z^{(2)\prime}| > 1\big\} \cup \Upsilon^{(2)},\\ S^{(3)} &= \big\{z^{(2)} \in \partial\Omega : z_3 = g_2(|z^{(2)\prime}|),\ |z^{(2)\prime}| > 1\big\} \cup \Upsilon^{(3)} \cup \partial D_3^*, \end{aligned}$$

where ∂H^* is a lateral surface of the cylinder H and ∂D_3^* is a lateral surface of the paraboloidal outlet D_3, $\Upsilon^{(1)}$ is a surface connecting ∂D_1 and ∂H^*, $\Upsilon^{(2)}$ is a surface connecting ∂D_2 and ∂H^*, $\Upsilon^{(3)}$ is a surface connecting ∂D_2 and ∂D_3^*.

Below we will use the following notation:

$$\begin{aligned} D_i(\tau_i) &= \big\{z^{(i)} \in D_i : |z^{(i)\prime}| < \tau_i\big\}, \quad i = 1,2,\\ D_3(\tau_3) &= \big\{z^{(3)} \in D_3 : z_3^{(3)} < \tau_3\big\},\\ \Omega(\tau_1,\tau_2,\tau_3) &= \Omega_0 \cup D_1(\tau_1) \cup D_2(\tau_2) \cup D_3(\tau_3),\\ \omega_i(\tau) &= \big\{z^{(i)} \in D_i : \tau < |z^{(i)\prime}| < 2\tau\big\}, \quad i = 1,2,\\ \omega_3(\tau) &= \left\{z^{(3)} \in D_3 : \tau - \frac{g_3(\tau)}{2L_3} < z_3^{(3)} < \tau\right\},\\ \sigma_i(\tau) &= D_i \cap \big\{z^{(i)} : |z^{(i)\prime}| = \tau\big\}, \quad i = 1,2,\\ \sigma_3(\tau) &= D_3 \cap \big\{z^{(3)} : z_3^{(3)} = \tau\big\}, \end{aligned}$$

i.e. $\sigma_j \subset \mathbb{R}^2$ are the cross-sections of the outlets D_j, $j = 1,2,3$.

In order to prove the existence of at least one weak solution, we use the methods proposed in [33], [23], [40]. Following these methods, we have to select a family of the bounded domains $\Omega(t)$ such that $\Omega(t)$ exhausts the domain Ω as $t \to \infty$. Such a family of the domains can be taken in the following form

$$\Omega(t) = \Omega(2h_1(t), 2h_2(t), h_3(t)),$$

where $h_j(t)$, $j = 1,2,3$, are the functions inverse to

$$\chi_i(\tau) = \int_1^\tau \frac{dr}{r\, g_i^{1/3}(r)}, \quad i = 1,2, \qquad \chi_3(\tau) = \int_1^\tau \frac{dr}{g_3^{4/3}(r)},$$

so that

$$t = \int_1^{h_i(t)} \frac{dr}{r\, g_i^{1/3}(r)}, \quad i = 1,2, \qquad t = \int_1^{h_3(t)} \frac{dr}{g_3^{4/3}(r)}.$$

If the integrals

$$\int_1^{2h_i(t)} \frac{dr}{r\, g_i^3(r)}, \quad i = 1,2, \quad \text{and} \quad \int_1^{h_3(t)} \frac{dr}{g_3^4(r)}$$

diverge, then the functions $\chi_j(\tau)$, $h_j(t)$, $j=1,2,3$, increase monotonically and tend to infinity as t, τ run through the interval $[1,\infty)$. Moreover, the following relations

$$h_i'(t) = h_i(t)\cdot g_i^{1/3}(h_i(t)), \quad i=1,2, \qquad h_3'(t) = g_3^{4/3}(h_3(t)) \tag{2.2}$$

hold.

Remark 2.1. In case that the integrals

$$\int_1^{2h_i(t)} \frac{dr}{r\,g_i^3(r)}, \quad i=1,2, \quad \text{and} \quad \int_1^{h_3(t)} \frac{dr}{g_3^4(r)}$$

are bounded, we get the existence of at least one weak solution from the Leray–Schauder Theorem and we do not need to control the corresponding Dirichlet integral (see the proof of existence).

2.3. Formulation of the problem. We consider the following problem

$$\begin{cases} -\nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = 0 & \text{in } \Omega, \\ \operatorname{div}\mathbf{u} = 0 & \text{in } \Omega, \\ \mathbf{u} = \mathbf{a} & \text{on } \partial\Omega, \\ \displaystyle\int_{\sigma_j(t)} \mathbf{u}\cdot\mathbf{n}\,dS = \mathcal{F}_j & \text{for } j=1,2,3, \end{cases} \tag{2.3}$$

where $\mathbf{n}$ is the unit vector of the normal to σ_j, $j=1,2,3$.

We suppose that the boundary value $\mathbf{a}\in W^{1/2,2}(\partial\Omega)$ has a compact support (3) and $\Lambda = \operatorname{supp}\mathbf{a}\cap S \subset S^{(3)}$ (see Figure 1). Let

$$\int_\Gamma \mathbf{a}\cdot\mathbf{n}\,dS = \mathbb{F}^{(\mathrm{inn})}, \qquad \int_\Lambda \mathbf{a}\cdot\mathbf{n}\,dS = \mathfrak{F}^{(\mathrm{out})}$$

be the fluxes of the boundary value $\mathbf{a}$ over the inner and the outer boundaries, respectively. Then the necessary flux compatibility condition can be written as

$$\mathbb{F}^{(\mathrm{inn})} + \mathfrak{F}^{(\mathrm{out})} + \mathcal{F}_1 + \mathcal{F}_2 + \mathcal{F}_3 = 0 \tag{2.4}$$

(the total flux is equal to zero).

The main purpose of the paper is to construct an appropriate extension of the boundary data which gives the possibility to reduce the nonhomogeneous boundary conditions to the homogeneous ones. This extension is constructed as the sum

$$\mathbf{A} = \mathbf{B}^{(\mathrm{inn})} + \mathbf{B}^{(\mathrm{out})} + \mathbf{B}^{(\mathrm{flux})},$$

where $\mathbf{B}^{(\mathrm{inn})}$ extends the boundary value $\mathbf{a}$ from the inner boundary Γ, $\mathbf{B}^{(\mathrm{out})}$ extends $\mathbf{a}$ from the connected component $S^{(3)}$ of the noncompact outer boundary S, and $\mathbf{B}^{(\mathrm{flux})}$ has zero boundary value and "removes" the fluxes over the

(3) As in the paper [14], this assumption is made to insure that the flux $\mathfrak{F}^{(\mathrm{out})}$ of the boundary value $\mathbf{a}$ over the unbounded parts of the boundary has sense.

cross-sections σ_j, $j = 1, 2, 3$. The vector fields $\mathbf{B}^{(\mathrm{out})}$ and $\mathbf{B}^{(\mathrm{flux})}$ are constructed to satisfy so called Leray–Hopf inequality (see (2.8) below) which allows to obtain the a priori estimates of the solution for the arbitrary large fluxes $\mathfrak{F}^{(\mathrm{out})}$ and $\mathcal{F}_j$, $j = 1, 2, 3$. The construction of the vector fields $\mathbf{B}^{(\mathrm{out})}$ and $\mathbf{B}^{(\mathrm{flux})}$ is based on the methods proposed in [22], [42], [38] (see also [14] where such extensions are constructed in the domains having only paraboloidal outlets to infinity). Notice that in general Leray–Hopf inequality cannot be true for the vector field $\mathbf{B}^{(\mathrm{inn})}$. There are counterexamples of a bounded domain (see [44], [11], [3]), showing that for nonzero fluxes of the boundary value $\mathbf{a}$ over the connected components of the boundary Leray–Hopf inequality may be false whatever the choice of the solenoidal extension is taken. Therefore, we suppose that the flux $\mathbb{F}^{(\mathrm{inn})}$ of $\mathbf{a}$ over the inner boundary Γ is "sufficiently small". After the extension $\mathbf{A}$ with the above properties is constructed, the proof of the existence of a weak solution to problem (1.1) is based on the methods developed in [23], [33]. The construction of a suitable extension depends on the form of the outlet to infinity to which we "drain" the fluxes. In this paper we analyse in details the case, when we "transport" the fluxes from the bounded parts of $\partial\Omega$ to the layer-type outlet to infinity.

2.4. Weak solutions. The weak solution of problem (2.3) is a solenoidal vector field $\mathbf{u} \in W^{1,2}_{\mathrm{loc}}(\Omega)$ satisfying the boundary condition $\mathbf{u}|_{\partial\Omega} = \mathbf{a}$, the flux conditions

$$\int_{\sigma_j(t)} \mathbf{u}\cdot\mathbf{n}\, dS = \mathcal{F}_j, \quad j = 1, 2, 3, \tag{2.5}$$

and the integral identity

$$\nu\int_\Omega \nabla\mathbf{u} : \nabla\boldsymbol{\eta}\, dx - \int_\Omega (\mathbf{u}\cdot\nabla)\boldsymbol{\eta}\cdot\mathbf{u}\, dx = 0 \quad \text{for all } \boldsymbol{\eta} \in J_0^\infty(\Omega). \tag{2.6}$$

Assume that the necessary compatibility condition (2.4) is valid. Let $\mathbf{A} \in W^{1,2}_{\mathrm{loc}}(\Omega)$ be a solenoidal extension of the boundary value $\mathbf{a}$ satisfying the flux conditions (2.5):

$$\operatorname{div}\mathbf{A} = 0, \quad \mathbf{A}|_{\partial\Omega} = \mathbf{a}, \quad \int_{\sigma_j(t)} \mathbf{A}\cdot\mathbf{n}\, dS = \mathcal{F}_j, \quad j = 1, 2, 3.$$

We reduce problem (2.6) to the problem with the homogeneous boundary conditions and zero fluxes. After substituting $\mathbf{u} = \mathbf{v} + \mathbf{A}$ into (2.6), we look for the new unknown velocity field $\mathbf{v} \in W^{1,2}_{\mathrm{loc}}(\Omega)$ satisfying the conditions

$$\operatorname{div}\mathbf{v} = 0, \quad \mathbf{v}|_{\partial\Omega} = 0, \quad \int_{\sigma_j(t)} \mathbf{v}\cdot\mathbf{n}\, dS = 0, \quad j = 1, 2, 3,$$

and the integral identity

$$\nu\int_\Omega \nabla\mathbf{v} : \nabla\boldsymbol{\eta}\, dx - \int_\Omega (\mathbf{v}\cdot\nabla)\boldsymbol{\eta}\cdot\mathbf{v}\, dx - \int_\Omega (\mathbf{A}\cdot\nabla)\boldsymbol{\eta}\cdot\mathbf{v}\, dx \tag{2.7}$$

$$-\int_\Omega (\mathbf{v}\cdot\nabla)\boldsymbol{\eta}\cdot\mathbf{A}\,dx = \int_\Omega (\mathbf{A}\cdot\nabla)\boldsymbol{\eta}\cdot\mathbf{A}\,dx - \nu\int_\Omega \nabla\mathbf{A} : \nabla\boldsymbol{\eta}\,dx,$$

for all $\boldsymbol{\eta}\in J_0^\infty(\Omega)$. We construct the extension $\mathbf{A}$ which satisfies Leray–Hopf type inequalities

$$\begin{aligned}\int_{\Omega(t)} |\mathbf{A}|^2|\mathbf{w}|^2\,dx &\le \varepsilon\int_{\Omega(t)} |\nabla\mathbf{w}|^2\,dx,\\ \int_{\omega_j(h_j(t))} |\mathbf{A}|^2|\mathbf{w}|^2\,dx &\le \varepsilon\int_{\omega_j(h_j(t))} |\nabla\mathbf{w}|^2\,dx,\end{aligned}\tag{2.8}$$

where $\mathbf{w}\in W^{1,2}_{\mathrm{loc}}(\Omega)$ is an arbitrary solenoidal function with $\mathbf{w}|_{\partial\Omega}=0$ and ε can be chosen arbitrary small.

3. Construction of the extension of the boundary value

3.1. Construction of the extension $\mathbf{B}^{(\mathrm{inn})}$. We start with the construction of the “virtual drain” function $\mathbf{b}^{(\mathrm{inn})}$ which “transforms” the flux $\mathbb{F}^{(\mathrm{inn})}$ from the inner boundary Γ to infinity. Constructing the vector field $\mathbf{b}^{(\mathrm{inn})}$ (also $\mathbf{b}^{(\mathrm{out})}$ in the next subsection) one can arbitrary choose the outlet where the virtual drain function has nonzero flux. Generally speaking, choosing different outlets, different solutions of problem (2.3) may be obtained (a solution is unique only for small data). A solenoidal vector field with nonzero flux can have finite Dirichlet integral over the outlet only if it is “sufficiently wide” (see [33], [42]). Moreover, if the Dirichlet integral is infinite, growth of it over $D_1(\tau)$ depends on how fast the outlet D_1 is expanding at infinity. Therefore, constructing $\mathbf{b}^{(\mathrm{inn})}$ (and $\mathbf{b}^{(\mathrm{out})}$), we “drain” the flux to the “widest” outlet in order to minimize the dissipation of energy (Dirichlet integral). In this paper we suppose that such an outlet is of the layer type, say D_1.

First, we construct in D_1 a solenoidal vector field $\mathbf{b}_1^{(\mathrm{inn})}$ such that

$$\mathbf{b}_1^{(\mathrm{inn})}(x)|_{\partial D_1\cap\partial\Omega}=0,\qquad \int_{\sigma_1(t)}\mathbf{b}_1^{(\mathrm{inn})}\cdot\mathbf{n}\,dS=\mathbb{F}^{(\mathrm{inn})}.$$

Introduce the infinite layer $\mathbb{L}=\{y\in\mathbb{R}^3 : 0<y_3<g_1(|y'|),\ y'\in\mathbb{R}^2\}$ which for $|y'|>1$ coincides with the outlet D_1. Let $\gamma_1=\{y\in\mathbb{L} : |y'|=0\}$ (i.e. one can take $y=z^{(1)}$). Define in $\mathbb{L}$ a cut–off function

$$\zeta_1(y)=\Psi\Bigg(\ln\left(\frac{\rho(\delta(y))}{\Delta(y)}\right)\Bigg),\tag{3.1}$$

where $\delta(y)=\Delta_{\gamma_1\cup\{y_3=g_1(|y'|)\}}(y)$, $\Delta(y)=\Delta_{\partial\mathbb{L}\setminus\{y_3=g_1(|y'|)\}}(y)$, Ψ is a smooth monotone function, $0\le\Psi(t)\le 1$,

$$\Psi(t)=\begin{cases}0 & \text{for } t\le 0,\\ 1 & \text{for } t\ge 1,\end{cases}\tag{3.2}$$

$\rho(\tau)$ is a smooth monotone function with $\rho(\tau) = a_1 d_0/2$ for $\tau \le a_2 d_0/2$, $\rho(\tau) = \tau$ for $\tau \ge a_2 d_0$, where a_1, a_2 are the constants from inequality (2.1), d_0 is a small positive number.

LEMMA 3.1. *The function $\zeta_1(y)$ is equal to zero at those points of $\mathbb{L}\setminus\gamma_1$ where $\rho(\delta(y)) \le \Delta(y)$ and $\zeta_1(y) = 1$ if $\Delta(y) \le e^{-1}\rho(\delta(y))$. The following estimates*

$$\Big|\frac{\partial \zeta_1(y)}{\partial y_k}\Big| \le \frac{c}{\Delta(y)}, \qquad \Big|\frac{\partial^2 \zeta_1(y)}{\partial y_k \partial y_l}\Big| \le \frac{c}{\Delta^2(y)} \tag{3.3}$$

hold.

PROOF. The proof follows directly from the definition of the function ζ_1, the properties of the regularized distance and the fact that supp$\nabla\zeta_1$ is contained in the set where $\Delta(y) \le \rho(\delta(y))$ (see Lemma 2 in [42] for the details). □

Define

$$\widehat{\mathbf{b}}_1^{(\mathrm{inn})}(y) = \mathbb{F}^{(\mathrm{inn})}\mathrm{curl}\,(\zeta_1(y)\mathbf{b}_0(y)) = \mathbb{F}^{(\mathrm{inn})}\nabla\zeta_1(y) \times \mathbf{b}_0(y), \quad y \in \mathbb{L},$$

where

$$\mathbf{b}_0(y) = \frac{1}{2\pi}\Big(-\frac{y_2}{|y'|^2}, \frac{y_1}{|y'|^2}, 0\Big).$$

LEMMA 3.2. *The solenoidal vector field $\widehat{\mathbf{b}}_1^{(\mathrm{inn})}$ is infinitely differentiable for $y \in \mathbb{L} \setminus \{y : |y'| = 0\}$, vanishes near the set $\partial\mathbb{L} \cup \{y : |y'| = 0\}$ and satisfies the conditions*:

$$\int_{\sigma_1^{\mathbb{L}}} \widehat{\mathbf{b}}_1^{(\mathrm{inn})} \cdot \mathbf{n}\, dS = \mathbb{F}^{(\mathrm{inn})}, \tag{3.4}$$

$$|\widehat{\mathbf{b}}_1^{(\mathrm{inn})}(y)| \le \frac{c|\mathbb{F}^{(\mathrm{inn})}|}{d(y)}, \tag{3.5}$$

$$|\widehat{\mathbf{b}}_1^{(\mathrm{inn})}(y)| \le \frac{C|\mathbb{F}^{(\mathrm{inn})}|}{g_1(|y'|)\,|y'|}, \tag{3.6}$$

$$|\nabla\widehat{\mathbf{b}}_1^{(\mathrm{inn})}(y)| \le \frac{C|\mathbb{F}^{(\mathrm{inn})}|}{g_1^2(|y'|)\,|y'|} + \frac{C|\mathbb{F}^{(\mathrm{inn})}|}{g_1(|y'|)\,|y'|^2}, \quad y \in \mathbb{L} \setminus \Omega_0. \tag{3.7}$$

Here $d(y) = \mathrm{dist}(y, \partial\mathbb{L} \cap \partial\Omega \setminus \{y \in \partial\Omega : y_3 = g_1(|y'|)\})$.

PROOF. Since

$$(\nabla\zeta_1 \times \mathbf{b}_0) \cdot \mathbf{n}|_{\sigma_1^{\mathbb{L}}(|y'|)} = -\frac{1}{2\pi} \cdot \frac{\partial\zeta_1}{\partial y_3} \cdot \frac{1}{|y'|},$$

we get

$$\int_{\sigma_1^{\mathbb{L}}(t)} \widehat{\mathbf{b}}_1^{(\mathrm{inn})} \cdot \mathbf{n}\, dS = \mathbb{F}^{(\mathrm{inn})} \int_{\sigma_1^{\mathbb{L}}(t)} (\nabla\zeta_1 \times \mathbf{b}_0) \cdot \mathbf{n}\, dS$$
$$= -\mathbb{F}^{(\mathrm{inn})} \frac{t}{2\pi} \int_0^{2\pi} d\phi \int_0^{g_1(t)} \frac{\partial\zeta_1}{\partial y_3} \cdot \frac{1}{t}\, dy_3 = -\mathbb{F}^{(\mathrm{inn})} \int_0^{g_1(t)} \frac{\partial\zeta_1}{\partial y_3}\, dy_3$$

$$= -\mathbb{F}^{(\mathrm{inn})}(\zeta_1(y_1, y_2, g_1(t)) - \zeta_1(y_1, y_2, 0)) = \mathbb{F}^{(\mathrm{inn})},$$

where $\sigma_1^{\mathbb{L}}(t)$ is a cross section of the layer $\mathbb{L}$. From the definition of $\widehat{\mathbf{b}}_1^{(\mathrm{inn})}$ and (3.3) it follows that

$$\begin{aligned}
|\widehat{\mathbf{b}}_1^{(\mathrm{inn})}(y)| &\le |\mathbb{F}^{(\mathrm{inn})}||\nabla\zeta_1(y)||\mathbf{b}_0(y)| \le \frac{c|\mathbb{F}^{(\mathrm{inn})}|}{\Delta(y)\,|y'|},\\
|\nabla\widehat{\mathbf{b}}_1^{(\mathrm{inn})}(y)| &\le |\mathbb{F}^{(\mathrm{inn})}|(|\nabla(\nabla\zeta_1(y))||\mathbf{b}_0(y)| + |\nabla\zeta_1(y)||\nabla\mathbf{b}_0(y)|)\\
&\le c|\mathbb{F}^{(\mathrm{inn})}|\left(\frac{1}{\Delta^2(y)|y'|} + \frac{1}{|\Delta(y)||y'|^2}\right).
\end{aligned} \tag{3.8}$$

It is easy to see that for the points $y \in \operatorname{supp}\widehat{\mathbf{b}}_1^{(\mathrm{inn})}$ the inequalities

$$c_1 g_1(|y'|) \le \Delta(y) \le c_2 g_1(|y'|) \tag{3.9}$$

hold (see [33] for the details). Therefore, estimates (3.5)–(3.7) follow from (3.8), (3.9). □

Define

$$\mathbf{b}_1^{(\mathrm{inn})}(z^{(1)}) = \widehat{\mathbf{b}}_1^{(\mathrm{inn})}(z^{(1)})|_{D_1}. \tag{3.10}$$

LEMMA 3.3. *For any vector field* $\mathbf{w} \in W^{1,2}_{\mathrm{loc}}(D_1)$ *with* $\mathbf{w}|_{\partial D_1 \cap \partial\Omega} = 0$ *the following inequalities*:

$$\begin{aligned}
\int_{\Omega(t)} |\mathbf{b}_1^{(\mathrm{inn})}|^2|\mathbf{w}|^2\,dx &\le c\,|\mathbb{F}^{(\mathrm{inn})}|^2 \int_{\Omega(t)} |\nabla\mathbf{w}|^2\,dx,\\
\int_{\omega_1(h_1(t))} |\mathbf{b}_1^{(\mathrm{inn})}|^2|\mathbf{w}|^2\,dx &\le c\,|\mathbb{F}^{(\mathrm{inn})}|^2 \int_{\omega_1(h_1(t))} |\nabla\mathbf{w}|^2\,dx
\end{aligned} \tag{3.11}$$

hold. The constant c is independent of t.

PROOF. It is well known (see [20]) that the following inequality

$$\int_\Pi \frac{|\mathbf{w}|^2\,dx}{\operatorname{dist}^2(x, \mathcal{L})} \le c\int_\Pi |\nabla\mathbf{w}|^2\,dx \tag{3.12}$$

holds in a bounded domain Π and for any $\mathbf{w} \in W^{1,2}(\Pi)$ with $\mathbf{w}|_{\mathcal{L}} = 0$, where $\mathcal{L} \subseteq \partial\Pi$ has a positive surface measure. Therefore, estimates (3.11) follow from (3.5) and (3.12). For the detailed proof see [33] and [42]. □

Let us briefly describe the construction of the virtual drain function, which "removes" non-zero flux from inner component Γ (this construction is similar to that of the paper [14]). Let $x^{(1)} \in G$, be the point lying inside the "hole" G. Denote $q_1(x) = q(x - x^{(1)})$, where $q(x) = 1/(4\pi|x|)$ is the fundamental solution of the Laplace operator in $\mathbb{R}^3$, and let

$$\mathbf{b}_\sharp^{(\mathrm{inn})}(x) = \mathbb{F}^{(\mathrm{inn})}\nabla q_1(x).$$

Then

$$\operatorname{div} \mathbf{b}_{\sharp}^{(\mathrm{inn})} = 0, \qquad \int_{\Gamma} \mathbf{b}_{\sharp}^{(\mathrm{inn})} \cdot \mathbf{n}\, dS = \mathbb{F}^{(\mathrm{inn})}, \qquad \int_{\partial\Omega_0} \mathbf{b}_{\sharp}^{(\mathrm{inn})} \cdot \mathbf{n}\, dS = -\mathbb{F}^{(\mathrm{inn})}$$

Set

$$\boldsymbol{\beta}_1 = \begin{cases} 0 & \text{for } x \in \Gamma, \\ \mathbf{b}_1^{(\mathrm{inn})}|_{\partial\Omega_0 \cap \overline{D}_1} + \mathbf{b}_{\sharp}^{(\mathrm{inn})}|_{\partial\Omega_0 \cap \overline{D}_1} & \text{for } x \in \partial\Omega_0 \cap \overline{D}_1, \\ \mathbf{b}_{\sharp}^{(\mathrm{inn})}|_{\partial\Omega_0 \setminus \overline{D}_1} & \text{for } x \in \partial\Omega_0 \setminus (\overline{D}_1 \cup \Gamma). \end{cases}$$

We have

$$\int_{\partial\Omega_0} \boldsymbol{\beta}_1 \cdot \mathbf{n}\, dS = \int_{\partial\Omega_0 \cap \overline{D}_1} \mathbf{b}_1^{(\mathrm{inn})} \cdot \mathbf{n}\, dS + \int_{\partial\Omega_0} \mathbf{b}_{\sharp}^{(\mathrm{inn})} \cdot \mathbf{n}\, dS = \mathbb{F}^{(\mathrm{inn})} - \mathbb{F}^{(\mathrm{inn})} = 0.$$

Therefore, the function $\boldsymbol{\beta}_1$ can be extended inside the domain Ω_0 as a solenoidal vector field $\mathbf{b}_{01}^{(\mathrm{inn})} \in W^{1,2}(\Omega_0)$ and

$$\begin{aligned} \|\mathbf{b}_{01}^{(\mathrm{inn})}\|_{W^{1,2}(\Omega_0)} &\le c\|\boldsymbol{\beta}_1\|_{W^{1/2,2}(\partial\Omega_0)} \\ &\le c(\|\mathbf{b}_{\sharp}^{(\mathrm{inn})}\|_{W^{1/2,2}(\partial\Omega_0)} + \|\mathbf{b}_1^{(\mathrm{inn})}\|_{W^{1/2,2}(\partial\Omega_0 \cap \overline{D}_1)}) \le c|\mathbb{F}^{(\mathrm{inn})}|, \end{aligned}$$

where the constant c depends only on the domain Ω_0 (see, for example, [20], [21]). Now, define the virtual drain function

$$\mathbf{b}^{(\mathrm{inn})} = \begin{cases} \mathbf{b}_{\sharp}^{(\mathrm{inn})} + \mathbf{b}_{01}^{(\mathrm{inn})} & \text{for } x \in \Omega_0, \\ \mathbf{b}_1^{(\mathrm{inn})} & \text{for } x \in D_1, \\ 0 & \text{for } x \in D_2, D_3. \end{cases}$$

Set

$$\boldsymbol{\beta}_0 = \begin{cases} \mathbf{a} - \mathbf{b}_{\sharp}^{(\mathrm{inn})}|_{\Gamma} & \text{for } x \in \Gamma, \\ 0 & \text{for } x \in \partial\Omega_0 \setminus \Gamma. \end{cases}$$

Then

$$\int_{\Gamma} \boldsymbol{\beta}_0 \cdot \mathbf{n}\, dS = \int_{\Gamma} \mathbf{a} \cdot \mathbf{n}\, dS - \int_{\Gamma} \mathbf{b}_{\sharp}^{(\mathrm{inn})} \cdot \mathbf{n}\, dS = \mathbb{F}^{(\mathrm{inn})} - \mathbb{F}^{(\mathrm{inn})} = 0,$$

and, therefore, the function $\boldsymbol{\beta}_0$ can be extended inside Ω_0 in the form (see [20])

$$\mathbf{b}_0^{(\mathrm{inn})}(x) = \operatorname{curl}(\chi(x)\mathbf{E}(x)),$$

where $\mathbf{E} \in W^{2,2}(\Omega_0)$, $\operatorname{curl} \mathbf{E}|_{\partial\Omega_0} = \boldsymbol{\beta}_0$ and χ is a smooth cut-off function with $\chi(x) = 1$ on Γ, $\operatorname{supp} \chi$ is contained in a small neighbourhood of Γ and

$$|\nabla \chi_m(x)| \le \frac{c}{\operatorname{dist}(x, \Gamma)}.$$

Moreover, for any $\mathbf{w} \in W^{1,2}_{\mathrm{loc}}(\Omega)$ with $\mathbf{w}|_{\partial\Omega} = 0$ the following estimate

$$\int_{\Omega_0} |\mathbf{b}_0^{(\mathrm{inn})}(x)|^2 |\mathbf{w}(x)|^2\, dx \le c|\mathbb{F}^{(\mathrm{inn})}|^2 \int_{\Omega_0} |\nabla \mathbf{w}(x)|^2\, dx \tag{3.13}$$

holds. Finally, we put

$$\mathbf{B}^{(\mathrm{inn})} = \mathbf{b}^{(\mathrm{inn})} + \mathbf{b}_0^{(\mathrm{inn})}. \tag{3.14}$$

Lemma 3.4. *The vector field* $\mathbf{B}^{(\mathrm{inn})}$ *is solenoidal,* $\mathbf{B}^{(\mathrm{inn})}|_\Gamma = \mathbf{a}|_\Gamma$, $\mathbf{B}^{(\mathrm{inn})}|_{S^{(m)}} = 0$, $m = 1,2,3$, $\mathbf{B}^{(\mathrm{inn})} \in W^{1,2}_{\mathrm{loc}}(\Omega)$, $\mathbf{B}^{(\mathrm{inn})}(x) = 0$, $x \in D_2$, $x \in D_3$, $|x| \gg 1$. *For any* $\mathbf{w} \in W^{1,2}_{\mathrm{loc}}(\Omega)$ *with* $\mathbf{w}|_{\partial\Omega} = 0$ *the following estimates*:

$$\begin{aligned} \int_{\omega_j(h_j(t))} |\mathbf{B}^{(\mathrm{inn})}|^2|\mathbf{w}|^2\,dx \;&\le c\,|\mathbb{F}^{(\mathrm{inn})}|^2 \int_{\omega_j(h_j(t))} |\nabla\mathbf{w}|^2\,dx,\\ \int_{\Omega(t)} |\mathbf{B}^{(\mathrm{inn})}|^2|\mathbf{w}|^2\,dx \;&\le c\,|\mathbb{F}^{(\mathrm{inn})}|^2 \int_{\Omega(t)} |\nabla\mathbf{w}|^2\,dx, \end{aligned} \tag{3.15}$$

hold. Moreover,

$$\begin{aligned} |\mathbf{B}^{(\mathrm{inn})}(x)| &\le \frac{C|\mathbb{F}^{(\mathrm{inn})}|}{g_1(|z^{(1)\prime}|)|z^{(1)\prime}|}, && x \in D_1,\\ |\nabla\mathbf{B}^{(\mathrm{inn})}(x)| &\le \frac{C|\mathbb{F}^{(\mathrm{inn})}|}{g_1^2(|z^{(1)\prime}|)|z^{(1)\prime}|} + \frac{C|\mathbb{F}^{(\mathrm{inn})}|}{g_1(|z^{(1)\prime}|)|z^{(1)\prime}|^2}, && x \in D_1,\\ |\mathbf{B}^{(\mathrm{inn})}(x)| + |\nabla\mathbf{B}^{(\mathrm{inn})}(x)| &\le C|\mathbb{F}^{(\mathrm{inn})}|, && x \in \operatorname{supp}\mathbf{B}^{(\mathrm{inn})}. \end{aligned} \tag{3.16}$$

The proof of this lemma follows directly from the construction of $\mathbf{B}^{(\mathrm{inn})}$ and from estimates (3.11), (3.13), (3.6) and (3.7)

3.2. Construction of the extension $\mathbf{B}^{(\mathrm{out})}$. In this subsection we construct the vector field $\mathbf{B}^{(\mathrm{out})}$, extending the boundary value $\mathbf{a}$ from the outer boundary S. We start with the construction of the “flux carrier” $\mathbf{b}^{(\mathrm{out})}$ which “drains” the flux $\mathfrak{F}^{(\mathrm{out})}$ from the bounded part $\Lambda \subset S^{(3)}$ to infinity.

Let γ be a smooth simple contour which intersects $\partial\Omega$ at the points $x^{(\Lambda)} \in \Lambda \subset S^{(3)}$ and $x^{(1)} \in S^{(1)}$ and have the form

$$\gamma = \gamma_1 \cup l_1 \cup \widehat{\gamma} \cup l_\Lambda,$$

where γ_1 is a finite curve lying in D_1 and intersecting boundary $S^{(1)}$ at the point $x^{(1)}$, $\widehat{\gamma} \subset \Omega_0$ is a finite curve connecting γ_1 and the point $x^{(\Lambda)}$, l_1, $l_\Lambda \subset \mathbb{R}^3 \setminus \Omega$ are the semi-infinite curves which begin at the points $x^{(1)}$ and $x^{(\Lambda)}$, respectively (see Figure 2).

Assume that the direction of γ coincides with the direction of increase of the coordinate $z_3^{(1)}$ and that the $\operatorname{dist}(\gamma, S^{(3)} \setminus \Lambda) \ge d_0 > 0$, where d_0 is a sufficiently small number. Denote by

$$\widehat{\mathbf{b}}(x) = \frac{1}{4\pi} \oint_\gamma \frac{x-y}{|x-y|^3} \times d\mathbf{l}_y$$

a magnetic field generating, upon passage through γ, an electric flow of unit intensity. We take $\mathbf{b}(x) = \widehat{\mathbf{b}}(x)|_\Omega$.

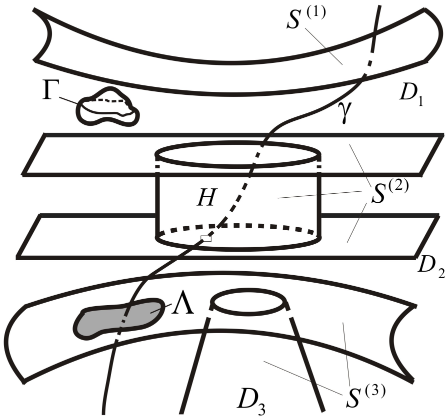

FIGURE 2. Contour γ.

LEMMA 3.5. *The vector field* $\boldsymbol{b}$ *is solenoidal in* $\mathbb{R}^3 \setminus \gamma$, $\operatorname{curl} \boldsymbol{b} = 0$, *and the circulation of* $\boldsymbol{b}$ *along any closed contour, enveloping* γ, *is equal to* -1, *if the direction of the integration along this contour and along* γ *are connected by the gimlet rule. If this contour does not envelop* γ, *then the circulation of* $\boldsymbol{b}$ *along it is equal to zero. At the points whose distance from* γ *is not less than* d_0, *we have*

$$|D_x^\alpha \boldsymbol{b}(x)| \le \frac{c(\alpha, d_0)}{d_\gamma^{1+|\alpha|}(x)}, \tag{3.17}$$

where $d_\gamma(x) = \operatorname{dist}(x, \gamma)$.

The proof of estimate (3.17) repeats the corresponding arguments from Lemma 1 in [42].

In the domain Ω we introduce the virtual drain function

$$\mathbf{b}^{(\mathrm{out})}(x,\varepsilon) = \mathfrak{F}^{(\mathrm{out})} \operatorname{curl}(\zeta(x,\varepsilon)\cdot \mathbf{b}(x)) = \mathfrak{F}^{(\mathrm{out})} \nabla\zeta(x,\varepsilon) \times \mathbf{b}(x), \tag{3.18}$$

where

$$\zeta(x,\varepsilon) = \Psi\biggl(\varepsilon \ln \frac{\delta(x)}{\Delta_{\partial\Omega\setminus(\Lambda\cup S^{(1)})}(x)}\biggr), \tag{3.19}$$

$$\delta(x) = \begin{cases} \rho_1(x)\Delta_{\gamma_1\cup S^{(1)}}(x) + \displaystyle\sum_{j=2}^{3} \rho_i(x)|x-x_0| & \\ \qquad + \biggl(1 - \displaystyle\sum_{j=1}^{3}\rho_j(x)\biggr)\Delta_{\gamma\cup S^{(1)}}(x) & \text{if } x \in \Omega \setminus (\gamma \cup S^{(1)}),\ x_0 \in \widehat{\gamma}, \\ 0 & \text{if } x \in \gamma \cup S^{(1)}, \end{cases}$$

$$\rho_j(x) = \begin{cases} 1 & \text{for } x \in D_j \setminus D_j(3), \\ 0 & \text{for } x \in (\Omega\setminus D_j) \cup D_j(2),\ j = 1,2,3. \end{cases}$$

The function $\delta(x)$ is continuous in the domain $\overline{\Omega}$ and infinitely differentiable in $\Omega \setminus (\gamma \cup S^{(1)})$, $|\nabla\delta(x)|$ is bounded. It is easy to see that

$$\delta(x) = \begin{cases} \Delta_{\gamma_1 \cup S^{(1)}}(x) & \text{for } x \in D_1 \setminus D_1(3), \\ |x - x_0|, \quad |x| \gg 1 & \text{for } x \in D_i,\ i = 2, 3, \\ \Delta_{\gamma \cup S^{(1)}}(x) & \text{for } x \in \Omega_0, \end{cases}$$

the function ζ is equal to zero on $S^{(1)}$ and $\zeta = 1$ on $\partial\Omega \setminus (\Lambda \cup S^{(1)})$.

Lemma 3.6. *The vector field $\boldsymbol{b}^{(\mathrm{out})}$ is infinitely differentiable and solenoidal, $\boldsymbol{b}^{(\mathrm{out})}$ vanishes near the surface $\partial\Omega \setminus \Lambda$, in a small neighbourhood of the curve $\gamma \cap \overline{\Omega}$ and for $x \in D_2$, $x \in D_3$, $|x| \gg 1$. The following estimates*:

$$|\boldsymbol{b}^{(\mathrm{out})}(x,\varepsilon)| \le \frac{c\varepsilon}{d_{\partial\Omega\setminus(\Lambda\cup S^{(1)})}(x) d_{\gamma\cup S^{(1)}}(x)}, \qquad x \in \Omega, \tag{3.20}$$

$$|\boldsymbol{b}^{(\mathrm{out})}(x,\varepsilon))| \le \frac{c(\varepsilon)|\mathfrak{F}^{(\mathrm{out})}|}{g_1(|z^{(1)\prime}|)\,|z^{(1)\prime}|}, \qquad x \in D_1, \tag{3.21}$$

$$|\nabla\boldsymbol{b}^{(\mathrm{out})}(x,\varepsilon)| \le \frac{c(\varepsilon)|\mathfrak{F}^{(\mathrm{out})}|}{g_1^2(|z^{(1)\prime}|)\,|z^{(1)\prime}|} + \frac{c(\varepsilon)|\mathfrak{F}^{(\mathrm{out})}|}{g_1(|z^{(1)\prime}|)\,|z^{(1)\prime}|^2}, \quad x \in D_1, \tag{3.22}$$

hold. The constant c in (3.20) *is independent of ε. Finally,*

$$\int_\Lambda \boldsymbol{b}^{(\mathrm{out})} \cdot \boldsymbol{n}\, dS = \mathfrak{F}^{(\mathrm{out})}.$$

Proof. The first statement of the lemma follows from definitions (3.18), (3.19) and from the properties of the regularized distance. Estimates (3.20)–(3.22) can be proved using definitions (3.18), (3.19) just in the same way as the analogous estimates in [33], [42], [38]. Since $\mathbf{b}^{(\mathrm{out})}(x,\varepsilon)|_{\partial\Omega\setminus\Lambda} = 0$, $\mathbf{b}^{(\mathrm{out})}(x,\varepsilon) = 0$ for $x \in D_2$, $x \in D_3$, $|x| \gg 1$, and $\zeta(x,\varepsilon) = 1$ on $\partial\Omega \setminus (\Lambda \cup S^{(1)})$ and $\zeta(x,\varepsilon) = 0$ on $S^{(1)}$, the Ostrogradsky–Gauss and the Stokes formulas yield

$$\begin{aligned} \int_\Lambda \mathbf{b}^{(\mathrm{out})} \cdot \mathbf{n}\, dS &= -\int_{\sigma_1(t)} \mathbf{b}^{(\mathrm{out})} \cdot \mathbf{n}\, dS = -\mathfrak{F}^{(\mathrm{out})} \int_{\sigma_1(t)} \operatorname{curl}(\zeta\mathbf{b}) \cdot \mathbf{n}\, dS \\ &= -\mathfrak{F}^{(\mathrm{out})} \int_{\partial\sigma_1(t)} (\zeta\mathbf{b}) \cdot d\mathbf{l} = -\mathfrak{F}^{(\mathrm{out})} \bigg(\int_{\alpha_0(t)} (\zeta\mathbf{b}) \cdot d\mathbf{l} + \int_{\alpha_1(t)} (\zeta\mathbf{b}) \cdot d\mathbf{l} \bigg) \\ &= -\mathfrak{F}^{(\mathrm{out})} \int_{\alpha_1(t)} \mathbf{b} \cdot d\mathbf{l} = \mathfrak{F}^{(\mathrm{out})}, \end{aligned}$$

where $\alpha_0(t) = \partial\sigma_1(t) \cap S^{(1)}$, $\alpha_1(t) = \partial\sigma_1(t) \cap S^{(2)}$ (4). □

Let $\boldsymbol{\beta}(x,\varepsilon) = \mathbf{a}(x)|_\Lambda - \mathbf{b}^{(\mathrm{out})}(x,\varepsilon)|_\Lambda$. Then

$$\int_\Lambda \boldsymbol{\beta} \cdot \mathbf{n}\, dS = \int_\Lambda \mathbf{a} \cdot \mathbf{n}\, dS - \int_\Lambda \mathbf{b}^{(\mathrm{out})} \cdot \mathbf{n}\, dS = 0.$$

(4) Notice that the contours $\alpha_0(t)$ and $\alpha_1(t)$ have the opposite directions.

Therefore, $\boldsymbol{\beta}$ can be extended inside Ω in the form

$$\mathbf{b}_0^{(\mathrm{out})}(x,\varepsilon) = \operatorname{curl}\,(\chi(x,\varepsilon)\mathbf{E}(x)),$$

where $\mathbf{E} \in W^{2,2}(\Omega_0)$, $\operatorname{curl}\mathbf{E}|_\Lambda = \boldsymbol{\beta}$ and χ is a Hopf's type cut-off function such that $\chi(x,\varepsilon) = 1$ on Λ, $\operatorname{supp}\chi$ is contained in a small neighbourhood of Λ, and

$$|\nabla\chi_m(x,\varepsilon)| \le \frac{\varepsilon c}{\operatorname{dist}(x, S^{(3)})} \tag{3.23}$$

(see [20]). Define

$$\mathbf{B}^{(\mathrm{out})}(x,\varepsilon) = \mathbf{b}^{(\mathrm{out})}(x,\varepsilon) + \mathbf{b}_0^{(\mathrm{out})}(x,\varepsilon).$$

Obviously,

$$\operatorname{div}\mathbf{B}^{(\mathrm{out})} = 0, \quad \mathbf{B}^{(\mathrm{out})}|_\Lambda = \mathbf{a}, \quad \mathbf{B}^{(\mathrm{out})}|_{\partial\Omega\setminus\Lambda} = 0,$$
$$\mathbf{B}^{(\mathrm{out})} = 0, \quad x \in D_2, \quad x \in D_3, \quad |x| \gg 1.$$

LEMMA 3.7. *The following estimates*

$$\int_{\omega_j(h_j(t))} |\mathbf{B}^{(\mathrm{out})}|^2|\mathbf{w}|^2\,dx \le \varepsilon c\,|\mathfrak{F}^{(\mathrm{out})}|^2 \int_{\omega_j(h_j(t))} |\nabla\mathbf{w}|^2\,dx, \quad j = 1,2,3, \tag{3.24}$$

$$\int_{\Omega(t)} |\mathbf{B}^{(\mathrm{out})}|^2|\mathbf{w}|^2\,dx \le \varepsilon c\,|\mathfrak{F}^{(\mathrm{out})}|^2 \int_{\Omega(t)} |\nabla\mathbf{w}|^2\,dx, \tag{3.25}$$

hold for any solenoidal $\mathbf{w} \in W^{1,2}_{\mathrm{loc}}(\Omega)$ *with* $\mathbf{w}|_{\partial\Omega} = 0$. *The constant c does not depend on ε and t. Moreover,*

$$|\mathbf{B}^{(\mathrm{out})}(x,\varepsilon)| \le \frac{C(\varepsilon)|\mathfrak{F}^{(\mathrm{out})}|}{g_1(|z^{(1)\prime}|)\,|z^{(1)\prime}|}, \qquad x \in D_1, \tag{3.26}$$

$$|\nabla\mathbf{B}^{(\mathrm{out})}(x,\varepsilon)| \le \frac{C(\varepsilon)|\mathfrak{F}^{(\mathrm{out})}|}{g_1^2(|z^{(1)\prime}|)\,|z^{(1)\prime}|} + \frac{C(\varepsilon)|\mathfrak{F}^{(\mathrm{out})}|}{g_1(|z^{(1)\prime}|)\,|z^{(1)\prime}|^2}, \quad x \in D_1, \tag{3.27}$$

$$|\mathbf{B}^{(\mathrm{out})}(x,\varepsilon)| + |\nabla\mathbf{B}^{(\mathrm{out})}(x,\varepsilon)| \le C(\varepsilon)|\mathfrak{F}^{(\mathrm{out})}|, \qquad x \in \operatorname{supp}\mathbf{B}^{(\mathrm{out})}. \tag{3.28}$$

Inequalities (3.24) and (3.25) follow from (3.20) and (3.23). Estimates (3.26)–(3.28) are the consequences of (3.21) and (3.22).

3.3. Construction of the vector field $\mathbf{B}^{(\mathrm{flux})}$. Now we need to compensate the fluxes over the cross-sections of the outlets to infinity, i.e. we have to construct a solenoidal vector field $\mathbf{B}^{(\mathrm{flux})}$ satisfying the flux conditions:

$$\begin{aligned}
&\int_{\sigma_1(t)} \mathbf{B}^{(\mathrm{flux})}\cdot\mathbf{n}\,dS = \mathcal{F}_1 + \mathbb{F}^{(\mathrm{inn})} + \mathfrak{F}^{(\mathrm{out})},\\
&\int_{\sigma_2(t)} \mathbf{B}^{(\mathrm{flux})}\cdot\mathbf{n}\,dS = \mathcal{F}_2,\\
&\int_{\sigma_3(t)} \mathbf{B}^{(\mathrm{flux})}\cdot\mathbf{n}\,dS = \mathcal{F}_3.
\end{aligned} \tag{3.29}$$

Note that the total flux is equal to zero:

$$\mathcal{F}_1 + \mathcal{F}_2 + \mathcal{F}_3 + \mathbb{F}^{(\mathrm{inn})} + \mathfrak{F}^{(\mathrm{out})} = 0. \tag{3.30}$$

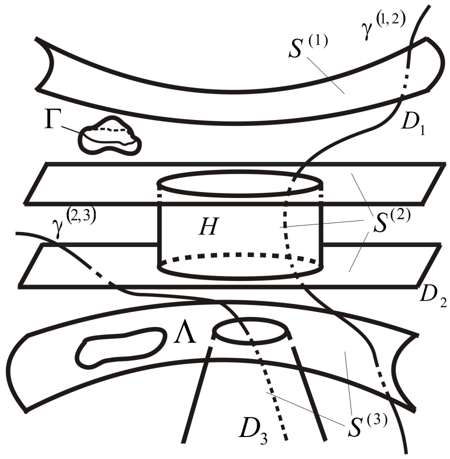

FIGURE 3. Contours $\gamma^{(1,2)}$ and $\gamma^{(2,3)}$.

Constructing $\mathbf{B}^{(\mathrm{flux})}$, first, we define the vectors

$$\mathbf{b}^{(j,j+1)}(x) = \frac{1}{4\pi} \oint_{\gamma^{(j,j+1)}} \frac{x-y}{|x-y|^3} \times d\mathbf{l}_y, \quad j = 1, 2,$$

describing the magnetic fields upon passage through the contours $\gamma^{(1,2)}$ and $\gamma^{(2,3)}$. Let us introduce $\gamma^{(1,2)}$ and $\gamma^{(2,3)}$.

1. Contour $\gamma^{(1,2)}$ goes through two layer type outlets D_1 and D_2, $\gamma^{(1,2)}$ is an infinite smooth simple curve which intersects $\partial\Omega$ at the points $x^{(1)} \in S^{(1)}$ and $x^{(2)} \in S^{(3)} \cap \partial D_2$. Contour $\gamma^{(1,2)}$ consists of a finite curve $\widehat{\gamma}_1^{(2)} \subset \Omega_0$ and semi-infinite curves $l_1,\ l_2 \subset \mathbb{R}^3 \setminus \Omega$ which begin at the points $x^{(1)}$, $x^{(2)}$, respectively (see Figure 3):

$$\gamma^{(1,2)} = \widehat{\gamma}_1^{(2)} \cup l_1 \cup l_2.$$

2. Contour $\gamma^{(2,3)}$ goes through two outlets, one of which is paraboloidal outlet D_3 and another D_2 is of the layer type:

$$\gamma^{(2,3)} = \widehat{\gamma}^{(2)} \cup \widehat{\gamma}^{(3)} \cup l_2 \cup \widehat{\gamma}_2^{(3)},$$

where $\widehat{\gamma}^{(3)} \subset D_3$ is a semi-infinite line, $\widehat{\gamma}^{(2)} \subset D_2$ is a finite line intersecting $\partial\Omega$ at the point $x^{(2)} \in S^{(2)} \cap \partial D_2$, $\widehat{\gamma}_2^{(3)} \subset \Omega_0$ is a curve joining $\widehat{\gamma}^{(3)}$ with $\widehat{\gamma}^{(2)}$, and $l_2 \subset \mathbb{R}^3 \setminus \Omega$ is a semi-infinite curve which starts at the point $x^{(2)}$ (see Figure 3).

We suppose that the distances $\operatorname{dist}(\gamma^{(1,2)}, \partial\Omega \setminus (S^{(1)} \cup (S^{(3)} \cap D_2))) \geq d_0$ and $\operatorname{dist}(\gamma^{(2,3)}, \partial\Omega \setminus (S^{(2)} \cap D_2)) \geq d_0$ are positive, where d_0 is a sufficiently small positive number (see Figure 3). The direction of the curves $\gamma^{(1,2)}$ and $\gamma^{(2,3)}$ coincides with the direction of increase of the axis $z_3^{(2)}$, $z_3^{(3)}$, respectively.

Denote

$$\begin{aligned}\mathbf{b}^{(\mathrm{flux})}_{j,j+1}(x,\varepsilon) &= \operatorname{curl}(\zeta_{j,j+1}(x,\varepsilon)\mathbf{b}^{(j,j+1)}(x)) = \nabla\zeta_{j,j+1}(x,\varepsilon)\times\mathbf{b}^{(j,j+1)}(x),\\ \zeta_{j,j+1}(x,\varepsilon) &= \Psi\biggl(\varepsilon\ln\frac{\delta^{(j,j+1)}(x)}{\Delta_{S^{(j,j+1)}\cup\Gamma}(x)}\biggr),\quad j=1,2,\end{aligned}$$

where

$$\delta^{(1,2)}(x)=\begin{cases}\rho_1(x)\Delta_{(\gamma\cup S^{(1)})\cap D_1}+\rho_2(x)\Delta_{(\gamma\cup S^{(3)})\cap D_2}+\rho_3(x)C_0\\ \quad+\Bigl(1-\sum\limits_{k=1}^{3}\rho_k(x)\Bigr)\Delta_{\gamma\cup S^{(1)}\cup S^{(3)}}(x), & x\in\Omega\setminus(\gamma\cup S^{(1)}\cup S^{(3)}),\\ 0, & x\in\gamma\cup S^{(1)}\cup S^{(3)},\end{cases}$$

$$\delta^{(2,3)}(x)=\begin{cases}\rho_2(x)\Delta_{(\gamma\cup S^{(2)})\cap D_2}+\rho_3(x)\Delta_{\gamma\cap D_3}+\rho_1(x)C_0\\ \quad+\Bigl(1-\sum\limits_{k=1}^{3}\rho_k(x)\Bigr)\Delta_{\gamma\cup S^{(1)}\cup S^{(2)}}(x), & x\in\Omega\setminus(\gamma\cup S^{(1)}\cup S^{(2)}),\\ 0, & x\in\gamma\cup S^{(1)}\cup S^{(2)},\end{cases}$$

with a sufficiently small positive constant C_0.

Note that in the first case (the contour $\gamma^{(1,2)}$ goes from the layer D_1 to the layer D_2) the function ζ is equal to zero on $S^{(1)}\cup S^{(3)}$ and $\zeta=1$ on $S^{(2)}$. In the second case (the contour $\gamma^{(2,3)}$ goes from the layer D_2 to the paraboloidal outlet D_3) the function ζ is equal to zero on $S^{(1)}\cup S^{(2)}$ and $\zeta=1$ on $S^{(3)}$.

The vector fields $\mathbf{b}^{(\mathrm{flux})}_{j,j+1}$ have the following properties (see [42], [38] for the case of the paraboloidal outlets and [33] for the case of the layer type outlets).

LEMMA 3.8. *The vector fields* $\boldsymbol{b}^{(\mathrm{flux})}_{1,2}(x,\varepsilon)$ *and* $\boldsymbol{b}^{(\mathrm{flux})}_{2,3}(x,\varepsilon)$ *are solenoidal,* $\boldsymbol{b}^{(\mathrm{flux})}_{1,2}|_{\partial\Omega}=0$, $\boldsymbol{b}^{(\mathrm{flux})}_{2,3}|_{\partial\Omega}=0$, $\boldsymbol{b}^{(\mathrm{flux})}_{1,2}(x,\varepsilon)=0$ *for* $x\in D_3$, $|x|\gg 1$, $\boldsymbol{b}^{(\mathrm{flux})}_{2,3}(x,\varepsilon)=0$ *for* $x\in D_1$, $|x|\gg 1$ *and*

$$\begin{aligned}\int_{\sigma_1(t)}\boldsymbol{b}^{(\mathrm{flux})}_{1,2}\cdot\mathbf{n}\,dS &= 1, & \int_{\sigma_2(t)}\boldsymbol{b}^{(\mathrm{flux})}_{1,2}\cdot\mathbf{n}\,dS &= -1,\\ \int_{\sigma_2(t)}\boldsymbol{b}^{(\mathrm{flux})}_{2,3}\cdot\mathbf{n}\,dS &= 1, & \int_{\sigma_3(t)}\boldsymbol{b}^{(\mathrm{flux})}_{2,3}\cdot\mathbf{n}\,dS &= -1.\end{aligned}$$

For any solenoidal $\mathbf{w}\in W^{1,2}_{\mathrm{loc}}(\Omega)$ *with* $\mathbf{w}|_{\partial\Omega}=0$ *the following estimates*:

$$\begin{aligned}\int_{\omega_s(h_s(t))}|\boldsymbol{b}^{(\mathrm{flux})}_{j,j+1}|^2|\mathbf{w}|^2\,dx &\le \varepsilon c\int_{\omega_s(h_s(t))}|\nabla\mathbf{w}|^2\,dx,\\ &\qquad j=1,2,\ s=1,2,3,\\ \int_{\Omega(t)}|\boldsymbol{b}^{(\mathrm{flux})}_{j,j+1}|^2|\mathbf{w}|^2\,dx &\le \varepsilon c\int_{\Omega(t)}|\nabla\mathbf{w}|^2\,dx\end{aligned}\tag{3.31}$$

hold with the constant c *independent of* ε *and* t*. Moreover,*

$$|\boldsymbol{b}^{(\mathrm{flux})}_{j,j+1}(x,\varepsilon)|\le\frac{C(\varepsilon)}{g_3^2(z_3^{(3)})},\quad |\nabla\boldsymbol{b}^{(\mathrm{flux})}_{j,j+1}(x,\varepsilon))|\le\frac{C(\varepsilon)}{g_3^3(z_3^{(3)})},\quad x\in D_3,$$

$$
\begin{aligned}
|\boldsymbol{b}^{(\mathrm{flux})}_{j,j+1}(x,\varepsilon)| &\le \frac{C(\varepsilon)}{g_i(|z^{(i)\prime}|)|z^{(i)\prime}|},\\
|\nabla \boldsymbol{b}^{(\mathrm{flux})}_{j,j+1}(x,\varepsilon))| &\le \frac{C(\varepsilon)}{g_i^2(|z^{(i)\prime}|)|z^{(i)\prime}|}+\frac{C(\varepsilon)}{g_i(|z^{(i)\prime}|)|z^{(i)\prime}|^2}, \quad x\in D_i,\ i=1,2,\\
|\boldsymbol{b}^{(\mathrm{flux})}_{j,j+1}(x,\varepsilon)| + |\nabla \boldsymbol{b}^{(\mathrm{flux})}_{j,j+1}(x,\varepsilon))| &\le C(\varepsilon), \qquad x\in \operatorname{supp} \boldsymbol{b}^{(\mathrm{flux})}_{j,j+1}.
\end{aligned}
$$

Let us define

$$
\mathbf{B}^{(\mathrm{flux})}(x,\varepsilon)=\alpha_1\mathbf{b}^{(\mathrm{flux})}_{1,2}(x,\varepsilon)+\alpha_2\mathbf{b}^{(\mathrm{flux})}_{2,3}(x,\varepsilon),
$$

where $\alpha_1=\mathcal{F}_1+\mathbb{F}^{(\mathrm{inn})}+\mathfrak{F}^{(\mathrm{out})}$, $\alpha_2=\mathcal{F}_1+\mathcal{F}_2+\mathbb{F}^{(\mathrm{inn})}+\mathfrak{F}^{(\mathrm{out})}$. The vector field $\mathbf{B}^{(\mathrm{flux})}$ satisfies flux conditions (3.29).

3.4. Solvability of problem (2.3)**.** We look for the solution $\mathbf{u}$ in the form

$$
\mathbf{u}(x)=\mathbf{A}(x,\varepsilon)+\mathbf{v}(x), \tag{3.32}
$$

where

$$
\mathbf{A}(x,\varepsilon)=\mathbf{B}^{(\mathrm{out})}(x,\varepsilon)+\mathbf{B}^{(\mathrm{inn})}(x)+\mathbf{B}^{(\mathrm{flux})}(x,\varepsilon).
$$

As it follows from (3.15), (3.24), (3.31), for any solenoidal $\mathbf{w}\in W^{1,2}_{\mathrm{loc}}(\Omega)$ with $\mathbf{w}|_{\partial\Omega}=0$ the following inequalities

$$
\begin{aligned}
\int_{\omega_j(h_j(t))}|\mathbf{A}|^2|\mathbf{w}|^2\,dx &\le c(\varepsilon|\overrightarrow{\mathcal{F}}|^2+\varepsilon|\mathfrak{F}^{(\mathrm{out})}|^2+|\mathbb{F}^{(\mathrm{inn})}|^2)\int_{\omega_j(h_j(t))}|\nabla\mathbf{w}|^2dx,\\
\int_{\Omega(t)}|\mathbf{A}|^2|\mathbf{w}|^2\,dx &\le c(\varepsilon|\overrightarrow{\mathcal{F}}|^2+\varepsilon|\mathfrak{F}^{(\mathrm{out})}|^2+|\mathbb{F}^{(\mathrm{inn})}|^2)\int_{\Omega(t)}|\nabla\mathbf{w}|^2\,dx,
\end{aligned}
\tag{3.33}
$$

hold, where $|\overrightarrow{\mathcal{F}}|=\sqrt{\mathcal{F}_1^2+\mathcal{F}_2^2+\mathcal{F}_3^2}$, $\overrightarrow{\mathcal{F}}=(\mathcal{F}_1,\mathcal{F}_2,\mathcal{F}_3)$, $j=1,2,3$.

In order to prove the existence of at least one weak solution, we need some known results.

Lemma 3.9 ([23]). *Let non-negative, non-decreasing smooth functions $y(t)$ and $\varphi(t)$ satisfy the inequalities*

$$
y(t)\le\theta(y'(t))+\frac{1}{2}\varphi(t), \tag{3.34}
$$

$$
\varphi(t)\ge 2\theta(\varphi'(t)), \qquad t\in[0,T], \tag{3.35}
$$

where $\theta(s)$ is a positive increasing function of a positive argument s. If $y(T)\le\varphi(T)$, then $y(t)\le\varphi(t)$ for all $t\in[0,T]$.

Lemma 3.10 (Poincaré inequality). *Let $u\in \mathring{W}^{1,2}(\Omega)$. Then the following inequalities*

$$
\int_{\omega_j(t)}|u(x)|^2\,dx\le c\,g_j^2(t)\int_{\omega_j(t)}|\nabla u(x)|^2\,dx, \quad j=1,2,3, \tag{3.36}
$$

hold, where the constant c is independent of u and t.

LEMMA 3.11. *Let $u \in \mathring{W}^{1,2}(\Omega)$. Then the following inequalities*

$$\|u\|_{L^4(\omega_j(t))} \le c\, g_j^{1/4}(t) \|\nabla u\|_{L^2(\omega_j(t))}, \quad j = 1,2,3, \tag{3.37}$$

hold, where the constant c is independent of u and t.

The proof of this lemma follows directly from the multiplicative inequality

$$\|u\|_{L^4(\omega_j(t))} \le c \|\nabla u\|^{3/4}_{L^2(\omega_j(t))} \cdot \|u\|^{1/4}_{L^2(\omega_j(t))}, \quad j = 1,2,3, \tag{3.38}$$

and Poincaré inequality (3.36). The constant c in (3.38) is independent of t.

LEMMA 3.12. *Let $f \in L^2(\omega_i(t))$ and*

$$\int_{\omega_i(t)} f\,dx = 0, \quad i = 1,2.$$

Then problem

$$\begin{cases} \operatorname{div} \boldsymbol{u} = f & \text{for } x \in \Omega, \\ \boldsymbol{u} = 0 & \text{for } x \in \partial\Omega, \end{cases} \tag{3.39}$$

admits a solution $\boldsymbol{u} \in \mathring{W}^{1,2}(\omega_i(t))$ satisfying the estimate

$$\|\nabla \boldsymbol{u}\|_{L^2(\omega_i(t))} \le c \frac{t}{g_i(t)} \|f\|_{L^2(\omega_i(t))} \tag{3.40}$$

with the constant c independent of $\boldsymbol{u}$, f and t.

LEMMA 3.13. *Let $f \in L^2(\omega_3(t))$ and*

$$\int_{\omega_3(t)} f\,dx = 0.$$

Then problem (3.39) *admits a solution $\boldsymbol{u} \in \mathring{W}^{1,2}(\omega_3(t))$ satisfying the estimate*

$$\|\nabla \boldsymbol{u}\|_{L^2(\omega_3(t))} \le c \|f\|_{L^2(\omega_3(t))} \tag{3.41}$$

with the constant c independent of $\boldsymbol{u}$, f and t.

The last two lemmas were proved in [21] and [33], respectively.

THEOREM 3.14. *Assume that the boundary value $\boldsymbol{a} \in W^{1/2,2}(\partial\Omega)$ has a compact support, the flux $\mathbb{F}^{(\mathrm{inn})}$ is sufficiently small and the necessary condition* (2.4) *holds. Then there exists at least one weak solution $\boldsymbol{u}$ of problem* (2.3) *satisfying the inequality*

$$\int_{\Omega(t)} |\nabla \boldsymbol{u}|^2\,dx \le c \cdot c(\boldsymbol{a}, \overrightarrow{\mathcal{F}}) \Big(1 + \sum_{i=1}^{2} \int_1^{2h_i(t)} \frac{dr}{r\, g_i^3(r)} + \int_1^{h_3(t)} \frac{dr}{g_3^4(r)} \Big), \tag{3.42}$$

where

$$c(\boldsymbol{a}, \overrightarrow{\mathcal{F}}) = \|\boldsymbol{a}\|^2_{W^{1/2,2}(\partial\Omega)} + \|\boldsymbol{a}\|^4_{W^{1/2,2}(\partial\Omega)} + |\overrightarrow{\mathcal{F}}|^2 + |\overrightarrow{\mathcal{F}}|^4, \quad |\overrightarrow{\mathcal{F}}|^2 = \mathcal{F}_1^2 + \mathcal{F}_2^2 + \mathcal{F}_3^2.$$

Proof. In every bounded domain $\Omega(T)$ there exists a vector field $\mathbf{v}^{(T)} \in H(\Omega(T))$ satisfying the integral identity

$$\nu \int_{\Omega(T)} \nabla \mathbf{v}^{(T)} : \nabla \boldsymbol{\eta}\, dx - \int_{\Omega(T)} ((\mathbf{A} + \mathbf{v}^{(T)}) \cdot \nabla)\boldsymbol{\eta} \cdot \mathbf{v}^{(T)}\, dx - \int_{\Omega(T)} (\mathbf{v}^{(T)} \cdot \nabla)\boldsymbol{\eta} \cdot \mathbf{A}\, dx = -\nu \int_{\Omega(T)} \nabla \mathbf{A} : \nabla \boldsymbol{\eta}\, dx + \int_{\Omega(T)} (\mathbf{A} \cdot \nabla)\boldsymbol{\eta} \cdot \mathbf{A}\, dx \tag{3.43}$$

for all $\boldsymbol{\eta} \in H(\Omega(T))$. Indeed, it is well known that in the bounded domain $\Omega(T)$ this integral identity is equivalent to the operator equation

$$\mathbf{v}^{(T)} = \widehat{\mathcal{L}}\mathbf{v}^{(T)}, \tag{3.44}$$

where $\widehat{\mathcal{L}}$ is a compact operator (e.g. [20]). By the Leray–Schauder Theorem, (3.44) admits at least one solution if the norms of all possible solutions of the equation

$$\mathbf{v}^{(T,\lambda)} = \lambda \widehat{\mathcal{L}}\mathbf{v}^{(T,\lambda)}, \quad \lambda \in [0, 1], \tag{3.45}$$

are bounded by the same constant independent of λ.

Taking $\boldsymbol{\eta} = \mathbf{v}^{(T)}$ ([5]) in (3.43) and using the Leray–Hopf (2.8) and Cauchy–Schwarz inequalities, we obtain

$$\begin{aligned} \nu \int_{\Omega(T)} |\nabla \mathbf{v}^{(T)}|^2\, dx &= \lambda \int_{\Omega(T)} (\mathbf{v}^{(T)} \cdot \nabla)\mathbf{v}^{(T)} \cdot \mathbf{A}\, dx \\ &\quad - \nu\lambda \int_{\Omega(T)} \nabla \mathbf{A} : \nabla \mathbf{v}^{(T)}\, dx + \lambda \int_{\Omega(T)} (\mathbf{A} \cdot \nabla)\mathbf{v}^{(T)} \cdot \mathbf{A}\, dx \\ &\le \Big(\int_{\Omega(T)} |\nabla \mathbf{v}^{(T)}|^2\, dx \Big)^{1/2} \cdot \Big(\int_{\Omega(T)} |\mathbf{v}^{(T)}|^2 |\mathbf{A}|^2\, dx \Big)^{1/2} \\ &\quad + \Big(\int_{\Omega(T)} |\nabla \mathbf{A}|^2\, dx \Big)^{1/2} \cdot \Big(\int_{\Omega(T)} |\nabla \mathbf{v}^{(T)}|^2\, dx \Big)^{1/2} \\ &\quad + \Big(\int_{\Omega(T)} |\mathbf{A}|^4\, dx \Big)^{1/2} \cdot \Big(\int_{\Omega(T)} |\nabla \mathbf{v}^{(T)}|^2\, dx \Big)^{1/2} \\ &\le c\sqrt{\varepsilon|\overrightarrow{\mathcal{F}}|^2 + \varepsilon|\mathfrak{F}^{(\mathrm{out})}|^2 + |\mathbb{F}^{(\mathrm{inn})}|^2} \int_{\Omega(T)} |\nabla \mathbf{v}^{(T)}|^2\, dx \\ &\quad + c\big(\|\nabla \mathbf{A}\|^2_{L^2(\Omega(T))} + \|\mathbf{A}\|^4_{L^4(\Omega(T))} \big) + \frac{\nu}{2} \|\nabla \mathbf{v}^{(T)}\|^2_{L^2(\Omega(T))}. \end{aligned} \tag{3.46}$$

The last inequality yields

$$\Big(\frac{\nu}{2} - c\sqrt{\varepsilon|\overrightarrow{\mathcal{F}}|^2 + c\varepsilon|\mathfrak{F}^{(\mathrm{out})}|^2 + |\mathbb{F}^{(\mathrm{inn})}|^2} \Big) \|\nabla \mathbf{v}^{(T)}\|^2_{L^2(\Omega(T))} \le c\big(\|\nabla \mathbf{A}\|^2_{L^2(\Omega(T))} + \|\mathbf{A}\|^4_{L^4(\Omega(T))} \big).$$

([5]) For simplicity, we omit the index λ.

Assuming that $|\mathbb{F}^{(\mathrm{inn})}| \le \nu/(4c)$ and taking ε sufficiently small we derive

$$\|\nabla \mathbf{v}^{(T)}\|^2_{L^2(\Omega(T))} \le c\big(\|\nabla \mathbf{A}\|^2_{L^2(\Omega(T))} + \|\mathbf{A}\|^4_{L^4(\Omega(T))}\big). \tag{3.47}$$

Since

$$\begin{aligned}\|\nabla \mathbf{A}\|^2_{L^2(\Omega(T))} + \|\mathbf{A}\|^4_{L^4(\Omega(T))} &\le c\big(\|\mathbf{a}\|^2_{W^{1/2,2}(\partial\Omega)} + \|\mathbf{a}\|^4_{W^{1/2,2}(\partial\Omega)} \\ &\quad + |\overrightarrow{\mathcal{F}}|^2 + |\overrightarrow{\mathcal{F}}|^4\big)\bigg(1 + \sum_{i=1}^{2}\int_1^{2h_i(t)} \frac{dr}{r\, g_i^3(r)} + \int_1^{h_3(t)} \frac{dr}{g_3^4(r)}\bigg) \\ &= c\cdot c(\mathbf{a}, \overrightarrow{\mathcal{F}})\bigg(1 + \sum_{i=1}^{2}\int_1^{2h_i(t)} \frac{dr}{r\, g_i^3(r)} + \int_1^{h_3(t)} \frac{dr}{g_3^4(r)}\bigg),\end{aligned}$$

from (3.47) it follows that

$$\|\nabla \mathbf{v}^{(T)}\|^2_{L^2(\Omega(T))} \le c\cdot c(\mathbf{a}, \overrightarrow{\mathcal{F}})\bigg(1 + \sum_{i=1}^{2}\int_1^{2h_i(t)} \frac{dr}{r\, g_i^3(r)} + \int_1^{h_3(t)} \frac{dr}{g_3^4(r)}\bigg). \tag{3.48}$$

Hence, for any fixed T the existence of a solution $\mathbf{v}^{(T)}$ of operator equation (3.44) follows from the Leray–Schauder Theorem. If the integrals

$$\int_1^{2h_i(t)} \frac{dr}{r\, g_i^3(r)}, \quad i = 1, 2, \quad \text{and} \quad \int_1^{h_3(t)} \frac{dr}{g_3^4(r)}$$

are bounded, then the right hand side of the above estimate is bounded by a constant uniformly independent of T. Let us assume that all the integrals are unbounded. Then we have to estimate the norm $\|\nabla \mathbf{v}^{(T)}\|_{L^2(\Omega(t))}$ for $t < T$. We introduce the cut-off function $\kappa(x,t)$:

$$\kappa(x,t) = \begin{cases} 1 & \text{for } x \in \Omega\Big(h_1(t), h_2(t), h_3(t) - \dfrac{g_3(h_3(t))}{2L_3}\Big), \\ \dfrac{2\,h_1(t) - |z^{(1)\prime}|}{h_1(t)} & \text{for } x \in \omega_1(h_1(t)), \\ \dfrac{2\,h_2(t) - |z^{(2)\prime}|}{h_2(t)} & \text{for } x \in \omega_2(h_2(t)), \\ 2L_3\dfrac{h_3(t) - z_3^{(3)}}{g_3(h_3(t))} & \text{for } x \in \omega_3(h_3(t)), \\ 0 & \text{for } x \in \Omega \setminus \Omega(t), \end{cases} \tag{3.49}$$

where

$$\begin{aligned}\omega_i(h_i(t)) &= \big\{z^{(i)} \in D_i : h_i(t) < |z^{(i)\prime}| < 2h_i(t)\big\}, \quad i = 1, 2, \\ \omega_3(h_3(t)) &= \Big\{z^{(3)} \in D_3 : h_3(t) - \frac{g_3(h_3(t))}{2L_3} < z_3^{(3)} < h_3(t)\Big\}.\end{aligned}$$

The derivatives of the function $\kappa(x,t)$ satisfy the estimates

$$\begin{aligned} &\frac{\partial\kappa}{\partial z_k^{(i)}} = -\frac{z_k^{(i)}}{h_i(t)\cdot|z^{(i)\prime}|}, && k=1,2, \\ &|\nabla\kappa| \le \frac{c}{h_i(t)}, \quad \frac{\partial\kappa}{\partial t} \ge g_i^{1/3}(h_i(t)), && x\in\omega_i(h_i(t)),\ i=1,2, \end{aligned} \tag{3.50}$$

$$\begin{aligned} &\frac{\partial\kappa}{\partial z_3^{(3)}} = -\frac{2L_3}{g_3(h_3(t))}, \\ &|\nabla\kappa| \le \frac{c}{g_3(h_3(t))}, \quad \frac{\partial\kappa}{\partial t} \ge g_3^{1/3}(h_3(t)), \quad x\in\omega_3(h_3(t)). \end{aligned} \tag{3.51}$$

Define the function

$$\mathbf{U}^{(T)}(x,t) = \begin{cases} \mathbf{v}^{(T)}(x), & x\in\Omega\Big(h_1(t),h_2(t),h_3(t)-\dfrac{g_3(h_3(t))}{2L_3}\Big), \\ \kappa(x,t)\mathbf{v}^{(T)}(x)+\displaystyle\sum_{j=1}^3\widehat{\mathbf{v}}_j^{(T)}(x), & x\in\displaystyle\bigcup_{j=1}^3\omega_j(h_j(t)), \\ 0, & x\in\Omega\setminus\Omega(t), \end{cases} \tag{3.52}$$

where $\widehat{\mathbf{v}}_j^{(T)}\in\mathring{W}^{1,2}(\omega_j(h_j(t)))$ are the solutions of the problems

$$\begin{aligned} \operatorname{div}\widehat{\mathbf{v}}_j^{(T)} &= -\nabla\kappa\cdot\mathbf{v}^{(T)} && \text{in } \omega_j(h_j(t)), \\ \widehat{\mathbf{v}}_j^{(T)} &= 0 && \text{on } \partial\omega_j(h_j(t)). \end{aligned} \tag{3.53}$$

Since

$$\begin{aligned} \int_{\omega_j(h_j(t))}\nabla\kappa\cdot\mathbf{v}^{(T)}\,dx &= \int_{\omega_j(h_j(t))}\operatorname{div}(\kappa\mathbf{v}^{(T)})\,dx \\ &= \int_{\partial\omega_j(h_j(t))}\kappa\mathbf{v}^{(T)}\cdot\mathbf{n}\,dx = \int_{\sigma_j(h_j(t^*))}\mathbf{v}^{(T)}\cdot\mathbf{n}\,dx = 0, \end{aligned}$$

for $j=1,2,3$, and

$$t^*=h_i(t),\quad i=1,2,\qquad t^*=h_3(t)-\frac{g_3(h_3(t))}{2L_3},$$

there exist the solutions $\widehat{\mathbf{v}}_j^{(T)}$ of problems (3.53) satisfying the estimates

$$\|\nabla\widehat{\mathbf{v}}_j^{(T)}\|_{L^2(\omega_j(h_j(t)))} \le C_j(\omega_j(h_j(t)))\|\nabla\kappa\cdot\mathbf{v}^{(T)}\|_{L^2(\omega_j(h_j(t)))}, \tag{3.54}$$

where

$$C_i(\omega_i(h_i(t))) \le \frac{c\,h_i(t)}{g_i(h_i(t))},\quad i=1,2,\qquad C_3(\omega_3(h_3(t)))\le c,$$

the constant c is independent of t (see Lemmas 3.12 and 3.13, respectively).

Using the estimates of the derivatives of the function $\kappa(x,t)$ and Poincaré inequality (3.36), from (3.54) we derive

$$\begin{aligned}\|\nabla\widehat{\mathbf{v}}_i^{(T)}\|_{L^2(\omega_i(h_i(t)))} &\le c\frac{h_i(t)}{g_i(h_i(t))}\|\nabla\kappa\cdot\mathbf{v}^{(T)}\|_{L^2(\omega_i(h_i(t)))}\\ &\le c\frac{h_i(t)}{g_i(h_i(t))}\,\frac{1}{h_i(t)}\|\mathbf{v}^{(T)}\|_{L^2(\omega_i(h_i(t)))}\le c\|\nabla\mathbf{v}^{(T)}\|_{L^2(\omega_i(h_i(t)))},\end{aligned}$$

for $i=1,2$, and

$$\begin{aligned}\|\nabla\widehat{\mathbf{v}}_3^{(T)}\|_{L^2(\omega_3(h_3(t)))} &\le c\|\nabla\kappa\cdot\mathbf{v}^{(T)}\|_{L^2(\omega_3(h_3(t)))}\\ &\le\frac{c}{g_3(h_3(t))}\|\mathbf{v}^{(T)}\|_{L^2(\omega_3(h_3(t)))}\le c\|\nabla\mathbf{v}^{(T)}\|_{L^2(\omega_3(h_3(t)))}.\end{aligned}$$

Hence,

$$\|\nabla\widehat{\mathbf{v}}_j^{(T)}\|_{L^2(\omega_j(h_j(t)))}\le c\|\nabla\mathbf{v}^{(T)}\|_{L^2(\omega_j(h_j(t)))},\quad j=1,2,3. \tag{3.55}$$

Taking in integral identity (3.43) $\boldsymbol{\eta}=\mathbf{U}^{(T)}$ and using the fact that

$$\int_{\Omega(t)}((\mathbf{v}^{(T)}+\mathbf{A})\cdot\nabla)\mathbf{U}^{(T)}\cdot\mathbf{U}^{(T)}\,dx=0,$$

we obtain

$$\begin{aligned}\nu\int_{\Omega(t)}&|\nabla\mathbf{v}^{(T)}|^2\kappa(x,t)\,dx\\ &=\sum_{j=1}^{3}\int_{\omega_j(h_j(t))}((\mathbf{v}^{(T)}+\mathbf{A})\cdot\nabla)\mathbf{U}^{(T)}\cdot(\mathbf{v}^{(T)}-\mathbf{U}^{(T)})\,dx\\ &\quad-\nu\sum_{j=1}^{3}\int_{\omega_j(h_j(t))}\nabla\mathbf{v}^{(T)}:\nabla\mathbf{U}^{(T)}\,dx+\int_{\Omega(t)}(\mathbf{v}^{(T)}\cdot\nabla)\mathbf{U}^{(T)}\cdot\mathbf{A}\,dx\\ &\quad-\nu\int_{\Omega(t)}\nabla\mathbf{A}:\nabla\mathbf{U}^{(T)}\,dx+\int_{\Omega(t)}(\mathbf{A}\cdot\nabla)\mathbf{U}^{(T)}\cdot\mathbf{A}\,dx.\end{aligned}\tag{3.56}$$

Using definition (3.52) of the function $\mathbf{U}^{(T)}$, Poincaré inequality (3.36) and estimates (3.55), we obtain

$$\begin{aligned}\|\mathbf{v}^{(T)}\|_{L^4(\omega_j(h_j(t)))}&\le c\,g_j^{1/4}(h_j(t))\|\nabla\mathbf{v}^{(T)}\|_{L^2(\omega_j(h_j(t)))},\\ \|\mathbf{v}^{(T)}-\mathbf{U}^{(T)}\|_{L^4(\omega_j(h_j(t)))}&\le c\|\mathbf{v}^{(T)}\|_{L^4(\omega_j(h_j(t)))}+c\|\widehat{\mathbf{v}}_j^{(T)}\|_{L^4(\omega_j(h_j(t)))}\\ &\le c\,g_j^{1/4}(h_j(t))\|\nabla\mathbf{v}^{(T)}\|_{L^2(\omega_j(h_j(t)))}\\ &\quad+c\,g_j^{1/4}(h_j(t))\|\nabla\widehat{\mathbf{v}}_j^{(T)}\|_{L^2(\omega_j(h_j(t)))}\\ &\le c\,g_j^{1/4}(h_j(t))\|\nabla\mathbf{v}^{(T)}\|_{L^2(\omega_j(h_j(t)))},\end{aligned}$$

for $j = 1,2,3$. Definition (3.52) of the function $\mathbf{U}^{(T)}$, estimates (3.50), (3.51), (3.55) and Poincaré inequality (3.36) yield

$$\begin{aligned}
&\|\nabla \mathbf{U}^{(T)}\|_{L^2(\omega_i(h_i(t)))} \le \|\nabla(\kappa \mathbf{v}^{(T)})\|_{L^2(\omega_i(h_i(t)))} + \|\nabla \widehat{\mathbf{v}}_i^{(T)}\|_{L^2(\omega_i(h_i(t)))}\\
&\le \frac{c}{h_i(t)}\|\mathbf{v}^{(T)}\|_{L^2(\omega_i(h_i(t)))} + c\|\nabla \mathbf{v}^{(T)}\|_{L^2(\omega_i(h_i(t)))} + \|\nabla \widehat{\mathbf{v}}_i^{(T)}\|_{L^2(\omega_i(h_i(t)))}\\
&\le \frac{cg_i(h_i(t))}{h_i(t)}\|\nabla \mathbf{v}^{(T)}\|_{L^2(\omega_i(h_i(t)))} + c\|\nabla \mathbf{v}^{(T)}\|_{L^2(\omega_i(h_i(t)))}\\
&\le c\|\nabla \mathbf{v}^{(T)}\|_{L^2(\omega_i(h_i(t)))},
\end{aligned}$$

for $i = 1,2$, and

$$\begin{aligned}
\|\nabla \mathbf{U}^{(T)}\|_{L^2(\omega_3(h_3(t)))} &\le \|\nabla(\kappa \mathbf{v}^{(T)})\|_{L^2(\omega_3(h_3(t)))} + \|\nabla \widehat{\mathbf{v}}_3^{(T)}\|_{L^2(\omega_3(h_3(t)))}\\
&\le \frac{c}{g_3(h_3(t))}\|\mathbf{v}^{(T)}\|_{L^2(\omega_3(h_3(t)))} + c\,\|\nabla \mathbf{v}^{(T)}\|_{L^2(\omega_3(h_3(t)))}\\
&\le c\|\nabla \mathbf{v}^{(T)}\|_{L^2(\omega_3(h_3(t)))}.
\end{aligned}$$

Hence,

$$\|\nabla \mathbf{U}^{(T)}\|_{L^2(\omega_j(h_j(t)))} \le c\|\nabla \mathbf{v}^{(T)}\|_{L^2(\omega_j(h_j(t)))}, \quad j = 1,2,3.$$

Therefore, we can estimate the integrals in the right-hand side of (3.56) as follows

$$\begin{aligned}
&\left|\int_{\omega_j(h_j(t))}((\mathbf{v}^{(T)} + \mathbf{A})\cdot\nabla)\mathbf{U}^{(T)}\cdot(\mathbf{v}^{(T)} - \mathbf{U}^{(T)})\,dx\right|\\
&\quad\le \|\mathbf{v}^{(T)}\|_{L^4(\omega_j(h_j(t)))}\,\|\mathbf{v}^{(T)} - \mathbf{U}^{(T)}\|_{L^4(\omega_j(h_j(t)))}\,\|\nabla \mathbf{U}^{(T)}\|_{L^2(\omega_j(h_j(t)))}\\
&\qquad + \|\nabla \mathbf{U}^{(T)}\|_{L^2(\omega_j(h_j(t)))}\left(\int_{\omega_j(h_j(t))}|\mathbf{A}|^2|\mathbf{v}^{(T)} - \mathbf{U}^{(T)}|^2\,dx\right)^{1/2}\\
&\quad\le cg_j^{1/2}(h_j(t))\|\nabla \mathbf{v}^{(T)}\|^3_{L^2(\omega_j(h_j(t)))}\\
&\qquad + c\left(\sqrt{\varepsilon|\mathfrak{F}^{(\mathrm{out})}|^2 + \varepsilon|\overrightarrow{\mathcal{F}}|^2 + |\mathbb{F}^{(\mathrm{inn})}|^2}\right)\|\nabla \mathbf{v}^{(T)}\|_{L^2(\omega_j(h_j(t)))}\\
&\qquad \times\left(\int_{\omega_j(h_j(t))}|\nabla(\mathbf{v}^{(T)}) - \mathbf{U}^{(T)}|^2\,dx\right)^{1/2}\\
&\quad\le cg_j^{1/2}(h_j(t))\|\nabla \mathbf{v}^{(T)}\|^3_{L^2(\omega_j(h_j(t)))}\\
&\qquad + c\left(\sqrt{\varepsilon|\mathfrak{F}^{(\mathrm{out})}|^2 + \varepsilon|\overrightarrow{\mathcal{F}}|^2 + |\mathbb{F}^{(\mathrm{inn})}|^2}\right)\|\nabla \mathbf{v}^{(T)}\|^2_{L^2(\omega_j(h_j(t)))},
\end{aligned}$$

for $j = 1,2,3$, and

$$\begin{aligned}
&\sum_{j=1}^{3}\left|\int_{\omega_j(h_j(t))}\nabla \mathbf{v}^{(T)} : \nabla \mathbf{U}^{(T)}\,dx\right|\\
&\le \sum_{j=1}^{3}\|\nabla \mathbf{v}^{(T)}\|_{L^2(\omega_j(h_j(t)))}\,\|\nabla \mathbf{U}^{(T)}\|_{L^2(\omega_j(h_j(t)))} \le c\sum_{j=1}^{3}\|\nabla \mathbf{v}^{(T)}\|^2_{L^2(\omega_j(h_j(t)))};
\end{aligned}$$

$$\begin{aligned}\biggl|\int_{\Omega(t)}(\mathbf{v}^{(T)}\cdot\nabla)\mathbf{U}^{(T)}\cdot\mathbf{A}\,dx\biggr| &\le \|\nabla\mathbf{U}^{(T)}\|_{L^2(\Omega(t))}\biggl(\int_{\Omega(t)}|\mathbf{v}^{(T)}|^2|\mathbf{A}|^2\,dx\biggr)^{1/2}\\
&\le c\Bigl(\sqrt{\varepsilon|\mathfrak{F}^{(\mathrm{out})}|^2+\varepsilon|\overrightarrow{\mathcal{F}}|^2+|\mathbb{F}^{(\mathrm{inn})}|^2}\Bigr)\\
&\quad\times\biggl(\int_{\Omega(t)}|\nabla\mathbf{v}^{(T)}|^2\kappa(x,t)\,dx+\sum_{j=1}^{3}\|\nabla\mathbf{v}^{(T)}\|^2_{L^2(\omega_j(h_j(t)))}\biggr);\end{aligned}$$

$$\begin{aligned}&\nu\biggl|\int_{\Omega(t)}\nabla\mathbf{A}:\nabla\mathbf{U}^{(T)}\,dx\biggr|+\biggl|\int_{\Omega(t)}(\mathbf{A}\cdot\nabla)\mathbf{U}^{(T)}\cdot\mathbf{A}\,dx\biggr|\\
&\quad\le c\bigl(\|\nabla\mathbf{A}\|_{L^2(\Omega(t))}+\|\mathbf{A}\|^2_{L^4(\Omega(t))}\bigr)\|\nabla\mathbf{U}^{(T)}\|_{L^2(\Omega(t))}\\
&\quad\le C(\mu)\bigl(\|\nabla\mathbf{A}\|_{L^2(\Omega(t))}+\|\mathbf{A}\|^2_{L^4(\Omega(t))}\bigr)^2+\mu\|\nabla\mathbf{U}^{(T)}\|^2_{L^2(\Omega(t))}\\
&\quad\le C(\mu)\cdot c\bigl(\mathbf{a},\overrightarrow{\mathcal{F}}\bigr)\biggl(1+\sum_{i=1}^{2}\int_1^{2h_i(t)}\frac{dr}{r\,g_i^3(r)}+\int_1^{h_3(t)}\frac{dr}{g_3^4(r)}\biggr)\\
&\qquad+\mu\biggl(\int_{\Omega(t)}|\nabla\mathbf{v}^{(T)}|^2\kappa(x,t)\,dx+\sum_{j=1}^{3}\|\nabla\mathbf{v}^{(T)}\|^2_{L^2(\omega_j(h_j(t)))}\biggr).\end{aligned}$$

Hence,

$$\begin{aligned}\nu\int_{\Omega(t)}|\nabla\mathbf{v}^{(T)}|^2\kappa(x,t)\,dx &\le c\sum_{j=1}^{3}g_j^{1/2}(h_j(t))\|\nabla\mathbf{v}^{(T)}\|^3_{L^2(\omega_j(h_j(t)))}\\
&\quad+c\Bigl(\sqrt{\varepsilon|\mathfrak{F}^{(\mathrm{out})}|^2+\varepsilon|\overrightarrow{\mathcal{F}}|^2+|\mathbb{F}^{(\mathrm{inn})}|^2}\Bigr)\sum_{j=1}^{3}\|\nabla\mathbf{v}^{(T)}\|^2_{L^2(\omega_j(h_j(t)))}\\
&\quad+\nu c\sum_{j=1}^{3}\|\nabla\mathbf{v}^{(T)}\|^2_{L^2(\omega_j(h_j(t)))}\\
&\quad+c\Bigl(\sqrt{\varepsilon|\mathfrak{F}^{(\mathrm{out})}|^2+\varepsilon|\overrightarrow{\mathcal{F}}|^2+|\mathbb{F}^{(\mathrm{inn})}|^2}\Bigr)\int_{\Omega(t)}|\nabla\mathbf{v}^{(T)}|^2\kappa(x,t)\,dx\\
&\quad+C(\mu)\cdot c\bigl(\mathbf{a},\overrightarrow{\mathcal{F}}\bigr)\biggl(1+\sum_{i=1}^{2}\int_1^{2h_i(t)}\frac{dr}{r\,g_i^3(r)}+\int_1^{h_3(t)}\frac{dr}{g_3^4(r)}\biggr)\\
&\quad+\mu\biggl(\int_{\Omega(t)}|\nabla\mathbf{v}^{(T)}|^2\kappa(x,t)\,dx+\sum_{j=1}^{3}\|\nabla\mathbf{v}^{(T)}\|^2_{L^2(\omega_j(h_j(t)))}\biggr).\end{aligned}$$

Assuming that $|\mathbb{F}^{(\mathrm{inn})}|\le\nu/(4c)$ and taking ε, μ sufficiently small we obtain

$$\begin{aligned}&\int_{\Omega(t)}|\nabla\mathbf{v}^{(T)}|^2\kappa(x,t)\,dx\\
&\quad\le c_1\sum_{j=1}^{3}g_j^{1/2}(h_j(t))\|\nabla\mathbf{v}^{(T)}\|^3_{L^2(\omega_j(h_j(t)))}+c_2\sum_{j=1}^{3}\|\nabla\mathbf{v}^{(T)}\|^2_{L^2(\omega_j(h_j(t)))}\end{aligned}$$

$$+ c_3 \cdot c\big(\mathbf{a}, \overrightarrow{\mathcal{F}}\big)\bigg(1 + \sum_{i=1}^{2} \int_1^{2h_i(t)} \frac{dr}{r\, g_i^3(r)} + \int_1^{h_3(t)} \frac{dr}{g_3^4(r)}\bigg).$$

Denote $y(t) = \int_{\Omega(t)} |\nabla \mathbf{v}^{(T)}|^2\, \kappa(x,t)\, dx$. Then the last inequality can be rewritten in the form

$$y(t) \le c_1 \sum_{j=1}^{3} g_j^{1/2}(h_j(t)) \|\nabla \mathbf{v}^{(T)}\|^3_{L^2(\omega_j(h_j(t)))} + c_2 \sum_{j=1}^{3} \|\nabla \mathbf{v}^{(T)}\|^2_{L^2(\omega_j(h_j(t)))}$$
$$+ c_3 \cdot c\big(\mathbf{a}, \overrightarrow{\mathcal{F}}\big)\bigg(1 + \sum_{i=1}^{2} \int_1^{2h_i(t)} \frac{dr}{r\, g_i^3(r)} + \int_1^{h_3(t)} \frac{dr}{g_3^4(r)}\bigg).$$

Since (6)

$$\frac{dy(t)}{dt} = \sum_{j=1}^{3} \int_{\omega_j(h_j(t))} |\nabla \mathbf{v}^{(T)}|^2 \frac{\partial \kappa}{\partial t} dx \ge c \sum_{j=1}^{3} g_j^{1/3}(h_j(t)) \|\nabla \mathbf{v}^{(T)}\|^2_{L^2(\omega_j(h_j(t)))}$$

we derive

$$y(t) \le c_* y'(t) + c_{**}(y'(t))^{3/2} + \frac{1}{2}\varphi(t),$$

where

$$\varphi(t) = 2\, c_3 \cdot c\big(\mathbf{a}, \overrightarrow{\mathcal{F}}\big)\bigg(1 + \sum_{i=1}^{2} \int_1^{2h_i(t)} \frac{dr}{y\, g_i^3(r)} + \int_1^{h_3(t)} \frac{dr}{g_3^4(r)}\bigg).$$

Using (2.2) we get

$$\varphi'(t) = c_3 \cdot c\big(\mathbf{a}, \overrightarrow{\mathcal{F}}\big)\bigg(\sum_{i=1}^{2} \frac{2h_i'(t)}{2h_i(t) \cdot g_i^3(2h_i(t))} + \frac{h_3'(t)}{g_3^4(h_3(t))}\bigg) \tag{3.57}$$
$$\le c_3\, c_4 \cdot c\big(\mathbf{a}, \overrightarrow{\mathcal{F}}\big) \sum_{j=1}^{3} g_j^{-8/3}(h_j(t)).$$

Then because of the condition (3.57) we can choose the constant c_3 in such a manner that condition (3.35) should hold, i.e.

$$2\,\theta(\varphi'(t)) = 2\, c_* \varphi'(t) + 2\, c_{**}(\varphi'(t))^{3/2}$$
$$\le 2\, c_*\, c_3\, c_4 \cdot c\big(\mathbf{a}, \overrightarrow{\mathcal{F}}\big) \sum_{j=1}^{3} g_j^{-8/3}(h_j(t)) + 2\, c_{**}\bigg(c_3\, c_4 \cdot c\big(\mathbf{a}, \overrightarrow{\mathcal{F}}\big) \sum_{j=1}^{3} g_j^{-8/3}(h_j(t))\bigg)^{3/2}$$
$$\le c_3\, c_4 \cdot c\big(\mathbf{a}, \overrightarrow{\mathcal{F}}\big)\bigg(2\, c_* \sum_{j=1}^{3} g_j^{-8/3}(h_j(t)) + 8 c_{**} \sqrt{c_3\, c_4 \cdot c\big(\mathbf{a}, \overrightarrow{\mathcal{F}}\big)} \sum_{j=1}^{3} g_j^{-4}(h_j(t))\bigg).$$

Note that $\sum_{j=1}^{3} g_j^{-8/3}(h_j(t))$ and $\sum_{j=1}^{3} g_j^{-4}(h_j(t))$ are bounded for every t and the integrals $\int_1^{2h_i(t)} dr/(r\, g_i^3(r))$, $i = 1, 2$, and $\int_1^{h_3(t)} dr/g_3^4(r)$ tend to infinity as

(6) This estimate follows from (3.50) and (3.51).

$t \to \infty$. Therefore, for sufficiently large t we get the following estimate

$$\theta(\varphi'(t)) \le c_3 \cdot c(\mathbf{a}, \overrightarrow{\mathcal{F}})\bigg(1 + \sum_{i=1}^{2} \int_1^{2h_i(t)} \frac{dr}{r\, g_i^3(r)} + \int_1^{h_3(t)} \frac{dr}{g_3^4(r)}\bigg).$$

Thus, $\varphi(t)$ satisfies conditions of Lemma 3.9 and we obtain the estimate

$$\begin{aligned} y(t) &= \int_{\Omega(t)} |\nabla \mathbf{v}^{(T)}|^2\, \kappa(x,t)\, dx \\ &\le c_3 \cdot c\big(\mathbf{a}, \overrightarrow{\mathcal{F}}\big)\bigg(1 + \sum_{i=1}^{2} \int_1^{2h_i(t)} \frac{dr}{r\, g_i^3(r)} + \int_1^{h_3(t)} \frac{dr}{g_3^4(r)}\bigg). \end{aligned} \tag{3.58}$$

Since for every bounded domain $\Omega(t)$ the embedding $W^{1,2}(\Omega(t)) \hookrightarrow L^4(\Omega(t))$ is compact, the estimate (3.58) guarantees the existence of a subsequence $\{\mathbf{v}^{(T_m)}\}$ which converges weakly in $\mathring{W}^{1,2}(\Omega(t))$ and strongly in $L^4(\Omega(t))$ for any t (such subsequence could be constructed by Cantor diagonal process). Taking in integral identity (3.43) an arbitrary test function $\boldsymbol{\eta}$ with a compact support, we can find such t that $\operatorname{supp} \boldsymbol{\eta} \subset \Omega(t)$ and, hence $\boldsymbol{\eta} \in H(\Omega(t))$. Extending $\boldsymbol{\eta}$ by zero into $\Omega \setminus \Omega(t)$, and considering all the integrals in (3.43) as the integrals over Ω, we can pass in (3.43) to a limit as $T_m \to \infty$. As a result we get for the limit vector function $\mathbf{v}$ integral identity (2.7). Therefore, the sum $\mathbf{u} = \mathbf{A} + \mathbf{v}$ is a weak solution of problem (2.3). Estimate (3.42) for $\mathbf{v}$ follows from (3.58). Since for $\mathbf{A}$ the analogous to (3.42) estimate is also valid, we obtain (3.42) for the sum $\mathbf{u} = \mathbf{A} + \mathbf{v}$. □

REMARK 3.15. All the results obtained in the paper remain valid for the nonhomogeneous Navier–Stokes system if the external force $\mathbf{f}$ have an appropriate decay at infinity.

Acknowledgements. The author expresses her special gratitude to K. Pileckas for posing the problem and useful discussions.

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Manuscript received April 23, 2014

accepted April 3, 2015

Kristina Kaulakytė
Faculty of Mathematics and Informatics
Naugarduko Str. 24
Vilnius University
03225 Vilnius, LITHUANIA

E-mail address: kristina.kaulakyte@mif.vu.lt