Topological **M**ethods in **N**onlinear **A**nalysis
Volume 46, No. 2, 2015, 755–771
DOI: 10.12775/TMNA.2015.067

SOLUTIONS TO A NONLINEAR SCHRÖDINGER EQUATION WITH PERIODIC POTENTIAL AND ZERO ON THE BOUNDARY OF THE SPECTRUM

Jarosław Mederski

Abstract. We study the following nonlinear Schrödinger equation

$$\begin{cases} -\Delta u + V(x)u = g(x,u) & \text{for } x \in \mathbb{R}^N, \\ u(x) \to 0 & \text{as } |x| \to \infty, \end{cases}$$

where $V\colon \mathbb{R}^N \to \mathbb{R}$ and $g\colon \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ are periodic in x. We assume that 0 is a right boundary point of the essential spectrum of $-\Delta + V$. The superlinear and subcritical term g satisfies a Nehari type monotonicity condition. We employ a Nehari manifold type technique in a strongly indefitnite setting and obtain the existence of a ground state solution. Moreover, we get infinitely many geometrically distinct solutions provided that g is odd.

1. Introduction

We are concerned with the following nonlinear Schrödinger equation

$$\begin{cases} -\Delta u + V(x)u = g(x,u) & \text{for } x \in \mathbb{R}^N, \\ u(x) \to 0 & \text{as } |x| \to \infty, \end{cases} \tag{1.1}$$

where $V\colon \mathbb{R}^N \to \mathbb{R}$ is a periodic potential and $g\colon \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ has superlinear growth. This equation appears in mathematical physics, e.g. when one studies

2010 *Mathematics Subject Classification.* Primary: 35Q55; Secondary: 35J10, 35J20, 58E05.

Key words and phrases. Schrödinger equation, ground state, variational methods, strongly indefinite functional, Nehari–Pankov manifold.

The study was supported by research grant NCN 2013/09/B/ST1/01963.

standing waves $\Phi(x,t) = u(x)e^{-iEt/\hbar}$ of the time-dependent Schrödinger equation of the form

$$i\hbar\frac{\partial\Phi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\Phi + W(x)\Phi - f(x,|\Phi|)\Phi.$$

If the potential V is periodic, then (1.1) is of particular interest since it has a wide range of physical applications, e.g. in photonic crystals, where one considers periodic optical nanostructures (see [18] and references therein). It is well-known that the spectrum $\sigma(-\Delta+V)$ of $-\Delta+V$ is purely continuous and may contain gaps, i.e. open intervals free of spectrum (see [21]). When $\inf\sigma(-\Delta+V) > 0$ or 0 lies in a gap of the spectrum $\sigma(-\Delta+V)$ then nonlinear Schrödinger equations have been widely investigated by many authors (see [8], [20], [1], [7], [26], [13], [9] and references therein) and nontrivial solutions to (1.1) have been obtained. Ground state solutions, i.e. nontrivial solutions with the least possible energy, play an important role in physics and their existence has been studied e.g. in [14], [18], [24], [15]. If $V=0$ then $\sigma(-\Delta+V) = [0,+\infty)$ and the problem has been investigated in a classical work [6] or in a recent one [2] (see also references therein). If V is constant and negative then 0 is an interior point of $\sigma(-\Delta+V)$ and solutions to (1.1) have been found in [10].

In the present work, we focus on the situation when 0 lies in the spectrum of $-\Delta+V$ and is the left endpoint of a spectral gap. As far as we know there are only three papers dealing with this case. In [4] Bartsch and Ding obtained a nontrivial solution to (1.1) assuming, among others, the following Ambrosetti–Rabinowitz condition:

$$g(x,u)u \geq \gamma G(x,u) > 0 \quad \text{for some } \gamma > 2 \text{ and all } u \in \mathbb{R}\setminus\{0\},\ x\in\mathbb{R}^N, \tag{1.2}$$

and a lower bound estimate:

$$G(x,u) \geq b|u|^\mu \quad \text{for some } b>0,\ \mu>2 \text{ and all } u\in\mathbb{R},\ x\in\mathbb{R}^N, \tag{1.3}$$

where G is the primitive of g with respect to u. Applying a generalized linking theorem due to Kryszewski and Szulkin [13], they proved that there is a solution in $H^2_{\mathrm{loc}}(\mathbb{R}^N)\cap L^t(\mathbb{R}^N)$ for $\mu \leq t \leq 2^*$, where $2^* = 2N/(N-2)$ if $N\geq 3$, and $2^*=\infty$ if $N=1,2$. If g is odd then the existence of infinitely many geometrically distinct solutions was obtained as well by means of an abstract critical point theory involving the $(\mathrm{PS})_I$-attractor concept (see Section 4 in [4] for details). In [28] Willem and Zou relaxed condition (1.2) and they dealt with the lack of boundedness of Palais–Smale sequences. The authors developed the so-called monotonicity trick for strongly indefinite problems and established weak linking results. Recently Yang, Chen and Ding in [29] considered a Nehari-type monotone condition (see (G5) below) instead of (1.2) and obtained a solution to (1.1) using a variant of weak linking due to Schechter and Zou [23]. The lower bound estimate (1.3) has been assumed so far.

In this paper, our first aim is to prove the existence of a ground state solution to (1.1) under the assumption that 0 lies in the spectrum of $-\Delta + V$ and is the left endpoint of a spectral gap. As far as we know this is the first paper dealing with ground states in this case. Moreover, neither (1.2) nor (1.3) are assumed. Namely, throughout the paper we impose the following conditions.

(V) $V \in C(\mathbb{R}^N, \mathbb{R})$, V is 1-periodic in x_i, $i = 1, \ldots, N$, $0 \in \sigma(-\Delta + V)$ and there exists $\beta > 0$ such that $(0, \beta] \cap \sigma(-\Delta + V) = \emptyset$.

(G1) $g \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$, g is 1-periodic in x_i, $i = 1, \ldots, N$.

(G2) There are $a > 0$ and $2 < \mu \leq p < 2^*$ such that

$$|g(x, u)| \leq a(|u|^{\mu-1} + |u|^{p-1}) \quad \text{for all } u \in \mathbb{R},\ x \in \mathbb{R}^N.$$

(G3) There is $b > 0$ such that

$$G(x, u) \geq b|u|^\mu \quad \text{for all } |u| \leq 1,\ x \in \mathbb{R}^N.$$

(G4) $G(x, u)/|u|^2 \to \infty$ uniformly in x as $|u| \to \infty$.

(G5) $u \mapsto g(x, u)/|u|$ is strictly increasing on $(-\infty, 0)$ and $(0, \infty)$.

We point out that (G3) and (G4) are substantially weaker than (1.3). Indeed, take a nonlinearity of the type

$$g(x, u) = q(x)u \ln(1 + |u|^{p-2}), \quad q(x) \geq \inf_{\mathbb{R}^N} q > 0,$$

where $q\colon \mathbb{R}^N \to \mathbb{R}$ is continuous and 1-periodic in x_i, $i = 1, \ldots, N$. Observe that conditions (G1)–(G5) are obeyed with $2 < \mu = p < 2^*$, but (1.3) does not hold. The above nonlinearity has recently attracted attention of many authors since the Ambrosetti–Rabinowitz condition (1.2) is not satisfied and thus Palais–Smale sequences do not have to be bounded (see e.g. [12], [16], [9], [17]).

Assumptions (V), (G1)–(G5) allow to find a function space $E_{2,\mu}$ (see Section 2) on which the energy functional associated to (1.1)

$$\mathcal{J}(u) := \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + V(x)|u|^2 \, dx - \int_{\mathbb{R}^N} G(x, u) \, dx \tag{1.4}$$

is a well-defined C^1-map. Moreover, critical points of $\mathcal{J}$ correspond to solutions to (1.1). In order to find ground state solutions we consider the Nehari–Pankov manifold $\mathcal{N} \subset E_{2,\mu}$ defined later by (4.1).

Our main results read as follows. For the precise definitions see the next sections.

THEOREM 1.1. *If assumptions* (V), (G1)–(G5) *are satisfied then* (1.1) *has a ground state solution* $u \in \mathcal{N}$ *such that* $\mathcal{J}(u) = \inf_{\mathcal{N}} \mathcal{J} > 0$. *Moreover* $u \in H^2_{\mathrm{loc}}(\mathbb{R}^N) \cap L^t(\mathbb{R}^N)$ *for* $\mu \leq t \leq 2^*$.

Furthermore, we establish the following multiplicity result and we would like to emphasize that (1.2) is not assumed as opposed to [4].

Theorem 1.2. *If assumptions* (V), (G1)–(G5) *are satisfied, g is odd in u, then* (1.1) *has infinitely many pairs $\pm u$ of geometrically distinct solutions in $H^2_{\mathrm{loc}}(\mathbb{R}^N)\cap L^t(\mathbb{R}^N)$ for $\mu\le t\le 2^*$.*

The paper is organized as follows. In the next section we formulate a variational approach to (1.1) and we define a function space $E_{2,\mu}$ such that the energy functional $\mathcal{J}\colon E_{2,\mu}\to\mathbb{R}$ associated with (1.1) is a well-defined C^1-map. Moreover, some embeddings results of $E_{2,\mu}$ are established. In Section 3 we recall the recently obtained critical point theory from [5] which allows to deal with the underlying geometry of $\mathcal{J}$. Next, in Section 4, we introduce the Nehari-Pankov manifold $\mathcal{N}\subset E_{2,\mu}$ on which we minimize $\mathcal{J}$ to find a ground state and we prove Theorem 1.1. Finally, in the last Section 5, the multiplicity result is obtained.

2. Variational setting

Let $H^1(\mathbb{R}^N)$ denote the Sobolev space with the norm $\|\cdot\|_{H^1}$. Let us consider a functional $\mathcal{J}\colon H^1(\mathbb{R}^N)\to\mathbb{R}$ given by formula (1.4). We note that $\mathcal{J}$ is of class C^1 and its critical points correspond to solutions to (1.1). By assumption (V), $H^1(\mathbb{R}^N)$ has the decomposition of the form $E^+\oplus E'$ corresponding to the decomposition of spectrum of $\sigma(S)$ into $\sigma(S)\cap[\beta,\infty)$ and $\sigma(S)\cap(-\infty,0]$, where $S:=-\Delta+V$ with domain $\mathcal{D}(S)=H^2(\mathbb{R}^N)$. We can define a new norm $\|\cdot\|_E$ on E^+ (resp. E') by setting

$$\|u^+\|_E^2:=\int_{\mathbb{R}^N}|\nabla u^+|^2+V(x)|u^+|^2\,dx$$

and

$$\|u'\|_E^2:=-\int_{\mathbb{R}^N}|\nabla u'|^2+V(x)|u'|^2\,dx$$

for $u^+\in E^+$ and $u'\in E'$. Then $\|\cdot\|_E$ is equivalent to $\|\cdot\|_{H^1}$ on E^+ and is weaker than $\|\cdot\|_{H^1}$ on E' (see [4]). Let E be the completion of $H^1(\mathbb{R}^N)$ with respect to $\|\cdot\|_E$. Then $H^1(\mathbb{R}^N)=E^+\oplus E'$ is continuously embedded in E and E is a Hilbert space with the inner product $\langle u,v\rangle_E:=\langle|S|^{1/2}u,|S|^{1/2}v\rangle_{L^2}$, where $\langle\,\cdot\,,\,\cdot\,\rangle_{L^2}$ is the usual inner product in $L^2(\mathbb{R}^N)$. Note that $\mathcal{J}$ can be written as follows

$$\mathcal{J}(u)=\frac12(\|u^+\|_E^2-\|u'\|_E^2)-\int_{\mathbb{R}^N}G(x,u)\,dx=\frac12\|u^+\|_E^2-I(u),$$

where

$$I(u):=\frac12\|u'\|_E^2+\int_{\mathbb{R}^N}G(x,u)\,dx$$

for any $u=u^++u'\in E^+\oplus E'$. We do not know if $\mathcal{J}$ has critical points in $H^1(\mathbb{R}^N)$. Moreover, I is not defined on E owing to our assumptions on $g(x,u)$. Therefore we are going to define a space $E_{2,\mu}$ such that there are continuous embeddings $H^1(\mathbb{R}^N)\subset E_{2,\mu}\subset E$, I is well-defined on $E_{2,\mu}$ and $\mathcal{J}$ admits critical points on $E_{2,\mu}$.

2.1. Function space. Let $(P_\lambda : L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N))_{\lambda\in\mathbb{R}}$ denote the spectral family of S. Let $L' := P_0(L^2(\mathbb{R}^N))$ and $L^+ := (\mathrm{id} - P_0)(L^2(\mathbb{R}^N))$. Then we have the orthogonal decomposition $L^2(\mathbb{R}^N) = L^+ \oplus L'$ and then $E^+ = H^1(\mathbb{R}^N) \cap L^+$, $E' = H^1(\mathbb{R}^N) \cap L'$ (see [21], [19]). Moreover,

$$\|u\|_E^2 = \int_{-\infty}^{\infty} |\lambda|\, d|P_\lambda u|_2^2,$$

where here and in the sequel, $|\cdot|_k$ denotes the usual norm in $L^k(\mathbb{R}^N)$ for any $k \geq 1$.

Let us assume that $2 \leq \nu \leq \mu$. By $L^{\nu,\mu}(\mathbb{R}^N) := L^\nu(\mathbb{R}^N) + L^\mu(\mathbb{R}^N)$ we denote the Banach space of all functions of the form $v = v_1 + v_2$, where $v_1 \in L^\nu(\mathbb{R}^N)$ and $v_2 \in L^\mu(\mathbb{R}^N)$, endowed with the following norm

$$|v|_{\nu,\mu} := \inf\{|v_1|_\nu + |v_2|_\mu |\ v = v_1 + v_2\}.$$

By [3, Proposition 2.5] the infimum in $|\cdot|_{\nu,\mu}$ is attained. Moreover, there is a continuous embedding

$$L^t(\mathbb{R}^N) \subset L^{\nu,\mu}(\mathbb{R}^N)$$

for any $\nu \leq t \leq \mu$ and, if $\nu = \mu$ then norms $|\cdot|_{\nu,\mu}$ and $|\cdot|_\mu$ are equivalent. Let $E'_{\nu,\mu}$ and E'_μ be the completions of E' with respect to the norms

$$\|\cdot\|_{\nu,\mu} = (\|\cdot\|_E^2 + |\cdot|_{\nu,\mu}^2)^{1/2} \quad \text{and} \quad \|\cdot\|_\mu = (\|\cdot\|_E^2 + |\cdot|_\mu^2)^{1/2},$$

respectively. Thus we have the following continuous embeddings

$$E' \subset E'_\mu \subset E'_{\nu,\mu} \subset E.$$

Space E'_μ has been introduced in [4] and note that, if $\nu = \mu$ then $E'_{\nu,\mu} = E'_\mu$ and the norms $\|\cdot\|_{\nu,\mu}$ and $\|\cdot\|_\mu$ are equivalent. In our setting, space $E'_{\nu,\mu}$ with $\nu = 2$ plays an important role because of superlinear growth conditions (G3) and (G4) (cf. Lemma 4.1). The following somewhat surprising observation is crucial for continuous embeddings of $E'_{\nu,\mu}$ into $L^t(\mathbb{R}^N)$ (see Lemma 2.2).

Lemma 2.1. *$E'_{\nu,\mu} = E'_\mu$ and norms $\|\cdot\|_{\nu,\mu}$, $\|\cdot\|_\mu$ are equivalent for any $2 \leq \nu \leq \mu \leq 2^*$.*

Proof. Note that it is enough to show the inclusion $E'_{\nu,\mu} \subset E'_\mu$. Let $u \in E'_{\nu,\mu}$ and we proceed as follows.

Step 1. For any $y \in \mathbb{R}^N$, $r > 0$ and $\varepsilon > 0$ we have $u \in H^2(B(y,r))$ and

$$\|u\|_{H^2(B(y,r))} \leq c(|u|_{L^2(B(y,r+\varepsilon))} + |Su|_{L^2(B(y,r+\varepsilon))}) \tag{2.1}$$

for some constant $c > 0$ depending on r and ε.

Indeed, similarly as in proof of [4, Lemma 2.1], take a sequence $(u_n)_{n\in\mathbb{N}} \subset E'$ such that $\|u_n - u\|_{\nu,\mu} \to 0$ as $n \to \infty$. Note that $E' \subset L' \subset \mathcal{D}(S) = H^2(\mathbb{R}^N)$

because the spectrum of S is bounded below. Since

$$\begin{aligned}|S(u_n-u_m)|_2^2 &= \int_{-\infty}^0 \lambda^2\, d|P_\lambda(u_n-u_m)|_2^2\\ &\le \alpha\int_\alpha^0 \lambda\, d|P_\lambda(u_n-u_m)|_2^2 = -\alpha\|u_n-u_m\|_E^2,\end{aligned}$$

where $\alpha < \inf\sigma(S) < 0$, then Su_n is a Cauchy sequence in $L^2(\mathbb{R}^N)$. Since $u_n\to u$ in $L^{\nu,\mu}(\mathbb{R}^N)$ then, by [3][Prop. 2.14], $u_n\to u$ in $L^\nu(\Omega)$ for any bounded and measurable $\Omega\subset\mathbb{R}^N$, hence the convergence holds in $L^2(\Omega)$ as well. In view of the Calderon–Zygmund inequality (see [11, Theorem 9.11]) there is a constant $c>0$ such that

$$\|u_n-u_m\|_{H^2(B(y,r))}\le c(|u_n-u_m|_{L^2(B(y,r+\varepsilon))}+|S(u_n-u_m)|_{L^2(B(y,r+\varepsilon))}).$$

Thus $u\in H^2(B(y,r))$ and again by the Calderon–Zygmund inequality (2.1) holds.

Step 2.

$$u\in L^{2^*}(\mathbb{R}^N). \tag{2.2}$$

In view of [3, Proposition 2.5] there are $u_1\in L^\nu(\mathbb{R}^N)$ and $u_2\in L^\mu(\mathbb{R}^N)$ such that $u=u_1+u_2$ and $|u|_{\nu,\mu,2^*}^{2^*}=|u_1|_\nu^{2^*}+|u_2|_\mu^{2^*}$ where

$$|v|_{\nu,\mu,k}=(\inf\{|v_1|_\nu^k+|v_2|_\mu^k|\, v=v_1+v_2,\ v_1\in L^\nu(\mathbb{R}^N),\ v_2\in L^\mu(\mathbb{R}^N)\})^{1/k}$$

defines a family of equivalent norms on $L^{\nu,\mu}(\mathbb{R}^N)$ for $k\ge 1$ (see also [3, Proposition 2.4]). Observe that from (2.1), for any $y\in\mathbb{R}^N$, $r>0$ and $\varepsilon>0$

$$\begin{aligned}|u|_{L^{2^*}(B(y,r))} &\le c_1(|u|_{L^2(B(y,r+\varepsilon))}+|Su|_{L^2(B(y,r+\varepsilon))})\\ &\le c_1(|u_1|_{L^2(B(y,r+\varepsilon))}+|u_2|_{L^2(B(y,r+\varepsilon))}+|Su|_{L^2(B(y,r+\varepsilon))})\\ &\le c_2(|u_1|_{L^\nu(B(y,r+\varepsilon))}+|u_2|_{L^\mu(B(y,r+\varepsilon))}+|Su|_{L^2(B(y,r+\varepsilon))}),\end{aligned}$$

for some constants $c_1,c_2>0$ depending on r and ε. Therefore

$$\begin{aligned}\int_{B(y,r)}|u|^{2^*}\,dx \le c_3\bigg(&|u_1|_\nu^{2^*-\nu}\int_{B(y,r+\varepsilon)}|u_1|^\nu\,dx\\ &+|u_2|_\mu^{2^*-\mu}\int_{B(y,r+\varepsilon)}|u_2|^\mu\,dx+|Su|_2^{2^*-2}\int_{B(y,r+\varepsilon)}|Su|^2\,dx\bigg)\end{aligned}$$

for some constant $c_3>0$. For any $r>0$ there is $\varepsilon>0$ and a covering of $\mathbb{R}^N$ by balls $\{B(y,r)\}_{y\in Y}$, where $Y\subset\mathbb{R}^N$ such that each point of $\mathbb{R}^N$ is contained in at most $N+1$ balls $B(y,r+\varepsilon)$. Therefore

$$\int_{\mathbb{R}^N}|u|^{2^*}\,dx\le(N+1)c_3\big(|u_1|_\nu^{2^*}+|u_2|_\mu^{2^*}+|Su|_2^{2^*}\big)=(N+1)c_3\big(|u|_{\nu,\mu,2^*}^{2^*}+|Su|_2^{2^*}\big).$$

Since norms $|\cdot|_{\nu,\mu,2^*}$ and $|\cdot|_{\nu,\mu,1}=|\cdot|_{\nu,\mu}$ are equivalent, then $u\in L^{2^*}(\mathbb{R}^N)$.

Step 3. $u\in E_\mu(\mathbb{R}^N)$.

Indeed, since $u \in L^{\nu,\mu}(\mathbb{R}^N)$ then by [3, Proposition 2.3] we obtain

$$u \in L^{\nu}(\Omega_u) \cap L^{\mu}(\Omega_u^c), \quad \text{where } \Omega_u := \{x \in \mathbb{R}^N \mid |u(x)| > 1\}$$

has finite Lebesgue measure. Since $u \in L^{2^*}(\Omega_u)$ then by the interpolation inequality we get $u \in L^{\mu}(\Omega_u)$. Hence $u \in L^{\mu}(\mathbb{R}^N)$ and $u \in E_{\mu}(\mathbb{R}^N)$. $\square$

From (2.1) and (2.2) or by [4, Lemma 2.1] we infer the following embeddings.

Lemma 2.2. *If $2 \le \nu \le \mu \le 2^*$ then $E'_{\nu,\mu}$ embeds continuously into $H^2_{\mathrm{loc}}(\mathbb{R}^N)$ and $L^t(\mathbb{R}^N)$ for $\mu \le t \le 2^*$, and compactly into $L^t_{\mathrm{loc}}(\mathbb{R}^N)$ for $2 \le t < 2^*$.*

Observe that we obtain continuous embeddings

$$H^1(\mathbb{R}^N) \subset E_{\nu,\mu} := E^+ \oplus E'_{\nu,\mu} \subset E$$

where $E_{\nu,\mu}$ is endowed with the norm

$$\|u\| := (\|u^+\|_E^2 + \|u'\|_{\nu,\mu}^2)^{1/2} \quad \text{for } u = u^+ + u' \in E^+ \oplus E'_{\nu,\mu}.$$

Since $|\cdot|_{\nu,\mu}$ is uniformly convex (see [3, Proposition 2.6]), then $E_{\nu,\mu}$ is reflexive and bounded sequences in $E_{\nu,\mu}$ are relatively weakly compact. In view of the Sobolev embeddings, Lemma 2.2 holds also for $E_{\nu,\mu}$ and $\mathcal{J}\colon E_{\nu,\mu} \to \mathbb{R}$ given by (1.4) is a well-defined C^1-map. Moreover, from Lemma 2.1 and [4, Corollary 2.3] we get that a solution to (1.1) in $E_{\nu,\mu}$ vanishes at infinity.

Corollary 2.3. *If $u \in E_{\nu,\mu}$ solves* (1.1) *then $u(x) \to 0$ as $|x| \to \infty$.*

3. Abstract setting

In this section we are going recall the recent abstract result obtained in [5] which seems to be appropriate in dealing with the geometry and the regularity of energy functional $\mathcal{J}$.

For the purpose of this section we assume that X is an arbitrary reflexive Banach space with norm $\|\cdot\|$ such that $X = X^+ \oplus X'$, X^+, X' are closed subspaces of X and $X^+ \cap X' = \{0\}$. If $u \in X$ then there is the unique decomposition $u = u^+ + u'$ where $u^+ \in X^+$ and $u' \in X'$. We may also assume that $\|u\|^2 = \|u^+\|^2 + \|u'\|^2$. In order to ensure that a unit sphere in X^+

$$S^+ := \{u \in X^+ \mid \|u\| = 1\}$$

is a C^1-submanifold of X^+, we assume that X^+ is a Hilbert space with the scalar product $\langle \cdot, \cdot \rangle$ such that $\langle u, u \rangle = \|u\|^2$ for any $u \in X^+$. In addition to the norm topology we need the topology $\mathcal{T}$ on X which is the product of the norm topology in X^+ and the weak topology in X'. In particular, $u_n \xrightarrow{\mathcal{T}} u$ provided that $u_n^+ \to u^+$ and $u_n' \rightharpoonup u'$.

We define the following Nehari–Pankov manifold (cf. [18])

$$\mathcal{N} := \{u \in X \setminus X' \mid \mathcal{J}'(u)(u) = 0,\ \mathcal{J}'(u)(h') = 0 \text{ for any } h' \in X'\}.$$

We say that $\mathcal{J}$ satisfies the $(\mathrm{PS})_c^{\mathcal{T}}$-condition in $\mathcal{N}$ if every $(\mathrm{PS})_c$-sequence in $\mathcal{N}$ has a subsequence which converges in $\mathcal{T}$:

$$u_n \in \mathcal{N},\ J'(u_n) \to 0,\ \mathcal{J}(u_n) \to c \ \Rightarrow\ u_n \xrightarrow{\mathcal{T}} u \in X \text{ along a subsequence.}$$

Theorem 3.1 (see [5]). *Let $J \in C^1(X, \mathbb{R})$ be a map of the form*

$$\mathcal{J}(u) = \frac{1}{2}\|u^+\|^2 - I(u) \tag{3.1}$$

for any $u = u^+ + u' \in X^+ \oplus X'$ such that:

- (J1) *$I(u) \geq I(0) = 0$ for any $u \in X$ and, I is $\mathcal{T}$-sequentially lower semicontinuous, i.e. if $u_n \xrightarrow{\mathcal{T}} u_0$ then $\liminf\limits_{n\to\infty} I(u_n) \geq I(u_0)$.*
- (J2) *If $u_n \xrightarrow{\mathcal{T}} u_0$ and $I(u_n) \to I(u_0)$ then $u_n \to u_0$.*
- (J3) *If $u \in \mathcal{N}$ then $\mathcal{J}(u) > \mathcal{J}(tu + h')$ for any $t \geq 0$, $h' \in X'$ such that $tu + h' \neq u$.*
- (J4) *$0 < \inf\limits_{u\in X^+,\ \|u\|=r} \mathcal{J}(u)$.*
- (J5) *$\|u^+\| + I(u) \to \infty$ as $\|u\| \to \infty$.*
- (J6) *$I(t_n u_n)/t_n^2 \to \infty$ if $t_n \to \infty$ and $u_n^+ \to u_0^+$ for some $u_0^+ \neq 0$ as $n \to \infty$.*

Then:

- (a) *$c := \inf\limits_{\mathcal{N}} \mathcal{J} > 0$ and there exists a $(\mathrm{PS})_c$-sequence $(u_n)_{n\in\mathbb{N}} \subset \mathcal{N}$, i.e. $\mathcal{J}(u_n) \to c$ and $\mathcal{J}'(u_n) \to 0$ as $n \to \infty$. If $\mathcal{J}$ satisfies the $(\mathrm{PS})_c^{\mathcal{T}}$-condition in $\mathcal{N}$ then c is achieved by a critical point of $\mathcal{J}$.*
- (b) *There is a homeomorphism $n\colon S^+ \to \mathcal{N}$ such that $n^{-1}(u) = u^+/\|u^+\|$, $n(u)$ is the unique maximum of $\mathcal{J}$ on $\mathbb{R}^+ u \oplus X'$ for $u \in \mathcal{N}$ and $\mathcal{J} \circ n\colon S^+ \to \mathbb{R}$ is of class C^1. Moreover, a sequence $(u_n) \subset S^+$ is a Palais–Smale sequence for $\mathcal{J} \circ n$ if and only if $n(u_n) \subset \mathcal{N}$ is a Palais–Smale sequence for $\mathcal{J}$, and $u \in S^+$ is a critical point of $\mathcal{J} \circ n$ if and only if $n(u)$ is a critical point of $\mathcal{J}$.*

Proof of Theorem 3.1 is based on the Ekeland's variational applied to a map $\mathcal{J} \circ n\colon S^+ \to \mathbb{R}$. Some steps of the proof are enlisted in (b) since they play a crucial role in Section 5 (see [5], cf. [25]).

4. Ground state solutions

We are going to look for critical points of $\mathcal{J}\colon E_{2,\mu} \to \mathbb{R}$ on the following *Nehari–Pankov manifold*

$$\mathcal{N} := \{u \in E_{2,\mu} \setminus E'_{2,\mu} \mid \mathcal{J}'(u)(u) = 0,\ \mathcal{J}'(u)(h') = 0 \text{ for any } h' \in E'_{2,\mu}\}. \tag{4.1}$$

The idea to consider a Nehari-type manifold for indefinite problems was firstly observed by Pankov in [18]. If $\mathcal{J} \in C^2(E_{2,\mu}, \mathbb{R})$ and under some additional assumptions, $\mathcal{N}$ is a C^1-submanifold of $E_{2,\mu}$ (see [18], [25]). However we assume only that $\mathcal{J}$ is of C^1-class and $\mathcal{N}$ does not have to be a C^1-submanifold of $E_{2,\mu}$.

In order to find a minimizing Palais–Smale sequence we need to check assumptions (J1)–(J6) of Theorem 3.1 by setting $X^+ := E^+$ and $X' := E'_{2,\mu}$. Firstly observe that the following inequality holds.

Lemma 4.1. *There is a constant $c > 0$ such that for any $u \in E_{2,\mu}$*

$$\int_{\mathbb{R}^N} G(x,u)\,dx \geq c\min\{|u|_{2,\mu}^2, |u|_{2,\mu}^\mu\}. \tag{4.2}$$

Proof. Note that by (G2) and (G5) we know that $G(x,u) > 0$ if $u \neq 0$. Therefore (G3) and (G4) imply that there is $b' > 0$ such that

$$G(x,u) \geq b'\min\{|u|^2, |u|^\mu\} \tag{4.3}$$

for all $u \in \mathbb{R}$ and $x \in \mathbb{R}^N$. Then we infer that for $u \in E_{2,\mu}$

$$\int_{\mathbb{R}^N} G(x,u)\,dx \geq b'\bigg(\int_{\Omega_u} |u|^2\,dx + \int_{\Omega_u^c} |u|^\mu\,dx\bigg) = b'(|u\chi_{\Omega_u}|_2^2 + |u\chi_{\Omega_u^c}|_\mu^\mu),$$

where χ denoted the characteristic function and $\Omega_u := \{x \in \mathbb{R}^N \mid |u(x)| > 1\}$ is bounded. In view of [3, Proposition 2.4])

$$|u|_{2,\mu,\infty} := \inf\{\max\{|u_1|_2, |u_2|_\mu\} \mid u = u_1 + u_2,\ u_1 \in L^2(\mathbb{R}^N),\ u_2 \in L^\mu(\mathbb{R}^N)\}$$

defines a norm on $L^{2,\mu}(\mathbb{R}^N)$ equivalent with $|\cdot|_{2,\mu}$.

Observe that if $|u\chi_{\Omega_u}|_2 \geq |u\chi_{\Omega_u^c}|_\mu$ then

$$|u\chi_{\Omega_u}|_2^2 + |u\chi_{\Omega_u^c}|_\mu^\mu \geq (\max\{|u\chi_{\Omega_u}|_2, |u\chi_{\Omega_u^c}|_\mu\})^2 \geq |u|_{2,\mu,\infty}^2$$

and if $|u\chi_{\Omega_u}|_2 < |u\chi_{\Omega_u^c}|_\mu$ then

$$|u\chi_{\Omega_u}|_2^2 + |u\chi_{\Omega_u^c}|_\mu^\mu \geq (\max\{|u\chi_{\Omega_u}|_2, |u\chi_{\Omega_u^c}|_\mu\})^\mu \geq |u|_{2,\mu,\infty}^\mu.$$

Therefore

$$\int_{\mathbb{R}^N} G(x,u)\,dx \geq b'\min\{|u|_{2,\mu,\infty}^2, |u|_{2,\mu,\infty}^\mu\} \geq c\min\{|u|_{2,\mu}^2, |u|_{2,\mu}^\mu\},$$

for some constant $c > 0$. $\square$

The following lemma shows that (J4)–(J6) hold for $\mathcal{J}$.

Lemma 4.2. *The following conditions hold*:

(a) $0 < \inf_{u \in E^+,\ \|u\| = r} \mathcal{J}(u)$.

(b) $\|u^+\| + I(u) \to \infty$ *as* $\|u\| \to \infty$.

(c) $I(t_n u_n)/t_n^2 \to \infty$ *if* $u_n^+ \to u_0^+$ *for some* $u_0^+ \neq 0$ *and* $t_n \to \infty$ *as* $n \to \infty$.

Proof. (a) If $u \in E^+$ then, by (G2),

$$\mathcal{J}(u) \geq \frac{1}{2}\|u\|_E^2 - \frac{a}{\mu}|u|_\mu^\mu - \frac{a}{p}|u|_p^p.$$

Since E^+ is continuously embedded in $L^\mu(\mathbb{R}^N)$ and in $L^p(\mathbb{R}^N)$ then

$$\mathcal{J}(u) \geq \frac{1}{2}\|u\|_E^2 - C_1(\|u\|_E^\mu + \|u\|_E^p)$$

for some constant $C_1 > 0$. Thus we get the inequality in (a).

(b) Suppose that $\|u_n\| \to \infty$ as $n \to \infty$ and $(\|u_n^+\|_E)_{n\in\mathbb{N}}$ is bounded. Then $(|u_n^+|_{2,\mu})_{n\in\mathbb{N}}$ is bounded and

$$\|u_n'\|_{2,\mu}^2 = \|u_n'\|_E^2 + |u_n'|_{2,\mu}^2 \to \infty \quad \text{as } n \to \infty.$$

If along a subsequence $\|u_n'\|_E \to \infty$ then obviously $I(u_n) \to \infty$. Assume that $(\|u_n'\|_E)_{n\in\mathbb{N}}$ is bounded. Then $|u_n'|_{2,\mu} \to \infty$ and, by (4.2), $I(u_n) \to \infty$ as $n \to \infty$.

(c) Suppose that, up to a subsequence, $I(t_n u_n)/t_n^2$ is bounded, $u_n^+ \to u_0^+$ for some $u_0^+ \in E^+ \setminus \{0\}$ and $t_n \to \infty$ as $n \to \infty$. Note that by (4.2)

$$\frac{I(t_n u_n)}{t_n^2} \geq \frac{1}{2}\|u_n'\|_E^2 + c\min\{|u_n|_{2,\mu}^2, t_n^{\mu-2}|u_n|_{2,\mu}^{\mu}\},$$

and then $(\|u_n'\|_{2,\mu})_{n\in\mathbb{N}}$ is bounded. In view of Lemma 2.2 we may assume that $u_n' \rightharpoonup u_0'$ in $E_{2,\mu}'$ and $u_n'(x) \to u_0'(x)$ almost everywhere on $\mathbb{R}^N$. If the Lebesgue measure $|\Omega| > 0$, where $\Omega := \{x \in \mathbb{R}^N \mid u_0^+(x) + u_0'(x) \neq 0\}$, then by (G4) and Fatou's lemma

$$\int_{\mathbb{R}^N} \frac{G(x, t_n u_n)}{t_n^2}\, dx \to \infty.$$

Thus we obtain that $I(t_n u_n)/t_n^2 \to \infty$ which is a contradiction. Therefore $|\Omega| = 0$ and $u_0' = -u_0^+$ almost everywhere on $\mathbb{R}^N$. Since $\langle u_0', u_0^+\rangle_E = 0$ then $u_0^+ = 0$. The obtained contradiction implies that $I(t_n u_n)/t_n^2 \to \infty$. □

We recall that $u_n \xrightarrow{\mathcal{T}} u_0$ provided that $u_n^+ \to u_0^+$ in E^+ and $u_n' \rightharpoonup u_0'$ in $E_{2,\mu}'$ (see Section 3).

LEMMA 4.3. *The following conditions hold*:

(a) *$I(u) \geq 0$ for any $u \in E_{2,\mu}$ and I is $\mathcal{T}$-sequentially lower semicontinuous.*
(b) *If $u_n \xrightarrow{\mathcal{T}} u_0$ and $I(u_n) \to I(u_0)$ then $u_n \to u_0$.*
(c) *If $u \in \mathcal{N}$ then $\mathcal{J}(u) > \mathcal{J}(tu + h')$ for any $t \geq 0$, $h' \in E_{2,\mu}'$ such that $tu + h' \neq u$.*

PROOF. (a) Let $u_n \xrightarrow{\mathcal{T}} u_0$. Since $E_{2,\mu}$ is compactly embedded in $L^2_{\mathrm{loc}}(\mathbb{R}^N)$, then we may assume that $u_n \to u_0$ in $L^2_{\mathrm{loc}}(\mathbb{R}^N)$ and $u_n(x) \to u(x)$ almost everywhere in $\mathbb{R}^N$. In view of the Fatou's lemma and the weakly sequentially lower semicontinuity of the map $E' \ni u' \mapsto \|u'\|_E^2/2 \in \mathbb{R}$, we get $\liminf_{n\to\infty} I(u_n) \geq I(u_0)$.

(b) Let $u_n \xrightarrow{\mathcal{T}} u_0$ and $I(u_n) \to I(u_0)$. Since $E' \ni u' \mapsto \|u'\|_E^2/2 \in \mathbb{R}$ is weakly sequentially lower semicontinuous and $E_{2,\mu} \ni u \mapsto \int_{\mathbb{R}^N} G(x,u)\, dx \in \mathbb{R}$ is $\mathcal{T}$-sequentially lower semicontinuous, then $\lim_{n\to\infty} \|u_n'\|_E^2 = \|u_0'\|_E^2$ and

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} G(x, u_n)\, dx = \int_{\mathbb{R}^N} G(x, u_0)\, dx. \tag{4.4}$$

Note that, along a subsequence,

$$\|u_n' - u_0'\|_E^2 = \|u_n'\|_E^2 - \|u_0'\|_E^2 - 2\langle u_n' - u_0', u_0'\rangle_E \to 0.$$

Hence $u_n = u_n^+ + u_n' \to u_0 = u_0^+ + u_0'$ in E. Thus we need to show that $u_n' \to u_0'$ in $L^{2,\mu}(\mathbb{R}^N)$. Since $E_{2,\mu}$ is compactly embedded in $L^2_{\mathrm{loc}}(\mathbb{R}^N)$, then we may assume that $u_n(x) \to u_0(x)$ almost everywhere in $\mathbb{R}^N$. Observe that

$$\begin{aligned}(4.5)\qquad \int_{\mathbb{R}^N} G(x,u_n) - G(x,u_n-u_0)\,dx &= \int_{\mathbb{R}^N}\int_0^1 \frac{d}{dt}G(x,u_n-u_0+tu_0)\,dt\,dx \\ &= \int_0^1\int_{\mathbb{R}^N} g(x,u_n-u_0+tu_0)u_0\,dx\,dt.\end{aligned}$$

Thus by (G2) for any $\Omega \subset \mathbb{R}^N$

$$\begin{aligned}\int_\Omega |g(x,u_n-u_0+tu_0)u_0|\,dx \le{}& a|u_n-u_0+tu_0|_\mu^{\mu-1}|u_0\chi_\Omega|_\mu \\ &+ a|u_n-u_0+tu_0|_p^{p-1}|u_0\chi_\Omega|_p\end{aligned}$$

In view of Lemma 2.2 we obtain that $(u_n-u_0+tu_0)_{n\in N}$ is bounded in $L^\mu(\mathbb{R}^N)$ and in $L^p(\mathbb{R}^N)$. Therefore, for any $\varepsilon > 0$, there is $\delta > 0$ such that, for any Ω with the Lebesgue measure $|\Omega| < \delta$, we have

$$\int_\Omega |g(x,u_n-u_0+tu_0)u_0|\,dx < \varepsilon$$

for any $n \in \mathbb{N}$. Thus $(g(x,u_n-u_0+tu_0)u_0)_{n\in\mathbb{N}}$ is uniformly integrable. Moreover, for any $\varepsilon > 0$ there is $\Omega \subset \mathbb{R}^N$, $|\Omega| < +\infty$, such that for any $n \in \mathbb{N}$

$$\int_{\mathbb{R}^N\setminus\Omega} |g(x,u_n-u_0+tu_0)u_0|\,dx < \varepsilon.$$

Hence a family $(g(x,u_n-u_0+tu_0)u_0)_{n\in\mathbb{N}}$ is tight over $\mathbb{R}^N$.

Since $g(u_n-u_0+tu_0)u_0 \to g(tu_0)u_0$ almost everywhere in $\mathbb{R}^N$, then in view of the Vitali convergence theorem $g(x,tu_0)u_0$ is integrable and

$$\int_{\mathbb{R}^N} g(x,u_n-u_0+tu_0)u_0\,dx \to \int_{\mathbb{R}^N} g(x,tu_0)u_0\,dx$$

as $n\to\infty$. By (4.5) we obtain

$$\int_{\mathbb{R}^N} G(x,u_n) - G(x,u_n-u_0)\,dx \to \int_0^1\int_{\mathbb{R}^N} g(x,tu_0)u_0\,dx\,dt = \int_{\mathbb{R}^N} G(x,u_0)\,dx$$

as $n\to\infty$. Taking into account (4.4) we get

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} G(x,u_n-u_0)\,dx = 0$$

and by (4.2) we have $u_n \to u_0$ in $L^{2,\mu}(\mathbb{R}^N)$. Hence $u_n' \to u_0'$ in $L^{2,\mu}(\mathbb{R}^N)$.

(c) Let $u \in \mathcal{N}$. Note that for any $t \ge 0$ and $h' \in E_{2,\mu}'$

$$\mathcal{J}(tu+h') - \mathcal{J}(u) = \frac{t^2-1}{2}\|u^+\|^2 + I(u) - I(tu+h').$$

Since $u \in \mathcal{N}$ and $\mathcal{J}'(u)(u) = \|u^+\|^2 - I'(u)(u)$, then for $u \neq tu + h'$

$$
\begin{aligned}
&\mathcal{J}(tu+h') - \mathcal{J}(u) = I'(u)\bigg(\frac{t^2-1}{2}u + th'\bigg) + I(u) - I(tu+h') \\
&= -\frac{1}{2}\|h'\|_E^2 + \int_{\mathbb{R}^N} g(x,u)\bigg(\frac{t^2-1}{2}u + th'\bigg) + G(x,u) - G(x,tu+h')\,dx < 0,
\end{aligned}
$$

where the last inequality follows from [24, Lemma 2.2]. □

Since $E^+ \subset H^1(\mathbb{R}^N)$ and $H^1(\mathbb{R}^N)$ is not compactly embedded in $L^\mu(\mathbb{R}^N)$ and $L^p(\mathbb{R}^N)$, then we do not know if $\mathcal{J}$ satisfies $(\mathrm{PS})_c^{\mathcal{T}}$-condition in $\mathcal{N}$ (see Section 3, cf. [5]). Moreover, Palais–Smale sequences do not have to be bounded since we do not assume (1.2) (cf. [12]). However the boundedness is attainable on $\mathcal{N}$.

LEMMA 4.4. *$\mathcal{J}$ is coercive on $\mathcal{N}$, i.e. $\mathcal{J}(u) \to \infty$ as $\|u\| \to \infty$, $u \in \mathcal{N}$.*

PROOF. Suppose that $\|u_n\| \to \infty$ as $n \to \infty$, $u_n \in \mathcal{N}$ and $\mathcal{J}(u_n) \leq c_1$ for some constant $c_1 > 0$. Let $v_n := u_n/\|u_n\|$. Since $E_{2,\mu}$ is reflexive and compactly embedded in $L^2_{\mathrm{loc}}(\mathbb{R}^N)$ then, up to a subsequence, $v_n \rightharpoonup v$ in $E_{2,\mu}$ and $v_n(x) \to v(x)$ almost everywhere in $\mathbb{R}^N$. Moreover, there is a sequence $(y_n)_{n\in\mathbb{N}} \subset \mathbb{R}^N$ such that

$$
\liminf_{n\to\infty} \int_{B(y_n,1)} |v_n^+|^2\,dx > 0. \tag{4.6}
$$

Otherwise, in view of Lions lemma (see [27, Lemm 1.21]) we get that $v_n^+ \to 0$ in $L^t(\mathbb{R}^N)$ for $2 < t < 2^*$. By (G2) we get

$$
\int_{\mathbb{R}^N} G(x, sv_n^+)\,dx \to 0 \quad \text{for any } s \geq 0.
$$

Let us fix $s \geq 0$. Hence, by Lemma 4.3(c),

$$
c_1 \geq \limsup_{n\to\infty} \mathcal{J}(u_n) \geq \limsup_{n\to\infty} \mathcal{J}(sv_n^+) = \frac{s^2}{2}\limsup_{n\to\infty} \|v_n^+\|^2. \tag{4.7}
$$

By (4.2) and in view of Theorem 3.1(a) we have

$$
\frac{1}{2}(\|u_n^+\|^2 - \|u_n'\|_E^2) - c\min\{|u_n|_{2,\mu}^2, |u_n|_{2,\mu}^\mu\} \geq \mathcal{J}(u_n) \geq c_{\mathrm{inf}} := \inf_{\mathcal{N}} \mathcal{J} > 0.
$$

If $\liminf_{n\to\infty} |u_n|_{2,\mu} = 0$ then, up to a subsequence, $|u_n|_{2,\mu} \to 0$ and for sufficiently large n,

$$
\begin{aligned}
2\|u_n^+\|^2 &\geq \|u_n^+\|^2 + \|u_n'\|_E^2 + 2c_{\mathrm{inf}} + 2c\min\{|u_n|_{2,\mu}^2, |u_n|_{2,\mu}^\mu\} \\
&\geq \|u_n^+\|^2 + \|u_n'\|_E^2 + |u_n|_{2,\mu}^2 = \|u_n\|^2.
\end{aligned}
$$

If $\liminf_{n\to\infty} |u_n|_\mu > 0$ then there is $c_2 \in (0,1)$ such that, for sufficiently large n,

$$
\begin{aligned}
2\|u_n^+\|^2 &\geq \|u_n^+\|^2 + \|u_n'\|_E^2 + 2c_{\mathrm{inf}} + 2c\min\{|u_n|_{2,\mu}^2, |u_n|_{2,\mu}^\mu\} \\
&\geq c_2(\|u_n^+\|^2 + \|u_n'\|_E^2 + |u_n|_{2,\mu}^2) = c_2\|u_n\|^2.
\end{aligned}
$$

Therefore, passing to a subsequence if necessary, $c_3 := \inf_{n\in\mathbb{N}} \|v_n^+\|^2 > 0$ and, by (4.7), $c_1 \geq s^2 c_3/2$ for any $s \geq 0$. The obtained contradiction shows that (4.6) holds. Then we may assume that $(y_n) \in \mathbb{Z}^N$ and

$$\liminf_{n\to\infty} \int_{B(y_n,r)} |v_n^+|^2\,dx > 0$$

for some $r > 1$. Since $\mathcal{J}$ and $\mathcal{N}$ are invariant under translations of the form $u \mapsto u(\,\cdot - k)$, $k \in \mathbb{Z}^N$, then we may assume that $v_n^+ \to v^+$ in $L^2_{\mathrm{loc}}(\mathbb{R}^N)$ and $v^+ \neq 0$. Note that if $v(x) \neq 0$ then $u_n(x) = v_n(x)\|u_n\| \to \infty$ and, by (G4),

$$\frac{G(x,u_n(x))}{\|u_n\|^2} = \frac{G(x,u_n(x))}{|u_n(x)|^2}|v_n(x)|^2 \to \infty \quad \text{as } n \to \infty.$$

Therefore by Fatou's lemma

$$\frac{\mathcal{J}(u_n)}{\|u_n\|^2} = \frac{1}{2}(\|v_n^+\|^2 - \|v_n'\|_E^2) - \int_{\mathbb{R}^N} \frac{G(x,u_n(x))}{\|u_n\|^2}\,dx \to -\infty.$$

Thus we get a contradiction and we conclude the coercivity. □

Proof of Theorem 1.1. In view of Theorem 3.1(a) $c_{\mathrm{inf}} = \inf_{\mathcal{N}} \mathcal{J} > 0$ and there exists a $(\mathrm{PS})_{c_{\mathrm{inf}}}$-sequence $(u_n)_{n\in\mathbb{N}} \subset \mathcal{N}$, i.e. $\mathcal{J}(u_n) \to c_{\mathrm{inf}}$ and $\mathcal{J}'(u_n) \to 0$ as $n \to \infty$. By Lemma 4.4 we get that $(u_n)_{n\in\mathbb{N}}$ is bounded and after passing to a subsequence $u_n \rightharpoonup u$ in $E_{2,\mu}$. Then there is a sequence $(y_n) \in \mathbb{R}^N$ such that

$$\liminf_{n\to\infty} \int_{B(y_n,1)} |u_n^+|^2\,dx > 0. \tag{4.8}$$

Otherwise, in view of Lions lemma (see [27, Lemma 1.21]), $u_n^+ \to 0$ in $L^t(\mathbb{R}^N)$ for $2 < t < 2^*$. By (G2) we obtain

$$\|u_n^+\|^2 = \mathcal{J}'(u_n)(u_n^+) + \int_{\mathbb{R}^N} g(x,u_n)u_n^+\,dx \to 0$$

as $n \to \infty$. Hence

$$0 < c_{\mathrm{inf}} = \lim_{n\to\infty} \mathcal{J}(u_n) \leq \lim_{n\to\infty} \frac{1}{2}\|u_n^+\|^2 = 0$$

and we get a contradiction. Therefore (4.8) holds and we may assume that there is a sequence $(y_n) \in \mathbb{Z}^N$ such that

$$\liminf_{n\to\infty} \int_{B(y_n,r)} |u_n^+|^2\,dx > 0 \tag{4.9}$$

for some $r > 1$. Since $\|u_n(\,\cdot + y_n)\| = \|u_n\|$, then there is $u \in E_{2,\mu}$ such that, up to a subsequence, $u_n(\,\cdot + y_n) \rightharpoonup u$ in $E_{2,\mu}$, $u_n(x+y_n) \to u(x)$ almost everywhere on $\mathbb{R}^N$ and $u_n^+(\,\cdot + y_n) \to u^+$ in $L^2_{\mathrm{loc}}(\mathbb{R}^N)$. By (4.9) we get $u^+ \neq 0$ and then $u \neq 0$. Since $\mathcal{J}$ and $\mathcal{N}$ are invariant under translations of the form $u \mapsto u(\,\cdot + y)$, $y \in \mathbb{Z}^N$, then $\mathcal{J}'(u) = 0$. Observe that $u \in \mathcal{N}$, and by (G2) and (G5)

$$\frac{1}{2}g(x,u_n(x+y_n))u_n(x+y_n) - G(x,u_n(x+y_n)) \geq 0.$$

Therefore, in view of the Fatou's lemma,

$$\begin{aligned} c_{\mathrm{inf}} &= \lim_{n\to\infty} \mathcal{J}(u_n(\,\cdot\,+y_n)) \\ &= \lim_{n\to\infty} \bigg(\mathcal{J}(u_n(\,\cdot\,+y_n)) - \frac{1}{2}\mathcal{J}'(u_n(\,\cdot\,+y_n))u_n(\,\cdot\,+y_n)\bigg) \geq \mathcal{J}(u). \end{aligned}$$

Thus we get $\mathcal{J}(u) = c_{\mathrm{inf}}$. Since $u \in E_{2,\mu}$ is a solution to (1.1), then by Corollary 2.3 we get $u(x) \to 0$ as $|x| \to \infty$. $\square$

5. Multiple solutions

Note that if $u \in E_{2,\mu}$ is a critical point of $\mathcal{J}$ then the orbit under the action of $\mathbb{Z}^N$, $\mathcal{O}(u) := \{u(\,\cdot\,-k) \mid k \in \mathbb{Z}^N\}$ consists of critical points. Two critical points $u_1, u_2 \in E_{2,\mu}$ are said to be *geometrically distinct* if $\mathcal{O}(u_1) \cap \mathcal{O}(u_2) = \emptyset$. In view of Theorem 3.1(b) we know that $\Psi := \mathcal{J} \circ n\colon S^+ \to \mathbb{R}$ is a C^1 map. Observe that in order to prove Theorem 1.2 it is enough to show that Ψ has infinitely many geometrically distinct critical points (see Theorem 3.1(b)). The following lemma is crucial in the consideration of the multiplicity of critical points (cf. [24, Lemma 2.14]).

Lemma 5.1. *Let $d \geq c_{\mathrm{inf}}$. If (u_n^1), $(u_n^2) \subset \Psi^d := \{u \in S^+ \mid \Psi(u) \leq d\}$ are two Palais–Smale sequences for Ψ, then either $\|u_n^1 - u_n^2\| \to 0$ as $n \to \infty$ or*

$$\begin{aligned} \text{(5.1)}\quad \limsup_{n\to\infty} &\|u_n^1 - u_n^2\| \\ &\geq \rho(d) \inf\{\|u_1 - u_2\| \mid \Psi'(u_1) = \Psi'(u_2) = 0,\ u_1 \neq u_2 \in S^+\}, \end{aligned}$$

where $\rho(d) > 0$ depends on d but not on the particular choice of Palais–Smale sequences.

Proof. Let (u_n^1), $(u_n^2) \subset \Psi^d := \{u \in S^+ \mid \Psi(u) \leq d\}$ be two Palais–Smale sequences for Ψ. Let us consider two cases.

Case 1. $|n(u_n^1)^+ - n(u_n^2)^+|_\mu \to 0$ and $|n(u_n^1)^+ - n(u_n^2)^+|_p \to 0$. Observe that by (G2)

$$\begin{aligned} \|n(u_n^1)^+ &- n(u_n^2)^+\|^2 \\ &= \mathcal{J}'(n(u_n^1))(n(u_n^1)^+ - n(u_n^2)^+) - \mathcal{J}'(n(u_n^2))(n(u_n^1)^+ - n(u_n^2)^+) \\ &\quad + \int_{\mathbb{R}^N} (g(x, n(u_n^1)) - g(x, n(u_n^2)))(n(u_n^1)^+ - n(u_n^2)^+)\,dx \\ &\leq \mathcal{J}'(n(u_n^1))(n(u_n^1)^+ - n(u_n^2)^+) - \mathcal{J}'(n(u_n^2))(n(u_n^1)^+ - n(u_n^2)^+) \\ &\quad + a(|n(u_n^1)|_\mu^{\mu-1} + |n(u_n^2)|_\mu^{\mu-1}) \cdot |n(u_n^1)^+ - n(u_n^2)^+|_\mu \\ &\quad + a(|n(u_n^1)|_p^{p-1} + |n(u_n^2)|_p^{p-1}) \cdot |n(u_n^1)^+ - n(u_n^2)^+|_p. \end{aligned}$$

By Theorem 3.1(b) we know that $(n(u_n^1))_{n\in\mathbb{N}}$ and $(n(u_n^2))_{n\in\mathbb{N}}$ are Palais–Smale sequences for $\mathcal{J}$ and, by Lemma 4.4, they are bounded in $E_{2,\mu}$. Since $E_{2,\mu}$ is continuously embedded in $L^\mu(\mathbb{R}^N)$ and in $L^p(\mathbb{R}^N)$, then

$$\|n(u_n^1)^+ - n(u_n^2)^+\| \to 0.$$

Observe that, if $u = u^+ + u' \in \mathcal{N}$ then inequality $\mathcal{J}(u) \geq c_{\inf}$ implies that

$$\|u^+\| \geq \max\{\sqrt{2c_{\inf}}, \|u'\|_E\}. \tag{5.2}$$

Therefore, similarly as in [24, Lemma 2.13], we infer that

$$\|u_n^1 - u_n^2\| = \left\| \frac{n(u_n^1)^+}{\|n(u_n^1)^+\|} - \frac{n(u_n^1)^+}{\|n(u_n^1)^+\|} \right\| \leq \sqrt{\frac{2}{c_{\inf}}} \|n(u_n^1)^+ - n(u_n^2)^+\|.$$

Thus $\|u_n^1 - u_n^2\| \to 0$.

Case 2. $|n(u_n^1)^+ - n(u_n^2)^+|_\mu \nrightarrow 0$ or $|n(u_n^1)^+ - n(u_n^2)^+|_p \nrightarrow 0$.

In view of Lions lemma [27, Lemma 1.21] there is a sequence $(y_n) \in \mathbb{Z}^N$ and $r > 1$ such that

$$\liminf_{n\to\infty} \int_{B(y_n,r)} |n(u_n^1)^+ - n(u_n^2)^+|^2 \, dx > 0. \tag{5.3}$$

Then we may assume that, up to a subsequence,

$$\begin{aligned} n(u_n^1)(\,\cdot\, + y_n) &\rightharpoonup v_1, & n(u_n^2)(\,\cdot\, + y_n) &\rightharpoonup v_2 \quad \text{in } E_{2,\mu}, \\ n(u_n^1)^+(\,\cdot\, + y_n) &\to v_1^+, & n(u_n^2)^+(\,\cdot\, + y_n) &\to v_2^+ \quad \text{in } L^2_{\mathrm{loc}}(\mathbb{R}^N) \end{aligned}$$

and $\|n(u_n^1)^+(\,\cdot\, + y_n)\| \to \alpha_1$, $\|n(u_n^2)^+(\,\cdot\, + y_n)\| \to \alpha_2$ for some α_1, $\alpha_2 \geq \sqrt{2c_{\inf}}$. From (5.3) we infer that $v_1^+ \neq v_2^+$ and thus $v_1 \neq v_2$. Since n, n^{-1}, $\mathcal{J}'$, $(\mathcal{J} \circ n)'$ are equivariant with respect to $\mathbb{Z}^N$-action, then $\mathcal{J}'(v_1) = \mathcal{J}'(v_2) = 0$. Observe that if $v_1 \neq 0$ and $v_2 \neq 0$ then v_1, $v_2 \in \mathcal{N}$ and

$$\begin{aligned} \liminf_{n\to\infty} \|u_n^1 - u_n^2\| &= \liminf_{n\to\infty} \|(u_n^1 - u_n^2)(\,\cdot\, + y_n)\| \\ &= \liminf_{n\to\infty} \left\| \frac{n(u_n^1)^+(\,\cdot\, + y_n)}{\|n(u_n^1)^+(\,\cdot\, + y_n)\|} - \frac{n(u_n^2)^+(\,\cdot\, + y_n)}{\|n(u_n^2)^+(\,\cdot\, + y_n)\|} \right\| \\ &\geq \left\| \frac{v_1^+}{\alpha_1} - \frac{v_2^+}{\alpha_2} \right\| \geq \min\left\{ \frac{\|v_1^+\|}{\alpha_1}, \frac{\|v_2^+\|}{\alpha_2} \right\} \left\| \frac{v_1^+}{\|v_1^+\|} - \frac{v_2^+}{\|v_2^+\|} \right\| \\ &\geq \frac{\sqrt{2c_{\inf}}}{s(d)} \|n^{-1}(v_1) - n^{-1}(v_2)\|, \end{aligned}$$

where $s(d) := \sup\{\|u^+\| \mid u \in \mathcal{N},\ \mathcal{J}(u) \leq d\}$. By Theorem 3.1, $n^{-1}(v_1)$, $n^{-1}(v_2)$ are critical points of Ψ and we get (5.1). Note that if $v_1 = 0$ or $v_2 = 0$ then, similarly as above, we show that

$$\liminf_{n\to\infty} \|u_n^1 - u_n^2\| \geq \frac{\sqrt{2c_{\inf}}}{s(d)}$$

and again (5.1) holds. □

Proof of Theorem 1.2. Let g be odd. In view of Theorem 3.1(b) we get that n is equivariant with respect to the $\mathbb{Z}^N$-action given by $u \mapsto u(\cdot - k)$ for $k \in \mathbb{Z}^N$. Moreover, $\mathcal{J}$ is even and n is odd. Therefore Ψ is even and invariant with respect to the $\mathbb{Z}^N$-action. Let $\mathcal{F}$ be the set of geometrically distinct critical points of Ψ and assume that $\mathcal{F}$ is finite. Then, similarly as in [24, Lemma 2.13], we show that

$$\inf\{\|u_1 - u_2\| \mid \Psi'(u_1) = \Psi'(u_2) = 0,\ u_1 \neq u_2 \in S^+\} > 0.$$

The obtained discreteness of Palais–Smale sequences in Lemma 5.1 allows us to repeat the following arguments: Lemmas 2.15, 2.16 and proof of Theorem 1.2 from [24]. In fact, we show that for any $k \in \mathbb{N}$ there is $u \in S^+$ such that $\Psi'(u) = 0$ and $\Psi(u) = c_k$, where $c_k := \inf\{d \in \mathbb{R} \mid \gamma(\Psi^d) \geq k\}$ and γ denotes the usual Krasnosel'skiĭ genus (see [22]). Moreover, $c_k < c_{k+1}$ for any $k \in \mathbb{Z}$ and thus we get the contradiction (see [24] for detailed arguments). In view of Theorem 3.1(b) we obtain the existence of infinitely many geometrically distinct solutions to (1.1). □

Acknowledgments. The author would like to thank the referee for the valuable comments and suggestions helping to improve the paper.

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Manuscript received March 11, 2014

accepted June 10, 2104

Jarosław Mederski
Faculty of Mathematics and Computer Science
Nicolaus Copernicus University
Chopina 12/18
87-100 Toruń, POLAND

E-mail address: jmederski@mat.umk.pl