THE R_∞ PROPERTY FOR ABELIAN GROUPS

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ABSTRACT. It is well known there is no finitely generated abelian group which has the R_∞ property. We will show that also many non-finitely generated abelian groups do not have the R_∞ property, but this does not hold for all of them! In fact we construct an uncountable number of infinite countable abelian groups which do have the R_∞ property. We also construct an abelian group such that the cardinality of the Reidemeister classes is uncountable for any automorphism of that group.

1. Introduction

Let G be a group and φ be an endomorphism of G. Then two elements x, y of G are said to be Reidemeister equivalent (with respect to φ), if there exists an element $z \in G$ such that $y = zx\varphi(z)^{-1}$. The equivalence classes are called the Reidemeister classes or twisted conjugacy classes.

DEFINITION 1.1. The Reidemeister number of a homomorphism φ, denoted by $R(\varphi)$, is the cardinality of the Reidemeister classes of φ.

We remark here that most authors define the Reidemeister number as either a positive integer or ∞. This latter definition of course coincides with ours in the finite case, but does not allow to make a distinction between the various infinite cases.

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The Reidemeister number is a relevant ingredient in connection with many parts of mathematics. See for example [4] and references therein. This is for instance also the case in the study of the fixed point properties of the homotopy class of a self map on a topological space. In this situation, the group G will be the fundamental group $\pi_1(X)$ of the space and the homomorphism $\varphi = f_\sharp$ is the one which is induced by the map f on the fundamental group G. Under certain hypothesis the Reidemeister number $R(\varphi)$ is then exactly the number of essential fixed point classes of f if $R(\varphi)$ is finite and the number of essential fixed point classes of f is zero if $R(\varphi)$ is infinite. See [7] and [12] and the references therein for more information.

A group G has the R_∞ property if for every automorphism φ of G the Reidemeister number is not finite. In recent years many works have studied the question of which groups G have the R_∞ property. We refer to [4] for an overview of the results which have been obtained in this direction. Some more recent results, e.g. dealing with (extensions of) linear groups and lattices in semisimple Lie groups, can be found in [3], [9], [10], [11]. In [1], we treated the case of free nilpotent and free solvable groups. It turns out that in this sitution, infinitely generated groups behave quite differently than finitely generated groups.

The present work will also give a contribution for this problem, where we will consider infinite abelian groups. If an abelian group A is finitely generated then it is well known that A does not have the R_∞ property, since it is easy to see that the automorphism $\varphi\colon A \to A\colon a \mapsto -a$ has a finite Reidemeister number in this case ([2]). So, in this paper, we will focus on abelian groups which are not finitely generated. For information about infinite abelian groups in general we refer to [5], [6] and [8].

To the best of our knowledge, up till now, there is no example in literature of an abelian group having the R_∞ property. In this paper we do construct an uncountable number of countable abelian groups which do have the R_∞ property.

Before we announce the main results of this paper, let us fix some notation:

• Let p be a prime, then with $\mathbb{Z}_p$, we will denote the additive group of p-adic integers.

• For any positive integer $n \geq 2$, $\mathbb{Z}/n\mathbb{Z}$ will denote the additive group of integers modulo n.

• Let $\mathcal{P}$ be any set of primes, then $\mathbb{Z}_{\mathcal{P}}$ denotes the additive group of rational numbers which can be written as a fraction whose denominator is relative prime with all primes in $\mathcal{P}$. When p is a prime, then $\widehat{p}$ is the set of all primes which are different from p and hence $\mathbb{Z}_{\widehat{p}}$ is the group of all rational numbers whose denominator is a power of p.

• Finally, when p is a prime $\mathbb{Z}(p^\infty)$ is the Prüfer group $\mathbb{Z}_{\widehat{p}}/\mathbb{Z}$.

Recall that a group G is divisible if and only if for any $x \in G$ and any positive integer n, there is a $y \in G$ such that $y^n = x$.

We can now formulate the main results of this note:

PROPOSITION 3.1. *The following abelian groups do not have the R_∞ property*:
(a) *Abelian divisible groups.*
(b) *The groups $\mathbb{Z}_{\mathcal{P}}$ for any set of primes $\mathcal{P}$.*
(c) *The p-adic groups $\mathbb{Z}_p$ for any prime p.*
(d) *Any abelian torsion group without 2-torsion elements.*

In the formulation of the following result we use the Reidemeister spectrum of a group A which is the set $\{R(\varphi) \mid \varphi \in \mathrm{Aut}(A)\}$:

PROPOSITION 3.6. *For any prime $p \neq 2$, the spectrum of $\mathbb{Z}_{\widehat{p}}$ is*

$$\{2\} \cup \{p^m + 1 \mid m \in \mathbb{N}\} \cup \{p^m - 1 \mid m \in \mathbb{N}\} \cup \{\infty\}.$$

Hence if $\varphi \colon \mathbb{Z}_{\widehat{p}} \to \mathbb{Z}_{\widehat{p}}$ is an automorphism and $p \neq 2$, then $R(\varphi) \neq 1$.

THEOREM 4.2. *Let $\mathcal{P}$ be an infinite set of primes and consider the group $\bigoplus \mathbb{Z}_{\widehat{p}}$ as p runs over the set $\mathcal{P}$. Then any automorphism of this group has infinite Reidemeister number.*

THEOREM 4.4. *Let $\mathcal{P}$ be an infinite set of primes and consider the group $\prod \mathbb{Z}_{\widehat{p}}$ as p runs over the set $\mathcal{P}$. Then any automorphism of this group has the property that the set of Reidemeister classes is uncountable. In particular this group also has the R_∞ property.*

This work is divided into 3 sections besides the introduction. In Section 2 we recall a few elementary properties of the infinite abelian groups. In Section 3 we show that many infinite abelian groups do not have the R_∞ property. Groups constructed using standard constructions like direct sums and direct products are analysed. In Section 4 we provide examples of countable and uncountable abelian groups which have the R_∞ property. Finally we present an example of a group having the property that for any automorphism the Reidemeister number is always uncountable.

2. Preliminaries about infinite abelian groups

In this section we recall some known results about infinite abelian groups and prove some elementary facts about these groups which are used in our study. Let A be an abelian group and $\varphi \colon A \to A$ a homomorphism of A. Whenever we need to have φ an automorphism we make this explicit.

From [8, Theorem 3, p. 9] we have:

THEOREM 2.1. *Any abelian group A has a unique largest divisible subgroup M and $A = M \oplus N$ where N has no non-zero divisible subgroups.*

This theorem shows the relevance of the divisible groups for the description of the infinite abelian groups. An abelian group having no non-zero divisible subgroup is called a reduced group. When M is the maximal divisible subgroup of an abelian group A we will call A/M the reduced part of A.

Because the groups in question are abelian groups, it follows that the Reidemeister number of an endomorphism φ of such an abelian group A coincides with the cardinality of the quotient group $A/\mathrm{Im}(\varphi - \mathrm{Id}_A)$ (or $A/\mathrm{Im}(\mathrm{Id}_A -\varphi)$).

Now we prove a lemma which is on the one hand very simple but on the other hand very useful to show that many abelian groups do not have the R_∞ property.

LEMMA 2.2. *Let A be an abelian group and consider the homomorphism $2\colon A \to A\colon a \mapsto 2(a) = a + a$ (so multiplication by 2). Then:*

(a) *The Reidemeister number of this homomorphism is 1.*
(b) *If $2(A)$, the image of the homomorphism, has finite index in A, then the automorphism $\tau\colon A \to A$ given by $x \mapsto -x$ has Reideimeister number equal to the index of $2(A)$ in A.*

PROOF. Part (a) follows straightforward from the definition of the Reidemeister classes since for any a we have $a = -a + 0 - 2(-a) = a$ so a is in the same Reidemeister class as 0 for any a. The second part follows from the fact that for abelian groups the Reidemeister classes for τ correspond with the cosets of the image of the homomorphism $\mathrm{Id} -\tau\colon A \to A\colon a \mapsto a - (-a) = 2a$.   □

COROLLARY 2.3. *If multiplication by 2 is an automorphism, then not only does A not have the R_∞ property, but also A admits automorphisms, which are multiplication by -1 and multiplication by 2, which have Reidemeister number 1.*

PROOF. Follows promptly from the lemma above.   □

In the rest of this paper we will also need the following lemma.

LEMMA 2.4. *Let $\mathcal{P}$ be any set of primes and let $m > 1$ be a positive integer whose prime decomposition only consists of primes in $\mathcal{P}$. Then, the index $[\mathbb{Z}_\mathcal{P} \colon m\mathbb{Z}_\mathcal{P}]$ equals m.*

PROOF. It suffices to show that any element of $\mathbb{Z}_\mathcal{P}$ belongs to exactly one of the cosets

$$i + m\mathbb{Z}_\mathcal{P} \text{ with } i \in \{0, 1, \ldots, m-1\}.$$

Let $x \in \mathbb{Z}_\mathcal{P}$. If $x = 0$, then $x \in 0 + m\mathbb{Z}_\mathcal{P}$, otherwise $x = q/r$ where r is 1 or a product of primes not belonging to $\mathcal{P}$ and $q \in \mathbb{Z}$. As $\gcd(m, r) = 1$, it follows from Bézout's identity that there exists integers α and β with $q = \alpha r + \beta m$. Then

$$x + m\mathbb{Z}_\mathcal{P} = \frac{q}{r} + \mathbb{Z}_\mathcal{P} = \alpha + m\frac{\beta}{r} + m\mathbb{Z}_\mathcal{P} = \alpha + m\mathbb{Z}_\mathcal{P}.$$

Now, write $\alpha = i + m\alpha'$ for some $i \in \{0, 1, \ldots, m - 1\}$ and $\alpha' \in \mathbb{Z}$. It follows immediately that

$$x \in x + m\mathbb{Z}_\mathcal{P} = \alpha + m\mathbb{Z}_\mathcal{P} = i + m\mathbb{Z}_\mathcal{P},$$

so that x belongs to at least one of these cosets $i + m\mathbb{Z}_\mathcal{P}$. It is also easy to see that all of these cosets are different, which finishes the proof. $\square$

3. Abelian groups which do not have the R_∞ property

In this section we show that many abelian groups do not have the R_∞ property and in some cases in fact we compute the Reidemeister spectrum (i.e. the set of all possible cardinals which are the Reidemeister number for some automorphism of the group). The calculation of the spectrum is useful for Section 4.

3.1. Divisible groups, the p-adic integers and torsion groups.

PROPOSITION 3.1. *The following abelian groups do not have the R_∞ property*:
(a) *abelian divisible groups.*
(b) *The groups $\mathbb{Z}_\mathcal{P}$, where $\mathcal{P}$ is any subset of the set of all primes.*
(c) *The p-adic integers $\mathbb{Z}_p$ for any prime p.*
(d) *Any abelian torsion group without 2-torsion elements.*

PROOF. Part (a) follows promptly from Lemma 2.2(b).

Part (b) follows from Corollary 2.3 if $2 \notin \mathcal{P}$ and from Lemma 2.2(b) and Lemma 2.4 otherwise.

Part (c) follows from Corollary 2.3 if p is odd and from Lemma 2.2(b) for $p = 2$.

Part (d) follows from Corollary 2.3. $\square$

Note that in case $\mathcal{P}$ is the set of all primes, then the group $\mathbb{Z}_\mathcal{P}$ is exactly the group $\mathbb{Z}$, which certainly does not have the R_∞ property, but this group is finitely generated.

For divisible groups we can even say more:

PROPOSITION 3.2. *Let A be a divisible abelian group and $\varphi\colon A \to A$ be any homomorphism. If $R(\varphi)$ is finite, then $R(\varphi) = 1$.*

PROOF. If $R(\varphi)$ is finite, then the group $\mathrm{Im}(\varphi - \mathrm{Id}_A)$ is a subgroup of finite index in A. However, the only subgroup of finite index in a divisible group A is the group A itself. Therefore $A = \mathrm{Im}(\varphi - \mathrm{Id}_A)$ and hence $R(\varphi) = 1$. $\square$

In fact, divisible groups can be totally ignored when studying the R_∞ property of abelian groups. We make this precise in the following proposition.

PROPOSITION 3.3. *Let A be an abelian group. Then A has the R_∞ property if and only if the reduced part of A has the R_∞ property.*

PROOF. Let M be the unique maximal divisible subgroup of A, then $A = M \oplus N$ where $A/M \cong N$ is the reduced part of A. Let φ be any automorphism of A, then φ restricts to an automorphism φ' of M and induces an automorphism $\overline{\varphi}$ of the quotient A/M.

It is easy to see that when $R(\overline{\varphi})$ is infinite, then also $R(\varphi)$ is infinite. Hence, if A/M has the R_∞ property, then also A has the R_∞ property.

On the other hand, assume that A has the R_∞ property and consider any automorphism $\overline{\varphi}$ of A/M. We can lift this automorphism, to an automorphism φ of A by defining

$$\varphi \colon M \oplus N \to M \oplus N, \qquad (m, n) \mapsto (-m, \overline{\varphi}(n)).$$

Recall that $R(\varphi)$ equals the index of $\mathrm{Im}(\mathrm{Id}_A - \varphi)$ in A. Since

$$\mathrm{Im}(\mathrm{Id}_A - \varphi) = 2M \oplus \mathrm{Im}(\mathrm{Id}_N - \overline{\varphi}) = M \oplus \mathrm{Im}(\mathrm{Id}_N - \overline{\varphi})$$

we have that $R(\varphi) = R(\overline{\varphi})$ and so this Reidemeister number is infinite, since A has the R_∞ property. Hence A/M also has this property.   □

It follows that, from the point of view of the R_∞ property, we are left to the study of reduced abelian groups.

Also in the case the groups are torsion, it suffices to study the $2-$torsion groups. Indeed, any abelian torsion group A can be decomposed as a direct sum $A = \bigoplus_{p \text{ prime}} A_p$, where A_p is the p-primary part of A, i.e. the subgroup of A all elements of p-power order ([8, Theorem 1]). As all of these subgroups A_p are characteristic in A, we have that $\mathrm{Aut}(A) = \prod_{p \text{ prime}} \mathrm{Aut}(A_p)$. Since for any $p \neq 2$ there is an automorphism $\varphi_p \in \mathrm{Aut}(A_p)$ with Reidemeister number $R(\varphi_p) = 1$ (e.g. φ_p is multiplication with 2), it follows that A has property R_∞ if and only if A_2 has property R_∞.

REMARK 3.4. We do not know an example of an abelian 2-torsion group which has the R_∞ property.

In Section 3.4 we continue our study of torsion groups.

3.2. Direct sum and product of any abelian group.

PROPOSITION 3.5. *If A is an arbitrary abelian group, then for any finite integer $n > 1$, there is an automorphism $\varphi \colon A^n \to A^n$, that has Reidemeister number 1. Furthermore, in the case α is an infinite cardinal, the same result holds for both the direct sum $\bigoplus_\alpha A$ (weak direct product) and the direct product $\prod_\alpha A$. Hence none of these groups has the R_∞ property.*

PROOF. If n is either 2 or 3, it is easy to find an element $\theta_n \in \mathrm{GL}(n, \mathbb{Z})$ such that $\det(\theta_n - \mathrm{Id}_n) = 1$, i.e. it has Reidemeister number 1. E.g. we can take

$$\theta_2 = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad \theta_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

Then given an arbitrary integer $n > 1$, using the result for $n = 2$ and $n = 3$, we can construct a blocked diagonal element $\theta_n \in \mathrm{GL}(n, \mathbb{Z})$ such that $\det(\theta_n - \mathrm{Id}_n) = 1$, i.e. it has Reidemeister number 1. Now we use this matrix, in the obvious way, to define an automorphism φ of A^n. Then the homomorphisms $\varphi - \mathrm{Id}$ is surjective and then we have $R(\varphi) = 1$.

Now let α be an infinite cardinal. Then $\alpha = \alpha + \alpha$ and hence

$$\prod_\alpha A \cong \left(\prod_\alpha A \right) \oplus \left(\prod_\alpha A \right).$$

Now, we can apply the proposition for the case $n = 2$.

The case of an infinite direct sum is completely analogous. $\qquad\square$

3.3. The subgroups of the rationals. Now we compute for any prime p the spectrum of the group $\mathbb{Z}_{\widehat{p}}$, the integers localized at the set of primes $\widehat{p}$. As already mentioned before, this is the set of fractions where the denominators are powers of p. We already saw that these groups do not have the R_∞ property (Proposition 3.1), but the calculation of the spectrum will be useful for Section 4.

PROPOSITION 3.6. *The spectrum of $\mathbb{Z}_{\widehat{p}}$ is*

$$\{2\} \cup \{p^m + 1 \mid m \in \mathbb{N}\} \cup \{p^m - 1 \mid m \in \mathbb{N}\} \cup \{\infty\},$$

in case $p \neq 2$ and is

$$\{2^m + 1 \mid m \in \mathbb{N}\} \cup \{2^m - 1 \mid m \in \mathbb{N}\} \cup \{\infty\},$$

for $p = 2$. Hence if $\varphi \colon \mathbb{Z}_{\widehat{p}} \to \mathbb{Z}_{\widehat{p}}$ is an automorphism and $p \neq 2$, then $R(\varphi) \neq 1$.

PROOF. Let us consider an automorphism φ of the group $\mathbb{Z}_{\widehat{p}}$. Then $\varphi(1) = a/p^n$ for some integers a and n and φ is just multiplication with a/p^n. Since φ is an automorphism, 1 must be in the image of φ, and hence there must exist a $b/p^k \in \mathbb{Z}_{\widehat{p}}$ such that

$$\varphi\left(\frac{b}{p^k}\right) = \frac{b}{p^k}\frac{a}{p^n} = 1.$$

Hence, the only prime which possibly divides a is the prime p. It follows that $\varphi(1) = \pm p^m$ for some integer m. Since $R(\varphi) = R(\varphi^{-1})$, we may assume that $m \geq 0$. Recall that $R(\varphi)$ is the index of $\mathrm{Im}(\varphi - \mathrm{Id})$ in $\mathbb{Z}_{\widehat{p}}$. We distinguish four cases:

- $\varphi(1) = 1$ (first case where $m = 0$). In this case $\varphi - \mathrm{Id}$ is the zero homomorphism and $R(\varphi) = \infty$.

- $\varphi(1) = -1$ (second case where $m = 0$). In this case $\varphi - \mathrm{Id}$ is multiplication by -2. For $p = 2$, this is an automorphism of $\mathbb{Z}_{\widehat{2}}$, which leads to $R(\varphi) = 1 = 2^1 - 1$. When $p \neq 2$, it follows from Lemma 2.4 that $R(\varphi) = 2$.
- $\varphi(1) = p^m$ (with $m > 0$). Now $\varphi - \mathrm{Id}$ is multiplication with $p^m - 1$ and since $p^m - 1$ is relative prime to p, Lemma 2.4 implies that $R(\varphi) = p^m - 1$.
- $\varphi(1) = -p^m$ (with $m > 0$). Then $\mathrm{Im}(\varphi - \mathrm{Id}) = \mathrm{Im}(\mathrm{Id} - \varphi)$ and since $\mathrm{Id} - \varphi$ is the same as multiplication with $p^m + 1$, Lemma 2.4 again implies that $R(\varphi) = p^m + 1$.

This finishes our computation of the spectrum.

The fact that in case $p \neq 2$, we always have that $R(\varphi) \neq 1$ is clear from the first part. $\qquad\square$

3.4. Torsion groups and direct products. Among torsion groups we have the Prüfer groups $\mathbb{Z}(p^\infty)$ where p is any prime number. From Section 5, *Divisible groups*, in [8] it follows that the Prüfer groups are divisible so they do not have the R_∞ property by Proposition 3.1.

Another easy way of constructing torsion groups is to take direct sums of finite cyclic groups. This situation is completely dealt with in the next two propositions.

PROPOSITION 3.7. *Let $n_1, n_2, \ldots$ be an increasing set of positive integers. Then both*

$$\bigoplus_{i \in \mathbb{N}} \mathbb{Z}/2^{n_i}\mathbb{Z} \quad and \quad \prod_{i \in \mathbb{N}} \mathbb{Z}/2^{n_i}\mathbb{Z}$$

admit an automorphism with Reidemeister number equal to 1. In particular these groups and also the torsion subgroup of $\prod\limits_{i \in \mathbb{N}} \mathbb{Z}/2^{n_i}\mathbb{Z}$ do not have the R_∞ property.

PROOF. We begin this proof with the case of the direct product.
A general element of $A = \prod_{i \in \mathbb{N}} \mathbb{Z}/2^{n_i}\mathbb{Z}$ can be written in the form

$$(a_1 + 2^{n_1}\mathbb{Z}, a_2 + 2^{n_2}\mathbb{Z}, a_3 + 2^{n_3}\mathbb{Z}, a_4 + 2^{n_4}\mathbb{Z}. \ldots).$$

for some integers $a_1, a_2, a_3, a_4, \ldots$. For simplicity we will write this shortly as

$$(\overline{a_1}, \overline{a_2}, \overline{a_3}, \overline{a_4}, \ldots)$$

Now define $\varphi \colon A \to A$ by

$$(\overline{a_1}, \overline{a_2}, \ldots)$$
$$\mapsto (\overline{a_1 + a_2 + a_3}, \overline{a_2 + a_3}, \overline{a_3 + a_4 + a_5}, \overline{a_4 + a_5}, \overline{a_5 + a_6 + a_7}, \overline{a_6 + a_7}, \ldots).$$

So the $(2k-1)$-th component of this image is $\overline{a_{2k-1} + a_{2k} + a_{2k+1}}$ and the $2k$-th component is $\overline{a_{2k} + a_{2k+1}}$.

As by assumption $n_1 \leq n_2 \leq \ldots$, the map φ is well defined and is an endomorphism of A. In fact, φ is an automorphism, since it is easy to check that the map $\psi \colon A \to A$:

$$(\overline{a_1}, \overline{a_2}, \ldots)$$
$$\mapsto (\overline{a_1 - a_2}, \overline{a_2 - a_3 + a_4}, \overline{a_3 - a_4}, \overline{a_4 - a_5 + a_6}, \overline{a_5 - a_6}, \overline{a_6 - a_7 + a_8}, \ldots)$$

is also well defined and is an endomorphism which is the inverse of φ. Moreover, the map $\varphi - \mathrm{Id} \colon A \to A$ is given by

$$(\overline{a_1}, \overline{a_2}, \overline{a_3}, \overline{a_4}, \ldots) \mapsto (\overline{a_2 + a_3}, \overline{a_3}, \overline{a_4 + a_5}, \overline{a_5}, \overline{a_6 + a_7}, \overline{a_7}, \ldots)$$

which is clearly surjectice. Hence $R(\varphi) = 1$.

It is clear that one can use the restriction of φ to the direct sum or the torsion subgroup of A to obtain the same result in these cases. $\square$

PROPOSITION 3.8. *Let A be any direct sum or any direct product of finite cyclic groups. Then A does not have the R_∞ property.*

PROOF. We will give the proof of the fact that any direct product of finite cyclic groups does not have the R_∞ property. The case for the direct sum is completely analogous and is left to the reader.

As any finite cyclic group is the direct product of cyclic p-groups (for different primes p), we can assume that

$$A = \prod_{i \in I} A_i$$

for some index set I and each A_i is a cyclic group of prime-power order. Now we divide I into two disjoint subsets $I = I_1 \cup I_2$, where

$$I_1 = \{i \in I \mid A_i \text{ is a 2-group }\} \quad \text{and} \quad I_2 = I \setminus I_1.$$

Let

$$A^{(1)} = \prod_{i \in I_1} A_i \quad \text{and} \quad A^{(2)} = \prod_{i \in I_2} A_i.$$

Then $A = A^{(1)} \times A^{(2)}$. Note that multiplication by 2 is an automorphism, say φ_2, of $A^{(2)}$ with Reidemeister number $R(\varphi_2) = 1$. It is now enough to show that also $A^{(1)}$ admits an automorphism φ_1 with finite Reidemeister number, for then the automorpshim $\varphi = \varphi_1 \times \varphi_2$ will have Reidmeister number $R(\varphi) = R(\varphi_1) \times R(\varphi_2) = R(\varphi_1)$.

So from now onwards we concentrate on $A^{(1)}$, and for any positive integer $n \in \mathbb{N}$ we let

$$I_1^{(n)} = \{i \in I_1 \mid \#A_i = 2^n\} \quad \text{and} \quad A_n^{(1)} = \prod_{i \in I_1^{(n)}} A_i.$$

and so $A^{(1)} = \prod\limits_{n \in \mathbb{N}} A_n^{(1)}$. For those $n \in \mathbb{N}$ for which $\#I_1^{(n)} > 1$, we know, by Proposition 3.5, that there exists an automorphism $\varphi_1^{(n)}$ of $A_n^{(1)}$ with Reidemeister number $R(\varphi_1^{(n)}) = 1$.

Now, let $N \subseteq \mathbb{N}$ be the subset of positive integers n such that $\#I_1^{(n)} = 1$ and let i_n denote the unique element in $I_1^{(n)}$. Then

$$A^{(1)} = \prod_{n \in N} A_{i_n} \times \prod_{n \in \mathbb{N} \setminus N} A_n^{(1)}.$$

As $\prod\limits_{n \in \mathbb{N} \setminus N} \varphi_1^{(n)}$ is an automorphism of $\prod\limits_{n \in \mathbb{N} \setminus N} A_n^{(1)}$ with Reidmeister number 1, it suffices to find an automorphism of $\prod\limits_{n \in N} A_{i_n}$ of finite Reidemeister number. If N is a finite set then any automorphism (e.g. the identity) will do. When N is infinite, the result follows from the previous proposition. $\square$

4. Abelian groups which have the R_∞ property

In this section we present for the first time an example of an abelian group which has the R_∞ property. In fact we will show that there are at least an uncountable number of abelian groups with this property.

Recall that for any prime p, $\mathbb{Z}_{\widehat{p}}$ is the subgroup of the rationals consisting of all fractions whose denominator is a power of p.

LEMMA 4.1. *If $p_1 \neq p_2$ (both primes) then $\mathrm{Hom}(\mathbb{Z}_{\widehat{p_1}}, \mathbb{Z}_{\widehat{p_2}})$ contains only one element, which is the trivial homomorphism.*

PROOF. Given a homomorphism $\varphi \in \mathrm{Hom}(\mathbb{Z}_{\widehat{p_1}}, \mathbb{Z}_{\widehat{p_2}})$ this homomorphism is determined completely by the value of $\varphi(1)$. But $1 \in \mathbb{Z}_{\widehat{p_1}}$ is divisible by p_1^n for all n, hence also $\varphi(1)$ must be divisible by all powers p_1^n. As there is no non-zero element in $\mathbb{Z}_{\widehat{p_2}}$ with this property ($p_1 \neq p_2$), we must have that $\varphi(1) = 0$. So the result follows. $\square$

THEOREM 4.2. *Let $\mathcal{P}$ be an infinite set of primes and consider the group*

$$A = \bigoplus_{p \in \mathcal{P}} \mathbb{Z}_{\widehat{p}}.$$

Then A has the R_∞ property.

PROOF. It follows from Lemma 4.1 that

$$\mathrm{Aut}(A) = \prod_{p \in \mathcal{P}} \mathrm{Aut}(\mathbb{Z}_{\widehat{p}})$$

i.e. any automorphism φ of A can be decomposed as a direct product $\varphi = \prod\limits_{p \in \mathcal{P}} \varphi_p$ of (auto)morphisms $\varphi_p \colon \mathbb{Z}_{\widehat{p}} \to \mathbb{Z}_{\widehat{p}}$. From the previous section (Proposition 3.6) any automorphism of $\mathbb{Z}_{\widehat{p}}$ has Reidemeister number greater than 1 if $p \neq 2$. Since

an infinite number of primes is different from 2, it follows that $R(\varphi) = \infty$. This shows that A has the R_∞ property. □

COROLLARY 4.3. *There is an uncountable number of abelian groups which have the R_∞ property.*

PROOF. Given two distinct sets of primes $\mathcal{P}$ and $\mathcal{P}'$, the corresponding groups $\bigoplus_{p \in \mathcal{P}} \mathbb{Z}_{\widehat{p}}$ and $\bigoplus_{p \in \mathcal{P}'} \mathbb{Z}_{\widehat{p}}$ are not isomorphic. This follows using similar arguments as in Lemma 4.1 above. As there are an uncountable number of infinite subsets of the set of all primes, the result follows from Theorem 4.2 above. □

Observe that for a given infinite set $\mathcal{P}$ of primes we can also construct the group $\prod_{p \in \mathcal{P}} \mathbb{Z}_{\widehat{p}}$ instead of $\bigoplus_{p \in \mathcal{P}} \mathbb{Z}_{\widehat{p}}$. A similar result hold for this group where now we can even say that the cardinality of the set of Reidemeister classes is indeed uncountable.

THEOREM 4.4. *Let $\mathcal{P}$ be an infinite set of primes and consider the group $\prod_{p \in \mathcal{P}} \mathbb{Z}_{\widehat{p}}$. Then, for any automorphism φ of this group, the set of Reidemeister classes is uncountable. In particular this group also has the R_∞ property.*

REMARK 4.5. It is easy to extend this results to obtain abelian groups of any infinite cardinality with the R_∞ property. Indeed, when taking a direct sum $M \oplus \bigoplus_{p \in \mathcal{P}} \mathbb{Z}_{\widehat{p}}$ or $M \oplus \prod_{p \in \mathcal{P}} Z_{\widehat{p}}$ where M is a divisible group (of any cardinality you like), the resulting group again has the R_∞ property by Proposition 3.3.

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