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ON A POWER-TYPE COUPLED SYSTEM OF MONGE–AMPÈRE EQUATIONS

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Abstract. We study an elliptic system coupled by Monge–Ampère equations:

$$\begin{cases} \det D^2 u_1 = (-u_2)^\alpha & \text{in } \Omega, \\ \det D^2 u_2 = (-u_1)^\beta & \text{in } \Omega, \\ u_1 < 0,\ u_2 < 0 & \text{in } \Omega, \\ u_1 = u_2 = 0 & \text{on } \partial\Omega, \end{cases}$$

here Ω is a smooth, bounded and strictly convex domain in $\mathbb{R}^N$, $N \geq 2$, $\alpha > 0$, $\beta > 0$. When Ω is the unit ball in $\mathbb{R}^N$, we use index theory of fixed points for completely continuous operators to get existence, uniqueness results and nonexistence of radial convex solutions under some corresponding assumptions on α, β. When $\alpha > 0$, $\beta > 0$ and $\alpha\beta = N^2$ we also study a corresponding eigenvalue problem in more general domains.

1. Introduction

Consider the following system coupled by Monge-Ampère equations:

$$\begin{cases} \det D^2 u_1 = (-u_2)^\alpha & \text{in } \Omega, \\ \det D^2 u_2 = (-u_1)^\beta & \text{in } \Omega, \\ u_1 < 0,\ u_2 < 0 & \text{in } \Omega, \\ u_1 = u_2 = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

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Here Ω is a smooth, bounded and strictly convex domain in $\mathbb{R}^N$, $N \geq 2$, $\alpha > 0$, $\beta > 0$; $\det D^2 u$ stands for the determinant of Hessian matrix $(\frac{\partial^2 u}{\partial x_i \partial x_j})$ of u.

Monge–Ampère equations are fully nonlinear second order PDEs, and there are important applications in geometry and other scientific fields. Monge–Ampère equations have been studied in the past years [1], [6], [9], [12], [16]. However, to our best knowledge, only a few works have been devoted to coupled systems. We refer the reader to [10] where the author established a symmetry result for a system, which arises in studying the relationship between two noncompact convex surfaces in $\mathbb{R}^3$. It seems to be H. Wang [13], [14] who first considered systems for Monge–Ampère equations. He investigated the following system of equations:

$$
\begin{cases}
\det D^2 u_1 = f(-u_2) & \text{in } B, \\
\det D^2 u_2 = g(-u_1) & \text{in } B, \\
u_1 = u_2 = 0 & \text{on } \partial B.
\end{cases}
\tag{1.2}
$$

Here and in the following $B := \{x \in \mathbb{R}^N : |x| < 1\}$. By reducing it to a system coupled by ODEs and using the fixed point index, the author obtained the following results:

Theorem 1.1 ([13, Theorem 1.1]). *Suppose $f, g\colon [0,\infty) \to [0,\infty)$ are continuous.*

(a) *If $f_0 = g_0 = 0$ and $f_\infty = g_\infty = \infty$, then* (1.2) *has at least one nontrivial radial convex solution.*
(b) *If $f_0 = g_0 = \infty$ and $f_\infty = g_\infty = 0$, then* (1.2) *has at least one nontrivial radial convex solution.*

The notations were

$$
f_0 := \lim_{x \to 0^+} \frac{f(x)}{x^N}, \qquad f_\infty := \lim_{x \to \infty} \frac{f(x)}{x^N}.
$$

The above theorem implies the solvability of (1.2) is related to the asymptotic behavior of f, g at zero and at infinity. Obviously, it asserts the existence of a radial convex solution for system (1.1) if $\Omega = B$ and one of the following cases holds:

(1) $\alpha > N$, $\beta > N$,
(2) $\alpha < N$, $\beta < N$.

What we are curious about is, for the sublinear-superlinear case, i.e. $\alpha < N$, $\beta > N$, does system (1.1) *admits a radial convex solution when $\Omega = B$?*

We obtain that:

Theorem 1.2. *Let $\Omega = B$, then* (1.1) *has a radial convex solution if $\alpha > 0$, $\beta > 0$ and $\alpha\beta \neq N^2$.*

THEOREM 1.3. *Let $\Omega = B$, $\alpha > 0$, $\beta > 0$ and $\alpha\beta < N^2$, then* (1.1) *has a unique radial convex solution.*

THEOREM 1.4. *Let $\Omega = B$, $\alpha > 0$, $\beta > 0$ and $\alpha\beta = N^2$, then* (1.1) *admits no radial convex solution.*

We also give new existence results for the more general system (1.2) in Remark 2.2. Our main tool is the fixed point index in a cone used in [13]. However, based on the idea of decoupling method we will consider a composite operator. Besides, solutions in our theorems are classical, see Remark 2.3.

As $\alpha\beta = N^2$, for the eigenvalue problem

$$
\begin{cases}
\det D^2 u_1 = \lambda(-u_2)^{\alpha} & \text{in } \Omega,\\
\det D^2 u_2 = \mu(-u_1)^{\beta} & \text{in } \Omega,\\
u_1 < 0,\ u_2 < 0 & \text{in } \Omega,\\
u_1 = u_2 = 0 & \text{on } \partial\Omega,
\end{cases}
\tag{1.3}
$$

with positive parameters λ and μ, we have:

THEOREM 1.5. *Suppose $\Omega \subset \mathbb{R}^N$ is a bounded, smooth and strictly convex domain. If $\alpha > 0$, $\beta > 0$ and $\alpha\beta = N^2$, then system* (1.3) *admits a convex solution if and only if $\lambda\mu^{\alpha/N} = C$, where C is a positive constant depending on N, α and Ω.*

We will use the decoupling technique again to prove the assertion. The solution operator is chosen to be of abstract form which will be specified in Section 3. What's more, a generalized Krein–Rutman theorem ([8]) is used. As to regularity, by Theorem 1.2 and second paragraph of p. 1253 of [11]), we see any eigenvector (admissible weak solution) of (1.3) belongs to $C^1(\overline{\Omega}) \times C^1(\overline{\Omega})$.

Recall the eigenvalue problem of the Monge–Ampère operator,

$$
\begin{cases}
\det D^2 u = |\lambda u|^N & \text{in } \Omega,\\
u = 0 & \text{on } \partial\Omega.
\end{cases}
$$

In [8], [9], [12], the authors proved by different methods that the above equation has a unique positive eigenvalue, called the principal eigenvalue of the Monge–Ampère operator. Now we consider

$$
\begin{cases}
\det D^2 u = |\lambda v|^N & \text{in } \Omega,\\
\det D^2 v = |\lambda u|^N & \text{in } \Omega,\\
u = v = 0 & \text{on } \partial\Omega.
\end{cases}
\tag{1.4}
$$

By Theorem 1.5, we immediately obtain the following result.

Corollary 1.6. *The system* (1.4) *admits nontrivial solutions if and only if* $|\lambda| = \lambda_1(\Omega)$, *where* $\lambda_1(\Omega)$ *is the principal eigenvalue of the Monge–Ampère operator corresponding to* Ω.

This paper is organized as follows. In Section 2 we give the proofs of Theorems 1.2–1.4. The eigenvalue problem (1.3) is discussed in Section 3 and we prove Theorem 1.5 there.

2. Results concerning radial solutions

When $\Omega = B$, let us search radial convex classical ($C^2(\Omega)$) solutions of (1.1). One can convert it to the following system of ODEs (see Appendix A.2 of [5] or [7]):

$$\begin{cases} ((u_1'(t))^N)' = Nt^{N-1}(-u_2(t))^\alpha & \text{for } 0 < t < 1, \\ ((u_2'(t))^N)' = Nt^{N-1}(-u_1(t))^\beta & \text{for } 0 < t < 1, \\ u_1 < 0,\ u_2 < 0 & \text{for } 0 \le t < 1, \\ u_1'(0) = u_2'(0) = 0,\ u_1(1) = u_2(1) = 0. \end{cases} \tag{2.1}$$

In fact, the conversion is reversible if we choose a suitable working space. However, we would rather look for solutions of (2.1) in $C^1[0,1] \times C^1[0,1]$ first and discuss the regularity later in Remark 2.3. Solutions of problem (2.1) are equivalent to fixed points of a certain operator, and we can tackle more general systems. Equivalently, we seek positive concave solutions for convenience by letting $v_1 = -u_1$, $v_2 = -u_2$, and we can transform the above system to

$$\begin{cases} ((-v_1'(t))^N)' = Nt^{N-1}(v_2(t))^\alpha & \text{for } 0 < t < 1, \\ ((-v_2'(t))^N)' = Nt^{N-1}(v_1(t))^\beta & \text{for } 0 < t < 1, \\ v_1 > 0,\ v_2 > 0, & \text{for } 0 \le t < 1, \\ v_1'(0) = v_2'(0) = 0,\ v_1(1) = v_2(1) = 0. \end{cases} \tag{2.2}$$

Below we will keep most notations used in [13]. Recall the following lemma about fixed point index in a cone.

Lemma 2.1 ([2]). *Let E be a Banach space, K a cone in E. For $r > 0$, define $K_r = \{u \in K : \|u\| < r\}$. Assume $T\colon \overline{K_r} \to K$ is completely continuous, satisfying $Tx \neq x$, for all $x \in \partial K_r = \{u \in K : \|u\| = r\}$.*

(a) *If $\|Tx\| \ge \|x\|$, for all $x \in \partial K_r$, then $i(T, K_r, K) = 0$.*
(b) *If $\|Tx\| \le \|x\|$, for all $x \in \partial K_r$, then $i(T, K_r, K) = 1$.*

Now take the Banach space to be $C[0,1] := X$ with supremum norm. Let $K \subset X$ be

$$K := \Big\{ v \in X : v(t) \ge 0,\ t \in [0,1], \min_{1/4 \le t \le 3/4} v(t) \ge \|v\|/4 \Big\},$$

which is a cone in X. Denote $K_r = \{u \in K : \|u\| < r\}$ as in Lemma 2.1. We introduce two solution operators. For $v \in K$, define $T_i \colon K \to X$ $(i = 1, 2)$ to be

$$\begin{aligned} T_1(v)(t) &= \int_t^1 \left(\int_0^s N\tau^{N-1} v^\alpha(\tau)\, d\tau \right)^{1/N} ds, \quad t \in [0,1], \\ T_2(v)(t) &= \int_t^1 \left(\int_0^s N\tau^{N-1} v^\beta(\tau)\, d\tau \right)^{1/N} ds, \quad t \in [0,1]. \end{aligned}$$

Note the image of each operator is a nonnegative concave C^1-function on $[0,1]$, so by Lemma 2.2 in [13], the above two operators map K into itself. Besides, both operators are completely continuous by standard arguments.

Define a composite operator $T = T_1 T_2$, which is also completely continuous from K to itself. Calculation shows that $(v_1, v_2) \in C^1[0,1] \times C^1[0,1]$ solves (2.2) if and only if (v_1, v_2) belongs to $K \setminus \{0\} \times K \setminus \{0\}$ and satisfies $v_1 = T_1 v_2$, $v_2 = T_2 v_1$. Thus, if $v_1 \in K \setminus \{0\}$ is a fixed point of T, define $v_2 = T_2 v_1$, then $v_2 \in K \setminus \{0\}$ so that $(v_1, v_2) \in C^1[0,1] \times C^1[0,1]$ solves (2.2); conversely, if $(v_1, v_2) \in C^1[0,1] \times C^1[0,1]$ solves (2.2), then v_1 must be a nonzero fixed point of T in K. So our task is to search nonzero fixed points of T.

We are in a position to give the following proof of Theorem 1.2.

PROOF OF THEOREM 1.2. Let Γ be the positive number given by

$$\Gamma = \int_{1/4}^{3/4} \left(\int_{1/4}^s N\tau^{N-1}\, d\tau \right)^{1/N} ds. \tag{2.3}$$

For each $v \in K$,

$$\begin{aligned} \|T_2(v)\| &= \int_0^1 \left(\int_0^s N\tau^{N-1} v^\beta(\tau)\, d\tau \right)^{1/N} ds \\ &\geq \int_{1/4}^{3/4} \left(\int_{1/4}^s N\tau^{N-1} v^\beta(\tau)\, d\tau \right)^{1/N} ds \\ &\geq \int_{1/4}^{3/4} \left(\int_{1/4}^s N\tau^{N-1} \left(\frac{1}{4}\|v\| \right)^\beta d\tau \right)^{1/N} ds = \Gamma \left(\frac{1}{4}\|v\| \right)^{\beta/N}. \end{aligned}$$

Similarly, we obtain $\|T_1(v)\| \geq \Gamma(\|v\|/4)^{\alpha/N}$. Hence

$$\|T(v)\| = \|T_1 T_2(v)\| \geq \Gamma \left(\frac{1}{4}\|T_2(v)\| \right)^{\alpha/N} \geq \Gamma \left(\frac{1}{4}\Gamma \left(\frac{1}{4}\|v\| \right)^{\beta/N} \right)^{\alpha/N},$$

which yields

$$\|T(v)\| \geq \Gamma_1 \|v\|^{\alpha\beta/N^2}. \tag{2.4}$$

where Γ_1 is a positive number that depends on α, β and N.

On the other hand, for each $v \in K$,

$$\begin{aligned}\|T_2(v)\| &= \int_0^1 \bigg(\int_0^s N\tau^{N-1} v^\beta(\tau)\,d\tau\bigg)^{1/N} ds \\ &\le \bigg(\int_0^1 N\tau^{N-1} v^\beta(\tau)\,d\tau\bigg)^{1/N} \le \bigg(\int_0^1 N\tau^{N-1}\|v\|^\beta\,d\tau\bigg)^{1/N} = \|v\|^{\beta/N}.\end{aligned}$$

Similarly, $\|T_1(v)\| \le \|v\|^{\alpha/N}$, thus

$$\|T(v)\| \le \|T_2(v)\|^{\alpha/N} \le \|v\|^{\alpha\beta/N^2}. \tag{2.5}$$

We take into account the following two cases.

Case 1. $\alpha\beta > N^2$.

Choose r_1 such that $0 < r_1 < 1$. For $v \in K$ satisfying $\|v\| = r_1$, we have $\|Tv\| < \|v\|$ by (2.5). On the other hand, by the estimate (2.4), we can take r_2 large such that $r_2 > r_1$, and for each $v \in K$ satisfying $\|v\| = r_2$ it holds $\|Tv\| > \|v\|$. By Lemma 2.1,

$$i(T, K_{r_1}, K) = 1, \qquad i(T, K_{r_2}, K) = 0.$$

We obtain $i(T, K_{r_2} \setminus \overline{K_{r_1}}, K) = -1$ due to the additivity of the fixed point index. Then by the existence property of the fixed point index, T has a fixed point say v_1 in $K_{r_2} \setminus \overline{K_{r_1}}$. Denote $v_2 = T_2 v_1$, then $(-v_1, -v_2)$ is the desired solution of (2.1). Considering regularity (see Remark 2.3 below), we get a classical solution for system (1.1) when $\Omega = B$.

Case 2. $\alpha\beta < N^2$.

By (2.4), we can choose $r_3 > 0$ small enough such that for each $v \in K$ satisfying $\|v\| = r_3$, it holds $\|Tv\| > \|v\|$. On the other hand, the estimate (2.5) ensures the existence of r_4 such that $r_4 > r_3$ and for each $v \in K$ satisfying $\|v\| = r_4$, we have $\|Tv\| < \|v\|$. By Lemma 2.1, we get

$$i(T, K_{r_4}, K) = 1, \qquad i(T, K_{r_3}, K) = 0.$$

The rest of the proof is similar to that in Case 1 and we omit it. □

Remark 2.2. The right hand side of each equation in system (2.1) is of particular form, while we can handle more general ones, i.e.

$$\begin{cases} ((u_1'(t))^N)' = Nt^{N-1} f(-u_2)(t) & \text{for } 0 < t < 1, \\ ((u_2'(t))^N)' = Nt^{N-1} g(-u_1)(t) & \text{for } 0 < t < 1, \\ u_1 < 0,\ u_2 < 0, & \text{for } 0 \le t < 1, \\ u_1'(0) = u_2'(0) = 0,\ u_1(1) = u_2(1) = 0. \end{cases} \tag{2.6}$$

Similar arguments go through and we can get the following conclusion: If $f, g\colon [0,\infty) \to [0,\infty)$ are continuous, both nondecreasing, then (2.6) admits a solution if one of the following cases is satisfied:

(1) $\lim_{x\to 0^+} f^{1/N}(g^{1/N}(x))/x = 0$ and $\lim_{x\to\infty} f^{1/N}(g^{1/N}(x))/x = \infty$;
(2) $\lim_{x\to\infty} f^{1/N}(g^{1/N}(x))/x = 0$ and $\lim_{x\to 0^+} f^{1/N}(g^{1/N}(x))/x = \infty$.

Remark 2.3. The solutions we obtained in Remark 2.2 are in $C^1[0,1] \times C^1[0,1]$. Suppose $(\overline{u}_1, \overline{u}_2)$ is a solution of system (2.6), can we get classical solutions for system (1.2) by letting $u_1(x) = \overline{u}_1(|x|)$, $u_2(x) = \overline{u}_2(|x|)$? This is the case if $(\overline{u}_1, \overline{u}_2)$ has higher order regularity, say belongs to $(C^2[0,1) \cap C^1[0,1]) \times (C^2[0,1) \cap C^1[0,1])$. To see this, we refer the reader to Lemma 3.1 of [15], which states that if $u(x) = \widetilde{u}(|x|)$ in B, then $u \in C^2(B)$ if and only if $\widetilde{u} \in C^2[0,1)$ and $\widetilde{u}'(0) = 0$. So, let us explore further the regularity of $(\overline{u}_1, \overline{u}_2)$. Since it is supposed a solution of system (2.6), we have

$$\begin{aligned} \overline{u}_1(t) &= -\int_t^1 \left(\int_0^s N\tau^{N-1} f(-\overline{u}_2(\tau))\, d\tau \right)^{1/N} ds, \quad t \in [0,1], \\ \overline{u}_1'(t) &= \left(\int_0^t N\tau^{N-1} f(-\overline{u}_2(\tau))\, d\tau \right)^{1/N}, \qquad t \in [0,1], \end{aligned}$$

and

$$\overline{u}_1''(t) = \frac{1}{N} \left(\int_0^t N\tau^{N-1} f(-\overline{u}_2(\tau))\, d\tau \right)^{1/N-1} (Nt^{N-1} f(-\overline{u}_2(t))). \tag{2.7}$$

Similarly, we can obtain

$$\overline{u}_2''(t) = \frac{1}{N} \left(\int_0^t N\tau^{N-1} g(-\overline{u}_1(\tau))\, d\tau \right)^{1/N-1} (Nt^{N-1} g(-\overline{u}_1(t))). \tag{2.8}$$

By (2.7) and (2.8), if $f(x) > 0$ and $g(x) > 0$ for arbitrary $x > 0$, then calculation shows $(\overline{u}_1,\ \overline{u}_2)$ belongs to $C^2[0,1] \times C^2[0,1]$. Thus we get a nontrivial convex classical solution of system (1.2), by letting $u_1(x) = \overline{u}_1(|x|)$, $u_2(x) = \overline{u}_2(|x|)$ on $\overline{B}$.

We turn to the proof of the uniqueness result. Fix $\alpha > 0, \beta > 0$ such that $\alpha\beta < N^2$ in system (2.1). We only need to show T has at most one fixed point in K. With this in mind, we will give a sketch of the proof, since the rest of it's idea is similar to that used in [7] where uniqueness for one single equation was established.

Definition 2.4 ([4] or [7, Definition 3.1]). Let P be a cone from a real Banach space Y. With some $u_0 \in P$ positive, $A\colon P \to P$ is called u_0-sublinear if:

(a) for any $x > 0$, there exists $\theta_1 > 0, \theta_2 > 0$ such that $\theta_1 u_0 \le Ax \le \theta_2 u_0$,
(b) for any $\theta_1 u_0 \le x \le \theta_2 u_0$ and $t \in (0,1)$, there always exists some $\eta = \eta(x,t) > 0$ such that $A(tx) \ge (1+\eta)tAx$.

Lemma 2.5 ([4] or [7, Lemma 3.3]). *An increasing and u_0-sublinear operator A can have at most one positive fixed point.*

Now we choose the Banach space to be $Y = X = C[0,1]$ as before, but we work in a new cone $P := \{v \in Y : v(t) \geq 0,\ t \in [0,1]\}$. Since $K \subset P$, we only need to show that T has at most one fixed point in P.

Proof of Theorem 1.3. It is readily seen that T_1, T_2 are increasing operators with respect to the partial order induced by P. So is $T = T_1T_2$. By Lemma 2.5, we only need to verify that T is u_0-sublinear for some u_0 positive in Y. Since $\alpha\beta < N^2$, we can assume $\alpha < N$ without loss of generality (otherwise consider the operator $\overline{T} := T_2T_1$). Under this assumption, take $u_0 = 1 - t$, then T_1 satisfies (a) of Definition 2.4, which is a consequence of Lemma 3.4 in [7]. From this we know $T = T_1T_2$ also satisfies (a) of Definition 2.4. The proof is complete if T satisfies (b) of Definition 2.4. To this end, let $\theta_1 u_0 \leq x \leq \theta_2 u_0$, $\xi \in (0,1)$, then direct calculation give $T_2(\xi x) = \xi^{\beta/N} T_2(x)$, $T_1(\xi x) = \xi^{\alpha/N} T_1(x)$. Thus $T(\xi x) = T_1(\xi^{\beta/N} T_2(x)) = \xi^{\alpha\beta/N^2} T_1 T_2(x) \geq (1+\eta)\xi T x$ for some $\eta > 0$. The last inequality holds because $\xi \in (0,1)$ and $\alpha\beta < N^2$. □

Finally in this section, we prove the nonexistence result.

Proof of Theorem 1.4. As analyzed previously, we only need to show that T has no positive fixed point in K. For each $v \in K$, we have

$$\begin{aligned} \|T_2(v)\| &= \int_0^1 \bigg(\int_0^s N\tau^{N-1} v^\beta(\tau)\,d\tau \bigg)^{1/N} ds \\ &\leq \bigg(\int_0^1 N\tau^{N-1} v^\beta(\tau)\,d\tau \bigg)^{1/N} \leq \bigg(\int_0^1 N\tau^{N-1} \|v\|^\beta\,d\tau \bigg)^{1/N} = \|v\|^{\beta/N}. \end{aligned} \tag{2.9}$$

Assume that T has a positive fixed point v_0 in K, then v_0 must be a concave function satisfying $v_0(1) = 0$ and $v_0(t) > 0$, $t \in [0,1)$. Thus if we take $v = v_0$ in the above estimate, we know the last inequality in (2.9) must be strict. Thus $\|T_2 v_0\| < \|v_0\|^{\beta/N}$. Similarly we have $\|T_1 v\| \leq \|v\|^{\alpha/N}$, for all $v \in K$. Therefore, we obtain that $\|Tv_0\| < \|v_0\|^{\alpha\beta/N^2} = \|v_0\|$. This contradicts the assumption $v_0 = Tv_0$. □

3. A corresponding eigenvalue problem

Checking the proof of Theorem 1.4, we see the argument would not go through if the radius of the ball is larger than 1. This observation leads us to consider the following system

$$\begin{cases} \det D^2 u_1 = (-u_2)^\alpha & \text{in } B_R, \\ \det D^2 u_2 = (-u_1)^\beta & \text{in } B_R, \\ u_1 < 0,\ u_2 < 0 & \text{in } B_R, \\ u_1 = u_2 = 0 & \text{on } \partial B_R. \end{cases} \tag{3.1}$$

Here B_R denotes the ball of radius R centered at zero, $\alpha, \beta > 0$ are such that $\alpha\beta = N^2$. By scaling, the solvability of (3.1) is equivalent to that of the following problem:

$$\begin{cases} \det D^2 u_1 = \lambda(-u_2)^\alpha & \text{in } B, \\ \det D^2 u_2 = \mu(-u_1)^\beta & \text{in } B, \\ u_1 < 0,\ u_2 < 0 & \text{in } B, \\ u_1 = u_2 = 0 & \text{on } \partial B, \end{cases} \tag{3.2}$$

where λ and μ are positive parameters. By Theorem 1.4, (3.2) admits no radial convex solution when $\lambda = \mu = 1$. Further calculations show that if (3.2) has a radial solution, then λ, μ should be in a suitable range. Indeed, let X be $C[0,1]$ and the cone K as in Section 2. Now we consider the new operators $\widetilde{T_1}$, $\widetilde{T_2}$ defined as:

$$\begin{aligned} \widetilde{T_1}(v)(t) &= \int_t^1 \Big(\int_0^s N\tau^{N-1} \lambda v^\alpha(\tau)\, d\tau \Big)^{1/N} ds, \quad t \in [0,1],\ v \in K, \\ \widetilde{T_2}(v)(t) &= \int_t^1 \left(\int_0^s N\tau^{N-1} \mu v^\beta(\tau)\, d\tau \right)^{1/N} ds, \quad t \in [0,1],\ v \in K. \end{aligned}$$

We also define $\widetilde{T} := \widetilde{T_1}\widetilde{T_2}$, and we will investigate the fixed points of $\widetilde{T}$.

Notice that $\|\widetilde{T_2}(v)\| \le \mu^{1/N}\|v\|^{\beta/N}$, $\|\widetilde{T_1}(v)\| \le \lambda^{1/N}\|v\|^{\alpha/N}$, which yield

$$\|\widetilde{T}(v)\| \le \lambda^{1/N}\|\widetilde{T_2}(v)\|^{\alpha/N} \le \lambda^{1/N}\mu^{\alpha/N^2}\|v\|.$$

So, if $v \neq 0$ is a fixed point of $\widetilde{T}$, we have necessarily $\lambda\mu^{\alpha/N} \ge 1$, which implies $\lambda\mu^{\alpha/N}$ can't be too small. On the other hand, with Γ defined in (2.3), we have for each $v \in K$,

$$\begin{aligned} \|\widetilde{T_2}(v)\| &= \mu^{1/N} \int_0^1 \left(\int_0^s N\tau^{N-1} v^\beta(\tau)\, d\tau \right)^{1/N} ds \\ &\ge \mu^{1/N} \int_{1/4}^{3/4} \left(\int_{1/4}^s N\tau^{N-1} v^\beta(\tau)\, d\tau \right)^{1/N} ds \\ &\ge \mu^{1/N} \int_{1/4}^{3/4} \left(\int_{1/4}^s N\tau^{N-1} \left(\frac{1}{4}\|v\|\right)^\beta d\tau \right)^{1/N} ds = \mu^{1/N}\Gamma\left(\frac{1}{4}\|v\|\right)^{\beta/N}. \end{aligned}$$

Similarly, $\|\widetilde{T_1}(v)\| \ge \lambda^{1/N}\Gamma(\|v\|/4)^{\alpha/N}$, hence

$$\begin{aligned} \|\widetilde{T}(v)\| &= \|\widetilde{T_1}\widetilde{T_2}(v)\| \ge \lambda^{1/N}\Gamma\left(\frac{1}{4}\|\widetilde{T_2}(v)\|\right)^{\alpha/N} \\ &\ge \lambda^{1/N}\Gamma\left(\frac{1}{4}\right)^{\alpha/N}\left(\mu^{1/N}\Gamma\left(\frac{1}{4}\|v\|\right)^{\beta/N}\right)^{\alpha/N} = \lambda^{1/N}\mu^{\alpha/N^2}\left(\frac{1}{4}\Gamma\right)^{1+\alpha/N}\|v\|. \end{aligned}$$

So, if $v \neq 0$ is a fixed point of $\widetilde{T}$, we have necessarily $\lambda^{1/N}\mu^{\alpha/N^2}(\Gamma/4)^{1+\alpha/N} \le 1$, which implies $\lambda\mu^{\alpha/N}$ can't be too large.

Is equation (3.2) solvable for suitable λ and μ? The answer is positive and the domain need not even be symmetric, as asserted by Theorem 1.5. Our tool for (1.3) is a generalized Krein–Rutman theorem developed in [8], where the author discussed eigenvalue problems for a broader class of fully nonlinear elliptic operators, including the Monge–Ampère operator.

Recall some concepts first(see [8] for details). Let E be a real Banach space with a cone $M \subset E$. The partial order induced by M is written: $u \preceq v \Leftrightarrow v-u \in M$. Let $A\colon E \to E$. A is said to be homogeneous if it is positively homogeneous with degree 1. A is monotone if it satisfies $x \preceq y \Rightarrow A(x) \preceq A(y)$. A is called positive if $A(M) \subseteq M$. Finally, a positive operator $A\colon E \to E$ is called strong (relative to M), if for all $u, v \in \mathrm{Im}(A) \cap M \setminus \{0\}$, there exist positive constants ρ and τ (which may depend on u, v), such that $u - \rho v \in M$ and $v - \tau u \in M$. The main content of the generalized Krein–Rutman theorem given in [8] is as follows.

Lemma 3.1 ([8, Theorem 2.7]). *Let E contain a cone M. Let $A\colon E \to E$ be a completely continuous operator with $A|_M\colon M \to M$ homogeneous, monotone, and strong. Furthermore, assume that there exist nonzero elements w, $A(w) \in Im(A) \cap M$. Then there exists a constant $\lambda_0 > 0$ with the following properties*:

(a) *There exists $u \in M \setminus \{0\}$, with $u = \lambda_0 A(u)$;*

(b) *If $v \in M \setminus \{0\}$ and $\lambda > 0$ such that $v = \lambda A(v)$, then $\lambda = \lambda_0$.*

We also need the following lemmas to prove Theorem 1.5. By Theorem 1.2 and second paragraph of p. 1253 of [11], we have

Lemma 3.2 (A special case of Trudinger [11, Theorem 1.1]). *Let Ω be a strictly convex bounded domain in $\mathbb{R}^N$, $\psi \in C(\overline{\Omega})$ with $\psi \geq 0$, $\phi \in C(\overline{\Omega})$. Then there exists a unique admissible weak solution $u \in C^1(\overline{\Omega})$ of the equation*

$$\begin{cases} \det D^2 u = \psi & \text{in } \Omega, \\ u = \phi & \text{on } \partial\Omega. \end{cases} \tag{3.3}$$

The definition of admissible weak solution coincides with the Aleksandrov sense weak solution (please see page 1252-1253 in Trudinger [11]), so Lemma 3.2 is valid for the Aleksandrov sense weak solution in the following Remark 3.1, we use the Aleksandrov sense weak solution here and in the following part of this paper.

Remark 3.3. The admissible weak solution in Lemma 3.2 can be viewed as in Aleksandrov sense. Recall the notion of Aleksandrov solution (see [6], Definition 1.1.1, Theorem 1.1.13 and Definition 1.2.1). Let $\Omega \subset \mathbb{R}^N$ be an open subset and $u\colon \Omega \to \mathbb{R}$. The normal mapping of u, or subdifferential of u, is the set-valued function $\partial u\colon \Omega \to 2^{\mathbb{R}^N}$ defined by

$$\partial u(x_0) = \{p \in \mathbb{R}^N : u(x) \geq u(x_0) + p \cdot (x - x_0), \text{ for all } x \in \Omega\}.$$

Given $e \subset \Omega$, define $\partial u(e) = \bigcup_{x \in e} \partial u(x)$.

Let u be continuous, then the class $\mathcal{S} = \{e \subset \Omega : \partial u(e)$ is Lebesgue measurable$\}$ is a Borel σ-algebra. The set function $Mu \colon \mathcal{S} \to \overline{\mathbb{R}}$, $Mu(e) = |\partial u(e)|$ is a measure, finite on compacts, that is called the Monge–Ampère measure associated with the function u.

Let ν be a Borel measure defined in Ω, an open and convex subset of $\mathbb{R}^N$. The convex function $u \in C(\Omega)$ is called *a generalized solution or Aleksandrov solution* to the Monge–Ampère equation $\det D^2 u = \nu$ if the Monge-Ampère measure Mu associated with u equals ν.

LEMMA 3.4 (Comparison Principle, [6]). *Let Ω be a bounded convex domain in $\mathbb{R}^N$. Denote $\mu[u]$ the Monge–Ampère measure determined by u. Let $u, v \in C(\overline{\Omega})$ be two convex functions satisfying*

$$\begin{cases} \mu[u](e) \geq \mu[v](e) & \text{for all Borel } e \subset \Omega, \\ u \leq v & \text{on } \partial\Omega. \end{cases}$$

then $u(x) \leq v(x)$ for any $x \in \Omega$.

We are ready to give the proof of Theorem 1.5.

PROOF OF THEOREM 1.5. Let E be the Banach space $C(\overline{\Omega})$ with supremum norm. Choose the negative cone $M := \{u \in E : u(x) \leq 0, \text{ for all } x \in \Omega\}$. Notice the partial order induced by M reads: $u \preceq v$ if and only if $v(x) \leq u(x)$, for all $x \in \Omega$. Define $A_1 \colon E \to E$, $A_1(u) = v$, where v is the unique admissible weak solution (Aleksandrov solution) of the equation

$$\begin{cases} \det D^2 v = |u|^\alpha & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.4}$$

By Lemma 3.2, A_1 is well defined. Similarly we define $A_2 \colon E \to E$, $A_2(u) = v$, where v is the unique admissible weak solution (Aleksandrov solution) of the equation

$$\begin{cases} \det D^2 v = |u|^\beta & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.5}$$

By Lemma 3.2 for the admissible weak solutions of (3.4) and (3.5), we see $A_1 u \in C^1(\overline{\Omega})$, $A_2 u \in C^1(\overline{\Omega})$. Finally, we define a composite operator $A := A_1 A_2$.

Let us verify A satisfies the assumptions of Lemma 3.1.

Firstly, A_1, A_2 (thus A) are completely continuous by Proposition 3.2 of [8]. Since $A(E) \subseteq M$, A is positive. Let $t > 0$, we have $A_2(tu) = t^{\beta/N} A_2(u)$, $A_1(tv) = t^{\alpha/N} A_1(u)$. As $\alpha\beta = N^2$, we deduce

$$A(tu) = A_1 A_2(tu) = A_1(t^{\beta/N} A_2(u)) = t A_1 A_2(u) = t A(u),$$

which implies that A is homogeneous. Besides, it is easy to get that A_1, A_2 are monotone operators by Lemma 3.4, so is A.

To see A is strong, notice if $u \in Im(A) \cap M \setminus \{0\}$, then there exists a $v \in E \setminus \{0\}$ such that $u = A_1(A_2 v)$. Now $A_2 v$ is a nonzero convex function that is strictly negative in Ω, by Lemma 3.2, we see $A_2 v \in C^1(\overline{\Omega})$, $A_1(A_2 v) \in C^1(\overline{\Omega})$. Then Lemma 3.4 of [3] gives the exterior normal derivative satisfies $u_\nu > 0$, for all $x \in \partial\Omega$, since u is convex thus subharmonic and $u(x) < 0$ for $x \in \Omega$. Using these facts, one can get by definition that A is a strong operator.

Finally, $\mathcal{N}(A) = \{0\}$ where $\mathcal{N}(A) := \{u \in M : A(u) = 0\}$. We see all assumptions in Lemma 3.1 are satisfied, then there exist $u_* \in M \setminus \{0\}$ and $\lambda_0 > 0$ such that $u_* = \lambda_0 A(u_*)$.

If we define $v_* = A_2(u_*)$, then (u_*, v_*) must be a solution of the following system

$$\begin{cases} \det D^2\left(\dfrac{u}{\lambda_0}\right) = (-v)^\alpha & \text{in } \Omega, \\ \det D^2 v = (-u)^\beta & \text{in } \Omega, \\ u < 0,\ v < 0 & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega. \end{cases}$$

Furthermore, by the second conclusion of Lemma 3.1, if $u_1 \in M \setminus \{0\}$ and $\lambda > 0$ satisfy $u_1 = \lambda A(u_1)$, then $\lambda = \lambda_0$. So the following system

$$\begin{cases} \det D^2 u = \widetilde{\lambda}(-v)^\alpha & \text{in } \Omega, \\ \det D^2 v = (-u)^\beta & \text{in } \Omega, \\ u < 0, v < 0 & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.6}$$

admits a solution if and only if $\widetilde{\lambda} = \lambda_0^N$.

Now we show that (1.3) has a convex solution if and only if $\lambda\mu^{\alpha/N} = \lambda_0^N$, which implies the first conclusion of Theorem 1.5. Indeed, if (u, v) is a convex solution of (1.3), then from $\det D^2 v = \mu(-u)^\beta$ we have $\det D^2(\mu^{-1/N} v) = (-u)^\beta$.

Let $\widetilde{v} = \mu^{-1/N} v$, then $(-v)^\alpha = \mu^{\alpha/N}(-\widetilde{v})^\alpha$, and thus $\det D^2 u = \lambda(-v)^\alpha = \lambda\mu^{\alpha/N}(-\widetilde{v})^\alpha$. It is easily seen $(u, \widetilde{v})$ is a convex solution of (3.6) if $\widetilde{\lambda} = \lambda\mu^{\alpha/N}$. Since we have proved that (3.6) admits a convex solution only when $\widetilde{\lambda} = \lambda_0^N$, we get $\lambda\mu^{\alpha/N} = \lambda_0^N$.

On the other hand, assume $\lambda\mu^{\alpha/N} = \lambda_0^N$, set $\widetilde{\lambda} = \lambda\mu^{\alpha/N}$, then $\widetilde{\lambda} = \lambda_0^N$ and (3.6) admits a convex solution, say (u, v). Define $v^\star = \mu^{1/N} v$, then it is easy to show that $(u, v^\star)$ is a convex solution of (1.3). $\square$

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