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THE TRIVIAL HOMOTOPY CLASS OF MAPS FROM TWO-COMPLEXES INTO THE REAL PROJECTIVE PLANE

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Abstract. We study reasons related to two-dimensional CW-complexes which prevent an extension of the Hopf–Whitney Classification Theorem for maps from those complexes into the real projective plane, even in the simpler situation in which the complex has trivial second integer cohomology group. We conclude that for such a two-complex K, the following assertions are equivalent: (1) Every based map from K into the real projective plane is based homotopic to a constant map; (2) The skeleton pair (K, K^1) is homotopy equivalent to that of a model two-complex induced by a balanced group presentation; (3) The number of two-dimensional cells of K is equal to the first Betti number of its one-skeleton; (4) K is acyclic; (5) Every based map from K into the circle S^1 is based homotopic to a constant map.

1. Introduction and main theorem

Given topological spaces X and Y, we denote the set of homotopy classes of maps from X into Y by $[X;Y]$. If X and Y have base points x_* and y_*, respectively, let $[X;Y]_*$ denote the based homotopy classes of based maps from X into Y, that is, maps $(X,x_*) \to (Y,y_*)$. If the base point $x_* \in X$ is non-degenerated (in particular, if X is a CW-complex) and Y is path connected, then there exists a natural action of the fundamental group $\pi_1(Y,y_*)$ on the set

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$[X;Y]_*$ in such a way that $[X;Y]$ is the quotient set of $[X;Y]_*$ by this action. See [4, Theorem 6.57]. As a particular consequence, a based map is null-homotopic if and only if it is based null-homotopic. See [4, Corollary 6.58]. Furthermore, if Y is simply connected, there exists a bijection between $[X;Y]$ and $[X;Y]_*$. See [4, § 6.16] for details.

The Hopf–Whitney Classification Theorem [8, Corollary 6.19, p. 244] states that the set $[K;Y]$ of the homotopy classes of maps from an n-dimensional CW-complex K into an $(n-1)$-connected n-simple space Y is in one-to-one correspondence with the cohomology group $H^n(K;\pi_n(Y))$. We remark that for $n \geq 2$, the hypothesis of Y to be $(n-1)$-connected implies automatically that Y is also n-simple, since in this case Y is simply connected. Again under such conditions, we have a one-to-one correspondence between $[K;Y]$ and the set $[K;Y]_*$ of the based homotopy classes, as we noted at the end of the first paragraph. Therefore, for $n \geq 2$, we might replace $[K;Y]$ by $[K;Y]_*$ in the statement of the Hopf–Whitney Classification Theorem.

Obviously, the Hopf–Whitney Classification Theorem does not apply for a two-dimensional CW-complex K (shortly a two-complex K) and $Y = \mathbb{R}\mathrm{P}^2$, since the real projective plane $\mathbb{R}\mathrm{P}^2$ is not 1-connected (simply connected). In fact, we may provide easily examples in which the cohomology group

$$H^2(K;\pi_2(\mathbb{R}\mathrm{P}^2)) = H^2(K;\mathbb{Z}) = H^2(K)$$

is "smaller" than the set $[K;\mathbb{R}\mathrm{P}^2]_*$ of based homotopy classes of maps from K into $\mathbb{R}\mathrm{P}^2$. This happens, for instance, if K collapses to a bouquet $\bigvee^n S^1$ of $n \geq 1$ circles; in this case, $H^2(K) = 0$, although the set $[K;\mathbb{R}\mathrm{P}^2]_*$ has 2^n elements. For another example, one in which the two-complex K does not collapses to a one-complex, consider $K = S^1 \vee S^2$; in this case we have $H^2(K) \approx \mathbb{Z}$ whereas $[K;\mathbb{R}\mathrm{P}^2]_*$ contains isomorphic copies of $\mathbb{Z}$ and $\mathbb{Z}_2$. The fact $H^2(K)$ is "smaller" than $[K;\mathbb{R}\mathrm{P}^2]_*$ is general; in fact, Lemma 3.1 shows that for a two-complex K with fundamental group Π, there exists a one-to-one correspondence between $H^2(K)$ and $[K;\mathbb{R}\mathrm{P}^2]_*$ if and only if $\mathrm{Hom}(\Pi;\mathbb{Z}_2) = 0$; else, there exists a strictly injection from $H^2(K)$ into $[K;\mathbb{R}\mathrm{P}^2]_*$. In any case, we have

$$[K;\mathbb{R}\mathrm{P}^2]_* = 0 \;\Rightarrow\; H^2(K) = 0,$$

where the expression on the left side means that $[K;\mathbb{R}\mathrm{P}^2]_*$ contains only the based homotopy class of the constant map at the base point of $\mathbb{R}\mathrm{P}^2$.

In this article we study conditions on a finite and connected two-complex K to ensure the opposite implication, that is, we study the possible triviality of the set $[K;\mathbb{R}\mathrm{P}^2]_*$ when K is a finite and connected two-dimensional CW-complex with trivial second cohomology group. For such a two-complex, we do not have always $[K;\mathbb{R}\mathrm{P}^2]_* = 0$, but we may characterize the two-complexes for which this happens. Such characterization is our main theorem:

Theorem 1.1 (Main Theorem). *The following assertions are equivalent for a finite and connected two-dimensional CW-complex K with trivial second integer cohomology group*:

(a) *Every based map from K into $\mathbb{R}\mathrm{P}^2$ is based homotopic to a constant map.*
(b) *The skeleton pair (K, K^1) is homotopy equivalent to that of a model two-complex induced by a balanced group presentation, that is, a finite group presentation with the same number of generators and relators.*
(c) *The number of 2-cells of K is equal to the first Betti number of its one-skeleton K^1.*
(d) *K is acyclic, that is, K has the homology of a point.*
(e) *Every based map from K into the circle S^1 is based homotopic to a constant map.*

The assumption on the nullity of $H^2(K)$ is essential for all equivalences, except (a) $\Leftrightarrow$ (c), *which always holds true. Moreover, assertions* (a) *and* (d) *implies, each one, the assumption on the nullity of $H^2(K)$.*

In this theorem, as well as throughout the article, the base point of a complex will be always one of its 0-cells. In particular, we consider the real projective plane $\mathbb{R}\mathrm{P}^2$ with its natural cellular decomposition $\mathbb{R}\mathrm{P}^2 = e^0_* \cup e^1_* \cup e^2_*$ so that its base point is the 0-cell e^0_*.

The problem approached in this paper and answered in the Main Theorem is motivated not only by the Hopf–Whitney Classification Theorem but also by a specific problem in Topological Root Theory proposed by Claudemir Aniz and Daciberg Lima Gonçalves in the 2000's. See [1] and [2]. We explain: let X be a finite and connected n-dimensional CW-complex. If X is homotopy equivalent to a complex of dimension less than n, then every map $f\colon X \to Y$ from X into a closed n-manifold Y is homotopic to a non-surjective map; in this case we say that f is *root free* or f is not *strongly surjective.* Thus, in order to exists a strong surjection from X into a closed n-manifold Y, it is necessary (but not sufficient, in general) that X has *essential dimension* n, that is, X is not homotopy equivalent to a complex of dimension less than n, the case in which we say also that X is *essentially of dimension n.* The essentiality of the dimension of a n-complex X may be or not detected by the integer cohomology group $H^n(X)$ in the following sense: If $H^n(X) \neq 0$, then X is essentially of dimensional n and by the Hopf–Whitney Classification Theorem there exists a strong surjection from X into the n-dimensional sphere S^n. However, the reciprocal is not true, that is, if X is essentially of dimension n, the group $H^n(X)$ is not necessarily nontrivial. Therefore, the integer cohomology is not able to detect the essential dimension of a complex. Because of this, and despite that, it is natural to ask if $H^n(X)$ is able to detect the existence of a strong surjection from X into a closed n-manifold. Aniz and Gonçalves studied this question for dimension $n = 3$. In [1], Aniz proved that every map from a 3-complex X with $H^3(X) = 0$ into $S^1 \times S^2$ is root free, but for Y the non-orientable S^1-bundle over S^2, there exists a strong surjection $f\colon X \to Y$ from such a 3-complex. In [2], Aniz proved that there is no strong surjection from such a 3-complex into the orbit space of the 3-sphere S^3 with respect to the action of the quaternion group Q_8 determined by the inclusion $Q_8 \subset S^3$. There are not many other works on this subject, which shows that such a study may be far from complete. In this article we present a contribution for the problem in dimension two. It is noteworthy that the dimension two is often left out since it does not permit the use of special techniques as obstruction theory and others. Our approach is mainly based on the combinatorial group theory. We do not treat the root problem directly; in fact, we do not investigate the existence of strong surjection, but the existence of nontrivial maps (maps that are not homotopic to a constant map), and we consider just maps into the real projective plane. In such context, the nonexistence of nontrivial maps implies the nonexistence of strong surjection, but not the reciprocal. In this sense, the Main Theorem presents a class of two-dimensional CW-complexes with trivial second integer cohomology group for which there are no strong surjections into the projective plane.

Returning to the Main Theorem, we observe that, once proved that assertions (a) and (d) implies, each one, the assumption on the nullity of the cohomology group $H^2(K)$, there is no sense in to investigate equivalences with these assertions without the assumption $H^2(K) = 0$. Therefore, such assumption is automatically essential for all equivalences which involves assertions (a) and (d), since all the others assertions are possible, even in the absence of the assumption $H^2(K) = 0$, as we show throughout the paper.

We highlight assertion (e) to remember that for the based spaces (K, k_*) and (S^1, s_*), the set $[K; S^1]_*$ is also denoted by $\pi^1(K, k_*)$ and called the *first cohomotopy set* of (K, k_*). Since S^1 is the Eilenberg–MacLaner space $K(\mathbb{Z}, 1)$, we have one-to-one correspondences between the sets $[K; S^1]$ and $[K; S^1]_*$ and the cohomology group $H^1(K)$. This fact might be used to proof the equivalences of assertion (e) with each of (b)–(d). However, we will present a simpler proof for these equivalences in Section 4 using the asphericity of the circle S^1.

Assertion (b) exposes an appropriated approach for the proof of the more interesting equivalences presented in the Main Theorem, those that involve assertions (a) and (e). This approach considers the so-called model two-complexes induced by group presentations. We follows [7] to present and explore this concept in Section 2. We anticipate that by [7, Theorem 1.9, p. 61], *the skeleton pair of a finite and connected two-complex is homotopy equivalent to that of a model two-complex.* This result will be directly used in the proof of the Main Theorem. Moreover, it is very useful to build examples.

In Section 2, we prove the equivalences (b) $\Leftrightarrow$ (c) $\Leftrightarrow$ (d) using a characterization of the nullity of the second cohomology group $H^2(K)$ in terms of the number of cells of the two-complex K, which follows from the Universal Coefficient Theorem. We also prove that the equivalence (b) $\Leftrightarrow$ (c) is independent on the nullity of $H^2(K)$.

In Section 3 we prove the equivalence (a) $\Leftrightarrow$ (b). In order to do this, we prove a key lemma in which, for a given model two-complex $K_{\mathcal{P}}$ with fundamental group Π, we provide an isomorphism between the abelian group $\mathrm{Hom}(\Pi;\mathbb{Z}_2)$ and the addictive group $\mathcal{S}_{\mathcal{P}}$ of all solutions of a certain linear system induced by $K_{\mathcal{P}}$.

In Section 4 we first prove that each of the assertions (b)–(d) is equivalent to (e). Next, we present a direct proof for the implication (e) $\Rightarrow$ (a) as a consequence of the aforementioned Lemma 3.1 and a simple algebraic fact.

Integer coefficient cohomology is understood in whole paper. We simplify K is a two-dimensional CW-complex by K is a two-complex.

2. Proof of the equivalences (b) $\Leftrightarrow$ (c) $\Leftrightarrow$ (d)

Let K be a finite and connected two-complex. Since $H_2(K)$ is torsion free, the Universal Coefficient Theorem implies that $H^2(K) \approx H_2(K) \oplus \mathcal{T}_1(K)$, where $\mathcal{T}_1(K)$ is the torsion subgroup of $H_1(K)$. It follows that $H^2(K) = 0$ if and only if $H_2(K) = 0$ and $H_1(K)$ is torsion free. In this case, the exact homology sequence of the skeleton pair (K, K^1) becomes

$$0 \to H_2(K, K^1) \to H_1(K^1) \to H_1(K) \to 0.$$

Since $H_1(K)$ is torsion free, this sequence splits and so $H_1(K^1) \approx H_2(K, K^1) \oplus H_1(K)$. Defining $\#^2(K)$ to be the number of 2-cells of K and denoting the first Betti number by β_1, the last isomorphism gives $\mathbb{Z}^{\beta_1(K^1)} \approx \mathbb{Z}^{\#^2(K)} \oplus \mathbb{Z}^{\beta_1(K)}$. This proves the following result:

Proposition 2.1. *Let K be a finite and connected two-complex. If K has trivial second cohomology group, then $\#^2(K) = \beta_1(K^1) - \beta_1(K)$.*

This result is specially interesting for two-complexes with a single 0-cell, or more particularly for model two-complexes, since for such a two-complex K, the Betti number $\beta_1(K^1)$ coincides with the number of 1-cells.

A *model two-complexes* $K_{\mathcal{P}}$ induced by a group presentation $\mathcal{P} = \langle \mathbf{x} \,|\, \mathbf{r} \rangle$, with the set of generators $\mathbf{x} = \{x_1, \ldots, x_n\}$ and the set of (not necessarily reduced) relators $\mathbf{r} = \{r_1, \ldots, r_m\}$ is that whose one-skeleton $K^1_{\mathcal{P}} = \bigvee_j^n S^1_j = e^0 \cup e^1_1 \cup \ldots \cup e^1_n$ is the bouquet of n circles (so that $n = \beta_1(K^1_{\mathcal{P}})$) and with m two-dimensional cells (so that $m = \#^2(K_{\mathcal{P}})$), we say $e^2_1, \ldots, e^2_m$, which are attaching on the one-skeleton according to the relators $r_1, \ldots, r_m$. See [7, Section 1.2] for details.

The group presentation $\mathcal{P}$ is a presentation for the fundamental group $\pi_1(K_{\mathcal{P}})$ and we have a natural quotient homomorphism

$$\Omega\colon \pi_1(K^1_{\mathcal{P}}) \approx F(\mathbf{x}) \to \frac{F(\mathbf{x})}{N(\mathbf{r})} \approx \pi_1(K_{\mathcal{P}}),$$

where $F(\mathbf{x})$ is the free group generated by the alphabet $\mathbf{x}$ and $N(\mathbf{r})$ is its normal subgroup generated by the set of words $\mathbf{r}$.

Proposition 2.2 ([7, Theorem 1.9, p. 61]). *The skeleton pair of a finite and connected two-complex K is homotopy equivalent to that of the model two-complex $K_{\mathcal{P}}$ induced by a group presentation $\mathcal{P} = \langle \mathbf{x} \,|\, \mathbf{r}\rangle$.*

Obviously, if the skeleton pair (K, K^1) of the two-complex K is homotopy equivalent to that of the model two-complex $K_{\mathcal{P}}$ of a group presentation $\mathcal{P} = \langle x_1, \dots, x_n \,|\, r_1, \dots, r_m\rangle$, then $\beta_1(K^1) = \beta_1(K^1_{\mathcal{P}}) = n$. The next result follows from Proposition 2.1.

Corollary 2.3. *For a given group presentation $\mathcal{P} = \langle x_1, \dots, x_n \,|\, r_1, \dots, r_m\rangle$, the condition $H^2(K_{\mathcal{P}}) = 0$ implies $m = n - \beta_1(K_{\mathcal{P}})$. So $H^2(K_{\mathcal{P}}) \neq 0$ if $m > n$.*

Obviously, the reciprocal implications of the statements in Proposition 2.1 and Corollary 2.3 are not true. However, for model two-complexes, we have an interesting characterization of the nullity of the second cohomology group. By the way, this characterization implies Corollary 2.3. We show: let $K_{\mathcal{P}}$ be the model two-complex of the group presentation $\mathcal{P} = \langle x_1, \dots, x_n \,|\, r_1, \dots, r_m\rangle$. Consider all the relators writing in its generic form

$$r_1 = \Big(x_1^{\delta_{11}^{(1)}} \dots x_n^{\delta_{1n}^{(1)}}\Big) \dots \Big(x_1^{\delta_{11}^{(k_1)}} \dots x_n^{\delta_{1n}^{(k_1)}}\Big);$$
$$\vdots$$
$$r_m = \Big(x_1^{\delta_{m1}^{(1)}} \dots x_n^{\delta_{mn}^{(1)}}\Big) \dots \Big(x_1^{\delta_{m1}^{(k_m)}} \dots x_n^{\delta_{mn}^{(k_m)}}\Big).$$

For each $1 \leq i \leq m$ and $1 \leq j \leq n$, we define the integer $\delta_{ij} = \sum_{\lambda=1}^{k_i} \delta_{ij}^{(\lambda)}$, the sum of all powers of the letter x_j in the relator word r_i, and we consider the integer matrix $\Delta_{\mathcal{P}} = (\delta_{ij})_{m\times n}$. The Main Theorem in [5] states that a cellular map $f\colon K_{\mathcal{P}} \to S^2$ is homotopic to a constant map if and only if the diophantine linear system $\Delta_{\mathcal{P}} Y = \deg(f)$ has an integer solution, where $\deg(f) \in \mathbb{Z}^m$ is the so called *vector-degree* of the map f. Additionally, again by [5], every integer-vector $\mathbf{d} \in \mathbb{Z}^m$ is the vector-degree of some cellular map $f_{\mathbf{d}}\colon K_{\mathcal{P}} \to S^2$. Using these facts and the Hopf's Theorem [6, Theorem 11.5, p. 224], we conclude:

Proposition 2.4 ([5, Proposition 3.1]). *A model two-complex $K_{\mathcal{P}}$ has trivial second cohomology group if and only if the linear system $\Delta_{\mathcal{P}} Y = \mathbf{d}$ has an integer* *solution for all integer-vector* $\mathbf{d}$ *of appropriated dimension. In particular, for a group presentation* $\mathcal{P}$ *with* n *generators and* m *relators*:

(a) *If* $m > n$*, then* $H^2(K_{\mathcal{P}}) \neq 0$.
(b) *If* $m = n$*, then* $H^2(K_{\mathcal{P}}) = 0$ *if and only if* $\det(\Delta_{\mathcal{P}}) = \pm 1$.
(c) *If* $m < n$*, then it may occur* $H^2(K_{\mathcal{P}}) = 0$ *or* $H^2(K_{\mathcal{P}}) \neq 0$.

Assertion (b) of the Main Theorem relates to the following definition:

DEFINITION 2.5. A model two-complex $K_{\mathcal{P}}$ is called a *balanced model two-complex* if it has the same number of cells of dimension one and two. It is equivalent to say that $K_{\mathcal{P}}$ is induced by a group presentation $\mathcal{P} = \langle \mathbf{x} \,|\, \mathbf{r} \rangle$ having the same number of generators and relators, case in which we say that $\mathcal{P}$ is a *balanced presentation*.

We finish this section proving the equivalences (b) $\Leftrightarrow$ (c) $\Leftrightarrow$ (d) proposed in the Main Theorem. First we prove the equivalence (b) $\Leftrightarrow$ (c) independently on the assumption on the nullity of the second cohomology group and next we use the characterization of this nullity to prove the equivalence (c) $\Leftrightarrow$ (d).

PROOF OF THE EQUIVALENCE (b) $\Leftrightarrow$ (c). Let K be a finite and connected two-complex. By Proposition 2.2, there exists a group presentation $\mathcal{P} = \langle \mathbf{x} \,|\, \mathbf{r} \rangle$ such that (K, K^1) is homotopy equivalent to $(K_{\mathcal{P}}, K^1_{\mathcal{P}})$. In particular,

$$\#\mathbf{r} = \#^2(K_{\mathcal{P}}) = \operatorname{rank} H_2(K_{\mathcal{P}}, K^1_{\mathcal{P}}) = \operatorname{rank} H_2(K, K^1) = \#^2(K),$$

and

$$\#\mathbf{x} = \beta_1(K^1_{\mathcal{P}}) = \beta_1(K^1).$$

It follows that the number of 2-cells of K is equal to $\beta_1(K^1)$ if and only if the group presentation $\mathcal{P}$ is balanced (that is, $\#\mathbf{x} = \#\mathbf{r}$). □

PROOF OF THE EQUIVALENCE (c) $\Leftrightarrow$ (d). Let K be a finite and connected two-complex with trivial second cohomology group. By Proposition 2.1,

$$\#^2(K) = \beta_1(K^1) - \beta_1(K).$$

Since $H_2(K) = 0$ and $H_1(K)$ is torsion free, K is acyclic if and only if $\beta_1(K) = 0$, what in turn happens if and only if $\#^2(K) = \beta_1(K^1)$. □

Obviously, assertion (d) is impossible without the assumption $H^2(K) = 0$. On the other hand, assertions (b) and (c) are possible with $H^2(K) \neq 0$, for example, consider $K = S^1 \vee S^2$ with its minimal cellular decomposition. This shows that the assumption $H^2(K) = 0$ is essential for the equivalences (b)–(c) $\Leftrightarrow$ (d).

3. Proof of the equivalence (a) $\Leftrightarrow$ (b)

In this section, we present a proof for the equivalence (a) $\Leftrightarrow$ (b) proposed in the Main Theorem. We start with two key lemmas.

In the first one, we establish a relationship between $H^2(K)$ and $[K;\mathbb{R}\mathrm{P}^2]_*$ in terms of the nullity of the group $\mathrm{Hom}(\Pi;\mathbb{Z}_2)$, where Π is the fundamental group of the two-complex K. In the second one, we build, for model two-complexes, an isomorphism between $\mathrm{Hom}(\Pi;\mathbb{Z}_2)$ and the addictive group of all solutions of a specific homogeneous linear system induced by the complex. This isomorphism makes it easier to check the possible nullity of $\mathrm{Hom}(\Pi;\mathbb{Z}_2)$.

Lemma 3.1. *Let K be a finite and connected two-complex with fundamental group* Π. *There exists a one-to-one correspondence between $H^2(K)$ and $[K;\mathbb{R}\mathrm{P}^2]_*$ if and only if* $\mathrm{Hom}(\Pi;\mathbb{Z}_2)=0$. *Else, there exists a strictly injection from $H^2(K)$ into $[K;\mathbb{R}\mathrm{P}^2]_*$.*

Proof. Let $\mathfrak{p}\colon S^2\to\mathbb{R}\mathrm{P}^2$ be the double covering map. The proof follows from a simple analysis of the diagram

$$\begin{array}{ccc} & & S^2 \\ & \nearrow & \downarrow \mathfrak{p} \\ K & \longrightarrow & \mathbb{R}\mathrm{P}^2 \end{array}$$

By the homotopy lifting property and the unique lifting property, the correspondence $[\varphi]\mapsto[\mathfrak{p}\circ\varphi]$ gives an injection $[K;S^2]_*\hookrightarrow[K;\mathbb{R}\mathrm{P}^2]_*$.

If $\mathrm{Hom}(\Pi;\mathbb{Z}_2)=0$, then every based map $f\colon K\to\mathbb{R}\mathrm{P}^2$ lifts through $\mathfrak{p}$ to a based map $\widetilde{f}\colon K\to S^2$ and so $[f]\mapsto[\widetilde{f}]$ in an inverse correspondence to $[\varphi]\mapsto[\mathfrak{p}\circ\varphi]$.

If $\mathrm{Hom}(\Pi;\mathbb{Z}_2)\neq 0$, there exists a based map $f\colon K\to\mathbb{R}\mathrm{P}^2$ which does not lift through $\mathfrak{p}$ and so $[f]$ is not in the image of the correspondence $[\varphi]\mapsto[\mathfrak{p}\circ\varphi]$.

The proof follows from the Hopf–Whitney Classification Theorem. $\square$

In the next lemma, we use the following notation: Given a model two-complex $K_{\mathcal{P}}$ of a presentation $\mathcal{P}=\langle x_1,\dots,x_n\,|\,r_1,\dots,r_m\rangle$, we consider the matrix $\Delta_{\mathcal{P}}\in M_{m\times n}(\mathbb{Z})$, as in Section 2, and take its mod 2 quotient $\overline{\Delta}_{\mathcal{P}}=\Delta_{\mathcal{P}}$ mod 2, the $m\times n$ matrix $\overline{\Delta}_{\mathcal{P}}=(\overline{\delta}_{ij})\in M_{m\times n}(\mathbb{Z}_2)$, in which the bar indicates the integer mod 2 in the additive notation.

Lemma 3.2. *Let $K_{\mathcal{P}}$ be a model two-complex with fundamental group* Π. *The abelian group* $\mathrm{Hom}(\Pi;\mathbb{Z}_2)$ *is isomorphic to the additive group $\mathcal{S}_{\mathcal{P}}$ of all solutions of the homogeneous linear system $\overline{\Delta}_{\mathcal{P}}Y=\overline{0}$ over the field $\mathbb{Z}_2$.*

Proof. Let $\mathcal{P}=\langle x_1,\dots,x_n\,|\,r_1,\dots,r_m\rangle$ be the group presentation of $K_{\mathcal{P}}$, so that, in particular, $\mathcal{P}$ is a presentation for the group Π. Denote $F=$

$F(x_1,\ldots,x_n)$ the free group generated by $x_1,\ldots,x_n$ and let $\Omega\colon F \to \Pi$ be the obvious epimorphism.

It is well known that the homomorphisms $\tau\colon \Pi \to \mathbb{Z}_2$ are in one-to-one correspondence, via the relation $\tau \circ \Omega = h$, with the homomorphisms $h\colon F \to \mathbb{Z}_2$ which map all the relators $r_1,\ldots,r_n$ to $\overline{0}$. Let $\mathrm{Hom}(F;\mathbb{Z}_2|\,\mathbf{r})$ be the group of such homomorphisms. We prove that $\mathrm{Hom}(F;\mathbb{Z}_2|\,\mathbf{r})$ is in one-to-one correspondence with the solutions of the linear system $\overline{\Delta}_{\mathcal{P}}Y = \overline{0}$ over $\mathbb{Z}_2$. Let us suppose that $h\colon F \to \mathbb{Z}_2$ belongs to $\mathrm{Hom}(F;\mathbb{Z}_2|\,\mathbf{r})$. Since $\mathbb{Z}_2$ is abelian, for each $1 \le i \le m$, we have

$$h(r_i) = \overline{\delta}_{i1}h(x_1) + \ldots + \overline{\delta}_{in}h(x_n).$$

So the condition $h(r_i) = \overline{0}$ for all $1 \le i \le m$ implies that the vector

$$\mathbf{h} = (h(x_1),\ldots,h(x_n)) \in \mathbb{Z}_2^n$$

satisfies the equation $\overline{\Delta}_{\mathcal{P}}\mathbf{h} = \overline{0}$ over $\mathbb{Z}_2$. Of course, associated to different homomorphisms h_1 and h_2 in $\mathrm{Hom}(F;\mathbb{Z}_2|\,\mathbf{r})$, we have different solutions $\mathbf{h}_1$ and $\mathbf{h}_2$ for $\overline{\Delta}_{\mathcal{P}}Y = \overline{0}$, since a homomorphism from a free group is uniquely defined by its values on the generators. Conversely, given a solution $\mathbf{y} = (y_1,\ldots,y_n)$ for $\overline{\Delta}_{\mathcal{P}}Y = \overline{0}$ over $\mathbb{Z}_2$, associated to it we have a unique homomorphism $h \in \mathrm{Hom}(F;\mathbb{Z}_2|\,\mathbf{r})$ given by $h(x_i) = y_i$ for all $1 \le i \le m$. Moreover, for different solutions we have different homomorphisms.

We have provided a bijection $\Psi\colon \mathrm{Hom}(\Pi;\mathbb{Z}_2) \approx \mathrm{Hom}(F;\mathbb{Z}_2|\,\mathbf{r}) \approx \mathcal{S}_{\mathcal{P}}$ by defining $\Psi(\tau) = ((\tau\circ\Omega)(x_1),\ldots,(\tau\circ\Omega)(x_n))$, in which each intermediate one-to-one correspondence is a group isomorphism. Therefore, Ψ is a group isomorphism.□

PROPOSITION 3.3. *Let $K_{\mathcal{P}}$ be a balanced model two-complex with fundamental group* Π. *If $H^2(K_{\mathcal{P}}) = 0$, then* $\mathrm{Hom}(\Pi;\mathbb{Z}_2) = 0$. *The converse is not true.*

PROOF. The proof of the first statement follows directly from Lemma 3.2 and from the fact that $H^2(K_{\mathcal{P}}) = 0$ implies $\det\overline{\Delta}_{\mathcal{P}} = \overline{1}$, by Proposition 2.4. In this case, since $\mathbb{Z}_2$ is a field, the linear system $\overline{\Delta}_{\mathcal{P}}Y = \overline{0}$ has only the trivial solution, so that $\mathrm{Hom}(\Pi;\mathbb{Z}_2) \approx \mathcal{S}_{\mathcal{P}} = 0$.

In order to exemplify that the reciprocal is not true, we consider the balanced model two-complex $K_{\mathcal{P}}$ of the presentation $\mathcal{P} = \langle x, y \,|\, x^2y, xy^2\rangle$. We have $\det\Delta_{\mathcal{P}} = 3$ and so $\det\overline{\Delta}_{\mathcal{P}} = \overline{1}$. It follows that $H^2(K_{\mathcal{P}}) \neq 0$ although $\mathrm{Hom}(\Pi;\mathbb{Z}_2) = 0$. □

Proposition 3.3 is not true for non-balanced model two-complexes; for example, take $\mathcal{P} = \langle x, y \,|\, y\rangle$ so that $K_{\mathcal{P}} = S^1 \vee D^2$.

The proof of Proposition 3.3 shows also that for a balanced model two-complex $K_{\mathcal{P}}$ with fundamental group Π, we have $\mathrm{Hom}(\Pi;\mathbb{Z}_2) = 0$ if and only if $\det\Delta_{\mathcal{P}}$ is odd. In fact, if $\det\Delta_{\mathcal{P}}$ is even, then $\det\overline{\Delta}_{\mathcal{P}} = \overline{0}$ and so the system

$\overline{\Delta}_{\mathcal{P}} Y = \overline{0}$ has more than one solution. Thus we have proved the following corollary:

Corollary 3.4. *Let $K_{\mathcal{P}}$ be a balanced model two-complex. There exists a one-to-one correspondence between $H^2(K_{\mathcal{P}})$ and $[K_{\mathcal{P}}; \mathbb{R}\mathrm{P}^2]_*$ if and only if* $\det \Delta_{\mathcal{P}}$ *is odd.*

Now we prove (a) $\Leftrightarrow$ (b) using the main results of Sections 2 and 3.

Proof of the Equivalence (a) $\Leftrightarrow$ (b). Let K be a finite and connected two-complex with fundamental group Π, with trivial second cohomology group and such that the skeleton pair (K, K^1) is homotopy equivalent to that of a model two-complex $K_{\mathcal{P}}$. Let take the (unique) 0-cell $e^0 \in K_{\mathcal{P}}$ to be the base point. Choose inverse homotopy equivalences $\varphi\colon (K_{\mathcal{P}}, K^1_{\mathcal{P}}) \to (K, K^1)$ and $\psi\colon (K, K^1) \to (K_{\mathcal{P}}, K^1_{\mathcal{P}})$, according to [7, Theorem 1.9, p. 61], so that ψ corresponds to a 3-deformation which standardizes the one-skeleton of K to the bouquet of circles corresponding to the one-skeleton $K^1_{\mathcal{P}}$. In particular, there exists a 0-cell $c^0 \in K$ such that $\psi(c^0) = e^0$ and $\varphi(e^0) = c^0$. Let take c^0 to be the base point in K.

(b) $\Rightarrow$ (a). Suppose that $K_{\mathcal{P}}$ is balanced. Then Proposition 2.4 implies that $\det \Delta_{\mathcal{P}} = \pm 1$, since $H^2(K_{\mathcal{P}}) = 0$ by assumption. By Corollary 3.4, we have $[K_{\mathcal{P}}; \mathbb{R}\mathrm{P}^2]_* = 0$. We will prove that also $[K; \mathbb{R}\mathrm{P}^2]_* = 0$. Let $f\colon K \to \mathbb{R}\mathrm{P}^2$ be a based map. Then $f \circ \varphi\colon K_{\mathcal{P}} \to \mathbb{R}\mathrm{P}^2$ is also a based map and so it is based homotopic to the constant map at e^0_*. Since $\varphi \circ \psi$ is homotopic to the identity map of K, it follows that f is homotopic to the constant map at e^0_*. It follows by [4, Corollary 6.58] that f is based homotopic to the constant map at e^0_*.

(a) $\Rightarrow$ (b). Suppose that $K_{\mathcal{P}}$ is not balanced. Since $H^2(K_{\mathcal{P}}) = 0$, Corollary 2.3 implies that $\beta_1(K_{\mathcal{P}}) \neq 0$, so that $H_1(K_{\mathcal{P}})$ is free and nontrivial. It follows that $\mathrm{Hom}(H_1(K_{\mathcal{P}}); \mathbb{Z}_2) \neq 0$ and so also $\mathrm{Hom}(\Pi; \mathbb{Z}_2) \neq 0$. By Lemma 3.1, we have a strictly injection of $H^2(K)$ into $[K; \mathbb{R}\mathrm{P}^2]_*$ and so $[K; \mathbb{R}\mathrm{P}^2]_* \neq 0$. □

Since assertion (a) is impossible without the assumption $H^2(K) = 0$, but (b) is possible (for example, $K = S^1 \vee S^2$), such assumption is essential to the equivalence (a) $\Leftrightarrow$ (b).

4. The equivalences with assertion (e)

As we have said in the introduction of the article, instead of using cohomotopy approach to prove the equivalences with assertion (e), we use simply the asphericity of the circle. We prove all at once the equivalences of assertion (e) with each of assertions (b)–(d).

Proof of the Equivalences (b)–(d) $\Leftrightarrow$ (e). Let K be a finite and connected two-complex with fundamental group Π and suppose that $H^2(K) = 0$, so that $H_2(K) = 0$ and $H_1(K)$ is torsion free. Since the circle S^1 is aspherical, the correspondence $f \mapsto f_\#$ induces a one-to-one correspondence between $[K; S^1]_*$ and $\mathrm{Hom}(\pi_1(K); \pi_1(S^1)) \approx \mathrm{Hom}(\Pi, \mathbb{Z})$. See [8, Theorem 4.3 on p. 225].

Since $\mathbb{Z}$ is abelian, the composition with the abelianization homomorphism $\rho\colon \Pi \to H_1(K)$ provides an isomorphism between the groups $\mathrm{Hom}(H_1(K), \mathbb{Z})$ and $\mathrm{Hom}(\Pi, \mathbb{Z})$. Thus, we have a one-to-one correspondence between the group $\mathrm{Hom}(H_1(K), \mathbb{Z})$ and the set $[K; S^1]_*$. Since $H_1(K)$ is torsion free, we have

$$[K; S^1]_* = 0 \Leftrightarrow \mathrm{Hom}(H_1(K), \mathbb{Z}) = 0 \Leftrightarrow H_1(K) = 0 \Leftrightarrow \beta_1(K) = 0.$$

The equivalence (d) $\Leftrightarrow$ (e) follows, since $H_2(K) = 0$ by assumption. Furthermore, the equivalences (b) $\Leftrightarrow$ (e) and (c) $\Leftrightarrow$ (e) follow from Proposition 2.1. □

We remark that the assumption on the nullity of the second cohomology group of K is essential for the equivalences between each of (b)–(d) and (e). Regarding the equivalence (d) $\Leftrightarrow$ (e), this is obvious, since (d) implies $H^2(K) = 0$, but (e) does not; for example, take $K = S^2$.

As for the equivalences (b)–(c) $\Leftrightarrow$ (e), note that for $K = S^2$, assertion (e) holds true but (b) and (c) do not. On the other hand, for $K = S^1 \vee S^2$ assertions (b) and (c) hold true, but (e) does not.

We note that we have concluded the proof of the Main Theorem; in fact, we have proved the link of equivalences (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c) $\Leftrightarrow$ (d) $\Leftrightarrow$ (e). However, we consider interesting to present a direct proof of the implication (e) $\Rightarrow$ (a). Next we present such a proof in which we use the following simple algebraic fact:

LEMMA 4.1. *For a group* Π *with abelianization* Π^{ab}*, we have* $\mathrm{Hom}(\Pi; \mathbb{Z}_2) = 0$ *if* $2\Pi^{ab} = \Pi^{ab}$.

PROOF. Since $\mathbb{Z}_2$ is abelian, given $h \in \mathrm{Hom}(\Pi; \mathbb{Z}_2)$, there exists a (unique) homomorphism $\overline{h}\colon \Pi^{ab} \to \mathbb{Z}_2$ such that $h = \overline{h} \circ \mathfrak{q}$, where $\mathfrak{q}\colon \Pi \to \Pi^{ab}$ is the abelianization homomorphism. We have obviously $2\Pi^{ab} \subset \ker \overline{h}$, so that the assumption $2\Pi^{ab} = \Pi^{ab}$ implies that $\overline{h}$, and so h, is the trivial homomorphism. □

PROOF OF THE IMPLICATION (e) $\Rightarrow$ (a). Let K be a finite and connected two-complex with fundamental group Π and trivial second cohomology group. Then $\Pi^{ab} \approx H_1(K)$ is torsion free and so $H^1(K) \approx H_1(K) \approx \Pi^{ab}$. Since $[K; S^1]_*$ is in one-to-one correspondence with $H^1(K)$, assertion (5) implies that $\Pi^{ab} = 0$. By Lemma 4.1 we have $\mathrm{Hom}(\Pi; \mathbb{Z}_2) = 0$ and so $[K; \mathbb{R}\mathrm{P}^2]_* = 0$ by Lemma 3.1. □

The last two proofs show also that:

COROLLARY 4.2. *For a finite and connected two-complex* K *with fundamental group* Π *and trivial second cohomology group, we have* $\mathrm{Hom}(\Pi; \mathbb{Z}_2) = 0$ *if* $\mathrm{Hom}(\Pi; \mathbb{Z}) = 0$.

An interesting example might be obtained considering a finite and connected two-complex K with trivial second cohomology group and finite fundamental group Π. For such a two-complex we have $\mathrm{Hom}(\Pi;\mathbb{Z}) = 0$, so that also $\mathrm{Hom}(\Pi;\mathbb{Z}_2) = 0$, by Corollary 4.2, and all the assertions (a)–(e) in the Main Theorem hold true. This happens, for instance, with the balanced model two-complex induced by the group presentation $\mathcal{P} = \langle x, y \,|\, x^3y^{-5}, (xy)^2y^{-5}\rangle$ of the Poincaré's binary icosahedral group.

Two-complexes with finite fundamental group are really specially interesting in the approach of the Main Theorem. In fact, for such a two-complex, assertion (e) is trivial and so, if we assume the nullity of the second cohomology group, then we conclude all the assertions (a)–(e). The curios here is that, as assertion (e), also assertion (b) might be concluded easily for a (finite and connected) two-complex with finite fundamental group and trivial second cohomology group. Indeed, this conclusion follows from Proposition 2.4 and the algebraic fact that every presentation of a finite group has number of relators greater than or equal to the number of generators. In fact: consider a presentation $\mathcal{P} = \langle x_1, \dots, x_n \,|\, r_1, \dots, r_m\rangle$ of a given group Π, so that the abelianization Π^{ab} of Π is presented by

$$\mathcal{P}_{ab} = \langle x_1, \dots, x_n \,|\, r_1, \dots, r_m, [x_j, x_k],\, 1 \leq j, k \leq n\rangle.$$

For each $1 \leq i \leq m$, let s_i be the reduced word obtained from r_i by commuting all the letters $x_1, \dots, x_n$. Then each relation $s_i = \mathbb{1}$ is a consequence of the relations $r_i = \mathbb{1}$ and $[x_j, x_k] = \mathbb{1}$ for all $1 \leq j, k \leq n$. On the other hand, each relation $r_i = \mathbb{1}$ is a consequence of the relations $s_i = \mathbb{1}$ and $[x_j, x_k] = \mathbb{1}$ for all $1 \leq j, k \leq n$. Hence, a sequence of m Tietze transformation of the type I followed by m Tietze tranformation of type I' (we use terminology and notation of [3, Chapter IV]) provides an equivalence between the presentation $\mathcal{P}_{ab}$ and the presentation

$$\mathcal{P}'_{ab} = \langle x_1, \dots, x_n \,|\, s_1, \dots, s_m, [x_j, x_k],\, 1 \leq j, k \leq n\rangle.$$

Thus, the group Π^{ab} is isomorphic to

$$\mathbb{Z}^n/\langle \mathfrak{q}(s_1), \dots, \mathfrak{q}(s_m)\rangle, \quad \text{where } \mathfrak{q}\colon F(x_1, \dots, x_n) \to \mathbb{Z}^n$$

is the abelianization homomorphism. Therefore, Π^{ab} and so Π is infinite if $m < n$.

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