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# PERIODIC SOLUTIONS FOR SECOND ORDER SINGULAR DIFFERENTIAL SYSTEMS WITH PARAMETERS

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Abstract. In this paper we consider the existence of periodic solutions of one-parameter and two-parameter families of second order singular differential equations.

## 1. Introduction

We say that a vector valued function $f\colon \mathbb{R}\times D\to\mathbb{R}^N$, $D\subseteq\mathbb{R}^N$, is singular if for a non-empty subset $\Omega\subset\partial D$ and any $x_0\in\Omega$

$$\lim_{x\to x_0}\|f(t,x)\|=\infty,\quad \text{uniformly in } t\in\mathbb{R}.$$

Equivalently, we say that the differential equation

$$\ddot{x}+a(t)x=f(t,x)+e(t) \tag{1.1}$$

is singular if the nonlinear term $f$ is singular, where $a,e\in C(\mathbb{R},\mathbb{R}^N)$ are $T$-periodic functions, $f\colon\mathbb{R}\times D\to\mathbb{R}^N$ is continuous for some $D\subseteq\mathbb{R}^N$ and periodic in $t$ with period $T$.

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During the last two decades, the existence of nontrivial periodic solutions of (1.1) has been studied by many researchers in the nonsingular case as well as in the singular case. See [2], [11], [12], [15], [16], [18], [23] for the scalar case and [3], [4], [6], [7], [17] for the higher dimensional case. Usually, the proof is based on either the method of upper and lower solutions [1], [10], [15], fixed point theorems [6], [7], [16]–[18], alternative principle of Leray–Schauder [3], [11] or topological degree theory [23], [26].

On the other hand, second-order nonlinear differential equations or systems with parameters have also been studied by some researchers. See, for example [8], [13], [19]–[22], [25] and the references therein. Based on a fixed point theorem in cones, under different combinations of superlinearity and sublinearity of the function $f$, Graef, Kong and Wang in [8] studied the existence, multiplicity, and nonexistence results for positive solutions of the following scalar nonsingular periodic boundary value problem

$$\begin{cases} \ddot{y} - \rho^2 y + \lambda g(t) f(y) = 0, \\ y(0) = y(2\pi), \quad \dot{y}(0) = \dot{y}(2\pi), \end{cases} \tag{1.2}$$

for different parameter values $\lambda \in \mathbb{R}^+ = (0, \infty)$. Later, Wang in [21] extended the similar idea to the singular periodic systems. For systems with two parameters, one nice result was proved in [22] by Wu and Wang. They studied the existence of periodic solutions of the following system with two parameters

$$\begin{cases} \ddot{u} + a_1(t) u = \lambda b_1(t) f_1(u, v), \\ \ddot{v} + a_2(t) v = \mu b_2(t) f_2(u, v), \end{cases} \tag{1.3}$$

where $(\lambda, \mu) \in (\mathbb{R}^+)^2$ and $a_i$, $b_i$, $f_i$, $i = 1, 2$, satisfy some additional conditions (see [22, conditions $(\mathrm{H}_1)$–$(\mathrm{H}_5)$]). By employing fixed point index theory, they show that there exist three nonempty subsets of $(\mathbb{R}^+)^2$: $\Gamma$, $\Delta_1$, $\Delta_2$ such that $(\mathbb{R}^+)^2 = \Gamma \cup \Delta_1 \cup \Delta_2$ and (1.3) has at least two positive periodic solutions for $(\lambda, \mu) \in \Delta_1$, one positive periodic solution for $(\lambda, \mu) \in \Gamma$ and no positive periodic solutions for $(\lambda, \mu) \in \Delta_2$. Note that the results in [22] can only be applied to the nonsingular case. Yang in [24] established the existence results for $2m$-order differential systems with two parameters.

Motivated by these recent developments, in this paper, we investigate the existence and multiplicity of $T$-periodic solutions of the following special case of system (1.3),

$$\begin{cases} \ddot{x} + a_1(t) x = \lambda f_1(x, y), \\ \ddot{y} + a_2(t) y = \mu f_2(x, y), \end{cases} \tag{1.4}$$

where $\lambda, \mu \in \mathbb{R}^+$. However, we consider (1.4) in the singular case, which is the main difference between our results and those in the literature.

In this paper, we always assume that the following condition is satisfied:

(H1) The function $a_i$ is continuous, positive, $T$-periodic and the linear equation $\ddot{x} + a_i(t)x = 0$ has a positive Green's function $G_i(t,s)$, i.e.

$$G_i(t,s) > 0 \quad \text{for all } (t,s) \in [0,T] \times [0,T],\ i = 1,2.$$

When (H1) holds, it is obvious that [16, Section 2] $(x,y)$ is a $T$-periodic solution of (1.4) if and only if

$$\begin{aligned} x(t) &= \lambda \int_0^T G_1(t,s) f_1(x(s),y(s))\,ds, \\ y(t) &= \mu \int_0^T G_2(t,s) f_2(x(s),y(s))\,ds. \end{aligned} \tag{1.5}$$

Torres in [16] has found sufficient conditions for (H1) to be satisfied. In fact, if $a_i(t) = k^2$, then (H1) holds if $k \in (0, \pi/T)$. If $a_i(t)$ is not a constant, then (H1) is valid if the $L^p$ norm of $a_i(t)$ is bounded from above by a constant, which depends on $p$ and $T$ (see [16]). When (H1) is satisfied, for $i = 1,2$, we denote

$$m_i = \min_{0\le s,t\le T} G_i(t,s), \quad M_i = \max_{0\le s,t\le T} G_i(t,s), \quad \delta_i = m_i/M_i. \tag{1.6}$$

Obviously, $M_i > m_i > 0$ and $0 < \delta_i < 1$.

The paper is organized as follows. In Section 2, under (H1) and two additional conditions (see (H2) and (H3) in next section), we establish the existence and multiplicity results of (1.4) in the case $\lambda = \mu$. The proof is based on a vector version of Krasnosel'skiĭ's fixed point theorem (Lemma 2.1). Note that we only assume that $f_1$ is singular along the $x$-axis and $f_2$ is singular along the $y$-axis (see condition (H3) in next section). However, in the literature, it was assumed that both $f_1$ and $f_2$ are singular at the origin (see [3], [4], [6], [7], [17] and the references therein). Therefore, we can deal with systems which are not covered in the literature (see Example 2.2).

Finally, in Section 3, we establish the existence results for the system (1.4) when (1.4) presents a repulsive singularity at the origin, which means that

$$\lim_{(x,y)\to(0,0)} f_i(x,y) = +\infty, \quad i = 1,2. \tag{1.7}$$

We show that there exists a bounded and continuous curve $\Gamma$ separating $(\mathbb{R}^+)^2$ into two disjoint subsets $\Theta_1$ and $\Theta_2$ such that (1.4) has at least two positive periodic solutions for $(\lambda,\mu) \in \Theta_1$, one positive periodic solution for $(\lambda,\mu) \in \Gamma$, and no positive periodic solution for $(\lambda,\mu) \in \Theta_2$. The proof is based on a well-known fixed point theorem in cones (Lemma 3.1), together with the method of upper and lower solutions. Compared with the results in [22], we deal with the singular case.

## 2. Existence results of (1.4) with $\lambda = \mu$

For the convenience to state the results, we rewrite (1.4) with $\lambda = \mu$ as the following system

$$\begin{cases} \ddot{x} + a_1(t)x = \lambda f_1(x,y), \\ \ddot{y} + a_2(t)y = \lambda f_2(x,y). \end{cases} \tag{2.1}$$

In this section, we establish the existence results of (2.1) under (H1) and the following two conditions:

- (H2) $f_1(x,y)$ is nondecreasing in $x$, nonincreasing in $y$, and $f_2(x,y)$ is nonincreasing in $x$, nondecreasing in $y$ for $x, y > 0$;
- (H3) $f_1\colon \{(x,y) \in \mathbb{R}^2 : y > 0\} \to \mathbb{R}^+$ and $f_2\colon \{(x,y) \in \mathbb{R} : x > 0\} \to \mathbb{R}^+$ are continuous and

$$\begin{aligned} \lim_{y\to 0^+} f_1(x,y) &= +\infty, \quad \text{for all } x \in \mathbb{R}, \\ \lim_{x\to 0^+} f_2(x,y) &= +\infty, \quad \text{for all } y \in \mathbb{R}. \end{aligned}$$

The proof is based on the following vector version of Krasnosel'skiĭ's fixed point theorem, which was proved by Precup [14]. First we recall some notations. Let $(X, |\cdot|)$ be a normed vector space and $K_1, K_2 \subset X$ be two cones. We use the symbol $\preceq$ and $\prec$ to denote the partial order relations and the strict order relations induced by $K_i$, $i = 1, 2$, in $X$ and $K = K_1 \times K_2$ in $X^2$. In particular, in $X^2$, the symbol $\preceq$ $(\prec)$ will have the following meaning: $u \preceq (\prec)\, v$ $(u, v \in X^2)$ if $u_i \preceq (\prec)\, v_i$, $i = 1, 2$.

Lemma 2.1. [14, Theorem 2.1] *Let $(X, |\cdot|)$ be a normed vector space, $K_1$, $K_2 \subset X$ be two cones, $K = K_1 \times K_2$, $r, R \in \mathbb{R}^2_+$ with $0 < r_i < R_i$ for $i = 1, 2$, $K_{r,R} := \{u = (u_1, u_2) \in K : r_i \le |u_i| \le R_i,\ i = 1, 2\}$ and $A = (A_1, A_2)$ be a compact map. Assume that for each $i \in \{1, 2\}$, one of the following conditions is satisfied in $K_{r,R}$,*

- (a) *$A_i(u) \succeq u_i$ if $|u_i| = r_i$, and $A_i(u) \preceq u_i$ if $|u_i| = R_i$,*
- (b) *$A_i(u) \preceq u_i$ if $|u_i| = r_i$, and $A_i(u) \succeq u_i$ if $|u_i| = R_i$.*

*Then $A$ has a fixed point $u$ in $K$ with $r_i \le |u_i| \le R_i$.*

To apply Lemma 2.1, let us denote $E$ the Banach space $C[0,T] \times C[0,T]$ and define the cone as $K = K_1 \times K_2$ with

$$K_1 = \{x(t) \in C[0,T] : x(t) \ge \delta_1 |x|_\infty\}, \quad K_2 = \{y(t) \in C[0,T] : y(t) \ge \delta_2 |y|_\infty\},$$

where $\delta_1, \delta_2$ are given as in (1.6) and $|\cdot|_\infty$ is the usual max-norm in $C[0,T]$. For $0 < r_i < R_i$, $i = 1, 2$, let

$$K_{r,R} := \{(x(t), y(t)) \in K : r_1 \le |x|_\infty \le R_1,\ r_2 \le |y|_\infty \le R_2\}.$$

Define the operator $A = (A_1, A_2)\colon K \to E$, where

$$A_i(x,y)(t) = \lambda \int_0^T G_i(t,s) f_i(x(s), y(s))\, ds, \quad 0 \le t \le T,\ i = 1,2.$$

Because of (1.5), we know that $(x,y)$ is a $T$-periodic solution of (2.1) if and only if $A(x,y) = (x,y)$.

Lemma 2.2. *Assume that* (H1) *and* (H2) *hold. Then for* $0 < r_i < R_i$, $i = 1,2$, *the operator* $A\colon K_{r,R} \to K$ *is completely continuous.*

Proof. We refer the reader to [21, Lemma 3.2] for details of this standard proof. □

Theorem 2.3. *Assume that* (H1)–(H3) *hold. Suppose further that*

($\mathrm{f_{sub}}$) $\lim\limits_{s\to+\infty} f_1(s, \delta_2 s)/s = 0,\ \lim\limits_{s\to+\infty} f_2(\delta_1 s, s)/s = 0.$

*Then* (2.1) *has at least one positive $T$-periodic solution for any* $\lambda > 0$.

Proof. Since (H2) holds, then for any $\lambda > 0$, there exists a $r > 0$ such that

$$f_1(x,y) \ge \Gamma y, \quad f_2(x,y) \ge \Gamma x, \quad \text{for } 0 < x \le r,$$

where $\Gamma$ is chosen such that $\Gamma \lambda T \max\{m_1\delta_2, m_2\delta_1\} > 1$. We choose $r_1 = r_2 = r$. Let $x \in K_1$ with $|x|_\infty = r$, then

$$\begin{aligned} A_1(x,y)(t) &= \lambda \int_0^T G_1(t,s) f_1(x(s), y(s))\, ds \ge \lambda m_1 \int_0^T f_1(x(s), y(s))\, ds \\ &\ge \lambda m_1 \Gamma \int_0^T y(s)\, ds \ge \lambda m_1 \Gamma T \delta_2 r > r \ge x(t), \end{aligned}$$

namely, $A_1(x,y)(t) \succ x(t)$, if $|x|_\infty = r$. In the similar way, if $y \in K_2$ with $|y|_\infty = r$, then $A_2(x,y)(t) \succ y(t)$.

On the other hand, using the condition ($\mathrm{f_{sub}}$), there exists a constant $\widehat{R} > 0$ such that

$$f_1(s, \delta_2 s) \le \varepsilon s, \quad f_2(\delta_1 s, s) \le \varepsilon s, \quad \text{for } s \ge \widehat{R},$$

where $\varepsilon$ satisfies $\varepsilon \lambda T \max\{M_1, M_2\} < \min\{\delta_1, \delta_2\}$.

We choose $R_1 = R_2 = \max\{\widehat{R}/\min\{\delta_1, \delta_2\}, 2r + 1\} = R$. Let $x \in K_1$ with $|x|_\infty = R$, then

$$\begin{aligned} A_1(x,y)(t) &= \lambda \int_0^T G_1(t,s) f_1(x(s), y(s))\, ds \\ &\le \lambda \int_0^T G_1(t,s) f_1(R, \delta_2 R)\, ds \le \lambda M_1 \int_0^T \varepsilon R\, ds \le \delta_1 R \le x(t), \end{aligned}$$

namely, $A_1(x,y)(t) \preceq x(t)$, if $|x|_\infty = R$. Similarly, if $y \in K_2$ with $|y|_\infty = R$, then $A_2(x,y)(t) \preceq y(t)$.

Now it follows from Lemma 2.1 that $A$ has a fixed point $(x, y) \in K_{r,R}$. Therefore, for any $\lambda > 0$, (2.1) has at least one positive $T$-periodic solution $(x, y)$ with $r \le |x|_\infty \le R$, $r \le |y|_\infty \le R$. □

Theorem 2.4. *Assume that* (H1)–(H3) *hold. Suppose further that*:

($f_{\sup}$) $\lim_{s\to+\infty} f_1(\delta_1 s, s)/s = +\infty$, $\lim_{s\to+\infty} f_2(s, \delta_2 s)/s = +\infty$.

*Then* (2.1) *has at least two positive $T$-periodic solutions for any $\lambda > 0$ sufficiently small.*

Proof. Let us fix a positive constant $\widehat{R} > r$. If $|x|_\infty = \widehat{R}$, then we have

$$\begin{aligned} A_1(x,y)(t) &= \lambda \int_0^T G_1(t,s) f_1(x(s),y(s))\,ds \le \lambda \int_0^T G_1(t,s) f_1(\widehat{R}, \delta_2\widehat{R})\,ds \\ &\le \lambda M_1 T f_1(\widehat{R}, \delta_2\widehat{R}) \le \frac{1}{2}\delta_1 R < \delta_1 R \le x(t), \end{aligned}$$

for sufficiently small $\lambda$, namely, $A_1(x,y)(t) \prec x(t)$, if $|x|_\infty = \widehat{R}$. In the similar way, if $|y|_\infty = \widehat{R}$, then $A_2(x,y)(t) \prec y(t)$. Therefore, it follows from Lemma 2.1 that the operator $A$ has a fixed point $(x,y) \in K_{r,\widehat{R}}$ for sufficiently small $\lambda$, which means that (2.1) has a positive $T$-periodic solution $(x,y)$ with $r \le |x|_\infty < \widehat{R}$, $r \le |y|_\infty < \widehat{R}$.

On the other hand, using the condition ($f_{\sup}$), there exists a $R > 2\widehat{R}$ such that

$$f_1(\delta_1 s, s) \ge \Lambda s, \quad f_2(s, \delta_2 s) \ge \Lambda s, \quad \text{for } s \ge R,$$

where $\Lambda$ satisfies $\Lambda\lambda T \max\{m_1, m_2\} > 1$.

For convenience, choose $R_1 = R_2 = R$. If $|x|_\infty = R$, then we have

$$\begin{aligned} A_1(x,y)(t) &= \lambda \int_0^T G_1(t,s) f_1(x(s),y(s))\,ds \ge \lambda m_1 \int_0^T f_1(\delta_1 R, R)\,ds \\ &\ge \lambda m_1 \int_0^T \Lambda R\,ds \ge \lambda m_1 \Lambda T R > R \ge x(t), \end{aligned}$$

namely, $A_1(x,y)(t) \succ x(t)$, if $|x|_\infty = R$. In the similar way, if $|y|_\infty = R$, we have $A_2(x,y)(t) \succ y(t)$, if $|y|_\infty = R$.

Now it follows from Lemma 2.1 that $A$ has another fixed point $(x,y) \in K_{\widehat{R},R}$, and therefore (2.1) has another positive periodic solution $(x,y)$ with $\widehat{R} < |x|_\infty \le R$, $\widehat{R} < |y|_\infty \le R$. □

Example 2.5. Consider the following system:

$$\begin{cases} \ddot{x} + a_1(t)x = \lambda\left(y^{-\alpha} + \dfrac{x^\nu}{1+y}\right), \\ \ddot{y} + a_2(t)y = \lambda\left(x^{-\beta} + \dfrac{y^\gamma}{1+x}\right), \end{cases} \tag{2.2}$$

with $a_1, a_2$ satisfying (H1) and $\alpha, \beta, \gamma, \nu > 0$. Then

(a) if $\gamma, \nu < 2$, then (2.2) has at least one positive $T$-periodic solution for all $\lambda > 0$.
(b) if $\gamma, \nu > 2$, then (2.2) has at least two positive $T$-periodic solutions for each $0 < \lambda < \lambda_1$, here $\lambda_1$ is some positive constant.

Proof. Let us take

$$f_1(x,y) = y^{-\alpha} + \frac{x^\nu}{1+y}, \qquad f_2(x,y) = x^{-\beta} + \frac{y^\gamma}{1+x}.$$

Since $\alpha, \beta, \gamma, \nu > 0$, (H2) and (H3) are satisfied. By easy computations, one may readily verify that $(\mathrm{f_{sub}})$ is satisfied if $\gamma, \nu < 2$; and $(\mathrm{f_{sup}})$ is satisfied if $\gamma, \nu > 2$. Now the results follow from Theorems 2.3 and 2.4. □

## 3. Existence results for (1.4)

In this section, we establish the existence results for (1.4) under condition (H1) and the following three additional conditions:

(H4) $f_i(x,y)\colon (\mathbb{R}^+)^2 \to \mathbb{R}^+$ is continuous and (1.7) holds, $i = 1, 2$.
(H5) $f_i(x,y)$ is superlinear growth at $+\infty$, which means that,

$$f_{i,\infty} = \lim_{|x|+|y| \to +\infty} \frac{f_i(x,y)}{|x|+|y|} = +\infty, \quad i = 1, 2.$$

(H6) There exists a $\overline{R} > 0$ such that

$$\frac{2}{\rho\delta^2} \max_{i=1,2} \sup_{(x,y):\delta\overline{R} \le x+y \le \overline{R}} f_i(x,y) \le \overline{R},$$

where $\delta = \{\delta_1, \delta_2\}$ and $\rho$ is given as in (3) below.

It follows from (H4) and (H5) that there exist constants $\rho_i > 0$ and $r_i > 0$ such that

$$f_i(x,y) \ge \rho_1(x+y), \quad \text{for } 0 < x+y < r_1,$$
$$f_i(x,y) \ge \rho_2(x+y), \quad \text{for } x+y > r_2.$$

In addition, since $f_i(x,y)/(x+y)$ is continuous on $\{(x,y) : r_1 \le x+y \le r_2\}$, its minimum exists and we denote

$$\rho_3 = \min_{i=1,2} \min_{(x,y):r_1 \le x+y \le r_2} \frac{f_i(x,y)}{x+y}.$$

Let $\rho = \min\{\rho_i : i = 1,2,3\} > 0$. Thus we have

$$f_i(x,y) \ge \rho(x+y), \quad \text{for all } (x,y) > (0,0).$$

The proof of the main result in this section is based on the method of upper and lower solutions, together with the following well-known fixed point theorem.

Lemma 3.1 ([9]). *Let $E$ be a Banach space, and $K \subset E$ be a cone in $E$. Assume that $\Omega_1$, $\Omega_2$ are open subsets of $E$ with $0 \in \Omega_1, \overline{\Omega}_1 \subset \Omega_2$, and let $A\colon K \cap (\overline{\Omega}_2 \setminus \Omega_1) \to K$ be a completely continuous operator such that either*

(a) $\|Au\| \le \|u\|$, $u \in K \cap \partial\Omega_1$ *and* $\|Au\| \ge \|u\|$, $u \in K \cap \partial\Omega_2$; *or*

(b) $\|Au\| \ge \|u\|$, $u \in K \cap \partial\Omega_1$ *and* $\|Au\| \le \|u\|$, $u \in K \cap \partial\Omega_2$.

*Then A has a fixed point in* $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

To apply Lemma 3.1, we define the operator

$$A_{\lambda,\mu}(x)(t) = (A_\lambda(x,y)(t), A_\mu(x,y)(t))$$

by

$$\begin{aligned} A_\lambda(x,y)(t) &= \lambda \int_0^T G_1(t,s) f_1(x(s),y(s))\,ds, \\ A_\mu(x,y)(t) &= \mu \int_0^T G_2(t,s) f_2(x(s),y(s))\,ds. \end{aligned}$$

We also use $E$ to denote the Banach space $C[0,T] \times C[0,T]$ with the norm

$$\|(x,y)\| = |x|_\infty + |y|_\infty.$$

Define a cone $\widetilde{K} \subset E$ as

$$\widetilde{K} = \{(x(t),y(t)) \in E : x(t),\ y(t) \ge 0 \text{ and } x(t)+y(t) \ge \delta\|(x,y)\|\}.$$

From a standard process, it is easy to prove that $A_{\lambda,\mu} \colon \widetilde{K} \setminus \{\overrightarrow{0}\} \to \widetilde{K}$ is completely continuous.

Now we give the definitions of upper and lower solutions. The method of upper and lower solutions is one of the most fruitful techniques in nonlinear analysis. We refer the reader to [5] for details.

Definition 3.2. Let $\alpha_i(t), \beta_i(t) \in C^2([0,T],\mathbb{R})$, $i = 1,2$. A function $\alpha(t) = (\alpha_1(t), \alpha_2(t))$ is a lower solution of (1.4) if $\alpha(t)$ satisfies

$$\begin{cases} \ddot{\alpha_1} + a_1(t)\alpha_1 \le \lambda f_1(\alpha_1,\alpha_2), \\ \ddot{\alpha_2} + a_2(t)\alpha_2 \le \mu f_2(\alpha_1,\alpha_2). \end{cases}$$

A function $\beta(t) = (\beta_1(t), \beta_2(t))$ is a upper solution of (1.4) if $\beta(t)$ satisfies

$$\begin{cases} \ddot{\beta_1} + a_1(t)\beta_1 \ge \lambda f_1(\beta_1,\beta_2), \\ \ddot{\beta_2} + a_2(t)\beta_2 \ge \mu f_2(\beta_1,\beta_2). \end{cases}$$

Lemma 3.3 ([5]). *Let $\alpha(t)$ and $\beta(t)$ be lower and upper solutions of* (1.4) *such that $\alpha(t) \le \beta(t)$, for $t \in [0,T]$. Then* (1.4) *has a solution $(x(t),y(t))$ satisfying $\alpha(t) \le (x(t),y(t)) \le \beta(t)$.*

Before stating the main result of this section, we first prove several lemmas.

Lemma 3.4. *Assume that* (H1), (H4), (H5) *hold and $\Sigma$ is a compact subset of $(\mathbb{R}^+)^2$. Then there exists a constant $C_\Sigma > 0$ such that for all $(\lambda,\mu) \in \Sigma$ and all possible positive $T$-periodic solutions $(x,y)$ of* (1.4) *at $(\lambda,\mu)$, one has $\|(x,y)\| \le C_\Sigma$.*

PROOF. On the contrary, suppose that there exists a sequence $\{(x_k, y_k)\}_{k=1}^{\infty}$ of positive $T$-periodic solutions of (1.4) at $\{(\lambda_k, \mu_k)\}_{k=1}^{\infty}$ such that $\{(\lambda_k, \mu_k)\}_{k=1}^{\infty} \subset \Sigma$ for all $k \in N$ and $\|(x_k, y_k)\| \to \infty$.

Since $\Sigma$ is compact, the sequence $\{(\lambda_k, \mu_k)\}_{k=1}^{\infty} \subset \Sigma$ has a convergent subsequence which we denote without loss of generality still by $\{(\lambda_k, \mu_k)\}_{k=1}^{\infty} \subset \Sigma$ such that

$$\lim_{k\to\infty} \lambda_k = \lambda^*, \qquad \lim_{k\to\infty} \mu_k = \mu^*.$$

We assume that $\lambda^* > 0$ holds. Hence for $k$ sufficiently large, we have $\lambda^* \geq \lambda^*/2 > 0$.

Since $f_{1,\infty} = +\infty$, there exists a constant $R > 0$ such that

$$f_1(x, y) \geq L_\lambda(x + y), \quad \text{for all } x + y \geq R,$$

where $L_\lambda$ satisfies $\lambda^* m_1 L_\lambda \delta/2 > 1$. Furthermore, we have

$$\begin{aligned}\max_{t\in[0,T]} |x_k(t)| \geq x_k(t) = \lambda_k \int_0^T G_1(t, s) f_1(x_k(s), y_k(s))\, ds \\ \geq m_1\lambda_k L_\lambda \delta\|(x_k, y_k)\| > \frac{\lambda^*}{2} m_1 L_\lambda \delta \|(x_k, y_k)\| > \|(x_k, y_k)\|\end{aligned}$$

for all $k$ sufficiently large. This yields a contradiction. When $\mu^* > 0$, the argument is the same using the fact $f_{2,\infty} = +\infty$. □

LEMMA 3.5. *Assume that* (H1), (H4), (H5) *hold and* (1.4) *has a positive $T$-periodic solution at $(\overline{\lambda}, \overline{\mu})$. Then* (1.4) *has a positive $T$-periodic solution at $(\lambda, \mu) \in (\mathbb{R}^+)^2$ for all $(\lambda, \mu) \leq (\overline{\lambda}, \overline{\mu})$.*

PROOF. Suppose that $(\overline{x}, \overline{y})$ is the fixed point of $A_{\overline{\lambda},\overline{\mu}}$. Then, for any $(\lambda, \mu) \leq (\overline{\lambda}, \overline{\mu})$, we have

$$\begin{aligned}\overline{x}(t) &= \overline{\lambda} \int_0^T G_1(t, s) f_1(\overline{x}(s), \overline{y}(s))\, ds \geq \lambda \int_0^T G_1(t, s) f_1(\overline{x}(s), \overline{y}(s))\, ds,\\ \overline{y}(t) &= \overline{\mu} \int_0^T G_2(t, s) f_2(\overline{x}(s), \overline{y}(s))\, ds \geq \mu \int_0^T G_2(t, s) f_2(\overline{x}(s), \overline{y}(s))\, ds.\end{aligned}$$

Therefore, $(\overline{x}(t), \overline{y}(t))$ and $(0, 0)$ are the upper and lower solutions of $A_{\lambda,\mu}$, respectively. Furthermore, we can find a function $(x, y)$ satisfying $(0, 0) < (x, y) \leq (\overline{x}, \overline{y})$, which corresponds to the positive $T$-periodic solution of (1.4) at $(\lambda, \mu)$ with $0 < \lambda \leq \overline{\lambda}$, $0 < \mu \leq \overline{\mu}$. □

LEMMA 3.6. *Assume that* (H1), (H4), (H5) *hold. Then there exists a* $(\lambda^*, \mu^*) > (0, 0)$ *such that* (1.4) *has a positive $T$-periodic solution for all $(\lambda, \mu) \leq (\lambda^*, \mu^*)$.*

PROOF. Let $\beta(t) = (\beta_1(t), \beta_2(t))$ be the unique $T$-periodic solution of

$$\begin{cases}\ddot{x}(t) + a_1(t)x = 1,\\ \ddot{y}(t) + a_2(t)y = 1.\end{cases}$$

Let $F_i = \max_{t\in[0,T]} f_i(\beta(t))$. It follows from (H4) that $F_i > 0$. Choosing $(\lambda^*, \mu^*) = (1/F_1, 1/F_2)$, then we have

$$\begin{aligned}\ddot{\beta}_1(t) + a_1(t)\beta_1 - \lambda^* f_1(\beta(t)) &= 1 - \lambda^* f_1(\beta(t)) \geq 0,\\ \ddot{\beta}_2(t) + a_2(t)\beta_2 - \mu^* f_2(\beta(t)) &= 1 - \mu^* f_2(\beta(t)) \geq 0,\end{aligned}$$

which implies that $\beta(t)$ is an upper solution of (1.4) at $(\lambda^*, \mu^*)$. On the other hand, $(0,0)$ ia a lower solution of (1.4). Thus (1.4) has a positive $T$-periodic solution at $(\lambda^*, \mu^*)$. Now Lemma 3.5 implies the conclusion of Lemma 3.6. □

Define a set $S$ by

$$S = \{(\lambda, \mu) \in (\mathbb{R}^+)^2 : (1.4) \text{ has a positive } T\text{-periodic solution at } (\lambda, \mu)\}.$$

It follows from Lemma 3.6 that $S \neq \emptyset$ and $(S, \leq)$ is a partially ordered set.

Lemma 3.7. *Assume* (H1), (H4) *and* (H5) *hold. Then* $(S, \leq)$ *is bounded from above.*

Proof. By (H4) and (H5), there exists a $\rho > 0$ such that $f_i(x, y) \geq \rho(x+y)$, for all $x, y \geq 0$. Let $(\lambda, \mu) \in S$ and $(x(t), y(t))$ be a positive $T$-periodic solution of (1.4) at $(\lambda, \mu)$. Then we get

$$x(t) = \lambda \int_0^T G_1(t,s) f_1(x(s), y(s))\, ds \geq \lambda m_1 T \rho\delta \|(x,y)\|.$$

Since $x(t) \leq |x|_\infty$, $|x|_\infty \neq 0$, we obtain $\lambda m_1 T\rho\delta \leq 1$, that is $\lambda \leq 1/(m_1 T\rho\delta)$. In a similar way, we have $\mu \leq 1/(m_2 T\rho\delta)$. Therefore $S$ is bounded above by

$$(\overline{\lambda}, \overline{\mu}) = \left(\frac{1}{m_1 T\rho\delta}, \frac{1}{m_2 T\rho\delta}\right). \qquad \square$$

Next we present three lemmas and omit their proofs because the proofs are similar as in [19, Lemma 4.6–4.8].

Lemma 3.8. *Assume that* (H1), (H4) *and* (H5) *hold. Then every chain in* $S$ *has a unique supremum in* $S$.

Lemma 3.9. *Assume that* (H1), (H4) *and* (H5) *hold. Then there exists* $\widetilde{\lambda} \in [\lambda^*, \overline{\lambda}]$ *such that* (1.4) *has a positive solution at* $(\lambda, 0)$ *for all* $0 < \lambda \leq \widetilde{\lambda}$, *no solution at* $(\lambda, 0)$ *for all* $\lambda > \widetilde{\lambda}$. *Similarly, there exists* $\widetilde{\mu} \in [\mu^*, \overline{\mu}]$ *such that* (1.4) *has a positive solution at* $(0, \mu)$ *for all* $0 < \mu \leq \widetilde{\mu}$, *no solution at* $(0, \mu)$ *for all* $\mu > \widetilde{\mu}$.

Lemma 3.10. *Assume that* (H1), (H4) *and* (H5) *hold. Then there exists a continuous curve* $\Gamma$ *separating* $(\mathbb{R}^+)^2$ *into two disjoint subsets* $\Theta_1$ *and* $\Theta_2$ *such that* $\Theta_1$ *is bounded and* $\Theta_2$ *is unbounded,* (1.4) *has at least one solution for* $(\lambda, \mu) \in \Theta_1 \cup \Gamma$ *and no solution for* $(\lambda, \mu) \in \Theta_2$. *The function* $\mu = \mu(\lambda)$ *is nonincreasing, that is, if* $\lambda \leq \lambda' \leq \widetilde{\lambda}$, *then* $\mu(\lambda) \geq \mu(\lambda')$.

Now we are in a position to state the main result of this section.

THEOREM 3.11. *Assume that* (H1), (H4), (H5), (H6) *hold. Then there exists a bounded and continuous curve* $\Gamma$ *separating* $(\mathbb{R}^+)^2$ *into two disjoint subsets* $\Theta_1$ *and* $\Theta_2$ *such that* (1.4) *has at least two positive* $T$*-periodic solutions for* $(\lambda,\mu)\in\Theta_1$*, one positive* $T$*-periodic solution for* $(\lambda,\mu)\in\Gamma$*, and no positive* $T$*-periodic solution for* $(\lambda,\mu)\in\Theta_2$*. Moreover, let* $\Gamma_+\cup\Gamma_0$ *be the parametric representation of* $\Gamma$*, where* $\Gamma_+:\mu=\mu(\lambda)>0$ *and* $\Gamma_0:\mu=\mu(\lambda)=0$*. Then on* $\Gamma_+$ *the function* $\mu(\lambda)$ *is continuous and nonincreasing in* $\mathbb{R}^+$*, that is, if* $\lambda_1\le\lambda_2$*, then* $\mu(\lambda_1)\ge\mu(\lambda_2)$*.*

PROOF. From the above lemmas, we only need to establish the existence of the second positive $T$-periodic solution of the problem (1.4) for $(\lambda,\mu)\in\Theta_1$. Let $(\lambda,\mu)\in\Theta_1$, then from Lemma 3.7 and 3.10, it is clear to see that $\lambda<1/(m_1T\rho\delta)$, $\mu<1/(m_2T\rho\delta)$. Set $\Omega_{\overline{R}}=\{(x,y)\in E:\|(x,y)\|<\overline{R}\}$, where $\overline{R}$ is given by (H6). Then, for any $(x,y)\in\partial\Omega_{\overline{R}}\cap\widetilde{K}$, we have

$$\begin{aligned}
A_\lambda(x,y)(t) &= \lambda\int_0^T G_1(t,s)f_1(x(s),y(s))\,ds\\
&< \frac{1}{m_1T\rho\delta}\int_0^T G_1(t,s)f_1(x(s),y(s))\,ds\\
&\le \frac{1}{m_1T\rho\delta}M_1T\Phi_1(\overline{R})\le\frac{1}{2}\overline{R}=\frac{1}{2}\|(x,y)\|,\\
A_\mu(x,y)(t) &= \mu\int_0^T G_2(t,s)f_2(x(s),y(s))\,ds\\
&< \frac{1}{m_2T\rho\delta}\int_0^T G_2(t,s)f_2(x(s),y(s))\,ds\\
&\le \frac{1}{m_2T\rho\delta}M_2T\Phi_2(\overline{R})\le\frac{1}{2}\overline{R}=\frac{1}{2}\|(x,y)\|,
\end{aligned}$$

where $\Phi_i(\overline{R})$ is defined as $\Phi_i(R)=\max\{f_i(x,y):\delta R\le x+y\le R\}$. So, for any $(x,y)\in\partial\Omega_{\overline{R}}\cap\widetilde{K}$, we have $\|A_{\lambda,\mu}(x,y)\|<\|(x,y)\|$.

On the one hand, since $\lim\limits_{(x,y)\to(0,0)^+}f_i(x,y)=+\infty$, there exists a constant $r$ with $0<r<\overline{R}$ such that $f_i(x,y)\ge\eta_i(x+y)$, for $0<x+y<r$, where $\eta_i>0$ satisfies $\eta_1\lambda\delta Tm_1>1/2$, $\eta_2\mu\delta Tm_2>1/2$. Then, for any $(x,y)\in\partial\Omega_r\cap\widetilde{K}$, we have

$$\begin{aligned}
A_\lambda(x,y)(t) &= \lambda\int_0^T G_1(t,s)f_1(x(s),y(s))\,ds\\
&\ge \lambda m_1\int_0^T \eta_1(x(s)+y(s))\,ds\ge\lambda m_1\eta_1\delta T\|(x,y)\|>\frac{1}{2}\|(x,y)\|,
\end{aligned}$$

$$\begin{aligned}A_\mu(x,y)(t) &= \mu\int_0^T G_2(t,s)f_2(x(s),y(s))\,ds\\ &\geq \mu m_2\int_0^T \eta_2(x(s)+y(s))\,ds \geq \mu m_2\eta_1\delta T\|(x,y)\| > \frac{1}{2}\|(x,y)\|.\end{aligned}$$

Therefore, for any $(x,y)\in\partial\Omega_r\cap\widetilde{K}$, we have $\|A_{\lambda,\mu}(x,y) > \|(x,y)\|$. On the other hand, it follows from (H5) that there exists a $\widehat{R}>0$ such that

$$f_i(x,y)\geq\vartheta_i(x+y),\quad \text{for all } x+y\geq\widehat{R},$$

where $\vartheta_i$ satisfies $\vartheta_1\lambda\delta m_1T>1/2$ and $\vartheta_2\mu\delta m_2T>1/2$.

Let $R=\max\{C_\Sigma,\delta^{-1}\widehat{R},\overline{R}\}+1$, where $C_\Sigma$ is given by Lemma 3.5 with $\Sigma=\overline{\Omega_1\cup\Gamma}$. Then, for any $(x,y)\in\partial\Omega_R\cap\widetilde{K}$, we have

$$\begin{aligned}A_\lambda(x,y)(t) &= \lambda\int_0^T G_1(t,s)f_1(x(s),y(s))\,ds\\ &\geq \lambda m_1\int_0^T \vartheta_1(x(s)+y(s))\,ds \geq \lambda m_1\vartheta_1\delta T\|(x,y)\| > \frac{1}{2}\|(x,y)\|,\\ A_\mu(x,y)(t) &= \mu\int_0^T G_2(t,s)f_2(x(s),y(s))\,ds\\ &\geq \mu m_2\int_0^T \vartheta_2(x(s)+y(s))\,ds \geq \mu m_2\vartheta_1\delta T\|(x,y)\| > \frac{1}{2}\|(x,y)\|.\end{aligned}$$

So, for any $(x,y)\in\partial\Omega_R\cap\widetilde{K}$, we have $\|A_{\lambda,\mu}(x,y)\|>\|(x,y)\|$. Therefore, it follows from Lemma 3.1 that $A_{\lambda,\mu}$ has two fixed points $(x_1(t),y_1(t))$ and $(x_2(t),y_2(t))$ for $\lambda\in\Theta_1$, moreover $(x_1(t),y_1(t))\in\overline{\Omega}_{\overline{R}}\setminus\Omega_r$ and $(x_2(t),y_2(t))\in\overline{\Omega}_R\setminus\Omega_{\overline{R}}$. □

EXAMPLE 3.12. Consider the following system

$$\left\{\begin{aligned}\ddot{x}+a_1(t)x &= \lambda\bigg[\frac{\tau_1}{(x+y)^{\alpha_1}}+(\kappa_1(x+y)+1)^{\beta_1}\bigg],\\ \ddot{y}+a_2(t)y &= \mu\bigg[\frac{\tau_2}{(x+y)^{\alpha_2}}+(\kappa_2(x+y)+1)^{\beta_2}\bigg].\end{aligned}\right. \tag{3.1}$$

Assume that $a_1,a_2$ satisfy (H1) and $\alpha_i,\tau_i,\kappa_i>0$, $\beta_i>1$, $i=1,2$. Then the results of Theorem 3.11 hold for (3.1) if $\tau_i$, $\kappa_i$ are small enough and $\beta_i$ are large enough, $i=1,2$.

PROOF. Let us take

$$f_i(x,y)=\frac{\tau_i}{(x+y)^{\alpha_i}}+(\kappa_i(x+y)+1)^{\beta_i},\quad i=1,2.$$

Since $\alpha_i,\tau_i,\kappa_i>0,\beta_i>1,i=1,2$, one may readily verify that (H4) and (H5) are satisfied.

Take $\rho = \min\{\kappa_1, \kappa_2\}$. Now, for some constant $\overline{R} > 0$, condition (H6) becomes

$$\frac{2}{\rho\delta^2} \sup_{\delta\overline{R} \le x+y \le \overline{R}} f_i(x,y) \le \frac{2}{\rho\delta^2}\left(\frac{\tau_i}{(\delta_i\overline{R})^{\alpha_i}} + \kappa_i^{\beta_i}(\overline{R}+1)^{\beta_i}\right) \le \overline{R}, \quad i = 1,2. \tag{3.2}$$

Note that, if we chose $\kappa_i$, $i = 1,2$ small enough and $\overline{R}$ large enough such that $\kappa_i < 1/(\overline{R}+1)$ for $i = 1,2$, then (3.2) hold since $\beta_i > 1$, $i = 1,2$. Now we have the result. □

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