WEAK AND STRONG CONVERGENCE THEOREMS
FOR m-GENERALIZED HYBRID MAPPINGS
IN HILBERT SPACES

Sattar Alizadeh — Fridoun Moradlou

ABSTRACT. In this paper, we prove a weak convergence theorem of Ishikawa's type for m-generalized hybrid mappings in a Hilbert space. Further, by using a new modification of Ishikawa iteration, we prove a strong convergence theorem for m-generalized hybrid mappings in a Hilbert space.

1. Introduction

Throughout this paper, we denote by $\mathbb{N}$ and $\mathbb{R}$ the sets of positive integers and real numbers, respectively. Let C be a nonempty, closed and convex subset of a real Hilbert space H. Then the self mapping T of C is called:

(i) *nonexpansive*, if $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in C$;
(ii) *firmly nonexpansive*, if $\|Tx - Ty\|^2 \le \langle x - y, Tx - Ty \rangle$ for all $x, y \in C$;
(iii) *nonspreading*, if $2\|Tx - Ty\|^2 \le \|Tx - y\|^2 + \|Ty - x\|^2$ for all $x, y \in C$;
(iv) *hybrid*, if $3\|Tx - Ty\|^2 \le \|x - y\|^2 + \|Tx - y\|^2 + \|Ty - x\|^2$ for all $x, y \in C$.

We denote by $F(T)$ the set of fixed points of T.

There exist some iteration processes which is often used to approximate a fixed point of a nonexpansive mapping: Picard iteration, Krasnosel'skiĭ iteration, Halpern iteration, Mann iteration and Ishikawa iteration. During the

recent years, Mann and Ishikawa iterative schemes [8], [15] have been studied by
a number of authors.

In 1953, Mann [15] defined the following iteration procedure:

$$(1.1) \qquad \begin{cases} x_0 \in C \quad \text{chosen arbitrarily,} \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \end{cases}$$

where $0 \leq \alpha_n \leq 1$ for all $n \in \mathbb{N} \cup \{0\}$.

In 1974, Ishikawa [8] defined the following iteration procedure:

$$(1.2) \qquad \begin{cases} x_0 \in C \quad \text{chosen arbitrarily,} \\ y_n = \beta_n x_n + (1 - \beta_n)Tx_n, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Ty_n, \end{cases}$$

where $0 \leq \alpha_n \leq \beta_n \leq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and he proved strong convergence of
the sequence $\{x_n\}$ generated by the above iterative scheme if $\lim_{n \to \infty} \beta_n = 1$ and
$\sum_{n=1}^{\infty} (1 - \alpha_n)(1 - \beta_n) = \infty$. By taking $\beta_n = 1$ for all $n \geq 0$ in (1.2), Ishikawa
iteration process reduces to Mann iteration process.

Process (1.2) is indeed more general than process (1.1). But research has been
done on the latter due probably to reasons that the formulation of process (1.1)
is simpler than that of (1.2) and that a convergence theorem for process (1.1)
may lead to a convergence theorem for process (1.2) provided that $\{\beta_n\}$ satisfies
certain appropriate conditions. On the other hand, the process (1.1) may fail to
converge while process (1.2) can still converge for a Lipschitz pseudocontractive
mapping in a Hilbert space [3]. Actually, Mann and Ishikawa iteration processes
have only weak convergence, in general (see [4]).

Recently, to obtain strong convergence, many mathematicians have been ex-
tensively considered modified processes. Nakajo and Takahashi [19] proposed the
following modification of the Mann's iteration for a nonexpansive self mapping
T of C in a Hilbert space H:

$$\begin{cases} x_0 \in C \quad \text{chosen arbitrarily,} \\ y_n = \alpha_n x_n + (1 - \alpha_n)Tx_n, \\ C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \end{cases}$$

where P_K denotes the metric projection from H onto a closed convex subset K
of H. They proved strong convergence of the sequence $\{x_n\}$, if the sequence
$\{\alpha_n\}$ is bounded above from one.

In 2006, Martinez-Yanes and Xu [17] introduced the following modified Ishi-
kawa iteration process for a nonexpansive self mapping T of C with $F(T) \neq \emptyset$

in a Hilbert space H:

$$\begin{cases} x_0 \in C \quad \text{chosen arbitrarily,} \\ z_n = \beta_n x_n + (1 - \beta_n) T x_n, \\ y_n = \alpha_n x_n + (1 - \alpha_n) T z_n, \\ C_n = \{ v \in C : \|y_n - v\|^2 \le \|x_n - v\|^2 \\ \qquad\qquad + (1 - \alpha_n)(\|z_n\|^2 - \|x_n\|^2 + 2\langle x_n - z_n, v \rangle \ge 0) \}, \\ Q_n = \{ v \in C : \langle x_n - v, x_0 - x_n \rangle \ge 0 \}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \end{cases}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$. They proved that if $\{\alpha_n\}$ bounded above from one and $\lim\limits_{n \to \infty} \beta_n = 1$, then the sequence $\{x_n\}$ converges strongly to $P_{F(T)} x_0$.

Very recently, Hojo et al. [6] proved some strong convergence theorems by hybrid methods for 2-generalized hybrid mappings in a Hilbert space.

In this paper, employing the idea of Nakajo and Takahashi [19], we modify Ishikawa iteration process for m-generalized hybrid mappings in a Hilbert space. This paper is organized as follows: In Section 2, we give some preliminaries which will used in next sections. In Section 3, by using Banach limits, we prove a fixed point theorem for m-generalized hybrid mappings in a Hilbert space. In Section 4, we prove some weak and strong convergence theorems for m-generalized hybrid mappings in a Hilbert space.

2. Preliminaries

Assume that H be a (real) Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. We denote the weak convergence and the strong convergence of $\{x_n\}$ to $x \in H$ by $x_n \rightharpoonup x$ and $x_n \to x$, respectively.

Now we recall some basic properties of Hilbert spaces which we will use in next section. For $x, y \in H$, we have

$$(2.1) \qquad \|\alpha x + (1 - \alpha) y\|^2 = \alpha \|x\|^2 + (1 - \alpha) \|y\|^2 - \alpha(1 - \alpha) \|x - y\|^2,$$

for all $\alpha \in \mathbb{R}$,

$$(2.2) \qquad \|x + y\|^2 \le \|x\|^2 + 2\langle y, x + y \rangle,$$

and

$$(2.3) \qquad \|x - y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x - y, y \rangle.$$

It is well known that a Hilbert space H satisfies the Opial condition, i.e. if $\{x_n\}$ is a sequence in H such that $x_n \rightharpoonup x$ and $x \ne y$, then

$$(2.4) \qquad \liminf_{n \to \infty} \|x_n - x\| < \liminf_{n \to \infty} \|x_n - y\|.$$

A self mapping T of C with $F(T) \neq \emptyset$ is called *quasi-nonexpansive* if $\|x - Ty\| \leq \|x - y\|$ for all $x \in F(T)$ and $y \in C$. It is well-known that for a *quasi-nonexpansive* mapping T, $F(T)$ is closed and convex [9]. Let K be a closed and convex subset of H and let P_K be metric (or nearest point) projection from H onto K (i.e. for $x \in H, P_K x$ is the only point in K such that $\|x - P_K x\| = inf\{\|x - z\| : z \in K\}$). Let $x \in H$ and $z \in K$, then $z = P_K x$ if and only if there holds the relation:

$$(2.5) \qquad \langle x - z, y - z \rangle \leq 0$$

for all $y \in K$. For more details we refer readers to [1], [21].

Using Opial condition, we can deduce the following lemma:

LEMMA 2.1 ([27]). *Let H be a Hilbert space and $\{x_n\}$ be a sequence in H such that there exists a nonempty subset $E \subset H$ satisfying*

(a) *For every $x^* \in E$, $\lim_{n \to \infty} \|x_n - x^*\|$ exists.*
(b) *If a subsequence $\{x_{n_j}\} \subset \{x_n\}$ converges weakly to x^*, then $x^* \in E$,*

then there exists $x_0 \in E$ such that $x_n \rightharpoonup x_0$.

We will use the following lemmas in the proof of our main results in next section.

LEMMA 2.2 ([23]). *Let H be a Hilbert space and E be a nonempty, closed and convex subset of H. Let $\{x_n\}$ be a sequence in H. If $\|x_{n+1} - x\| \leq \|x_n - x\|$ for all $n \in \mathbb{N}$ and $x \in E$, then $\{P_E(x_n)\}$ converges strongly to some $z \in E$, where P_E stands for the metric projection on H onto E.*

LEMMA 2.3 ([17]). *Let H be a real Hilbert space. Given a closed and convex subset $C \subset H$ and points $x, y, z \in H$. Given also a real number $a \in \mathbb{R}$. The set*

$$D := \{u \in C : \|y - u\|^2 \leq \|x - u\|^2 + \langle z, u \rangle + a\}$$

is convex (and closed).

Let C be a nonempty, closed and convex subset of H. A self mapping T of C is called *generalized hybrid* [12] if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$(2.6) \qquad \alpha \|Tx - Ty\|^2 + (1 - \alpha)\|x - Ty\|^2 \leq \beta \|Tx - y\|^2 + (1 - \beta)\|x - y\|^2$$

for all $x, y \in C$. We call such a mapping an (α, β)-*generalized hybrid* mapping. It easy to see that

- $(1, 0)$-generalized hybrid mapping is nonexpansive;
- $(2, 1)$-generalized hybrid mapping is nonspreading;
- $(3/2, 1/2)$-generalized hybrid mapping is hybrid.

Let $m \in \mathbb{N}$, a self mapping T of C is called *m-generalized hybrid* [18] if there exist $\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_m \in \mathbb{R}$ such that

$$\sum_{k=1}^{m} \alpha_k \|T^{m+1-k}x - Ty\|^2 + \left(1 - \sum_{k=1}^{m} \alpha_k\right)\|x - Ty\|^2$$

$$\leq \sum_{k=1}^{m} \beta_k \|T^{m+1-k}x - y\|^2 + \left(1 - \sum_{k=1}^{m} \beta_k\right)\|x - y\|^2$$

for all $x, y \in C$. Therefore, T is 2-*generalized hybrid mapping* if $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$ such that

$$\alpha_1 \|T^2 x - Ty\|^2 + \alpha_2 \|Tx - Ty\|^2 + (1 - \alpha_1 - \alpha_2)\|x - Ty\|^2$$

$$\leq \beta_1 \|T^2 x - y\|^2 + \beta_2 \|Tx - y\|^2 + (1 - \beta_1 - \beta_2)\|x - y\|^2$$

for all $x, y \in C$. In [6], Hojo et al. give two examples of 2-generalized hybrid mapping which are not generalized hybrid mappings. Therefore, class of 2-generalized hybrid mappings is broader than generalized hybrid mappings.

Also, one can easily show that m-generalized hybrid mapping is quasi-non-expansive if the set of it's fixed points is nonempty.

3. Fixed point theorem by using Banach limit

In this section, we recall the notion of *Banach limits* and some properties of them. For more details, we refer to readers to [1], [21].

Denote by l^∞ the set of all bounded sequences equipped with supremum norm.

DEFINITION 3.1. A continuous linear functional μ on l^∞ is called a *Banach limit* if

 (a) $\mu(e) = \|e\| = 1$, where $e = (1, 1, 1, \ldots)$,
 (b) $\mu_n(x_n) = \mu_n(x_{n+1})$ for all $x = (x_1, x_2, \ldots) \in l^\infty$, where $\mu_n(x_{n+m}) = \mu(x_{m+1}, \ldots, x_{m+n}, \ldots)$.

As usual, we denote by $\mu_n(x_n)$ the value of μ at $x = (x_1, x_2, \ldots)$. It is well known that there exists a Banach limit on l^∞. Let μ be a Banach limit. Then

$$\liminf_{n\to\infty} x_n \leq \mu_n(x_n) \leq \limsup_{n\to\infty} x_n.$$

Moreover, if $x_n \to a$, then $\mu_n(x_n) = a$. Takahashi and Yao [24], by using Banach limits, proved the following theorem:

THEOREM 3.2. *Let H be a Hilbert space and C be a nonempty, closed and convex subset of H and let T be a self mapping of C. Assume that there exists an element $x \in C$ such that $\{T^n x\}$ is bounded and*

$$\mu_n \|T^n x - Ty\|^2 \leq \mu_n \|T^n x - y\|^2,$$

for all $y \in C$ and for some Banach limit μ. Thus, T has a fixed point in C.

Now, we prove main result of this section.

Theorem 3.3. *Let C be a nonempty, closed and convex subset of a Hilbert space H and T be m-generalized hybrid self mapping of C. If $\{T^n z\}$ is bounded for some $z \in C$, then T has a fixed point in C and vice versa.*

Proof. Since T is a m-generalized hybrid self mapping of C, then

$$\sum_{k=1}^{m} \alpha_k \|T^{m+1-k}x - Ty\|^2 + \left(1 - \sum_{k=1}^{m} \alpha_k\right)\|x - Ty\|^2$$
$$\leq \sum_{k=1}^{m} \beta_k \|T^{m+1-k}x - y\|^2 + \left(1 - \sum_{k=1}^{m} \beta_k\right)\|x - y\|^2,$$

for all $x, y \in C$. Let $z \in C$ such that $\{T^n z\}$ is bounded. Assume that μ is a Banach limit. Thus, for all $y \in C$ and $n \in \mathbb{N} \cup \{0\}$, we get

$$(3.1) \quad \sum_{k=1}^{m} \alpha_k \|T^{n+m+1-k}z - Ty\|^2 + \left(1 - \sum_{k=1}^{m} \alpha_k\right)\|T^n z - Ty\|^2$$
$$\leq \sum_{k=1}^{m} \beta_k \|T^{n+m+1-k}z - y\|^2 + \left(1 - \sum_{k=1}^{m} \beta_k\right)\|T^n z - y\|^2,$$

for all $y \in C$. By using boundedness of $\{T^n z\}$, we can take a Banach limit μ to both sides of the inequality (3.1). So, we get

$$\sum_{k=1}^{m} \alpha_k \mu_n \|T^{n+m+1-k}z - Ty\|^2 + \left(1 - \sum_{k=1}^{m} \alpha_k\right)\mu_n \|T^n z - Ty\|^2$$
$$\leq \sum_{k=1}^{m} \beta_k \mu_n \|T^{n+m+1-k}z - y\|^2 + \left(1 - \sum_{k=1}^{m} \beta_k\right)\mu_n \|T^n z - y\|^2,$$

and therefore

$$\left(\sum_{k=1}^{m} \alpha_k\right)\mu_n\left\|T^n z - Ty\|^2 + \left(1 - \sum_{k=1}^{m} \alpha_k\right)\mu_n \|T^n z - Ty\|^2$$
$$\leq \left(\sum_{k=1}^{m} \beta_k\right)\mu_n \|T^n z - y\|^2 + \left(1 - \sum_{k=1}^{m} \beta_k\right)\mu_n \|T^n z - y\|^2.$$

This yields

$$\mu_n \|T^n z - Ty\|^2 \leq \mu_n \|T^n z - y\|^2,$$

for all $y \in C$. By Theorem 3.2, we have a fixed point in C.

Conversely, let $F(T)$ be nonempty, then there exists a $z \in F(T)$ and therefore $\{T^n z\} = \{z\}$. Thus $\{T^n z\}$ is bounded. $\square$

Since the class of m-generalized hybrid mappings contain the class of nonexpansive mappings, nonspreading mappings, hybrid mappings and generalized hybrid mappings, we have the following corollaries.

COROLLARY 3.4. *Let C be a nonempty, closed and convex subset of a Hilbert space H and let T be a generalized hybrid self mapping of C. Then T has a fixed point in C if and only if $\{T^n x\}$ is bounded for some $x \in C$.*

COROLLARY 3.5. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let T be a nonexpansive self mapping of C. Then T has a fixed point in C if and only if $\{T^n x\}$ is bounded for some $x \in C$.*

COROLLARY 3.6 ([13]). *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let T be a nonspreading self mapping of C. Then T has a fixed point in C if and only if $\{T^n x\}$ is bounded for some $x \in C$.*

COROLLARY 3.7 ([22]). *Let H be a Hilbert space and C be a nonempty, closed and convex subset of H. Let T be a hybrid self mapping of C. Then T has a fixed point in C if and only if $\{T^n x\}$ is bounded for some $x \in C$.*

4. Weak and strong convergence theorems of Ishikawa's type

In this section, we prove some weak and strong convergence theorems of Ishikawa's type related to m-generalized hybrid mappings.

THEOREM 4.1. *Let C be a nonempty, closed and convex subset of a real Hilbert space H and T be a m-generalized hybrid self mapping of C with $F(T) \neq \phi$ and $||T^2 x - Tx|| \leq ||Tx - x||$ for all $x \in C$. Assume that $\{x_n\}$ is a sequence generated by*

$$\begin{cases} x_1 = x \in C, \\ y_n = (1 - \lambda_n)x_n + \lambda_n Tx_n, \\ x_{n+1} = (1 - \gamma_n)x_n + \gamma_n Ty_n, \end{cases}$$

where $0 \leq \gamma_n, \lambda_n \leq 1$ for all $n \in \mathbb{N}$, $\liminf_{n \to \infty} \lambda_n(1 - \lambda_n) > 0$ and $0 < a < \gamma_n < b < 1$. Then $x_n \rightharpoonup x_0 \in F(T)$, where $x_0 = \lim_{n \to \infty} P_{F(T)}(x_n)$.

PROOF. By hypothesis, T is a m-generalized hybrid mapping such that $F(T) \neq \phi$, so T is quasi-nonexpansive. Then, for all $q \in F(T)$ and all $n \in \mathbb{N}$, we have

(4.1) $\|y_n - q\|^2 = (1 - \lambda_n)\|x_n - q\|^2$

$$+ \lambda_n \|Tx_n - q\|^2 - \lambda_n(1 - \lambda_n)\|x_n - Tx_n\|^2$$

$$\leq (1 - \lambda_n)\|x_n - q\|^2 + \lambda_n \|x_n - q\|^2 - \lambda_n(1 - \lambda_n)\|x_n - Tx_n\|^2$$

$$= \|x_n - q\|^2 - \lambda_n(1 - \lambda_n)\|x_n - Tx_n\|^2,$$

and hence

(4.2) $\|x_{n+1} - q\|^2 = \|(1 - \gamma_n)x_n + \gamma_n Ty_n - q\|^2$

$$= (1 - \gamma_n)\|x_n - q\|^2 + \gamma_n \|Ty_n - q\|^2 - \gamma_n(1 - \gamma_n)\|x_n - Ty_n\|^2$$

$$\leq (1 - \gamma_n)\|x_n - q\|^2 + \gamma_n\|y_n - q\|^2 - \gamma_n(1 - \gamma_n)\|x_n - Ty_n\|^2$$
$$\leq (1 - \gamma_n)\|x_n - q\|^2 + \gamma_n\|x_n - q\|^2$$
$$- \gamma_n\lambda_n(1 - \lambda_n)\|x_n - Tx_n\|^2 - \gamma_n(1 - \gamma_n)\|x_n - Ty_n\|^2$$
$$\leq \|x_n - q\|^2 - \gamma_n\lambda_n(1 - \lambda_n)\|x_n - Tx_n\|^2 \leq \|x_n - q\|^2.$$

So, we can conclude that $\lim_{n\to\infty}\|x_n - q\|$ exists. This yields that $\{x_n\}$, $\{y_n\}$ and $\{Ty_n\}$ are bounded. It follows from (4.2) that

$$\|x_{n+1} - q\|^2 \leq \|x_n - q\|^2 - \gamma_n\lambda_n(1 - \lambda_n)\|x_n - Tx_n\|^2.$$

By using $0 < a < \gamma_n < b < 1$, it is easy to see that

$$\|x_{n+1} - q\|^2 \leq \|x_n - q\|^2 - a\lambda_n(1 - \lambda_n)\|x_n - Tx_n\|^2.$$

Also, we have

$$0 \leq a\lambda_n(1 - \lambda_n)\|x_n - Tx_n\|^2 \leq \|x_n - q\|^2 - \|x_{n+1} - q\|^2 \to 0$$

as $n \to \infty$, since $\liminf_{n\to\infty}\lambda_n(1 - \lambda_n) > 0$. Therefore

(4.3) $$\lim_{n\to\infty}\|x_n - Tx_n\| = 0.$$

Now, boundedness of $\{x_n\}$ implies that there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup x^* \in C$. Since T is a m-generalized hybrid mapping, then

$$\sum_{k=1}^{m}\alpha_k\|T^{m+1-k}x - Ty\|^2 + \left(1 - \sum_{k=1}^{m}\alpha_k\right)\|x - Ty\|^2$$
$$\leq \sum_{k=1}^{m}\beta_k\|T^{m+1-k}x - y\|^2 + \left(1 - \sum_{k=1}^{m}\beta_k\right)\|x - y\|^2,$$

hence

$$0 \leq \sum_{k=1}^{m}\beta_k\|T^{m+1-k}x - y\|^2 + \left(1 - \sum_{k=1}^{m}\beta_k\right)\|x - y\|$$
$$- \sum_{k=1}^{m}\alpha_k\|T^{m+1-k}x - Ty\|^2 - \left(1 - \sum_{k=1}^{m}\alpha_k\right)\|x - Ty\|,$$

replacing x and y by x_n and x^* in above inequality, respectively, we get

$$0 \leq \sum_{k=1}^{m}\beta_k(\|T^{m+1-k}x_n\|^2 - 2\langle T^{m+1-k}x_n, x^*\rangle + \|x^*\|^2)$$
$$+ \left(1 - \sum_{k=1}^{m}\beta_k\right)(\|x_n\|^2 - 2\langle x_n, x^*\rangle + \|x^*\|^2)$$
$$- \sum_{k=1}^{m}\alpha_k(\|T^{m+1-k}x_n\|^2 - 2\langle T^{m+1-k}x_n, Tx^*\rangle + \|Tx^*\|^2)$$

$$-\left(1-\sum_{k=1}^{m}\alpha_k\right)(\|T^{n+1-k}x_n\|^2 - 2\langle x_n, Tx^*\rangle + \|Tx^*\|^2)$$

$$= \|x^*\|^2 - \|Tx^*\|^2 + \sum_{k=1}^{m}(\beta_k - \alpha_k)(\|T^{m+1-k}x_n\|^2 - \|x_n\|^2)$$

$$+ 2\sum_{k=1}^{m}\alpha_k\langle T^{m+1-k} - x_n, Tx^*\rangle$$

$$- 2\sum_{k=1}^{m}\beta_k\langle T^{m+1-k}x_n - x_n, x^*\rangle + 2\langle x_n, Tx^* - x^*\rangle$$

$$\leq \|x^*\|^2 - \|Tx^*\|^2 + \sum_{k=1}^{m}(\beta_k - \alpha_k)(\|T^{m+1-k}x_n\|$$

$$+ \|x_n\|)(\|T^{m+1-k}x_n - x_n\|) + 2\sum_{k=1}^{m}\alpha_k\langle T^{m+1-k}x_n - x_n, Tx^*\rangle$$

$$- 2\sum_{k=1}^{m}\beta_k\langle T^{m+1-k}x_n - x_n, x^*\rangle + 2\langle x_n, Tx^* - x^*\rangle.$$

Now, substituting n by n_i, we have

$$(4.4) \quad 0 \leq \|x^*\|^2 - \|Tx^*\|^2 + \sum_{k=1}^{m}(\beta_k - \alpha_k)(\|T^{m+1-k}x_{n_i}\|$$

$$+ \|x_{n_i}\|)(\|T^{m+1-k}x_{n_i} - x_{n_i}\|) + 2\sum_{k=1}^{m}\alpha_k\langle T^{m+1-k}x_{n_i} - x_{n_i}, Tx^*\rangle$$

$$- 2\sum_{k=1}^{m}\beta_k\langle T^{m+1-k}x_{n_i} - x_{n_i}, x^*\rangle + 2\langle x_{n_i}, Tx^* - x^*\rangle$$

for all $i \in \mathbb{N}$. Since $x_{n_i} \rightharpoonup x^*$ as $i \to \infty$, it follows from (4.3) and (4.4) that

$$0 \leq \|x^*\|^2 - \|Tx^*\|^2 + 2\langle x^*, Tx^* - x^*\rangle$$
$$= 2\langle x^*, Tx^*\rangle - \|x^*\|^2 - \|Tx^*\|^2 = -\|x^* - Tx^*\|^2.$$

So, we have $Tx^* = x^*$, i.e. $x^* \in F(T)$. Therefore the condition (b) of Lemma 2.1 satisfies for $E = F(T)$. On the other hand, we see that $\lim_{n\to\infty}\|x_n - q\|$ exists for $q \in F(T)$. Hence, it follows from Lemma 2.1 that there exists $x_0 \in F(T)$ such that $\{x_n\}$ is weakly convergent to x_0 as $n \to \infty$. In addition, for all $q \in F(T)$, we have $\|x_{n+1} - q\| \leq \|x_n - q\|$, for all $n \in \mathbb{N}$. So, Lemma 2.2 implies that there exists some $x \in F(T)$ such that $P_{F(T)}(x_n) \to x$. Then

$$\langle x_0 - P_{F(T)}(x_n), x_n - P_{F(T)}(x_n)\rangle \leq 0.$$

Hence, we get $\langle x_0 - x, x_0 - x\rangle = \|x_0 - x\|^2 \leq 0$. Therefore $x_0 = x$, i.e. $x_n \rightharpoonup x_0 = \lim_{n\to\infty} P_{F(T)}(x_n)$.   □

Theorem 4.2. *Let C be a nonempty, closed and convex subset of a real space H and T be a m-generalized hybrid self mapping of C with $F(T) \neq \phi$. Assume that $\{\gamma_n\}$ and $\{\lambda_n\}$ are sequences in $[0,1]$ such that $0 \leq \gamma_n \leq \delta < 1$ and $\lambda_n \to 1$ as $n \to \infty$. If $\{x_n\}$ in C is a sequence generated by*

$$
\begin{cases}
x_1 = x \in C, \\[2mm]
y_n = \gamma_n x_n + \dfrac{1 - \gamma_n}{n} \displaystyle\sum_{k=0}^{n-1} T^k z_n, \\[2mm]
z_n = \lambda_n x_n + (1 - \lambda_n) T x_n, \\[2mm]
C_n = \{u \in C : \\[1mm]
\qquad \|y_n - u\|^2 \leq \|x_n - u\|^2 + (1 - \gamma_n)(\|z_n\|^2 - \|x_n\|^2 + 2\langle x_n - z_n, u\rangle)\}, \\[2mm]
Q_n = \{u \in C : \langle x_n - u, x_n - x\rangle \leq 0\}, \\[2mm]
x_{n+1} = P_{C_n \cap Q_n} x,
\end{cases}
$$

then $\{x_n\}$ converges strongly to $P_{F(T)} x$.

Proof. Since T is a m-generalized hybrid mapping such that $F(T) \neq \phi$, T is quasi-nonexpansive. So $F(T)$ is closed and convex. Therefore we can define the metric projection from H onto $F(T)$. It is easily seen that C_n is closed and Q_n is closed and convex for all $n \in \mathbb{N}$. Also, by Lemma 2.3, C_n is convex. So $C_n \cap Q_n$ is closded and convex for all $n \in \mathbb{N}$. Let $u \in F(T)$ and put $w_n = (1/n) \sum_{k=0}^{n-1} T^k z_n$ for all $n \in \mathbb{N}$. Since T is quasi-nonexpansive, we have

$$
\begin{aligned}
\|w_n - u\| &= \left\| \frac{1}{n} \sum_{k=0}^{n-1} T^k z_n - u \right\| \\[2mm]
&\leq \frac{1}{n} \sum_{k=0}^{n-1} \|T^k z_n - u\| \leq \frac{1}{n} \sum_{k=0}^{n-1} \|z_n - u\| = \|z_n - u\|.
\end{aligned}
$$

Thus

$$
\begin{aligned}
\|y_n - u\|^2 &= \|\gamma_n(x_n - u) + (1 - \gamma_n)(w_n - u)\|^2 \\
&\leq \gamma_n \|x_n - u\|^2 + (1 - \gamma_n)\|w_n - u\|^2 \\
&\leq \gamma_n \|x_n - u\|^2 + (1 - \gamma_n)\|z_n - u\|^2 \\
&= \gamma_n \|x_n - u\|^2 + (1 - \gamma_n)(\|z_n - u\|^2 - \|x_n - u\|^2) \\
&= \gamma_n \|x_n - u\|^2 + (1 - \gamma_n)(\|z_n\|^2 - \|x_n\|^2 + 2\langle x_n - z_n, u\rangle),
\end{aligned}
$$

so $u \in C_n$ for all $n \in \mathbb{N}$.

Now by using induction we will show that $F(T) \subset C_n \cap Q_n$. For $n = 1$, we have $x_1 = x \in C$ and $Q_1 = C$, since $F(T) \subset Q_1$ and hence $F(T) \subset C_1 \cap Q_1$. Assume that $F(T) \subset C_n \cap Q_n$ for some n. Since $x_{n+1} = P_{C_n \cap Q_n} x$, It follows

from (2.5) that

$$(4.5) \qquad \langle x_{n+1} - u, x - x_{n+1} \rangle \geqslant 0,$$

for all $u \in C_n \cap Q_n$. By the induction assumption, we have $F(T) \subset C_n \cap Q_n$, so we can conclude that for all $u \in F(T)$ the inequality (4.5) holds. On the other hand, by the definition of Q_{n+1}, we obtain $F(T) \subset Q_{n+1}$. Therefore, $F(T) \subset C_{n+1} \cap Q_{n+1}$. So $\{x_n\}$ is well-defined. Notice that the definition of Q_n implies $x_n = P_{Q_n} x$. Moreover, $F(T) \subset Q_n$, by using (2.3), we have

$$\|x - x_n\| \leq \|x - u\|, \quad \text{for all } u \in F(T).$$

This implies that $\{x_n\}$ is bounded. Since $x_{n+1} = P_{C_n \cap Q_n} x \in Q_n$, we have $x_{n+1} \in Q_n$, i.e. $\langle x_{n+1} - x_n, x_n - x \rangle \geq 0$. So, by using (2.3), we obtain

$$\begin{aligned}
\|x_{n+1} - x_n\|^2 &= \|(x_{n+1} - x) - (x_n - x)\|^2 \\
&= \|x_{n+1} - x\|^2 - \|x_n - x\|^2 - 2\langle x_{n+1} - x_n, x_n - x \rangle \\
&\leq \|x_{n+1} - x\|^2 - \|x_n - x\|^2.
\end{aligned}$$

This yields that

$$(4.6) \qquad \|x_{n+1} - x_n\| \to 0.$$

From $x_{n+1} \in C_n$, we get

$$(4.7) \quad \|y_n - x_{n+1}\|^2 \leq \|x_n - x_{n+1}\|^2 + (1 - \gamma_n)(\|z_n\|^2 - \|x_n\|^2 + 2\langle x_n - z_n, x_{n+1}\rangle).$$

Since $\lambda_n \to 1$ and $\{x_n\}$ is bounded, we have

$$(4.8) \qquad \|z_n - x_n\| = (1 - \lambda_n)\|x_n - Tx_n\| \to 0,$$

so

$$(4.9) \quad \|z_n\|^2 - \|x_n\|^2 + 2\langle x_n - z_n, x_{n+1}\rangle = \|z_n - x_n\|^2 + 2\langle x_n - z_n, x_{n+1} - x_n\rangle \to 0.$$

It follows from (4.6)–(4.9) that $\|y_n - x_{n+1}\| \to 0$. From (4.6), we obtain

$$\|y_n - x_n\| \leq \|y_n - x_{n+1}\| + \|x_{n+1} - x_n\| \to 0.$$

Also from $\|x_n - y_n\| = \|x_n - \gamma_n x_n - (1 - \gamma_n)w_n\| = (1 - \gamma_n)\|x_n - w_n\|$ and $0 \leq \gamma_n \leq \delta < 1$, we have $\|w_n - x_n\| \to 0$. It follows from boundedness of $\{x_n\}$ that there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup v$. From $\|w_n - x_n\| \to 0$, we obtain $w_{n_i} \rightharpoonup v$. Since T is a m-generalized hybrid mapping, we get

$$\begin{aligned}
0 \leq{} & \sum_{j=1}^{m} \beta_j \|T^{k+m+1-j} x_n - y\|^2 + \left(1 - \sum_{j=1}^{m} \beta_j\right) \|T^k x_n - y\|^2 \\
& - \sum_{j=1}^{m} \alpha_j \|T^{k+m+1-j} x_n - Ty\|^2 - \left(1 - \sum_{j=1}^{m} \alpha_j\right) \|T^k x_n - Ty\|^2
\end{aligned}$$

$$= \sum_{j=1}^{m} \beta_j (\|T^{k+m+1-j}x_n - Ty\|^2$$

$$+ 2\langle T^{k+m+1-j}x_n - Ty, Ty - y\rangle + \|Ty - y\|^2)$$

$$+ \left(1 - \sum_{j=1}^{m} \beta_j\right)(\|T^k x_n - Ty\|^2$$

$$+ 2\langle T^k x_n - Ty, Ty - y\rangle + \|Ty - y\|^2)$$

$$- \sum_{j=1}^{m} \alpha_j \|T^{k+m+1-j}x_n - Ty\|^2 - \left(1 - \sum_{k=1}^{m} \alpha_j\right)\|T^k x_n - Ty\|^2$$

$$= \|Ty - y\|^2 + 2\left\langle \sum_{j=1}^{m} \beta_j T^{k+m+1-j}x_n + \left(1 - \sum_{j=1}^{m} \beta_j\right)T^k x_n - Ty, Ty - y\right\rangle$$

$$+ \sum_{j=1}^{m} (\beta_j - \alpha_j)(\|T^{k+m+1-j}x_n - Ty\|^2 - \|T^k x_n - Ty\|^2),$$

for all $y \in C$ and $k \in \mathbb{N} \cup \{0\}$. Therefore, we obtain

$$0 \le \|Ty - y\|^2 + 2\langle T^k x_n - Ty, Ty - y\rangle$$

$$+ 2\left\langle \sum_{j=1}^{m} \beta_j (T^{k+m+1-j}x_n - T^k x_n), Ty - y\right\rangle$$

$$+ \sum_{j=1}^{m} (\beta_j - \alpha_j)(\|T^{k+m+1-j}x_n - Ty\|^2 - \|T^k x_n - Ty\|^2),$$

for all $y \in C$ and $k \in \mathbb{N} \cup \{0\}$. Summing these inequalities from $k = 0$ to $n-1$ and dividing by n, we get

$$0 \le \|Ty - y\|^2 + 2\langle w_n - Ty, Ty - y\rangle$$

$$+ 2\left\langle \frac{1}{n}\sum_{k=0}^{n-1}\sum_{j=1}^{m} \beta_j (T^{k+m+1-j}x_n - T^k x_n), Ty - y\right\rangle$$

$$+ \frac{1}{n}\sum_{k=0}^{n-1}\sum_{j=1}^{m} (\beta_j - \alpha_j)(\|T^{k+m+1-j}x_n - Ty\|^2 - \|T^k x_n - Ty\|^2).$$

Substituting n by n_i in last inequality, we obtain

$$0 \le \|Ty - y\|^2 + 2\langle w_{n_i} - Ty, Ty - y\rangle$$

$$+ 2\left\langle \frac{1}{n_i}\sum_{k=0}^{n_i-1}\sum_{j=1}^{m} \beta_j (T^{k+m+1-j}x_{n_i} - T^k x_{n_i}), Ty - y\right\rangle$$

$$+ \frac{1}{n_i}\sum_{k=0}^{n_i-1}\sum_{j=1}^{m} (\beta_j - \alpha_j)(\|T^{k+m+1-j}x_{n_i} - Ty\|^2 - \|T^k x_{n_i} - Ty\|^2).$$

By taking the limit as $i \to \infty$, we obtain

$$(4.10) \qquad 0 \leq \|Ty - y\|^2 + 2\langle v - Ty, Ty - y\rangle,$$

since $w_{n_i} \rightharpoonup v$. By putting $y = v$ in (4.10), we get $0 \leq -\|Tv - v\|^2$ and thus $Tv = v$. Let $u_0 = P_{F(T)}x$. Since $u_0 = P_{F(T)}x \in C_n \cap Q_n$ and $x_{n+1} = P_{C_n \cap Q_n}x$, we have

$$(4.11) \qquad \|x - x_{n+1}\|^2 \leq \|x - u_0\|^2.$$

Since $x_{n_i} \rightharpoonup v$ and $\|\cdot\|^2$ is weakly lower semicontinuous, we get

$$\|x - v\|^2 = \|x\|^2 - 2\langle x, v\rangle + \|v\|^2 \leq \liminf_{i \to \infty}(\|x\|^2 - 2\langle x, x_{n_i}\rangle + \|x_{n_i}\|^2)$$
$$= \liminf_{i \to \infty} \|x - x_{n_i}\|^2 \leq \|x - u_0\|^2.$$

Now, the definition of u_0, implies that $v = u_0$. Thus we obtain that $x_n \rightharpoonup u_0$.

Finally, we prove that $x_n \to u_0$. It follows from (2.3) that

$$\|u_0 - x_n\|^2 = \|u_0 - x\|^2 + \|x - x_n\|^2 + 2\langle u_0 - x, x - x_n\rangle,$$

for all $n \in \mathbb{N}$. So, by using (4.11), we get

$$\limsup_{n \to \infty} \|u_0 - x_n\|^2 = \limsup_{n \to \infty}(\|u_0 - x\|^2 + \|x - x_n\|^2 + 2\langle u_0 - x, x - x_n\rangle)$$
$$\leq \limsup_{n \to \infty}(\|u_0 - x\|^2 + \|x - u_0\|^2 + 2\langle u_0 - x, x - x_n\rangle)$$
$$= (\|u_0 - x\|^2 + \|x - u_0\|^2 + 2\langle u_0 - x, x - u_0\rangle) = \|u_0 - u_0\|^2 = 0.$$

This yields that $\lim_{n \to \infty} \|u_0 - x_n\| = 0$. Thus $\{x_n\}$ converges strongly to u_0. $\qquad \square$

References

[1] R.P. Agarwal, D. O'Regan and D.R. Sahu, *Fixed Point Theory for Lipschitzian-type Mappings with Applications*, Springer, 2009.

[2] K. Aoyama, Y. Kimura, W. Takahashi and M. Toyoda, *Approximation of common fixed points of a countable family of nonexpansive mappings a Banach space*, Nonlinear Anal. **67** (2007), 2350–2360.

[3] C.E. Chidume and S.A. Mutangadura, *An example on the Mann iteration method for Lipschitz pseudocontractions*, Proc. Amer. Math. Soc. **129** (2001), 2359–2363.

[4] A. Genel and J. Lindenstrass, *An example concerning fixed points*, Israel J. Math. **22** (1975), 81–86.

[5] B. Halpern, *Fixed points of nonexpanding maps*, Bull. Amer. Math. Soc. **73** (1967), 957–961.

[6] M. Hojo, W. Takahashi and I. Termwuttipong, *Strong convergence theorem for 2-generalized hybrid mappings in Hilbert spaces*, Nonlinear Anal. **75** (2012), 2166–2176.

[7] S. Iemoto and W. Takahashi, *Approximating fixed points of nonexpansive mappings and nonspreading mappings in a Hilbert space*, Nonlinear Anal. **71** (2009), 2080–2089.

[8] S. Ishikawa, *Fixed points by a new iteration method*, Proc. Amer. Math. Soc. **40** (1974), 147–150.

[9] S. Itoh and W. Takahashi, *The common fixed point theory of single-valued mappings and multi-valued mappings*, Pacific J. Math. **79** (1978), 493–508.

[10] T.H. Kim and H.K. Xu, *Strong convergence of modified Mann iterations*, Nonlinear Anal. **61** (2005), 51–60.

[11] T.H. Kim and H.K. Xu, *Strong convergence of modified Mann iterations for asymptotically nonexpansive mappings and semigroups*, Nonlinear Anal. **64** (2006), 1140–1152.

[12] P. Kocourek, W. Takahashi and J.-C. Yao, *Fixed point theorems and weak convergence theorems for generalized hybrid mappings in Hilbert spaces*, Taiwanese J. Math. **6** (2010), 2497–2511.

[13] F. Kohsaka and W. Takahashi, *Fixed point theorems for a class of nonlinear mappings related to maximal monotone operators in Bnach spaces*, Arch. Math., **91** (2008), 166–177.

[14] Y. Kurokawa and W. Takahashi, *Weak and convergence theorems for nonspreading mappings in Hilbert spaces*, Nonlinear. Anal. **73** (2010) 1562–1568.

[15] W.R. Mann, *Mean value methods in iteration*, Proc. Amer. Math. Soc. **4** (1953), 506–510.

[16] G. Marino and H.K. Xu, *Weak and strong convergence theorems for strict pseudo-contractions in Hilbert spaces*, J. Math. Anal. Appl. **329** (2007), 336–346.

[17] C. Martinez-Yanes and H.K. Xu, *Strong convergence of CQ method fixed point biteration processes*, Nonlinear Anal. **64** (2006), 2400–2411.

[18] T. Maruyama, W. Takahashi and M. Yao, *Fixed point and mean ergodic theorems for new nonlinear mappings in Hilbert spaces*, J. Nonlinear Convex Anal. **12** (2011), 185–197.

[19] K. Nakajo and W. Takahashi, *Strong convergence theorems for nonexpansive mappings and nonexpansive semigroups*, J. Math. Anal. Appl. **279** (2003), 372–379.

[20] M.O. Osilike and F.O. Isiogugu, *Weak and strong convergence theorems for non-spreading-type mappings in Hilbert spaces*, Nonlinear. Anal. **74** (2011), 1814–1822.

[21] W. Takahashi, *Introduction to Nonlinear and Convex Analysis*, Yokohoma Publishers, Yokohoma, 2009.

[22] _____ , *Fixed point theorems for a new nonlinear mappings in a Hilbert space*, J. Nonlinear Convex Anal. **11** (2010), 79–88.

[23] W. Takahashi and M. Toyoda, *Weak convergence theorems for nonexpansive mappings and monotone mappings*, J. Optim. Theory Appl. **118** (2003), 417–428.

[24] W. Takahashi and J.-C. Yao, *Fixed point theorems and ergodic theorems for nonlinear mappings in Hilbert spaces*, Taiwanese J. Math. **15** (2011), 457–472.

[25] W. Takahashi, J.-C. Yao and M.-H. Hsu, *Generalized hybrid mappings in Hilbert spaces and Banach spaces*, Taiwanese. J. Math. **16** (2012), 129–149.

[26] W. Takahashi, J.-C. Yao and F. Kohsaka, *The fixed point property and unbounded sets in Banach spaces*, Taiwanese J. Math. **14** (2010), 733–742.

[27] F. Yan, Y. Su and Q. Feng, *Convergence of Ishikawa method for generalized hybrid mappings*, Commun. Korean Math. Soc. **28** (2013), No. 1, 135–141.

[28] H. Zhou, *Convergence theorems of fixed points for Lipschits pseudo-contractions*, J. Math. Anal. Appl. **243** (2008), 546–556.

Sattar Alizadeh and Fridoun Moradlou
Department of Mathematics
Sahand University of Technology
Tabriz, IRAN
   E-mail address: sa_alizadeh@sut.ac.ir, moradlou@sut.ac.ir, fridoun.moradlou@gmail.com