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THE BLOW-UP AND GLOBAL EXISTENCE OF SOLUTIONS OF CAUCHY PROBLEMS FOR A TIME FRACTIONAL DIFFUSION EQUATION

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Abstract. In this paper, we investigate the blow-up and global existence of solutions to the following time fractional nonlinear diffusion equations

$$\begin{cases} {}_0^C D_t^\alpha u - \triangle u = |u|^{p-1}u, & x \in \mathbb{R}^N,\ t > 0, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^N, \end{cases}$$

where $0 < \alpha < 1$, $p > 1$, $u_0 \in C_0(\mathbb{R}^N)$ and ${}_0^C D_t^\alpha u = (\partial/\partial t)_0 I_t^{1-\alpha}(u(t,x) - u_0(x))$, ${}_0 I_t^{1-\alpha}$ denotes left Riemann–Liouville fractional integrals of order $1-\alpha$. We prove that if $1 < p < 1+2/N$, then every nontrivial nonnegative solution blow-up in finite time, and if $p \geq 1 + 2/N$ and $\|u_0\|_{L^{q_c}(\mathbb{R}^N)}$, $q_c = N(p-1)/2$ is sufficiently small, then the problem has global solution.

1. Introduction

This paper is concerned with the blow-up and global existence of solutions to the following Cauchy problems for time fractional diffusion equation

$$(1.1)\qquad {}_0^C D_t^\alpha u - \triangle u = |u|^{p-1}u, \quad x \in \mathbb{R}^N,\ t > 0,$$

$$(1.2)\qquad u(0,x) = u_0(x), \quad x \in \mathbb{R}^N,$$

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where $0 < \alpha < 1$, $p > 1$, $u_0 \in C_0(\mathbb{R}^N) = \Big\{u \in C(\mathbb{R}^N) \ \Big|\ \lim_{|x|\to\infty} u(x) = 0\Big\}$ and ${}_0^C D_t^\alpha u = (\partial/\partial t)_0 I_t^{1-\alpha}(u(t,x) - u_0(x))$, ${}_0 I_t^{1-\alpha}$ denotes left Riemann–Liouville fractional integrals of order $1-\alpha$ and is defined by

$$ {}_0 I_t^{1-\alpha} u = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} u(s)\, ds. $$

When $\alpha = 1$, the problem (1.1)–(1.2) reduces to the semilinear heat equation

$$ u_t - \triangle u = |u|^{p-1}u, \quad x \in \mathbb{R}^N,\ t > 0 \tag{1.3} $$

with (1.2). In his pioneering article [9], Fujita showed that if $1 < p < 1 + 2/N$ and $u_0 \not\equiv 0$, then every solution of (1.3)–(1.2) blows up in a finite time. If $p > 1 + 2/N$, then for initial values bounded by a sufficiently small Gaussian, that is for $\tau > 0$, there is $\varepsilon = \varepsilon(\tau) > 0$ such that if $0 \le u_0(x) \le \varepsilon G_\tau(x)$, then the solution of (1.3)–(1.2) is global. The critical case $p = 1+2/N$ was later proved to be in the blow-up category [10], [13] (see also [27], [1]). In [27], Weissler proved that if the initial value u_0 is small enough in $L^{q_c}(\mathbb{R}^N)$, $q_c = N(p-1)/2 > 1$, then the solution of (1.3)–(1.2) exists globally.

In [12], Kirane, Laskyi and Tatar studied the following evolution problem

$$ {}_0^C D_t^\alpha u + (-\triangle)^{\beta/2} u = h(x,t)|u|^{1+\widetilde{p}}, \quad x \in \mathbb{R}^N,\ t > 0, \tag{1.4} $$

with (1.2), where $0 < \alpha < 1$, $0 < \beta \le 2$, ${}_0^C D_t^\alpha u = (\partial/\partial t)_0 I_t^{1-\alpha}(u(t,x) - u_0(x))$, $\widetilde{p} > 0$, h satisfies $h(x,t) \ge C_h |x|^\sigma t^\rho$ for $x \in \mathbb{R}^N$, $t > 0$, $C_h > 0$, and σ, ρ satisfy some conditions. $(-\triangle)^{\beta/2} u = \mathscr{F}^{-1}(|\xi|^\beta \mathscr{F}(u))$, where $\mathscr{F}$ denotes Fourier transform and $\mathscr{F}^{-1}$ denotes its inverse. They obtained that if $0 < \widetilde{p} \le (\alpha(\beta+\sigma) + \beta\rho)/(\alpha N + \beta(1-\alpha))$, then the problem (1.4)–(1.2) admits no global weak nonnegative solution other than the trivial one.

In [4], Cazenave, Dickstein and Weissler considered the following heat equation with nonlinear memory,

$$ u_t - \triangle u = \int_0^t (t-s)^{-\gamma} |u|^{p-1} u ds, \quad x \in \mathbb{R}^N,\ t > 0 \tag{1.5} $$

with (1.2), where $p > 1$, $0 \le \gamma < 1$ and $u_0 \in C_0(\mathbb{R}^N)$.

Let $p_\gamma = 1 + 2(2-\gamma)/(N-2+2\gamma)_+$, $(N-2+2\gamma)_+ = \max\{0, N-2+2\gamma\}$ and $p_* = \max\{1/\gamma, p_\gamma\} \in (0,\infty]$. They obtained that if $p \le p_*$, $u_0 \ge 0$, $u_0 \not\equiv 0$, then the solution u of (1.5)–(1.2) blows up in finite time and if $p > p_*$ and $u_0 \in L^{q_{sc}}(\mathbb{R}^N)$, $q_{sc} = N(p-1)/(4-2\gamma)$ with $\|u_0\|_{L^{q_{sc}}(\mathbb{R}^N)}$ sufficiently small, then the solution exists globally.

In [8], Fino and Kirane discussed the following equation

$$ u_t + (-\triangle)^{\beta/2} u = \int_0^t (t-s)^{-\gamma} |u|^{p-1} u\, ds, \quad x \in \mathbb{R}^N,\ t > 0 \tag{1.6} $$

with (1.2), $0 < \beta \le 2$, they got the blow-up and global existence results by using the test function method. The method based on rescalings of a compactly support test function to prove the blow-up results which is used by Mitidieri and Pohozaev [19] to show the blow-up results.

Fractional Cauchy problems are useful to model anomalous diffusion, describe Hamiltonian chaos, etc. see [21], [25], [28], [18] and references therein. We refer to several works on the mathematical treatments for time fractional diffusion equation. Eidelman and Kochubei [7] studied an evolution equation with time fractional and a uniformly elliptic operator with variable coefficients in the spatial variables, where the fundamental solution was constructed and investigated. In [14], Kochubei considered the Cauchy problem for a linear evolution systems of partial differential equations with the fractional derivative in the time variable and constructed and investigated the Green matrix of the Cauchy problem. In [24], [17], the existence and properties of solutions for a time fractional equation in a bounded domain were considered by applying eigenfunction expansions. Recently, there are many papers about the existence and properties of solutions for the fractional abstract Cauchy problem, see for example [26], [15], [29], [20] and the references therein.

Motivated by the above results, in this paper, we study the problem (1.1)–(1.2). Fujita exponent is determined. We will show that

(i) For $u_0 \in C_0(\mathbb{R}^N)$, $u_0 \ge 0$, and $u_0 \not\equiv 0$, if $1 < p < 1 + 2/N$, then the solution of (1.1)–(1.2) blows up in finite time.
(ii) For $u_0 \in C_0(\mathbb{R}^N) \cap L^{q_c}(\mathbb{R}^N)$, where $q_c = N(p-1)/2$, if $p \ge 1 + 2/N$ and $\|u_0\|_{L^{q_c}}$ is sufficiently small, then (1.1)–(1.2) has a global solution.

Compare with the classical results of heat equation (1.3)–(1.2), the major difference between the time fractional equation (1.1)–(1.2) and the heat equation (1.3)–(1.2) is that in critical case, that is $p = 1 + 2/N$, the solution of (1.1)–(1.2) could exist globally.

In [4], the authors inferred that for (1.6)–(1.2), the Fujita critical exponent is not the one which would be predicted from the scaling properties of the equation, which is different from the heat equation (1.3)–(1.2). But for (1.1), despite it is also nonlocal about t, we can also obtain the Fujita critical exponent by the scaling properties of the equation (1.1). In fact, if $u(t,x)$ is a solution of (1.1) with the initial value $u_0(x)$, then, for every $\lambda > 0$, $\lambda^{2\alpha/(p-1)} u(\lambda^2 t, \lambda^\alpha x)$ is also a solution of (1.1) with initial value $\lambda^{2\alpha/(p-1)} u_0(\lambda^\alpha x)$. Since

$$\|\lambda^{2\alpha/(p-1)} u_0(\lambda^\alpha \cdot)\|_{L^q(\mathbb{R}^N)} = \lambda^{2\alpha/(p-1) - \alpha N/q} \|u_0\|_{L^q(\mathbb{R}^N)}, \tag{1.7}$$

it follows that the invariant Lebesgue norm for (1.7) is given by $q_c = N(p-1)/2$. If $q_c > 1$, then $p > 1 + 2/N$. Therefore, one predicts $1 + 2/N$ is the Fujita critical exponent. Our main results show $1 + 2/N$ is the Fujita critical exponent for the problem (1.1)–(1.2).

This paper is organized as follows. In Section 2, some preliminaries are presented. In Section 3, the local existence and uniqueness of mild solution of the problem (1.1)–(1.2) are established. In Section 4, we show that the blow-up and global existence of the solutions to the problem (1.1)–(1.2).

2. Preliminaries

In this section, we present some preliminaries that will be used in the next sections.

First, we list some properties of two special functions. The Mittag–Leffler function is defined for complex $z \in \mathbb{C}$ in [11], [23]

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \alpha, \beta \in \mathbb{C},\ \mathrm{Re}(\alpha) > 0$$

and its Riemann–Liouville fractional integral satisfies

$${}_0I_t^{1-\alpha}(t^{\alpha-1}E_{\alpha,\alpha}(\lambda t^\alpha)) = E_{\alpha,1}(\lambda t^\alpha) \quad \text{for } \lambda \in \mathbb{C},\ 0 < \alpha < 1.$$

We also need the following Wright type function which was considered by Mainardi [16] (see also [23])

$$\begin{aligned} \phi_\alpha(z) &= \sum_{k=0}^{\infty} \frac{(-z)^k}{k!\Gamma(-\alpha k + 1 - \alpha)} \\ &= \frac{1}{\pi} \sum_{k=0}^{\infty} \frac{(-z)^k \Gamma(\alpha(k+1)) \sin(\pi(k+1)\alpha)}{k!} \end{aligned} \tag{2.1}$$

for $0 < \alpha < 1$. ϕ_α is an entire function and has the following properties.

(a) $\phi_\alpha(\theta) \geq 0$ for $\theta \geq 0$ and $\displaystyle\int_0^\infty \phi_\alpha(\theta)\, d\theta = 1$.

(b) $\displaystyle\int_0^\infty \phi_\alpha(\theta)\theta^r\, d\theta = \frac{\Gamma(1+r)}{\Gamma(1+\alpha r)}$ for $r > -1$.

(c) $\displaystyle\int_0^\infty \phi_\alpha(\theta) e^{-z\theta}\, d\theta = E_{\alpha,1}(-z)$, $z \in \mathbb{C}$.

(d) $\displaystyle\alpha \int_0^\infty \theta\phi_\alpha(\theta) e^{-z\theta}\, d\theta = E_{\alpha,\alpha}(-z)$, $z \in \mathbb{C}$.

If ${}_0^C D_t^\alpha f \in L^1(0,T)$, $g \in C^1([0,T])$ and $g(T) = 0$, then we have the following formula of integration by parts

$$\int_0^T g\, {}_0^C D_t^\alpha f\, dt = \int_0^T (f(t) - f(0))\, {}_t^C D_T^\alpha g\, dt, \tag{2.2}$$

where ${}_t^C D_T^\alpha g = -\frac{d}{dt} {}_t I_T^{1-\alpha} g$,

$${}_t I_T^{1-\alpha} g = \frac{1}{\Gamma(1-\alpha)} \int_t^T (s-t)^{-\alpha} g(s)\, ds.$$

We need know the Caputo fractional derivative of the following function, which will be used in the next sections. For given $T > 0$ and $n > 0$, if we let

$$\varphi(t) = \begin{cases} (1 - t/T)^n, & t \le T, \\ 0, & t > T, \end{cases}$$

then

$${}_t^C D_T^\alpha \varphi(t) = \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)} T^{-\alpha} \left(1 - \frac{t}{T}\right)^{n-\alpha}, \quad t \le T,$$

(see for example [11]).

We denote $A = \triangle$ and it generates a semigroup $\{T(t)\}_{t\ge 0}$ on $C_0(\mathbb{R}^N)$ with domain

$$D(A) = \{u \in C_0(\mathbb{R}^N) \mid \triangle u \in C_0(\mathbb{R}^N)\}.$$

Then $T(t)$ is an analytic and contractive semigroup on $C_0(\mathbb{R}^N)$ [3], [22] and, for $t > 0$, $x \in \mathbb{R}^N$,

$$T(t)u_0 = \int_{\mathbb{R}^N} G(t, x-y) u_0(y)\, dy, \qquad G(t,x) = \frac{1}{(4\pi t)^{N/2}} e^{-|x|^2/(4t)}, \tag{2.3}$$

and $T(t)$ is a contractive semigroup on $L^q(\mathbb{R}^N)$ for $q \ge 1$ [5], and

$$\|T(t)u_0\|_{L^p(\mathbb{R}^N)} \le (4\pi t)^{-(N/2)(1/q-1/p)} \|u_0\|_{L^q(\mathbb{R}^N)} \tag{2.4}$$

for $u_0 \in L^q(\mathbb{R}^N)$, $q \le p \le +\infty$.

Define the operators $P_\alpha(t)$ and $S_\alpha(t)$ as

$$P_\alpha(t)u_0 = \int_0^\infty \phi_\alpha(\theta) T(t^\alpha \theta) u_0\, d\theta, \qquad t \ge 0, \tag{2.5}$$

$$S_\alpha(t)u_0 = \alpha \int_0^\infty \theta \phi_\alpha(\theta) T(t^\alpha \theta) u_0\, d\theta, \quad t \ge 0. \tag{2.6}$$

Consider the following linear time fractional equation

$$\begin{cases} {}_0^C D_t^\alpha u - \triangle u = f(t,x), & x \in \mathbb{R}^N,\ t > 0, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^N, \end{cases} \tag{2.7}$$

where $u_0 \in C_0(\mathbb{R}^N)$ and $f \in L^1((0,T), C_0(\mathbb{R}^N))$. If u is a solution of (2.7), then by [2] (see also [26]), we get

$$u(t,x) = P_\alpha(t)u_0 + \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) f(s,x)\, ds,$$

where $P_\alpha(t)$ and $S_\alpha(t)$ are given by (2.5) and (2.6), respectively.

We denote

$$K(t,x)=\int_0^\infty \phi_\alpha(\theta)G(t^\alpha\theta,x)\,d\theta,\quad x\in\mathbb{R}^N\setminus\{0\},\ t>0.$$

Note that for given $t>0$ and $x\in\mathbb{R}^N\setminus\{0\}$, $G(t^\alpha\theta,x)\to 0$ as $\theta\to 0$, so K is well defined. Since $\int_0^\infty \phi_\alpha(\theta)\,d\theta=1$, $\int_{\mathbb{R}^N} G(t,x)\,dx=1$, we know that

$$\|K(t,\cdot)\|_{L^1(\mathbb{R}^N)}=1\quad \text{for } t>0.$$

Lemma 2.1. *The operator $\{P_\alpha(t)\}_{t>0}$ has the following properties*

(a) *If $u_0\ge 0$, $u_0\not\equiv 0$, then $P_\alpha(t)u_0>0$ and $\|P_\alpha(t)u_0\|_{L^1(\mathbb{R}^N)}=\|u_0\|_{L^1(\mathbb{R}^N)}$.*
(b) *If $1\le p\le q\le+\infty$ and $1/r=1/p-1/q<2/N$, then*

$$\|P_\alpha(t)u_0\|_{L^q(\mathbb{R}^N)}\le(4\pi t^\alpha)^{-N/(2r)}\frac{\Gamma(1-N/(2r))}{\Gamma(1-\alpha N/(2r))}\|u_0\|_{L^p(\mathbb{R}^N)}. \tag{2.8}$$

Proof. (a) follows from $T(t)u_0>0$, $\phi_\alpha\ge 0$ and $\|K(t,\cdot)\|_{L^1(\mathbb{R}^N)}=1$.

For (b), by (2.4) and the properties of ϕ_α, we have

$$\begin{aligned}\bigg\|\int_0^{+\infty}\phi_\alpha(\theta)T(t^\alpha\theta)u_0d\theta\bigg\|_{L^q(\mathbb{R}^N)}&\le\int_0^{+\infty}\phi_\alpha(\theta)(4\pi t^\alpha\theta)^{-N/(2r)}\|u_0\|_{L^p(\mathbb{R}^N)}\,d\theta\\&=(4\pi t^\alpha)^{-N/(2r)}\int_0^{+\infty}\phi_\alpha(\theta)\theta^{-N/(2r)}\,d\theta\|u_0\|_{L^p(\mathbb{R}^N)}\\&=(4\pi t^\alpha)^{-N/(2r)}\frac{\Gamma(1-N/(2r))}{\Gamma(1-\alpha N/(2r))}\|u_0\|_{L^p(\mathbb{R}^N)}.\end{aligned}$$

Hence, we derive (2.8) holds. □

Lemma 2.2. *For the operator $\{S_\alpha(t)\}_{t>0}$, we have the following results.*

(a) *If $u_0\ge 0$ and $u_0\not\equiv 0$, then*

$$S_\alpha(t)u_0>0\quad\text{and}\quad \|S_\alpha(t)u_0\|_{L^1(\mathbb{R}^N)}=\frac{1}{\Gamma(\alpha)}\|u_0\|_{L^1(\mathbb{R}^N)}.$$

(b) *For $1\le p\le q\le+\infty$, let $1/r=1/p-1/q$, if $1/r<4/N$, then*

$$\|S_\alpha(t)u_0\|_{L^q(\mathbb{R}^N)}\le\alpha(4\pi t^\alpha)^{-N/(2r)}\frac{\Gamma(2-N/(2r))}{\Gamma(1+\alpha-\alpha N/(2r))}\|u_0\|_{L^p(\mathbb{R}^N)}. \tag{2.9}$$

Proof. The proof is similar to that of Lemma 2.1, so we omit it. □

Lemma 2.3. *For $u_0\in C_0(\mathbb{R}^N)$, we have $P_\alpha(t)u_0\in D(A)$ for $t>0$, and*

$$\begin{aligned}{}_0^CD_t^\alpha P_\alpha(t)u_0&=AP_\alpha(t)u_0, && t>0,\\ \|AP_\alpha(t)u_0\|_{L^\infty(\mathbb{R}^N)}&\le\frac{C}{t^\alpha}\|u_0\|_{L^\infty(\mathbb{R}^N)}, && t>0,\end{aligned}$$

for some constant $C>0$.

Proof. Let $X = C_0(\mathbb{R}^N)$. First, we prove if $u_0 \in X$, then $P_\alpha(t)u_0 \in D(A)$. In fact, for $u_0 \in X$,

$$P_\alpha(t)u_0 = \int_0^\infty \phi_\alpha(\theta)T(t^\alpha\theta)u_0\,d\theta = \int_0^1 (\phi_\alpha(\theta) - \phi_\alpha(0))T(t^\alpha\theta)u_0\,d\theta$$
$$+ \phi_\alpha(0)\int_0^1 T(t^\alpha\theta)u_0\,d\theta + \int_1^\infty \phi_\alpha(\theta)T(t^\alpha\theta)u_0\,d\theta.$$

Clearly, $\int_0^1 T(t^\alpha\theta)u_0\,d\theta \in D(A)$. Note that there exists positive constant C such that

$$\|AT(t^\alpha\theta)u_0\|_X \le C\frac{\|u_0\|_X}{t^\alpha\theta}, \quad t > 0,\ \theta > 0,$$

we get that

$$\int_1^\infty \phi_\alpha(\theta)T(t^\alpha\theta)u_0\,d\theta \in D(A).$$

Next, we show that

$$\int_0^1 (\phi_\alpha(\theta) - \phi_\alpha(0))T(t^\alpha\theta)u_0\,d\theta \in D(A).$$

In fact, for every $h > 0$,

$$\frac{1}{h}\bigg[T(h)\int_0^1 (\phi_\alpha(\theta) - \phi_\alpha(0))T(t^\alpha\theta)u_0\,d\theta - \int_0^1 (\phi_\alpha(\theta) - \phi_\alpha(0))T(t^\alpha\theta)u_0\,d\theta\bigg]$$
$$= \frac{1}{h}\int_0^1 (\phi_\alpha(\theta) - \phi_\alpha(0))(T(t^\alpha\theta + h) - T(t^\alpha\theta))u_0\,d\theta.$$

Since

$$\bigg\|\frac{(T(t^\alpha\theta + h) - T(t^\alpha\theta))u_0}{h}\bigg\|_X \le \frac{C}{t^\alpha\theta}\|u_0\|_X, \qquad \bigg|\frac{\phi_\alpha(\theta) - \phi_\alpha(0)}{\theta}\bigg| \le C,$$

for some constant $C > 0$ independent of θ and h, so, by dominated convergence theorem, we know

$$\int_0^1 (\phi_\alpha(\theta) - \phi_\alpha(0))T(t^\alpha\theta)u_0\,d\theta \in D(A).$$

Note that

$$\begin{aligned}AP_\alpha(t)u_0 &= A\int_0^1 (\phi_\alpha(\theta) - \phi_\alpha(0))T(t^\alpha\theta)u_0\,d\theta \\ &\quad + \phi_\alpha(0)A\int_0^1 T(t^\alpha\theta)u_0\,d\theta + A\int_1^\infty \phi_\alpha(\theta)T(t^\alpha\theta)u_0\,d\theta \\ &= \int_0^1 (\phi_\alpha(\theta) - \phi_\alpha(0))AT(t^\alpha\theta)u_0\,d\theta \\ &\quad + \frac{\phi_\alpha(0)(T(t^\alpha)u_0 - u_0)}{t^\alpha} + \int_1^\infty \phi_\alpha(\theta)AT(t^\alpha\theta)u_0\,d\theta.\end{aligned}$$

Therefore

$$\|AP_\alpha(t)u_0\|_X \leq \frac{C}{t^\alpha}\,\|u_0\|_X \tag{2.10}$$

for some positive constant C.

By dominated convergence theorem, we obtain that for $u_0 \in X$,

$$\frac{d}{dt}P_\alpha(t)u_0 = t^{\alpha-1}AS_\alpha(t)u_0, \quad t > 0.$$

Furthermore, if $u_0 \in D(A)$, then

$$\frac{d}{dt}P_\alpha(t)u_0 = t^{\alpha-1}S_\alpha(t)Au_0, \quad t > 0.$$

Since

$$_0I_t^{1-\alpha}(t^{\alpha-1}S_\alpha(t)Au_0) = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{s^{\alpha-1}}{(t-s)^\alpha}\int_0^\infty \alpha\theta\phi_\alpha(\theta)T(s^\alpha\theta)Au_0\,d\theta\,ds,$$

$$\int_0^\infty \alpha\theta\phi_\alpha(\theta)T(s^\alpha\theta)Au_0\,d\theta = \frac{1}{2\pi i}\int_{\Gamma'} E_{\alpha,\alpha}(\lambda s^\alpha)(\lambda - A)^{-1}Au_0\,d\lambda,$$

where Γ' is a path composed from two rays $\rho e^{i\tau}$, $\rho \geq 1$, $\pi/2 < \tau < \pi$ and $\rho e^{-i\tau}$ and a curve $e^{i\beta}$, $-\tau \leq \beta \leq \tau$,

$$_0I_t^{1-\alpha}(t^{\alpha-1}E_{\alpha,\alpha}(\lambda t^\alpha)) = E_{\alpha,1}(\lambda t^\alpha),$$

so,

$$_0I_t^{1-\alpha}(t^{\alpha-1}S_\alpha(t)Au_0) = P_\alpha(t)Au_0 = AP_\alpha(t)u_0. \tag{2.11}$$

Therefore, we get ${}_0^CD_t^\alpha P_\alpha(t)u_0 = AP_\alpha(t)u_0$ if $u_0 \in D(A)$, $t > 0$. Next, we prove that the conclusion also holds if $u_0 \in X$.

In fact, if $u_0 \in X$, then we can find $\{u_{0,n}\} \subset D(A)$ such that $u_{0,n} \to u_0$ in X. By (2.11) and Lemma 2.1, we know

$${}_0^CD_t^\alpha P_\alpha(t)u_{0,n} = AP_\alpha(t)u_{0,n} \quad \text{and} \quad \|P_\alpha(t)u_{0,n}\|_X \leq \|u_{0,n}\|_X.$$

We denote $u_n = P_\alpha(t)u_{0,n}$. Then, there exists $u \in X$ such that for every $T > 0$, $u_n \to u$ uniformly in X for $t \in [0,T]$ as $n \to \infty$. Since

$$\|{}_0I_t^{1-\alpha}u_n\|_X \leq \frac{T^{1-\alpha}}{(1-\alpha)\Gamma(1-\alpha)}\|u_n\|_{L^\infty((0,T),X)}, \quad t \in [0,T],$$

so we know ${}_0I_t^{1-\alpha}u_n \to {}_0I_t^{1-\alpha}u$ in X. By (2.10),

$$\|{}_0^CD_t^\alpha u_n\|_X \leq \frac{C}{t^\alpha}\|u_{0,n}\|_X, \text{ for some constant } C > 0, \quad t > 0.$$

Hence, for every $\delta > 0$, there exists $w \in C([\delta,\infty),X)$ such that ${}_0^CD_t^\alpha u_n \to w$ uniformly in X on $t \in [\delta,\infty)$.

Note that for $t \in [\delta,\infty)$,

$${}_0^CD_t^\alpha u_n = \frac{d}{dt}({}_0I_t^{1-\alpha}(P_\alpha(t)u_{0,n} - u_{0,n})) = Au_n,$$

so

$$w = \frac{d}{dt}{}_0I_t^{1-\alpha}(u-u_0) = {}_0^CD_t^\alpha u, \quad t \in [\delta, \infty).$$

Since A is closed, we have $w = Au$, that is ${}_0^CD_t^\alpha u = Au = AP_\alpha(t)u_0$, $t \in [\delta, \infty)$. By arbitrariness of δ, we have ${}_0^CD_t^\alpha u = AP_\alpha(t)u_0$, $t > 0$. □

Lemma 2.4. *Assume that* $f \in L^q((0,T), C_0(\mathbb{R}^N))$, $q > 1$. *Let*

$$w(t) = \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) f(s)\, ds,$$

then

$${}_0I_t^{1-\alpha} w = \int_0^t P_\alpha(t-s) f(s)\, ds.$$

Furthermore, if $q\alpha > 1$, *then* $w \in C([0,T], C_0(\mathbb{R}^N))$.

Proof. Let $X = C_0(\mathbb{R}^N)$. By Fubini theorem and (2.11), we have

$$\begin{aligned}
{}_0I_t^{1-\alpha} w &= \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \int_0^s (s-\tau)^{\alpha-1} S_\alpha(s-\tau) f(\tau)\, d\tau\, ds \\
&= \frac{1}{\Gamma(1-\alpha)} \int_0^t \int_\tau^t (t-s)^{-\alpha} (s-\tau)^{\alpha-1} S_\alpha(s-\tau) f(\tau)\, ds\, d\tau \\
&= \frac{1}{\Gamma(1-\alpha)} \int_0^t \int_0^{t-\tau} (t-s-\tau)^{-\alpha} s^{\alpha-1} S_\alpha(s) f(\tau)\, ds\, d\tau \\
&= \int_0^t P_\alpha(t-\tau) f(\tau)\, d\tau.
\end{aligned}$$

For every $h > 0$ and $t + h \le T$, we have $w(t+h) - w(t) = I_1 + I_2$, where

$$\begin{aligned}
I_1 &= \alpha \int_t^{t+h} \int_0^\infty \theta \phi_\alpha(\theta) (t+h-\tau)^{\alpha-1} T((t+h-\tau)^\alpha \theta) f(\tau)\, d\theta\, d\tau. \\
I_2 &= \alpha \int_0^t \int_0^\infty \theta \phi_\alpha(\theta) [(t+h-\tau)^{\alpha-1} T((t+h-\tau)^\alpha \theta) \\
&\qquad - (t-\tau)^{\alpha-1} T((t-\tau)^\alpha \theta)] f(\tau)\, d\theta\, d\tau,
\end{aligned}$$

By Hölder inequality, we have

$$\begin{aligned}
\|I_1\|_X &\le \alpha \int_t^{t+h} \int_0^\infty \theta \phi_\alpha(\theta) (t+h-\tau)^{\alpha-1} \|f(\tau)\|_X\, d\theta\, d\tau \\
&= \frac{1}{\Gamma(\alpha)} \int_t^{t+h} (t+h-\tau)^{\alpha-1} \|f(\tau)\|_X\, d\tau \\
&\le \frac{1}{\Gamma(\alpha)} \|f\|_{L^q((0,T),X)} \left(\int_t^{t+h} (t+h-\tau)^{q(\alpha-1)/(q-1)}\, d\tau \right)^{(q-1)/q} \\
&= \frac{1}{\Gamma(\alpha)} \left(\frac{q-1}{q\alpha-1} \right)^{(q-1)/q} \|f\|_{L^q((0,T),X)} h^{(q\alpha-1)/q}.
\end{aligned} \tag{2.12}$$

Note that, for $0 < \tau < t$,

$$\begin{aligned}\|(t+h-\tau)^{\alpha-1}T((t+h-\tau)^{\alpha}\theta)f(\tau)-(t-\tau)^{\alpha-1}T((t-\tau)^{\alpha}\theta)f(\tau)\|_X \\ \le 2(t-\tau)^{\alpha-1}\|f(\tau)\|_X\end{aligned}$$

and there exists constant $C > 0$ such that

$$\begin{aligned}\|[(t+h-\tau)^{\alpha-1}&T((t+h-\tau)^{\alpha}\theta)-(t-\tau)^{\alpha-1}T((t-\tau)^{\alpha}\theta)]f(\tau)\|_X \\ &\le |(t+h-\tau)^{\alpha-1}-(t-\tau)^{\alpha-1}|\|T((t+h-\tau)^{\alpha}\theta)f(\tau)\|_X \\ &\quad + (t-\tau)^{\alpha-1}\|(T((t+h-\tau)^{\alpha}\theta)-T((t-\tau)^{\alpha}\theta))f(\tau)\|_X \\ &\le C(t-\tau)^{\alpha-2}h\|f(\tau)\|_X.\end{aligned}$$

Therefore

$$\begin{aligned}\|I_2\|_X &\le C\int_0^t\int_0^{\infty}\alpha\theta\phi_{\alpha}(\theta)\min\bigg\{\frac{1}{(t-\tau)^{1-\alpha}},\frac{h}{(t-\tau)^{2-\alpha}}\bigg\}\,d\theta\,\|f(\tau)\|_X\,d\tau \\ &\le \frac{C}{\Gamma(\alpha)}\bigg(\int_0^t\bigg(\min\bigg\{\frac{1}{(t-\tau)^{1-\alpha}},\frac{h}{(t-\tau)^{2-\alpha}}\bigg\}\bigg)^{q/(q-1)}d\tau\bigg)^{(q-1)/q} \\ &\quad\cdot\|f\|_{L^q((0,T),X)}.\end{aligned}$$

Observe that

$$\begin{aligned}\int_0^t\bigg(\min\bigg\{&\frac{1}{(t-\tau)^{1-\alpha}},\frac{h}{(t-\tau)^{2-\alpha}}\bigg\}\bigg)^{q/(q-1)}d\tau \\ &= \int_0^t\bigg(\min\bigg\{\frac{1}{\tau^{1-\alpha}},\frac{h}{\tau^{2-\alpha}}\bigg\}\bigg)^{q/(q-1)}d\tau \\ &\le \int_0^{\infty}\bigg(\min\bigg\{\frac{1}{\tau^{1-\alpha}},\frac{h}{\tau^{2-\alpha}}\bigg\}\bigg)^{q/(q-1)}d\tau \\ &= \int_0^h\tau^{q(\alpha-1)/(q-1)}\,d\tau+\int_h^{\infty}h^{q/(q-1)}\tau^{q(\alpha-2)/(q-1)}\,d\tau \\ &= \frac{q(q-1)}{(q\alpha-1)(q+1-q\alpha)}h^{q\alpha-1/(q-1)},\end{aligned}$$

so,

$$\|I_2\|_X \le C\|f\|_{L^q((0,T),X)}h^{(q\alpha-1)/q}. \tag{2.13}$$

Hence, (2.12)–(2.13) imply that the conclusion hold. □

Remark 2.5. For $\alpha = 1$, the conclusion of Lemma 2.4 also holds (see Theorem 3.1 of Chapter 4 in [22]).

3. Local existence

In this section, we give the local existence and uniqueness of mild solution of the problem (1.1)–(1.2). First, we give the definition of mild solution of (1.1)–(1.2).

Definition 3.1. Let $u_0 \in C_0(\mathbb{R}^N)$, $T > 0$. We call that $u \in C([0,T], C_0(\mathbb{R}^N))$ is a mild solution of the problem (1.1)–(1.2) if u satisfies the following integral equation

$$u(t) = P_\alpha(t)u_0 + \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s)|u|^{p-1}u(s)\,ds, \quad t \in [0,T].$$

For the problem (1.1)–(1.2), we have the following local existence result.

Theorem 3.2. *Given $u_0 \in C_0(\mathbb{R}^N)$, then there exists a maximal time $T_{\max} = T(u_0) > 0$ such that the problem (1.1)–(1.2) has a unique mild solution u in $C([0,T), C_0(\mathbb{R}^N))$ and either $T_{\max} = +\infty$ or $T_{\max} < +\infty$ and $\|u\|_{L^\infty((0,t),C_0(\mathbb{R}^N))} \to +\infty$ as $t \to T_{\max}$. If, in addition, $u_0 \geq 0$, $u_0 \not\equiv 0$, then $u(t) > 0$ and $u(t) \geq P_\alpha(t)u_0$ for $t \in (0, T_{\max})$. Moreover, if $u_0 \in L^r(\mathbb{R}^N)$ for some $r \in [1,\infty)$, then $u \in C([0,T_{\max}), L^r(\mathbb{R}^N))$.*

Proof. For given $T > 0$ and $u_0 \in C_0(\mathbb{R}^N)$, let

$$E_T = \{u \mid u \in C([0,T], C_0(\mathbb{R}^N)),\ \|u\|_{L^\infty((0,T),L^\infty(\mathbb{R}^N))} \leq 2\|u_0\|_{L^\infty(\mathbb{R}^N)}\},$$

$$d(u,v) = \max_{t\in[0,T]} \|u(t) - v(t)\|_{L^\infty(\mathbb{R}^N)} \quad \text{for } u, v \in E_T.$$

Since $C([0,T], C_0(\mathbb{R}^N))$ is a Banach space, (E_T, d) is a complete metric space. We define the operator G on E_T as

$$G(u)(t) = P_\alpha(t)u_0 + \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s)|u(s)|^{p-1}u(s)\,ds, \quad u \in E_T,$$

then $G(u) \in C([0,T], C_0(\mathbb{R}^N))$ in view of Lemma 2.4. If $u \in E_T$, then by Lemma 2.1(b) and Lemma 2.2(b), for $t \in [0,T]$,

$$\begin{aligned}\|G(u)(t)\|_{L^\infty(\mathbb{R}^N)} &\leq \|u_0\|_{L^\infty(\mathbb{R}^N)} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\|u(s)\|^p_{L^\infty(\mathbb{R}^N)}\,ds \\ &\leq \|u_0\|_{L^\infty(\mathbb{R}^N)} + \frac{2^p T^\alpha}{\alpha\Gamma(\alpha)}\|u_0\|^p_{L^\infty(\mathbb{R}^N)}.\end{aligned}$$

Hence, we can choose T small enough such that

$$\frac{2^p T^\alpha}{\alpha\Gamma(\alpha)}\|u_0\|^{p-1}_{L^\infty(\mathbb{R}^N)} \leq 1,$$

so we get $\|G(u)\|_{L^\infty((0,T),L^\infty(\mathbb{R}^N))} \leq 2\|u_0\|_{L^\infty(\mathbb{R}^N)}$. Furthermore, for $u, v \in E_T$, we have for $t \in [0,T]$

$$\begin{aligned}&\|G(u)(t) - G(v)(t)\|_{L^\infty(\mathbb{R}^N)} \\ &\quad\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\||u(s)|^{p-1}u(s) - |v(s)|^{p-1}v(s)\|_{L^\infty(\mathbb{R}^N)}\,ds \\ &\quad\leq \frac{4^{(p-1)} p T^\alpha \|u_0\|^{p-1}_{L^\infty(\mathbb{R}^N)}}{\alpha\Gamma(\alpha)}\|u - v\|_{L^\infty((0,T),L^\infty(\mathbb{R}^N))}.\end{aligned}$$

We can choose T small enough such that

$$\frac{p\,4^{(p-1)}T^{\alpha}\|u_0\|^{p-1}_{L^{\infty}(\mathbb{R}^N)}}{\alpha\Gamma(\alpha)} \le \frac{1}{2},$$

then $\|G(u)(t) - G(v)(t)\|_{L^{\infty}(\mathbb{R}^N)} \le \|u - v\|_{L^{\infty}((0,T),C_0(\mathbb{R}^N))}/2$. Therefore, G is contractive on E_T. So, G has a fixed point $u \in E_T$ by the contraction mapping principle.

Now, we prove the uniqueness. Let $u, v \in C([0,T], C_0(\mathbb{R}^N))$ be the mild solutions of (1.1)–(1.2) for some $T > 0$, then there exists positive constant $C > 0$ such that

$$\begin{aligned}\|u(t) - v(t)\|_{L^{\infty}(\mathbb{R}^N)} &= \|G(u(t) - G(v)(t)\|_{L^{\infty}(\mathbb{R}^N)} \\ &\le C\int_0^t (t-s)^{\alpha-1}\|u(s) - v(s)\|_{L^{\infty}(\mathbb{R}^N)}\,ds.\end{aligned}$$

Hence, by Gronwall's inequality, we know $u = v$.

Next, using the uniqueness of solution, we conclude that the existence of solution on a maximal interval $[0, T_{\max})$, where

$$\begin{aligned}T_{\max} = \sup\{T > 0 \mid &\text{ there exists a mild solution } u \\ &\text{of (1.1)–(1.2) in } C([0,T], C_0(\mathbb{R}^N))\}.\end{aligned}$$

Assume that $T_{\max} < +\infty$ and there exists $M > 0$ such that for $t \in [0, T_{\max})$,

$$\|u(t)\|_{L^{\infty}(\mathbb{R}^N)} \le M.$$

Next, we will verify that $\lim_{t\to T_{\max}^-} u(t)$ exists in $C_0(\mathbb{R}^N)$. In fact, for $0 < t < \tau < T_{\max}$, by the proof of Lemma 2.4, there exists constant $C > 0$ such that

$$\begin{aligned}\|u(t) &- u(\tau)\|_{L^{\infty}(\mathbb{R}^N)} \le \|P_\alpha(t)u_0 - P_\alpha(\tau)u_0\|_{L^{\infty}(\mathbb{R}^N)} \\ &+ \bigg\|\int_0^t (t-s)^{\alpha-1}S_\alpha(t-s)|u(s)|^{p-1}u(s) \\ &- (\tau-s)^{\alpha-1}S_\alpha(\tau-s)|u(s)|^{p-1}u(s)\,ds\bigg\|_{L^{\infty}(\mathbb{R}^N)} \\ &+ \bigg\|\int_t^\tau (\tau-s)^{\alpha-1}S_\alpha(\tau-s)|u(s)|^{p-1}u(s)\,ds\bigg\|_{L^{\infty}(\mathbb{R}^N)} \\ \le{}& \|P_\alpha(t)u_0 - P_\alpha(\tau)u_0\|_{L^{\infty}(\mathbb{R}^N)} + \frac{M^p}{\Gamma(\alpha)}\int_t^\tau (\tau-s)^{\alpha-1}\,ds \\ &+ CM^p\int_0^t \min\{(t-s)^{\alpha-1}, (t-s)^{\alpha-2}(\tau-t)\}\,ds \\ \le{}& \|P_\alpha(t)u_0 - P_\alpha(\tau)u_0\|_{L^{\infty}(\mathbb{R}^N)} + \frac{M^p}{\alpha\Gamma(\alpha)}(\tau-t)^\alpha + CM^p\frac{1}{\alpha(1-\alpha)}(\tau-t)^\alpha.\end{aligned}$$

Since $P_\alpha(t)u_0$ is uniformly continuous in $[0, T_{\max}]$, so $\lim_{t\to T_{\max}^-} u(t)$ exists.

We denote $u_{T_{\max}} = \lim\limits_{t\to T_{\max}^-} u(t)$ and define $u(T_{\max}) = u_{T_{\max}}$. Hence, $u \in C([0,T_{\max}], C_0(\mathbb{R}^N))$ and then, by Lemma 2.4,

$$\int_0^t (t-s)^{\alpha-1} S_\alpha(t-s)|u(s)|^{p-1}u(s)\,ds \in C([0,T_{\max}], C_0(\mathbb{R}^N)).$$

For $h>0$, $\delta>0$, let

$$E_{h,\delta} = \{u \in C([T_{\max}, T_{\max}+h], C_0(\mathbb{R}^N)) \mid u(T_{\max}) = u_{T_{\max}}, d(u, u_{T_{\max}}) \le \delta\},$$

where $d(u,v) = \max\limits_{t\in[T_{\max},T_{\max}+h]} \|u(t)-v(t)\|_{L^\infty(\mathbb{R}^N)}$ for $u, v \in E_{h,\delta}$.

Via $C([T_{\max}, T_{\max}+h], C_0(\mathbb{R}^N))$ is a Banach space, we know $(E_{h,\delta}, d)$ is a complete metric space.

We define the operator G on $E_{h,\delta}$ as

$$\begin{aligned} G(v)(t) = P_\alpha(t)u_0 &+ \int_0^{T_{\max}} (t-\tau)^{\alpha-1} S_\alpha(t-\tau)|u|^{p-1}u(\tau)\,d\tau \\ &+ \int_{T_{\max}}^t (t-\tau)^{\alpha-1} S_\alpha(t-\tau)|v|^{p-1}v(\tau)\,d\tau, \quad v \in E_{h,\delta}. \end{aligned}$$

Clearly, $G(v) \in C([T_{\max}, T_{\max}+h], C_0(\mathbb{R}^N))$ and $G(v)(T_{\max}) = u_{T_{\max}}$.

If $v \in E_{h,\delta}$, then for $t \in [T_{\max}, T_{\max}+h]$,

$$\begin{aligned} &\|G(v)(t) - u_{T_{\max}}\|_{L^\infty(\mathbb{R}^N)} \\ &\qquad \le \|P_\alpha(t)u_0 - P_\alpha(T_{\max})u_0\|_{L^\infty(\mathbb{R}^N)} + \|I_3\|_{L^\infty(\mathbb{R}^N)} + \|I_4\|_{L^\infty(\mathbb{R}^N)}, \end{aligned}$$

where

$$\begin{aligned} I_3 &= \int_0^{T_{\max}} (t-\tau)^{\alpha-1} S_\alpha(t-\tau)|u(\tau)|^{p-1}u(\tau) \\ &\quad - (T_{\max}-\tau)^{\alpha-1} S_\alpha(T_{\max}-\tau)|u(\tau)|^{p-1}u(\tau)\,d\tau, \\ I_4 &= \int_{T_{\max}}^t (t-\tau)^{\alpha-1} S_\alpha(t-\tau)|v|^{p-1}v(\tau)\,d\tau. \end{aligned}$$

Taking h small enough such that

$$\begin{aligned} \|P_\alpha(t)u_0 - P_\alpha(T_{\max})u_0\|_{L^\infty(\mathbb{R}^N)} &\le \frac{\delta}{3} \quad \text{for } t \in [T_{\max}, T_{\max}+h], \\ \|I_3\|_{L^\infty(\mathbb{R}^N)} &\le \frac{\delta}{3}, \end{aligned}$$

$$\begin{aligned}\|I_4\|_{L^\infty(\mathbb{R}^N)} \le& \bigg\| \int_{T_{\max}}^t (t-\tau)^{\alpha-1} S_\alpha(T_{\max}-\tau)(|v|^{p-1}v(\tau) \\ &- |u_{T_{\max}}|^{p-1}u_{T_{\max}})\, d\tau \bigg\|_{L^\infty(\mathbb{R}^N)} \\ &+ \bigg\| \int_{T_{\max}}^t (t-\tau)^{\alpha-1} S_\alpha(T_{\max}-\tau)|u_{T_{\max}}|^{p-1}u_{T_{\max}}\, d\tau \bigg\|_{L^\infty(\mathbb{R}^N)} \\ \le& C\delta \int_{T_{\max}}^t (t-\tau)^{\alpha-1}\, d\tau + \|u_{T_{\max}}\|^p_{L^\infty(\mathbb{R}^N)} \frac{1}{\Gamma(\alpha)} \int_{T_{\max}}^t (t-\tau)^{\alpha-1}\, d\tau \\ =& \frac{C\delta}{\alpha}(t-T_{\max})^\alpha + \frac{\|u_{T_{\max}}\|^p_{L^\infty(\mathbb{R}^N)}}{\Gamma(\alpha+1)}(t-T_{\max})^\alpha \le \frac{\delta}{3}\end{aligned}$$

for $t \in [T_{\max}, T_{\max}+h]$. Then, we have $\|G(v)(t) - u_{T_{\max}}\|_{L^\infty(\mathbb{R}^N)} \le \delta$, $t \in [T_{\max}, T_{\max}+h]$.

Next, we will prove that G is contractive on $E_{h,\delta}$ for h small enough. In fact, for $w, v \in E_{h,\delta}$, $t \in [T_{\max}, T_{\max}+h]$,

$$\begin{aligned}&\|w(t) - v(t)\|_{L^\infty(\mathbb{R}^N)} \\ &\le \int_{T_{\max}}^t (t-\tau)^{\alpha-1} \|S_\alpha(t-\tau)(|w|^{p-1}w(\tau) - |v|^{p-1}v(\tau))\|_{L^\infty(\mathbb{R}^N)}\, d\tau \\ &\le \|w-v\|_{L^\infty((T_{\max},T_{\max}+h),L^\infty(\mathbb{R}^N))} (\|w\|_{L^\infty((T_{\max},T_{\max}+h),L^\infty(\mathbb{R}^N))} \\ &\quad + \|v\|_{L^\infty((T_{\max},T_{\max}+h),L^\infty(\mathbb{R}^N))})^{p-1} \frac{p}{\Gamma(\alpha)} \int_{T_{\max}}^t (t-\tau)^{\alpha-1}\, d\tau \\ &\le \frac{2^{p-1}p}{\Gamma(\alpha+1)} (\delta + \|u_{T_{\max}}\|_{L^\infty(\mathbb{R}^N)})^{p-1} (t-T_{\max})^\alpha\, d(w,v).\end{aligned}$$

Choosing h small enough such that

$$\frac{2^{p-1}p}{\Gamma(\alpha+1)} (\delta + \|u_T\|_{L^\infty(\mathbb{R}^N)})^{p-1} h^\alpha \le \frac{1}{2}.$$

Then, G is contractive on $E_{h,\delta}$. So, we know G has a fixed point $v \in E_{h,\delta}$. Since $v(T_{\max}) = G(v(T_{\max})) = u(T_{\max})$, if we let

$$\widetilde{u}(t) = \begin{cases} u(t), & t \in [0, T_{\max}), \\ v(t), & t \in [T_{\max}, T_{\max}+h], \end{cases}$$

then $\widetilde{u} \in C([0, T_{\max}+h], C_0(\mathbb{R}^N))$ and

$$\widetilde{u}(t) = P_\alpha(t)u_0 + \int_0^t (t-\tau)^{\alpha-1} S_\alpha(t-\tau) |\tilde{u}|^{p-1} \widetilde{u}(\tau)\, d\tau.$$

Therefore, $\widetilde{u}(t)$ is a mild solution of (1.1)–(1.2), which contradicts with the definition of $T_{\max}$.

If $u_0 \in L^r(\mathbb{R}^N)$ for some $1 \le r < \infty$, then repeating the above argument, we get the conclusion. Moreover, if $u_0 \ge 0$, then we can obtain the nonnegative solution of (1.1) applying the above argument in the set $E_T^+ = \{u \in E_T \mid u \geq 0\}$. Then, we know $u(t) \geq P_\alpha(t)u_0 > 0$ on $t \in (0, T_{\max})$. □

4. Blow-up and global existence

In this section, we prove the blow-up results and global existence of solutions of (1.1)–(1.2). First, we give the definition of weak solution of (1.1)–(1.2).

Definition 4.1. We call $u \in L^p((0,T), L^\infty_{\rm loc}(\mathbb{R}^N))$, for $u_0 \in L^\infty_{\rm loc}(\mathbb{R}^N)$ and $T > 0$, is a weak solution of (1.1) if

$$\int_{\mathbb{R}^N} \int_0^T (|u|^{p-1}u\varphi + u_0 \, {}_t^C D_T^\alpha \varphi) \, dt \, dx = \int_{\mathbb{R}^N} \int_0^T u(-\triangle\varphi) \, dt \, dx + \int_{\mathbb{R}^N} \int_0^T u \, {}_t^C D_T^\alpha \varphi \, dt \, dx$$

for every $\varphi \in C^{2,1}_{x,t}(\mathbb{R}^N \times [0,T])$ with $\operatorname{supp}_x \varphi \subset\subset \mathbb{R}^N$ and $\varphi(\,\cdot\,, T) = 0$.

Lemma 4.2. *Assume $u_0 \in C_0(\mathbb{R}^N)$, let $u \in C([0,T], C_0(\mathbb{R}^N))$ be a mild solution of* (1.1)–(1.2)*, then u is also a weak solution of* (1.1)–(1.2).

Proof. Assuming that $u \in C([0,T], C_0(\mathbb{R}^N))$ is a mild solution of (1.1)–(1.2), we have

$$u - u_0 = P_\alpha(t)u_0 - u_0 + \int_0^t (t-\tau)^{\alpha-1} S_\alpha(t-\tau)|u|^{p-1}u \, d\tau.$$

Note that by Lemma 2.4,

$${}_0I_t^{1-\alpha}\bigg(\int_0^t (t-\tau)^{\alpha-1} S_\alpha(t-\tau)|u|^{p-1}u(\tau) \, d\tau\bigg) = \int_0^t P_\alpha(t-s)|u|^{p-1}u(s) \, ds,$$

so, we know

$${}_0I_t^{1-\alpha}(u-u_0) = {}_0I_t^{1-\alpha}(P_\alpha(t)u_0 - u_0) + \int_0^t P_\alpha(t-\tau)|u|^{p-1}u(\tau) \, d\tau.$$

Then, for every $\varphi \in C^{2,1}_{x,t}(\mathbb{R}^N \times [0,T])$ with $\operatorname{supp}_x \varphi \subset\subset \mathbb{R}^N$ and $\varphi(x,T) = 0$, we get

$$\int_{\mathbb{R}^N} {}_0I_t^{1-\alpha}(u-u_0)\varphi \, dx = I_5(t) + I_6(t), \tag{4.1}$$

where

$$I_5(t) = \int_{\mathbb{R}^N} {}_0I_t^{1-\alpha}(P_\alpha(t)u_0 - u_0)\varphi \, dx, \quad I_6(t) = \int_{\mathbb{R}^N} \int_0^t P_\alpha(t-s)|u|^{p-1}u \, ds \, \varphi \, dx.$$

By Lemma 2.3,

$$\frac{dI_5}{dt} = \int_{\mathbb{R}^N} A(P_\alpha(t)u_0)\varphi \, dx + \int_{\mathbb{R}^N} {}_0I_t^{1-\alpha}(P_\alpha(t)u_0 - u_0)\varphi_t \, dx. \tag{4.2}$$

For every $h>0$, $t\in[0,T)$ and $t+h\le T$, we have

$$\frac{1}{h}(I_6(t+h)-I_6(t))=\frac{1}{h}\int_0^{t+h}\int_{\mathbb{R}^N}P_\alpha(t+h-s)|u|^{p-1}u\,ds\varphi(t+h,x)\,dx \\ -\frac{1}{h}\int_0^t\int_{\mathbb{R}^N}P_\alpha(t-s)|u|^{p-1}uds\varphi(t,x)\,dx=I_7+I_8+I_9,$$

where

$$I_7=\frac{1}{h}\int_{\mathbb{R}^N}\int_t^{t+h}\int_0^\infty\phi_\alpha(\theta)T((t+h-s)^\alpha\theta)|u|^{p-1}u(s)\,d\theta\,ds\,\varphi(t+h,x)\,dx,$$
$$I_8=\frac{1}{h}\int_{\mathbb{R}^N}\int_0^t\int_0^\infty\phi_\alpha(\theta)(T((t+h-s)^\alpha\theta)-T((t-s)^\alpha\theta))|u|^{p-1}u(s)\,d\theta\,ds\,\varphi(t,x)\,dx,$$
$$I_9=\frac{1}{h}\int_{\mathbb{R}^N}\int_0^t\int_0^\infty\phi_\alpha(\theta)T((t+h-s)^\alpha\theta)|u|^{p-1}u(s)\,d\theta\,ds(\varphi(t+h,x)-\varphi(t,x))\,dx.$$

By dominated convergence theorem, we conclude that

$$I_7\to\int_{\mathbb{R}^N}|u|^{p-1}u\varphi\,dx\quad\text{as }h\to0,$$
$$I_9\to\int_{\mathbb{R}^N}\int_0^t\int_0^\infty\phi_\alpha(\theta)T((t-s)^\alpha\theta)|u|^{p-1}u(s)\,d\theta\,ds\,\varphi_t\,dx \\ =\int_{\mathbb{R}^N}\int_0^t P_\alpha(t-s)|u|^{p-1}u(s)ds\varphi_t\,dx\quad\text{as }h\to0.$$

Since

$$I_8=\int_{\mathbb{R}^N}\int_0^t\int_0^\infty\int_0^1\alpha\theta\phi_\alpha(\theta)(t+\tau h-s)^{\alpha-1} \\ \cdot A(T((t+\tau h-s)^\alpha\theta))|u|^{p-1}u(s)\,d\tau\,d\theta\,ds\varphi\,dx \\ =\int_{\mathbb{R}^N}A\int_0^t\int_0^\infty\int_0^1\alpha\theta\phi_\alpha(\theta)(t+\tau h-s)^{\alpha-1} \\ \cdot T((t+\tau h-s)^\alpha\theta)|u|^{p-1}u(s)\,d\tau\,d\theta\,ds\,\varphi\,dx \\ =\int_{\mathbb{R}^N}\int_0^t\int_0^\infty\int_0^1\alpha\theta\phi_\alpha(\theta)(t+\tau h-s)^{\alpha-1} \\ \cdot T((t+\tau h-s)^\alpha\theta)|u|^{p-1}u(s)\,d\tau\,d\theta\,ds\,A\varphi\,dx,$$

by dominated convergence theorem, we know

$$I_8\to\int_{\mathbb{R}^N}\int_0^t(t-s)^{\alpha-1}S_\alpha(t-s)|u|^{p-1}u(s)\,ds\,A\varphi\,dx\quad\text{as }h\to0.$$

Hence, the right derivative of I_6 on $[0,T)$ is

$$\int_{\mathbb{R}^N}|u|^{p-1}u\varphi\,dx+\int_{\mathbb{R}^N}\int_0^t(t-s)^{\alpha-1}S_\alpha(t-s)|u|^{p-1}u(s)\,ds\,A\varphi\,dx \\ +\int_{\mathbb{R}^N}\int_0^t P_\alpha(t-s)|u|^{p-1}u(s)\,ds\,\varphi_t\,dx$$

and it is continuous in $[0,T)$. Therefore,

$$\begin{aligned}(4.3)\quad \frac{dI_6}{dt} &= \int_{\mathbb{R}^N} |u|^{p-1}u\varphi\,dx + \int_{\mathbb{R}^N}\int_0^t (t-s)^{\alpha-1}S_\alpha(t-s)|u|^{p-1}u(s)\,ds\,A\varphi\,dx \\ &\quad + \int_{\mathbb{R}^N}\int_0^t P_\alpha(t-s)|u|^{p-1}u(s)\,ds\,\varphi_t\,dx \\ &= \int_{\mathbb{R}^N} |u|^{p-1}u\varphi\,dx + \int_{\mathbb{R}^N}\int_0^t (t-s)^{\alpha-1}S_\alpha(t-s)|u|^{p-1}u(s)\,ds\,A\varphi\,dx \\ &\quad + \int_{\mathbb{R}^N} {}_0I_t^{1-\alpha}\left(\int_0^t (t-s)^{\alpha-1}S_\alpha(t-s)|u|^{p-1}u(s)\,ds\right)\varphi_t\,dx,\end{aligned}$$

for $t\in[0,T)$. It follows from (4.1)–(4.3) that

$$\begin{aligned}0 &= \int_0^T \frac{d}{dt}\int_{\mathbb{R}^N} {}_0I_t^{1-\alpha}(u-u_0)\varphi\,dx\,dt = \int_0^T \frac{dI_5}{dt} + \frac{dI_6}{dt}\,dt \\ &= \int_0^T\int_{\mathbb{R}^N} P_\alpha(t)u_0\triangle\varphi\,dx\,dt \\ &\quad - \int_0^T\int_{\mathbb{R}^N}(u-u_0){}_t^CD_T^\alpha\varphi\,dx\,dt + \int_0^T\int_{\mathbb{R}^N}|u|^{p-1}u\varphi\,dx\,dt \\ &\quad + \int_0^T\int_{\mathbb{R}^N}\int_0^t (t-s)^{\alpha-1}S_\alpha(t-s)|u|^{p-1}u\,ds\,\triangle\varphi\,dx\,dt \\ &= \int_0^T\int_{\mathbb{R}^N} u\triangle\varphi\,dx\,dt - \int_0^T\int_{\mathbb{R}^N}(u-u_0){}_t^CD_T^\alpha\varphi\,dx\,dt + \int_0^T\int_{\mathbb{R}^N}|u|^{p-1}u\varphi\,dx\,dt.\end{aligned}$$

Hence, we get the conclusion. □

We say the solution u of the problem (1.1)–(1.2) blows up in a finite time T if $\lim_{t\to T}\|u(t,\cdot)\|_{L^\infty(\mathbb{R}^N)} = +\infty$.

Now, we give a blow-up result of the problem (1.1)–(1.2).

THEOREM 4.3. *Let $u_0\in C_0(\mathbb{R}^N)$ and $u_0\ge 0$, if*

$$\int_{\mathbb{R}^N} u_0(x)\chi(x)dx > 1,$$

where

$$\chi(x) = \left(\int_{\mathbb{R}^N} e^{-\sqrt{N^2+|x|^2}}\,dx\right)^{-1} e^{-\sqrt{N^2+|x|^2}},$$

then the mild solutions of (1.1)–(1.2) *blow up in a finite time.*

PROOF. We take $\psi\in C_0^\infty(\mathbb{R})$ such that

$$\psi(x) = \begin{cases}1, & |x|\le 1,\\ 0, & |x|\ge 2,\end{cases}$$

and $0 \le \psi(x) \le 1$, $x \in \mathbb{R}$. Let $\psi_n(x) = \psi(x/n)$, $n = 1, 2, \ldots$ By Lemma 4.2, a mild solution of (1.1)–(1.2) is also a weak solution of it. So, using the definition of weak solution of (1.1)–(1.2), taking $\varphi_n(x,t) = \chi(x)\psi_n(x)\varphi_1(t)$ for $\varphi_1 \in C^1([0,T])$ with $\varphi_1(T) = 0$ and $\varphi_1 \ge 0$, we have

$$\int_{\mathbb{R}^N}\int_0^T u^p\varphi_n\,dx\,dt + \int_{\mathbb{R}^N}\int_0^T u_0 {}_t^C D_T^\alpha \varphi_n\,dx\,dt = \int_{\mathbb{R}^N}\int_0^T (-u\triangle\varphi_n + u {}_t^C D_T^\alpha\varphi_n)\,dx\,dt. \tag{4.4}$$

Since $\triangle(\chi\psi_n) = (\triangle\chi)\psi_n + 2\nabla\chi\cdot\nabla\psi_n + (\triangle\psi_n)\chi$ and $\triangle\chi \ge -\chi$, by (4.4) and the dominated convergence theorem, let $n \to \infty$, we have

$$\int_{\mathbb{R}^N}\int_0^T u^p\chi\varphi_1\,dx\,dt + \int_{\mathbb{R}^N}\int_0^T u_0\chi {}_t^C D_T^\alpha \varphi_1\,dx\,dt \le \int_{\mathbb{R}^N}\int_0^T (u\chi\varphi_1 + u\chi {}_t^C D_T^\alpha\varphi_1)\,dx\,dt. \tag{4.5}$$

Hence, by Jensen's inequality and (4.5), we have

$$\int_0^T \bigg(\int_{\mathbb{R}^N} u\chi\,dx\bigg)^p \varphi_1\,dt + \int_{\mathbb{R}^N}\int_0^T u_0\chi {}_t^C D_T^\alpha \varphi_1\,dx\,dt \le \int_{\mathbb{R}^N}\int_0^T (u\chi\varphi_1 + u\chi {}_t^C D_T^\alpha\varphi_1)\,dx\,dt.$$

So, if we denote $f(t) = \int_{\mathbb{R}^N} u\chi\,dx$, then

$$\int_0^T (f^p - f)\varphi_1\,dt \le \int_0^T (f - f(0)) {}_t^C D_T^\alpha\varphi_1\,dt. \tag{4.6}$$

We take $\varphi_1 = {}_tI_T^\alpha\widetilde{\psi}(t)$ where $\widetilde{\psi} \in C_0^1((0,T))$ and $\widetilde{\psi} \ge 0$, then (4.6) implies

$$\int_0^T {}_0I_t^\alpha(f^p - f)\widetilde{\psi}\,dt = \int_0^T (f^p - f) {}_tI_T^\alpha\widetilde{\psi}(t)\,dt \le \int_0^T (f - f(0))\widetilde{\psi}\,dt.$$

Hence,

$${}_0I_t^\alpha(f^p - f) + f(0) \le f. \tag{4.7}$$

In view of $f(0) = \int_{\mathbb{R}^N} u_0(x)\chi(x)\,dx > 1$ and the continuity of f, we obtain $f(t) > 1$ when t is small enough. Then (4.7) implies $f(t) \ge f(0) > 1$ for $t \in [0,T]$. Taking $\varphi_1(t) = (1 - t/T)^m$, $t \in [0,T]$ $m \ge \max\{1, p\alpha/(p-1)\}$, we know there exists constant $C > 0$ such that

$$\int_0^T (f^p - f)\varphi_1\,dt + Cf(0)T^{1-\alpha} \le \varepsilon\int_0^T f^p\varphi_1\,dt + C(\varepsilon)T^{1-p\alpha/(p-1)}.$$

Choosing ε small enough such that $f(0) > (1-\varepsilon)^{-1/(p-1)}$, we then have $f(0) \le CT^{\alpha - p\alpha/(p-1)}$ for some constant $C > 0$. If the solution of (1.1)–(1.2) exists globally, we get $f(0) = 0$ by taking $T \to \infty$, which contradicts with $f(0) > 1$. Hence, Theorem 3.2 guarantees that u blows up in a finite time. □

Next, we give the main result of this paper.

THEOREM 4.4. *Let $u_0 \in C_0(\mathbb{R}^N)$ and $u_0 \geq 0$, $u_0 \not\equiv 0$, then*

(a) *If $1 < p < 1 + 2/N$, then the mild solution of* (1.1)–(1.2) *blows up in a finite time.*

(b) *If $p \geq 1+2/N$ and $\|u_0\|_{L^{q_c}(\mathbb{R}^N)}$ is sufficiently small, where $q_c = N(p-1)/2$, then the solutions of* (1.1)–(1.2) *exist globally.*

PROOF. (a) Let $\Phi \in C_0^\infty(\mathbb{R})$ such that $\Phi(s) = 1$ for $|s| \leq 1$, $\Phi(s) = 0$ for $|s| > 2$ and $0 \leq \Phi(s) \leq 1$. For $T > 0$, we define

$$\varphi_1(x) = (\Phi(T^{-\alpha/2}|x|))^{2p/(p-1)}, \quad \varphi_2(t) = \left(1 - \frac{t}{T}\right)^m, \quad m \geq \max\left\{1, \frac{p\alpha}{p-1}\right\},$$

for $t \in [0, T]$. Assuming that u is a mild solution of (1.1), then, by Lemma 4.2, we have

$$\begin{aligned}(4.8) \quad \int_{\mathbb{R}^N} \int_0^T (u^p \varphi_1 \varphi_2 + u_0 \varphi_1 {}_t^C D_T^\alpha \varphi_2)\, dt\, dx \\ = \int_{\mathbb{R}^N} \int_0^T (u(-\triangle \varphi_1)\varphi_2 + u\varphi_1 {}_t^C D_T^\alpha \varphi_2)\, dt\, dx\end{aligned}$$

Note that

$$|(-\triangle \varphi_1)\varphi_2 + \varphi_1 {}_t^C D_T^\alpha \varphi_2| \leq C T^{-\alpha} \varphi_1^{1/p} \varphi_2^{1/p} \tag{4.9}$$

for some positive constant C independent of T. Then, by (4.8), (4.9) and Hölder inequality, we have

$$\begin{aligned}\int_{\mathbb{R}^N} \int_0^T (u^p \varphi_1 \varphi_2 + u_0 \varphi_1 {}_t^C D_T^\alpha \varphi_2)\, dt\, dx &\leq C T^{-\alpha} \int_{\mathbb{R}^N} \int_0^T u \varphi_1^{1/p} \varphi_2^{1/p}\, dt\, dx \\ &\leq C T^{-\alpha + (1+\alpha N/2)(p-1)/p} \left(\int_{\mathbb{R}^N} \int_0^T u^p \varphi_1 \varphi_2\, dt\, dx \right)^{1/p}.\end{aligned}$$

Hence

$$T^{1-\alpha} \int_{\mathbb{R}^N} u_0 \varphi_1\, dx \leq C T^{1+\alpha N/2 - p\alpha/(p-1)}.$$

It follows from $p < 1 + 2/N$ that $(N/2 + 1)\alpha - p\alpha/(p-1) < 0$. Therefore, if solution of (1.1)–(1.2) exists globally, then taking $T \to \infty$, we obtain

$$\int_{\mathbb{R}^N} u_0 \varphi_1\, dx = 0$$

and then $u_0 \equiv 0$. Hence, by Theorem 4.3, we know u blows up in a finite time.

(b) We construct the global solution of (1.1)–(1.2) by the contraction mapping principle.

Since $p \geq 1 + 2/N > 1 + 2\alpha/(\alpha N + 2 - 2\alpha)$, we know

$$\frac{\alpha N(p-1)}{2(p\alpha - p + 1)_+} > 1, \tag{4.10}$$

where $(p\alpha - p + 1)_+ = \max\{0, p\alpha - p + 1\}$.

In view of $p \geq 1 + 2/N > (4 - N + \sqrt{N^2 + 16})/4$, we have

$$\frac{N(p-1)}{2p(2-p)_+} > 1. \tag{4.11}$$

Hence, by (4.10), (4.11) and $(p-1)N/(2p) < (\alpha N(p-1))/(2(p\alpha - p + 1)_+)$, we can choose $q > p \geq 1 + 2/N$ such that

$$\frac{\alpha}{p-1} - \frac{1}{p} < \frac{\alpha N}{2q} < \frac{\alpha}{p-1} \tag{4.12}$$

and

$$\frac{\alpha}{p-1} - \alpha < \frac{\alpha N}{2q}. \tag{4.13}$$

Let

$$\beta = \frac{\alpha N}{2}\left(\frac{1}{q_c} - \frac{1}{q}\right) = \frac{\alpha}{p-1} - \frac{\alpha N}{2q}. \tag{4.14}$$

Using (4.12) and (4.14), one verifies that

$$0 < p\beta < 1, \qquad \alpha = \frac{\alpha N(p-1)}{2q} + (p-1)\beta. \tag{4.15}$$

Assume that the initial value u_0 satisfies

$$\sup_{t>0} t^{\beta}\|P_\alpha(t)u_0\|_{L^q(\mathbb{R}^N)} = \eta < +\infty. \tag{4.16}$$

Note that (4.13) implies $1/q_c - 1/q < 2/N$. If $u_0 \in L^{q_c}(\mathbb{R}^N)$, (2.8) implies (4.16) holds. If $u_0(x) \leq C|x|^{-2/(p-1)}$ for some constant $C > 0$, then $\|T(t)u_0\|_{L^q(\mathbb{R}^N)} \leq Ct^{N/(2q)-1/(p-1)}$. Hence,

$$\|P_\alpha(t)u_0\|_{L^q(\mathbb{R}^N)} \leq Ct^{\alpha(N/(2q)-1/(p-1))}\int_0^\infty \phi_\alpha(\theta)\theta^{N/(2q)-1/(p-1)}\,d\theta.$$

Since $N/(2q) - 1/(p-1) > -1$,

$$\int_0^\infty \phi_\alpha(\theta)\theta^{N/(2q)-1/(p-1)}\,d\theta < \infty.$$

Therefore, we also obtain that (4.16) is satisfied in this case.

Let $Y = \{u \in L^\infty((0,\infty), L^q(\mathbb{R}^N)) \mid \|u\|_Y < \infty\}$, where

$$\|u\|_Y = \sup_{t>0} t^{\beta}\|u(t)\|_{L^q(\mathbb{R}^N)}.$$

For $u \in Y$, we define

$$\Phi(u)(t) = P_\alpha(t)u_0 + \int_0^t (t-s)^{\alpha-1}S_\alpha(t-s)|u|^{p-1}u(s)ds.$$

Denote $B_M = \{u \in Y \mid \|u\|_Y \le M\}$. For any $u, v \in B_M$, $t \ge 0$,

$$t^\beta \|\Phi(u)(t) - \Phi(v)(t)\|_{L^q(\mathbb{R}^N)} \le t^\beta \int_0^t (t-s)^{\alpha-1} \|S_\alpha(t-s)(u^p(s) - v^p(s))\|_{L^q(\mathbb{R}^N)}\, ds. \tag{4.17}$$

Since $q > p > N(p-1)/4$, so $p/q - 1/q < 4/N$. Hence, Hölder inequality, Lemma 2.2, (4.15) and (4.17) imply that there exists constant $C > 0$ such that

$$\begin{aligned}
&t^\beta \|\Phi(u) - \Phi(v)\|_{L^q(\mathbb{R}^N)} \\
&\quad \le C t^\beta \int_0^t (t-s)^{\alpha-1-\alpha N(p/q-1/q)/2} \|u^p - v^p\|_{L^{\frac{q}{p}}(\mathbb{R}^N)}\, ds \\
&\quad \le C t^\beta \int_0^t (t-s)^{\alpha-1-\alpha N(p-1)/(2q)} (\|u\|^{p-1}_{L^q(\mathbb{R}^N)} + \|v\|^{p-1}_{L^q(\mathbb{R}^N)}) \|u-v\|_{L^q(\mathbb{R}^N)}\, ds \\
&\quad \le C t^\beta M^{p-1} \int_0^t (t-s)^{\alpha-1-\alpha N(p-1)/(2q)} s^{-p\beta}\, ds \|u-v\|_Y \\
&\quad = C M^{p-1} t^{\beta - p\beta - \alpha N(p-1)/(2q) + \alpha} \\
&\qquad \cdot \int_0^1 (1-\tau)^{-\alpha N(p-1)/(2q)+\alpha-1} \tau^{-p\beta}\, d\tau\, \|u-v\|_Y \\
&\quad = C M^{p-1} \int_0^1 (1-\tau)^{-\alpha N(p-1)/(2q)+\alpha-1} \tau^{-p\beta}\, d\tau\, \|u-v\|_Y \\
&\quad = C M^{p-1} \frac{\Gamma((p-1)\beta)\Gamma(1-p\beta)}{\Gamma(1-\beta)} \|u-v\|_Y.
\end{aligned}$$

If we choose M small enough such that

$$C M^{p-1} \frac{\Gamma((p-1)\beta)\Gamma(1-p\beta)}{\Gamma(1-\beta)} < \frac{1}{2},$$

then $\|\Phi(u) - \Phi(v)\|_Y \le \|u-v\|_Y/2$. Since

$$\begin{aligned}
t^\beta \|\Phi(u)(t)\|_{L^q(\mathbb{R}^N)} &\le \eta + C M^p t^\beta \int_0^t (t-s)^{-\alpha N(p/q-1/q)/2-1+\alpha} s^{-p\beta}\, ds \\
&\le \eta + C M^p \frac{\Gamma((p-1)\beta)\Gamma(1-p\beta)}{\Gamma(1-\beta)}, \quad t \in [0, +\infty),
\end{aligned}$$

we can choose η and M small enough such that

$$\eta + C M^p \frac{\Gamma((p-1)\beta)\Gamma(1-p\beta)}{\Gamma(1-\beta)} \le M.$$

Therefore, by contraction mapping principle we know Φ has a fixed point $u \in B_M$. Next, we will prove $u \in C([0,\infty), C_0(\mathbb{R}^N))$.

First, we prove that for $T > 0$ small enough, $u \in C([0,T], C_0(\mathbb{R}^N))$. In fact, the above proof shows that u is the unique solution in

$$B_{M,T} = \Big\{ u \in L^\infty((0,T), L^q(\mathbb{R}^N)) \;\Big|\; \sup_{0<t<T} t^\beta \|u(t)\|_{L^q(\mathbb{R}^N)} \le M \Big\}.$$

By Theorem 3.2 and $u_0 \in C_0(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$, we know that for T small enough, (1.1) has a unique solution $\widetilde{u} \in C([0,T], C_0(\mathbb{R}^N) \cap L^q(\mathbb{R}^N))$.

Hence, we can take T small enough such that $\sup_{0<t<T} t^\beta \|\widetilde{u}(t)\|_{L^q(\mathbb{R}^N)} \leq M$. Then, by uniqueness, we know $u \equiv \widetilde{u}$ for $t \in [0,T]$ and then $u \in C([0,T], C_0(\mathbb{R}^N)) \cap C([0,T], L^q(\mathbb{R}^N))$.

Next, we show that $u \in C([0,T], C_0(\mathbb{R}^N))$ by a bootstrap argument. For $t > T$, we have

$$
\begin{aligned}
u - P_\alpha(t)u_0 &= \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) u^p \, ds \\
&= \int_0^T (t-s)^{\alpha-1} S_\alpha(t-s) u^p \, ds + \int_T^t (t-s)^{\alpha-1} S_\alpha(t-s) u^p \, ds \\
&= I_{10} + I_{11}.
\end{aligned}
$$

It follows from $u \in C([0,T], C_0(\mathbb{R}^N))$ that

$$
I_{10} \in C([T,\infty), C_0(\mathbb{R}^N)) \cap C([T,\infty), L^q(\mathbb{R}^N)).
$$

For $T_1 > T$, we know $u^p \in L^\infty((T,T_1), L^{q/p}(\mathbb{R}^N))$. Note that $q > N(p-1)/2$, we can choose $r > q$ such that $N(p/q - 1/r)/2 < 1$. Then analogous to the proof of Lemma 2.4, we can show that $I_{11} \in C([T,T_1], L^r(\mathbb{R}^N))$. By the arbitrariness of T_1, we know $I_{11} \in C([T,\infty), L^r(\mathbb{R}^N))$ and so $u \in C([T,\infty), L^r(\mathbb{R}^N))$.

We take $r = q\chi^i$, $\chi > 1$ such that

$$
\frac{N}{2}\left(\frac{p}{q} - \frac{1}{q\chi^i}\right) < 1, \quad i = 1, 2, \ldots,
$$

then $u \in C([T,\infty), L^{q\chi^i}(\mathbb{R}^N))$. By finite steps, we have $p/(q\chi^i) < 2/N$, so $u \in C([0,\infty), C_0(\mathbb{R}^N))$. □

Remark 4.5. Theorems 4.3, 4.4 and (1.7) guarantee that $L^{q_c}(\mathbb{R}^N)$ is the only possible Lebesgue space where smallness of the initial value could imply global existence.

Remark 4.6. The Fujita exponent can also be obtained by equaling the time decay rate $N\alpha/2$ of $\|p_\alpha(t)u_0\|_{L^\infty(\mathbb{R}^N)}$ (see (2.8)) and the blow-up rate $\alpha/(p-1)$ of fractional ordinary differential equation ${}_0^C D_t^\alpha u = u^p$, $p > 1$, $u = u_0$ (see [6] for $\alpha = 1$).

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