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ON SOLUTIONS OF SEMILINEAR ELLIPTIC EQUATION WITH LINEAR GROWTH NONLINEARITY IN $\mathbb{R}^N$

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Abstract. We study nontrivial solutions for a class of semilinear elliptic equation which could be resonant at infinity. We establish the existence of solutions for the equation by considering the modified non-resonant problem associated with the original equation through Morse theory. Moreover, only linear growth assumption is imposed on the nonlinearity and condition on the potential is weaker than the coercive assumption.

1. Introduction

In the present paper, we are concerned with solutions of the following semilinear elliptic equation

$$-\Delta u + \alpha(x)u = f(x,u), \quad x \in \mathbb{R}^N \tag{1.1}$$

where the potential $\alpha(x) \in C(\mathbb{R}^N, \mathbb{R})$ satisfies $\alpha(x) > \inf_{\mathbb{R}^N} \alpha(x) > 0$. As shown in [3], [17], such a problem is motivated by the study of existence of standing wave solution for the nonlinear Schrödinger equation which arises in mathematical models from several physical phenomena, especially in nonlinear optics.

If we consider problem (1.1) in a bounded domain $\Omega \subset \mathbb{R}^N$, the compact condition will be assured by Sobolev Embedding Theorem for bounded domains. Such case was studied by many authors in many literatures and references cited

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there [16], [23], [12], [11], [18]. When the problem (1.1) is dealt with on entire $\mathbb{R}^N$, the main difficulty occurs for lack of compact condition, since $\mathbb{R}^N$ is unbounded. To overcome this difficulty, lots of assumptions were imposed on the potential $\alpha(x)$. Also the spectrum of the Schrödinger operator $\mathcal{L} = -\Delta + \alpha$ was studied in [1], [2], [7], [6], [14], [10], [21].

In [19], the author considered a similar equation to (1.1) and make a different condition assumption on the potential which insures the compactness. Some different assumptions on the potential are employed in [5], [8]. In [17], [5], $\alpha(x)$ is assumed coercive, i.e. $\alpha(x) \to \infty$ as $|x| \to \infty$. This condition was generalized in [3]. That is

$$\text{For every } M > 0,\ \operatorname{mes}(\{x \in \mathbb{R}^N \mid \alpha(x) \le M\}) < \infty, \tag{1.2}$$

where $\operatorname{mes}(\,\cdot\,)$ stands for the Lebesgue measure in $\mathbb{R}^N$. The above assumption was used in [8] as well. The nonlinearity f in [8] is asymptotically linear and satisfies some resonant conditions. In this paper, assumption on the potential $\alpha(x)$ is little different from [8] and we only need that f has linear growth. Precisely, we assume that

- (H_1) $\alpha(x) \in C(\mathbb{R}^N, \mathbb{R})$, $\inf\limits_{\mathbb{R}^N} \alpha(x) > 0$ and there exists $r > 0$ such that, for any $\gamma > 0$,
$$\lim_{|y|\to\infty} \operatorname{mes}(A_\gamma(y)) = 0$$
where $A_\gamma(y) = \{x \in \mathbb{R}^N \mid \alpha(x) < \gamma\} \cap B_r(y)$, and $B_r(y)$ is a ball centered at y with radius r.
- (H_2) $f(x,u) \in C^1(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ and there exists a constant $C > 0$ such that
$$|f(x,u)| \le C|u| \quad \text{for } x \in \mathbb{R}^N \text{ and } u \in \mathbb{R}.$$

Remark 1.1. Note that condition $\lim\limits_{|y|\to\infty} \operatorname{mes}(A_\gamma(y)) = 0$ in (H_1) is weaker than (1.2). (1.2) implies for any $\gamma > 0$, $\{x \in \mathbb{R}^N \mid \alpha(x) < \gamma\}$ is a bounded set. Therefore for $|y|$ large enough, $\{x \in \mathbb{R}^N \mid \alpha(x) < \gamma\} \cap B_r(y) = \emptyset$.

Remark 1.2. The condition (H_1) is easy to verify. Let us take an instance for $N = 1$. Take $\alpha(x) = e^{|x|}$. Then $\alpha(x) \in C(\mathbb{R}^1, \mathbb{R})$, $\inf\limits_{\mathbb{R}^1} \alpha(x) = 1 > 0$. Moreover, for $r = 1$ and any $\gamma > 0$, $\lim\limits_{|y|\to\infty} \operatorname{mes}(A_\gamma(y)) = 0$, since for $|y| > \ln\gamma + 1$, $\{x \in \mathbb{R}^1 \mid \alpha(x) < \gamma\} \cap B_r(y) = \emptyset$.

We would not impose any resonant or non-resonant condition on the nonlinearity at infinity. The main idea is to study the interaction between the nonlinearity at infinity and linear spectrum directly. This method was employed to study the problem (1.1) in a bounded domain $\Omega \subset \mathbb{R}^N$ and the potential vanishes in [13]. Therefore the results in this paper generalizes some results of [13].

We work in a subspace of $H^1(\mathbb{R}^N)$ defined by

$$E = \left\{ u \in H^1(\mathbb{R}^N) \;\Big|\; \int_{\mathbb{R}^N} (|\nabla u|^2 + \alpha(x)u^2)\,dx < \infty \right\}.$$

In order to state our main result, we first decompose E as follows. For some function $\beta(x) \in C(\mathbb{R}^N, \mathbb{R})$, we set

$$E = E^-(\beta) \oplus E^0(\beta) \oplus E^+(\beta),$$

where the operator $-\Delta - \beta$ is negatively definite on $E^-(\beta)$, positively definite on $E^+(\beta)$ and null on $E^0(\beta)$. Denote by $i(\beta)$ and $n(\beta)$ the dimension of $E^-(\beta)$ and $E^0(\beta)$, respectively.

Now the main result in this paper can be read as follows.

Theorem 1.3. *Assume that* (H_1), (H_2) *hold and* $n(f'_u(x,0)) = 0$. *Then* (1.1) *possesses at least one nontrivial solution, provided one of the following conditions holds*:

- (H_3) *There exists a constant* $\tau > 0$, *and some function* $\beta_\infty(x) \in C(\mathbb{R}^N, \mathbb{R})$ *such that* $f'_u(x,u) \le \beta_\infty(x) + \alpha(x)$ *for all* $x \in \mathbb{R}^N$ *and* u *with* $|u| \ge \tau$ *and* $i(\beta_\infty) \le i(f'_u(x,0)) - 2$.
- (H_4) *There exists a constant* $\tau > 0$, *and some function* $\beta_\infty(x) \in C(\mathbb{R}^N, \mathbb{R})$ *such that* $f'_u(x,u) \ge \beta_\infty(x) + \alpha(x)$ *for all* $x \in \mathbb{R}^N$ *and* u *with* $|u| \ge \tau$ *and* $i(\beta_\infty) \ge i(f'_u(x,0)) + 2$.

Remark 1.4. Resonant condition or non-resonant condition at infinity for the nonlinearity f are often supposed in the references. From Theorem 1.3, it is easy to see that we do not impose any resonant or non-resonant condition on the nonlinearity f at infinity.

2. Proof of the main result

It is well known that the space E equipped with the following inner product

$$\langle u, v\rangle = \int_{\mathbb{R}^N} (\nabla u \nabla v + \alpha(x)uv)\,dx$$

is a Hilbert space. Denote by $\|\cdot\|$ the associated norm, i.e.

$$\|u\|^2 = \int_{\mathbb{R}^N} (|\nabla u|^2 + \alpha(x)u^2)\,dx.$$

By (H_2) and [22], the variational functional

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + \alpha(x)u^2)\,dx - \int_{\mathbb{R}^N} F(x,u)\,dx$$

is well defined and of the class C^2, where $F(x,u) = \int_0^u f(x,s)\,ds$. Furthermore, the critical points of I are precisely the weak solutions of equation (1.1).

By the condition (1.2) and the Sobolev Embedding Theorem, the immersion $E \hookrightarrow L^p(\mathbb{R}^N)$ is compact for $2 \le p < 2^* = 2N/(N-2)$, which is proved in [3].

Modify the proof in [3], we can show the embedding $E \hookrightarrow L^p(\mathbb{R}^N)$ is also compact for $2 \le p < 2^*$ under the assumption (H_1).

Lemma 2.1. *Under the condition* (H_1)*, the embedding* $E \hookrightarrow L^p(\mathbb{R}^N)$ *is compact for* $2 \le p < 2^*$.

Proof. We first consider the case where $p = 2$. Let $\{u_n\} \subset E$ such that $\|u_n\| \le C_0$. Then up to a subsequence, we have u_n converges weakly to u in E as $n \to \infty$. We want to show that

$$u_n \to u \quad \text{in } L^2(\mathbb{R}^N), \quad \text{as } n \to \infty.$$

Define $v_n = u_n - u$. Then $v_n \rightharpoonup 0$ in E. We only need to prove $v_n \to 0$ in $L^2(\mathbb{R}^N)$. By the Sobolev Embedding Theorem for bounded domains, one has $v_n \to 0$ in $L^2(B_R)$. The remain is to estimate $\int_{B_R^c} v_n^2\,dx$. We first choose $\{y_j\} \subset \mathbb{R}^N$ such that $\mathbb{R}^N \subset \bigcup_{j=1}^{\infty} B_r(y_j)$ and each $x \in \mathbb{R}^N$ is covered by at most 2^N such balls. Therefore

$$\begin{aligned}\int_{B_R^c} v_n^2\,dx &\le \sum_{|y_j|>R-r}^{\infty} \int_{B_r(y_j)} v_n^2\,dx \\ &= \sum_{|y_j|>R-r}^{\infty} \int_{B_r(y_j)\cap\{x\in\mathbb{R}^N \mid \alpha(x)>\gamma\}} v_n^2\,dx + \int_{A_\gamma(y_j)} v_n^2\,dx.\end{aligned}$$

Then

$$\int_{B_r(y_j)\cap\{x\in\mathbb{R}^N \mid \alpha(x)>\gamma\}} v_n^2\,dx \le \frac{1}{\gamma}\int_{B_r(y_j)} \alpha(x) v_n^2\,dx.$$

and by Hölder inequality, one has

$$\begin{aligned}\int_{A_\gamma(y_j)} v_n^2\,dx &\le \bigg(\int_{A_\gamma(y_j)} v_n^{2N/(N-2)}\,dx\bigg)^{(N-2)/N} \bigg(\int_{A_\gamma(y_j)} 1\,dx\bigg)^{2/N} \\ &\le \|v_n\|^2_{L^{2^*}(B_r(y_j))} [\mathrm{mes}(A_\gamma(y_j))]^{2/N} \\ &\le \|v_n\|^2_{H^1(B_r(y_j))} \sup_{|y_j|\ge R-r} [\mathrm{mes}(A_\gamma(y_j))]^{2/N}.\end{aligned}$$

We have then

$$\begin{aligned}\int_{B_R^c} v_n^2\,dx \le& \sum_{|y_j|\ge R-r}^{\infty} \bigg[\frac{1}{\gamma}\int_{B_r(y_j)} \alpha(x) v_n^2\,dx \\ &+ C_1 \sup_{|y_j|\ge R-r} [\mathrm{mes}(A_\gamma(y_j))]^{2/N} \int_{B_r(y_j)} (|\nabla v_n|^2 + \alpha(x) v_n^2)\,dx\bigg] \\ \le& \frac{2^N}{\gamma}\int_{B_{R-2r}^c} \alpha(x) v_n^2\,dx\end{aligned}$$

$$+ 2^N C_1 \sup_{|y_j|\geq R-r} [\mathrm{mes}(A_\gamma(y_j))]^{2/N} \int_{B^c_{R-2r}} (|\nabla v_n|^2 + \alpha(x) v_n^2)\, dx$$
$$\leq \frac{2^N}{\gamma} C_0^2 + C_1 \sup_{|y_j|\geq R-r} [\mathrm{mes}(A_\gamma(y_j))]^{2/N} C_0^2.$$

For any $0 < \varepsilon \ll 1$, we chose γ such that $2^N C_0^2/\gamma < \varepsilon$. For such a fixed $\gamma > 0$, there exists $R > 0$ such that

$$C_1 \sup_{|y|\geq R-r} [\mathrm{mes}(A_\gamma(y_j))]^{2/N} C_0^2 < \varepsilon.$$

Since

$$\sup_{|y_j|\geq R-r} [\mathrm{mes}(A_\gamma(y_j))]^{2/N} \to 0 \quad \text{as } R \to \infty.$$

Thus for such $R > 0$

$$\int_{B^c_R} v_n^2\, dx \to 0 \quad \text{as } n \to \infty.$$

For $2 < p < 2^*$, by Hölder inequality and Sobolev inequality, one has

$$\|v_n\|_{L^p} \leq \|v_n\|\|^t_{L^p} \|v_n\|\|^{1-t}_{L^{2^*}} \leq C_1^{1-t} \|v_n\|\|^t_{L^2} \|v_n\|\|^{1-t} \leq (C_0 C_1)^{1-t} \|v_n\|\|^t_{L^2} \to 0$$

as $n \to \infty$, where $t \in (0,1)$ satisfies $1/p = t/2 + (1-t)/2^*$. Then the proof is complete. □

We recall that a sequence $\{u_n\} \subset E$ is said to be a (PS) sequence if $I(u_n)$ being bounded and $I'(u_n) \to 0$. And I satisfies (PS) condition if every (PS) sequence has a convergent subsequence. By Lemma 2.1, if we want to show I satisfies (PS) condition, we only need to verify that $\{u_n\} \subset E$ is bounded. For a non-resonant elliptic problem, this can be done by a standard argument.

Now, let us recall some definitions on Morse theory [4], [15] which will be used later. Let u be a critical point of I. The Morse index of u is defined as the supremum of the subspace of E on which $I''(u)$ is negative definite and is denoted by $\mu(u)$. The nullity of u is defined as dimension of $\ker I''(u)$ and is denoted by $\nu(u)$. For a constant $c \in \mathbb{R}$, the level set of I is defined as

$$I_c = \{u \in E \mid I(u) \leq c\}.$$

Let $H_q(A,B)$ be the q-th singular homology group of the pair (A,B). Then the Betti number B_q of the pair (A,B) is defined by $B_q = \mathrm{rank}(H_q(A,B))$. For $a, b \in \mathbb{R}$ with $a < b$, the Morse type number M_q of the pair (I_b, I_a) is defined by

$$M_q = \sum_{k=1}^{j} \dim C_q(I, u_k),$$

where $C_q(I,u_k)$ is the q-th critical group and $u_1, \ldots, u_j$ are critical points contained in $I^{-1}([a,b])$.

Lemma 2.2. *Assume that* (H_1) *and* (H_2) *hold. And there exists a constant* $\tau > 0$ *and some function* $\beta(x) \in C(\mathbb{R}^N, \mathbb{R})$ *such that* $f'_u(x,u) \le \beta(x) + \alpha(x)$ *for all* $x \in \mathbb{R}^N$ *and* u *with* $|u| \ge \tau$. *Then there exists a constant* $\kappa = \kappa(\tau, \beta, C)$ *such that for each solution* u *of* (1.1), $\|u\|_{L^\infty} \le \kappa$, *provided* $\mu(u) + \nu(u) \ge i(\beta) + n(\beta) + 1$.

Proof. If the result is not true, then for any n, there exists a function f_n and a u_n satisfying

$$-\Delta u_n + \alpha(x) u_n = f_n(x, u_n), \quad x \in \mathbb{R}^N \tag{2.1}$$

such that $\mu(u_n) + \nu(u_n) \ge i(\beta) + n(\beta) + 1$ and $\|u\|_{L^\infty} \ge n$. The corresponding functional is defined as

$$I_n(u_n) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u_n|^2 + \alpha(x) u_n^2)\, dx - \int_{\mathbb{R}^N} F_n(x, u_n)\, dx,$$

where $F_n(x,u) = \int_0^u f_n(x,s)\, ds$. Then by ($H_1$) and elliptic estimate, we have

$$\int_{\Omega \subset \mathbb{R}^N} |\nabla u_n|^2\, dx \to \infty \quad \text{as } n \to \infty.$$

Thus

$$\|u_n\| \ge \int_{\mathbb{R}^N} |\ \nabla u_n|^2\, dx \ge \int_{\Omega \subset \mathbb{R}^N} |\ \nabla u_n|^2\, dx \to \infty.$$

Write $w_n = u_n / \|u_n\|$. Then w_n is bounded with $\|w_n\| = 1$. By passing to a subsequence, we can assume that for some $w \in E$

$$\begin{aligned} w_n &\rightharpoonup w && \text{in } E, \\ w_n &\to w && \text{in } L^2(\mathbb{R}^N), \\ w_n(x) &\to w(x) && \text{a.e. } x \in \mathbb{R}^N. \end{aligned}$$

Then (2.1) multiplied by u_n and integrated in $\mathbb{R}^N$, one has by (H_2)

$$\|u_n\|^2 = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u_n|^2 + \alpha(x) u_n^2)\, dx = \int_{\mathbb{R}^N} f_n(x, u_n) u_n\, dx \le C \|u_n\|_{L^2}^2$$

which yields that $\|w\|_{L^2} \ge 1/\sqrt{C}$ for each n, since

$$\|w_n\|_{L^2}^2 = \frac{1}{\|u_n\|^2} \int_{\mathbb{R}^N} |u_n|^2 dx = \frac{1}{\|u_n\|^2} \|u_n\|_{L^2}^2 \ge \frac{\|u_n\|_{L^2}^2}{C \|u_n\|_{L^2}^2} = \frac{1}{C}.$$

By (H_2), for $u_n \ne 0$, $f(x, u_n)/u_n(x) \le C$. Thus

$$g_n(x) = \begin{cases} \dfrac{f_n(x, u_n)}{u_n} & \text{as } u_n \ne 0, \\ 0 & \text{as } u_n = 0, \end{cases}$$

is bounded in $L^\infty(\mathbb{R}^N)$. So by passing to a subsequence, we may assume

$$g_n(x) \to g(x) \quad \text{in } L^\infty(\mathbb{R}^N)$$

in weak* topology. Since u_n satisfies (2.1), we have for each $\varphi \in C_0^\infty(\mathbb{R}^N)$

$$\int_{\mathbb{R}^N} (\nabla u_n \nabla \varphi + \alpha(x) u_n \varphi)\, dx - \int_{\mathbb{R}^N} f_n(x, u_n)\varphi\, dx = 0.$$

Then, divided by $\|u_n\| \neq 0$, we have

$$\int_{\mathbb{R}^N} (\nabla w_n \nabla \varphi + \alpha(x) w_n \varphi)\, dx - \int_{\mathbb{R}^N} \frac{f_n(x, u_n)}{u_n} \frac{u_n}{\|u_n\|} \varphi\, dx = 0.$$

That is

$$\int_{\mathbb{R}^N} (\nabla w_n \nabla \varphi + \alpha(x) w_n \varphi)\, dx - \int_{\mathbb{R}^N} g_n(x) w_n \varphi\, dx = 0.$$

Let $n \to \infty$, one has

$$\int_{\mathbb{R}^N} (\nabla w \nabla \varphi + \alpha(x) w \varphi)\, dx - \int_{\mathbb{R}^N} g(x) w \varphi\, dx = 0.$$

It concludes that w solves the following linear equation

$$\begin{cases} -\Delta w + \alpha(x) w = g(x) w, \\ x \in \mathbb{R}^N. \end{cases}$$

By the unique continuation property in [9], we have $w(x) \neq 0$ almost everywhere in $\mathbb{R}^N$, since $w \neq 0$. This means $u_n(x) \to \infty$ almost everywhere in $\mathbb{R}^N$.

Next we prove that there exists n_0, for any $z \in E^+(\beta) \setminus \{0\}$

$$\langle I_n''(u_n) z, z \rangle > 0 \quad \text{as } n \geq n_0 \tag{2.2}$$

which means $\mu(u_n) + \nu(u_n) \leq i(\beta) + n(\beta)$. That is a contradiction. Now, if (2.2) is not true, then there exists $n_j \to \infty$ and $z_j \in E^+(\beta)$ with $\|z_j\| = 1$ such that $\langle I''_{n_j}(u_{n_j}) z_j, z_j \rangle \leq 0$, i.e.

$$\int_{\mathbb{R}^N} (|\nabla z_j|^2 + \alpha(x) z_j^2)\, dx - \int_{\mathbb{R}^N} f'_{n_j,u}(x, u_{n_j}(x)) z_j^2(x)\, dx \leq 0.$$

That is

$$\int_{\mathbb{R}^N} f'_{n_j,u}(x, u_{n_j}(x)) z_j^2(x)\, dx \geq \int_{\mathbb{R}^N} (|\nabla z_j|^2 + \alpha(x) z_j^2)\, dx = \|z_j\|^2 = 1.$$

Note that $\{z_j\}$ is bounded in E, by Lemma 2.1, we can assume that $z_j \to z$ in $L^2(\mathbb{R}^N)$. Then by Hölder inequality and Fatou's Lemma

$$\begin{aligned} &\limsup_{j\to\infty} \int_{\mathbb{R}^N} f'_{n_j,u}(x, u_{n_j}(x)) z_j^2(x)\, dx \\ &\quad = \limsup_{j\to\infty} \bigg[\int_{\mathbb{R}^N} f'_{n_j,u}(x, u_{n_j}(x))(z_j^2(x) - z^2(x))\, dx \\ &\qquad + \int_{\mathbb{R}^N} f'_{n_j,u}(x, u_{n_j}(x)) z^2(x)\, dx \bigg] \\ &\quad \leq 0 + \limsup_{j\to\infty} \int_{\mathbb{R}^N} f'_{n_j,u}(x, u_{n_j}(x)) z^2(x)\, dx \end{aligned}$$

$$\leq \int_{\mathbb{R}^N} \limsup_{j\to\infty} f'_{n_j,u}(x, u_{n_j}(x))z^2(x)\,dx$$
$$\leq \int_{\mathbb{R}^N} (\alpha(x)+\beta(x))z^2(x)\,dx < 1$$

since for $z \in E^+(\beta)$,

$$\int_{\mathbb{R}^N} ((-\Delta+\alpha(x)-\alpha(x)-\beta(x))z,z) = \int_{\mathbb{R}^N} ((-\Delta-\beta(x))z,z) > 0,$$

that is

$$\int_{\mathbb{R}^N} (\alpha(x)+\beta(x))z^2(x)\,dx < \int_{\mathbb{R}^N} ((-\Delta+\alpha(x))z,z)\,dx = \|z\|^2.$$

This contributes a contradiction and (2.2) is true. □

Lemma 2.3. *Assume* (H_1), (H_2) *and there exists a constant* $\tau > 0$ *and some function* $\beta(x) \in C(\mathbb{R}^N, \mathbb{R})$ *such that* $f'_u(x,u) \geq \beta(x)+\alpha(x)$ *for all* $x \in \mathbb{R}^N$ *and* u *with* $|u| \geq \tau$. *Then there is a constant* $\kappa = \kappa(\tau,\beta,C)$ *such that for each solution* u *of* (1.1), $\|u\|_{L^\infty} \leq \kappa$ *provided* $\mu(u) \leq i(\beta)-1$.

Proof. We also make a indirect argument. Assume for any n, there exists f_n as in Lemma 2.2 and u_n satisfies (2.1) such that $\mu(u_n) \leq i(\beta)-1$ and $\|u_n\|_{L^\infty} \geq n$. Then similar to the proof of Lemma 2.2, $u_n(x) \to \infty$, almost everywhere in $\mathbb{R}^N$. For each $z \in E^-(\beta)\setminus\{0\}$ and by Fatou's Lemma, we have

$$\begin{aligned}
&\limsup_{j\to\infty} \langle I''_n(u_n)z,z\rangle \\
&\quad= \int_{\mathbb{R}^N} (|\nabla z|^2+\alpha(x)z^2)\,dx + \limsup_{j\to\infty}\bigg(-\int_{\mathbb{R}^N} f'_{n,u}(x,u_n(x))z^2(x)\,dx\bigg) \\
&\quad= \int_{\mathbb{R}^N} (|\nabla z|^2+\alpha(x)z^2)\,dx - \liminf_{j\to\infty}\int_{\mathbb{R}^N} f'_{n,u}(x,u_n(x))z^2(x)\,dx \\
&\quad\leq \int_{\mathbb{R}^N} (|\nabla z|^2+\alpha(x)z^2)\,dx - \int_{\mathbb{R}^N} \liminf_{j\to\infty} f'_{n,u}(x,u_n(x))z^2(x)\,dx \\
&\quad\leq \int_{\mathbb{R}^N} (|\nabla z|^2+\alpha(x)z^2)\,dx - \int_{\mathbb{R}^N} (\alpha(x)+\beta(x))z^2\,dx \\
&\quad= \int_{\mathbb{R}^N} (|\nabla z|^2\,dx - \int_{\mathbb{R}^N} \beta(x)z^2\,dx < 0
\end{aligned}$$

Henceforth, for $n \geq n_0$, where n_0 large enough $\langle I''_n(u_n)z,z\rangle < 0$. Thus we have $\mu(u_n) \geq i(\beta)$ for $n \geq n_0$, a contradiction. □

Now we can give proof of Theorem 1.3.

Proof of Theorem 1.3. First we transform problem (1.1) into a non-resonant case at infinity. Then we study the non-resonant problem by Morse theory. Finally, combining Lemmas 2.2 and 2.3, we obtain solutions of problem (1.1). We consider the case where (H_3) holds at the first place.

Without loss of generality, we can assume that $n(\beta_\infty)=0$. Let $\{\xi_n\}$ be an increasing sequence satisfying $\xi_1>\tau$ and $\xi_n\to\infty$ as $n\to\infty$.

To modify problem (1.1), we define a function $f_n(x,u)$ as

$$f_n(x,u)=\int_0^u f'_{n,u}(x,s)\,ds$$

where

$$f'_{n,u}(x,u))=\begin{cases} f'_u(x,u), & |u|\le\xi_n,\\ \Big(2-\dfrac{u}{\xi_n}\Big)f'_u(x,u)+\Big(\dfrac{u}{\xi_n}-1\Big)\beta_\infty(x), & \xi_n<u<2\xi_n,\\ \Big(2+\dfrac{u}{\xi_n}\Big)f'_u(x,u)-\Big(\dfrac{u}{\xi_n}+1\Big)\beta_\infty(x), & -2\xi_n<u<-\xi_n,\\ \beta_\infty(x), & |u|\ge 2\xi_n.\end{cases}$$

Then $f_n(x,u)\in C(\mathbb{R}^N,\mathbb{R})$. Now consider the following equation

$$-\Delta u+\alpha(x)u=f_n(x,u),\quad x\in\mathbb{R}^N, \tag{2.3}$$

and the corresponding functional is

$$I_n(u)=\frac12\int_{\mathbb{R}^N}(|\nabla u|^2+\alpha(x)u^2)\,dx-\int_{\mathbb{R}^N}F_n(x,u))\,dx.$$

It is easy to see $u=0$ is a trivial critical point of I_n. Observe that $n(\beta_\infty)=0$, I_n satisfies (PS) condition by Lemma 2.1. Jointed by (H_3), I_n has nontrivial critical points. Let a, b with $a<b$ be numbers such that any critical point u of I_n satisfies $a<I_n(u)<b$. What we want to verify is that there exists a nontrivial critical point u_n of I_n whose Morse index satisfies

$$\mu(u_n)+\nu(u_n)\ge i(\beta_\infty)+1. \tag{2.4}$$

We prove this by an indirect method. First, if I_n has only a finite number of critical points, by (H_3), $i_0=i(f'_u(x,0))\ge i(\beta_\infty)+2$. By [4], [15], we have

$$B_q=\delta_{q,i(\beta_\infty)} \tag{2.5}$$

and

$$M_q=\begin{cases}1, & q=i_0,\\ 0, & q\in\{i(\beta_\infty)+1,i(\beta_\infty)+2,\dots\}\setminus\{i_0\}.\end{cases} \tag{2.6}$$

Morse inequality at the level (i_0+1)-th can be written as

$$M_{i_0+1}-M_{i_0}+\ldots+(-1)^{i_0+1}M_0\ge B_{i_0+1}-B_{i_0}+\ldots+(-1)^{i_0+1}B_0. \tag{2.7}$$

Also, by Morse inequality, we have

$$\begin{aligned} M_{i(\beta_\infty)}-M_{i(\beta_\infty)-1}+\ldots+(-1)^{i(\beta_\infty)}M_0&\\ \ge B_{i(\beta_\infty)}-B_{i(\beta_\infty)-1}+\ldots+(-1)^{i(\beta_\infty)}B_0,&\end{aligned} \tag{2.8}$$

$$(2.9)\quad M_{i(\beta_\infty)+1} - M_{i(\beta_\infty)} + \ldots + (-1)^{i(\beta_\infty)+1} M_0 \geq B_{i(\beta_\infty)+1} - B_{i(\beta_\infty)} + \ldots + (-1)^{i(\beta_\infty)+1} B_0.$$

Notice that $M_{i(\beta_\infty)+1} = B_{i(\beta_\infty)+1} = 0$. Thus (2.8) and (2.9) yield that

$$(2.10)\quad M_{i(\beta_\infty)} - M_{i(\beta_\infty)-1} + \ldots + (-1)^{i(\beta_\infty)} M_0 = B_{i(\beta_\infty)} - B_{i(\beta_\infty)-1} + \ldots + (-1)^{i(\beta_\infty)} B_0.$$

It follows from (2.5), (2.6) and (2.8)–(2.10) that (2.7) can be reduced to

$$M_{i_0+1} - M_{i_0} + \ldots + (-1)^{i_0 - i(\beta_\infty)-1} M_{i(\beta_\infty)+2} \geq B_{i_0+1} - B_{i_0} + \ldots + (-1)^{i_0 - i(\beta_\infty)-1} B_{i(\beta_\infty)+2}$$

That is $-1 \geq 0$, a contradiction.

Next, if I_n has infinitely many critical points. Let $\mathcal{K} = \{u \neq 0 \mid I_n'(u) = 0\}$. Then, by the Marino–Prodi argument in Section 3 of [20], for any $0 < \varepsilon, \eta \ll 1$, there exists a functional J such that

$$(2.11)\quad \|I_n - J\|_{C^2(H_0^1(\mathbb{R}^N))} < \varepsilon$$

$$(2.12)\quad I_n(u) = J(u), \quad u \in H_0^1(\mathbb{R}^N)) \setminus \mathcal{N}_{2\eta}(\mathcal{K})$$

$$(2.13)\quad I_n''(u) = J''(u), \quad u \in \mathcal{N}_\eta(\mathcal{K})$$

Moreover, J satisfies (PS) condition and has only a finite number of critical points, which are all non-degenerate and contained in $\mathcal{N}_\eta(\mathcal{K})$, where $\mathcal{N}_\eta(\mathcal{K})$ is the domain of $\mathcal{K}$ with radius η. Therefore the Morse index of any nontrivial critical point of J is at least $i(\beta_\infty)$ by (2.13). Since J satisfies (PS) condition, $\mathcal{K}$ is compact and we can also let η converge to zero. Thus if we use J, $a - \varepsilon$ and $b + \varepsilon$ in M_q and B_q instead of I_n, a and b respectively, then (2.5) and (2.6) still hold and a contradiction takes place by (2.11), (2.12) and the proof for I_n with finite number of critical points.

By above discussion, I_n has a nontrivial critical point u_n satisfying (2.4). Then by Lemma 2.2, there exists $\kappa > 0$ such that $\|u_n\|_{L^\infty} < \kappa$. Note that $|u_n| < \|u_n\|_{L^\infty} < \kappa$ and ξ_n defined above is an increasing sequence. Hence for some n, $\xi_n > \kappa$, which makes $f_{n,u}'(x,u) = f_u'(x,u)$ and u_n is a nontrivial critical point of I, i.e. u_n is a nontrivial solution for (1.1).

Most of the proof for assumption (H_4) is similar to the argument for assumption (H_3). Here we only give a skeleton of the proof. If the result is false, we can also get contradiction from Morse inequality. Therefore, there exists a nontrivial solution u_n for (2.3) satisfying assumptions in Lemma 2.3. Finally by Lemma 2.3, u_n must be a nontrivial solution for (1.1). □

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