Topological Methods in Nonlinear Analysis
Volume 46, No. 1, 2015, 29–43

ANALYTIC INVARIANT MANIFOLDS FOR NONAUTONOMOUS EQUATIONS

Luis Barreira — Claudia Valls

Abstract. We construct real analytic stable invariant manifolds for sufficiently small perturbations of a linear equation $v' = A(t)v$ admitting a nonuniform exponential dichotomy. As a byproduct of our approach we obtain an exponential control not only of the trajectories on the invariant manifolds, but also of all their derivatives.

1. Introduction

We establish the existence of real analytic stable invariant manifolds for the equation

$$v' = A(t)v + f(t, v), \tag{1.1}$$

assuming that the linear equation $v' = A(t)v$ admits a nonuniform exponential dichotomy, and that the perturbation f has an appropriate extension to the complex domain and is sufficiently small (essentially we require that it decays exponentially with a speed related to the nonuniformity of the exponential behavior).

Our work naturally belongs to the theory of nonuniformly hyperbolic dynamics, and in a certain sense this is the weakest possible setting in which one can construct (analytic) stable invariant manifolds. We refer to [1] for a detailed exposition of the theory. The classical notion of exponential dichotomy is very

2010 *Mathematics Subject Classification.* Primary: 37D10, 37D25.
Key words and phrases. Analytic invariant manifolds, nonuniform hyperbolicity.
Partially supported FCT/Portugal through UID/MAT/04459/2013.

stringent for the dynamics and it is of interest to look for more general types of hyperbolic behavior, that can be much more typical. This is precisely what happens with the notion of nonuniform exponential dichotomy. Stable invariant manifolds for nonuniformly hyperbolic trajectories were first constructed by Pesin in [6].

In the case of a nonuniformly hyperbolic holomorphic dynamics in $\mathbb{C}^n$, the existence of invariant stable complex manifolds was announced by Wu in [7] while referring to his doctoral thesis. In the case of polynomial automorphisms of $\mathbb{C}^2$ it was shown independently by Wu in [7] and Bedford, Lyubich and Smillie in [3] (both developing the approach of Pesin in the "classical" nonuniform hyperbolicity theory) that with respect to the unique measure μ of maximal entropy, the stable manifold of almost every point is conformally equivalent to the complex plane. The measure μ was introduced by Bedford, Smillie and Sibony (see [4]) and it was shown to be the unique measure of maximal entropy in [3]. The more general case of arbitrary holomorphic diffeomorphisms of a complex manifold was considered by Jonsson and Varolin in [5], where they showed that for each Lyapunov regular trajectory the stable manifolds are biholomorphic to a complex Euclidean space.

Here we consider instead the case of a real analytic dynamics, mimicking to the possible extent our work in [2] in the case of discrete time. To the best of our knowledge, it exists nowhere in the literature an analytic stable manifold theorem for nonautonomous differential equations in the nonuniformly hyperbolic setting. One could try to establish the existence of invariant manifolds using the results described above for holomorphic dynamics, but the fact that a given real analytic map may have singularities arbitrarily close the real Euclidean space in which it is defined prevents us to proceed in this manner, at least without further hypotheses or modifications.

2. Stable manifold theorem

2.1. Setup. Let $A(t)$ be $k \times k$ matrices varying continuously with t in some open neighbourhood of $\mathbb{R}_0^+$.

Given $s \geq 0$ and $v_s \in \mathbb{R}^k$, thesolution of the initial value problem

$$v' = A(t)v, \quad v(s) = v_s \tag{2.1}$$

is defined for all $t \geq 0$, and we it in the form $v(t) = T(t,s)v_s$, where $T(t,s)$ is the associated linear evolution operator. We say that equation (2.1) admits a (*strong*) *nonuniform exponential dichotomy* if there exist projections $P(t)$ varying continuously with $t \geq 0$, such that

$$P(t)T(t,s) = T(t,s)P(s) \quad \text{for every } t, s \geq 0,$$

and there exist constants

$$\underline{a} \le \overline{a} < 0 \le \underline{b} \le \overline{b}, \quad \varepsilon \ge 0 \quad \text{and} \quad D > 0$$

such that for every $t \ge s \ge 0$ we have

$$\|T(t,s)P(s)\| \le De^{\overline{a}(t-s)+\varepsilon s}, \qquad \|T(t,s)^{-1}P(t)\| \le De^{-\underline{a}(t-s)+\varepsilon t}, \tag{2.2}$$

and

$$\|T(t,s)Q(s)\| \le De^{\overline{b}(t-s)+\varepsilon s}, \qquad \|T(t,s)^{-1}Q(t)\| \le De^{-\underline{b}(t-s)+\varepsilon t}, \tag{2.3}$$

where $Q(t) = \mathrm{Id} - P(t)$ is the complementary projection of $P(t)$ for each $t \ge 0$. Then the stable and unstable subspaces at time t are defined by

$$E(t) = P(t)(\mathbb{R}^k) \quad \text{and} \quad F(t) = Q(t)(\mathbb{R}^k).$$

We also consider a continuous function $f\colon \mathbb{R}_0^+ \times \mathbb{R}^k \to \mathbb{R}^k$ such that $f(t,\,\cdot\,)\colon \mathbb{R}^k \to \mathbb{R}^k$ is analytic and $f(t,0) = 0$ for each $t \ge 0$. Given $s \ge 0$ and $v_s = (\xi, \eta) \in E(s) \times F(s)$ we denote by

$$(x(t), y(t)) = (x(t,s,v_s), y(t,s,v_s)) \in E(t) \times F(t)$$

the unique solution of equation (1.1) with $v(s) = v_s$, or equivalently of the system

$$x(t) = T(t,s)P(s)\xi + \int_s^t T(t,\tau)P(\tau)f(\tau, x(\tau), y(\tau))\,d\tau,$$

$$y(t) = T(t,s)Q(s)\eta + \int_s^t T(t,\tau)Q(\tau)f(\tau, x(\tau), y(\tau))\,d\tau.$$

The semiflow generated by equation (1.1) given by

$$\Psi_\tau(s,\xi,\eta) = (s+\tau, x(s+\tau,\xi,\eta), y(s+\tau,\xi,\eta)), \quad \tau \ge 0. \tag{2.4}$$

2.2. Lyapunov norms. Due to the nonuniform exponential behavior in (2.2) and (2.3), we introduce appropriate Lyapunov norms with respect to which the exponential behavior becomes uniform.

Let us fix $\varrho > 0$ and $\zeta \in (0, -\overline{a}/2)$ such that

$$\overline{a} - \underline{b} + 2\zeta < 0 \quad \text{and} \quad \underline{b} - \overline{b} + \zeta \ne 0.$$

Given $t > \varrho$ and $(u,v) \in E(t) \times F(t)$ we define new norms by

$$\begin{aligned} \|u\|_t' &= \int_t^\infty \|T(\sigma,t)P(t)u\| e^{-(\overline{a}+\zeta)(\sigma-t)}\,d\sigma, \\ \|v\|_t' &= \int_{t-\varrho}^t \|T(t,\sigma)^{-1}Q(t)v\| e^{-(\underline{b}+\zeta)(t-\sigma)}\,d\sigma, \end{aligned} \tag{2.5}$$

and

$$\|(u,v)\|_t' = \max\{\|u\|_t', \|v\|_t'\}.$$

By (2.2) the first integral in (2.5) is finite.

Lemma 2.1. *There exists a constant $C \ge 1$ such that for every $t > \varrho$ and $(u,v) \in E(t) \times F(t)$ we have*

$$C^{-1}e^{-\varepsilon t}\|u\| \le \|u\|'_t \le Ce^{\varepsilon t}\|u\|, \qquad C^{-1}e^{-\varepsilon t}\|v\| \le \|v\|'_t \le Ce^{\varepsilon t}\|v\|. \tag{2.6}$$

Proof. By (2.5) we have

$$\begin{aligned}\|u\|'_t &\le \int_t^\infty De^{\overline{a}(\sigma-t)+\varepsilon t}\|u\|e^{-(\overline{a}+\zeta)(\sigma-t)}\,d\sigma\\ &= De^{\varepsilon t}\|u\|\int_t^\infty e^{-\zeta(\sigma-t)}\,d\sigma = \frac{D}{\zeta}e^{\varepsilon t}\|u\|.\end{aligned}$$

For the inequality in the left we note that

$$\begin{aligned}\|u\|'_t &\ge \int_t^\infty \|T(\sigma,t)^{-1}P(\sigma)\|^{-1}\|u\|e^{-(\overline{a}+\zeta)(\sigma-t)}\,d\sigma\\ &\ge \int_t^\infty D^{-1}e^{\underline{a}(\sigma-t)-\varepsilon\sigma}\|u\|e^{-(\overline{a}+\zeta)(\sigma-t)}\,d\sigma\\ &= D^{-1}\|u\|e^{-(\underline{a}-\overline{a}-\zeta)t}\int_t^\infty e^{(\underline{a}-\overline{a}-\zeta-\varepsilon)\sigma}\,d\sigma = \frac{1}{D|\underline{a}-\overline{a}-\zeta-\varepsilon|}e^{-\varepsilon t}\|u\|\end{aligned}$$

(notice that $\underline{a}-\overline{a}-\zeta-\varepsilon < 0$). In a similar manner, we have

$$\begin{aligned}\|v\|'_t &\le \int_{t-\varrho}^t De^{-\underline{b}(t-\sigma)+\varepsilon t}\|v\|e^{-(\underline{b}+\zeta)(t-\sigma)}\,d\sigma\\ &= De^{\varepsilon t}\|v\|\int_{t-\varrho}^t e^{-\zeta(t-\sigma)}\,d\sigma \le \frac{D}{\zeta}(1-e^{-\zeta\varrho})e^{\varepsilon t}\|v\|,\end{aligned}$$

and for the last inequality we note that

$$\begin{aligned}\|v\|'_t &\ge \int_{t-\varrho}^t \|T(t,\sigma)^{-1}Q(t)\|^{-1}\|v\|e^{-(\underline{b}+\zeta)(t-\sigma)}\,d\sigma\\ &\ge \int_{t-\varrho}^t D^{-1}e^{-\overline{b}(\sigma-t)-\varepsilon t}\|v\|e^{-(\underline{b}+\zeta)(t-\sigma)}\,d\sigma\\ &= D^{-1}\|v\|e^{-(\underline{b}-\overline{b}+\zeta+\varepsilon)t}\int_{t-\varrho}^t e^{(\underline{b}-\overline{b}+\zeta)\sigma}\,d\sigma\\ &= \frac{1}{D(\underline{b}-\overline{b}+\zeta)}(1-e^{-(\underline{b}-\overline{b}+\zeta)\varrho})e^{-\varepsilon t}\|v\|.\end{aligned}$$

This completes the proof of the lemma. □

2.3. Stable manifold theorem. Given a subspace $F \subset \mathbb{R}^k$, we write

$$B_t(F) = \{x \in F : \|x\|'_t \le 1\}, \qquad \Delta_t(F) = \{z \in \widetilde{F} : \|z\|'_t \le 1\}, \tag{2.7}$$

where $\widetilde{F}$ is the complexification of F. Now we consider the space $\mathcal{H}$ of all continuous functions $f\colon \mathbb{R}_0^+ \times \mathbb{R}^k \to \mathbb{R}^k$ with

$$f(t,0) = 0 \quad \text{and} \quad (\partial f/\partial v)(t,0) = 0 \quad \text{for } t \in \mathbb{R}_0^+,$$

such that $f(t,\cdot)|B(\mathbb{R}^k)$ has a holomorphic extension $\widetilde{f}(t,\cdot)$ to the interior of the polydisk

$$\Delta_t(\mathbb{R}^k) = \{(z_1,\ldots,z_k) \in \mathbb{C}^k : \|z_i\|'_t \le 1 \text{ for } i = 1,\ldots,k\}$$

which is continuous on $\Delta_t(\mathbb{R}^k)$. We assume that there is a constant $\delta \in (0,1)$ such that

$$\sup\left\{\frac{\|\widetilde{f}(t,u)-\widetilde{f}(t,v)\|'_t}{\|u-v\|'_t} : u,v \in \Delta_t(\mathbb{R}^k) \text{ with } u \ne v\right\} \le \delta e^{-2\varepsilon t} \tag{2.8}$$

for $t \in \mathbb{R}^+_0$. Let also $\mathcal{X}$ be the space of all continuous functions

$$\varphi\colon \{(t,\xi) \in (\varrho,+\infty)\times\mathbb{R}^k : \xi \in B_t(E(t))\} \to \mathbb{R}^k$$

such that:

(1) $\varphi(t, B_t(E(t))) \subset F(t)$, with each map $\varphi(t,\cdot)|B_t(E(t))$ having a holomorphic extension $\widetilde{\varphi}(t,\cdot)$ to the interior of $\Delta_t(E(t))$ (see (2.7)) which is continuous on $\Delta_t(E(t))$;

(2) for every $t > \varrho$ we have $\varphi(t,0) = 0$, $(\partial\varphi/\partial v)(t,0) = 0$, and

$$\sup\left\{\frac{\|\widetilde{\varphi}(t,\xi)-\widetilde{\varphi}(t,\bar{\xi})\|'_t}{\|\xi-\bar{\xi}\|'_t} : \xi,\bar{\xi} \in \Delta_t(E(t)) \text{ with } \xi \ne \bar{\xi}\right\} \le 1. \tag{2.9}$$

Setting $\bar{\xi} = 0$ in (2.9), we obtain $\|\widetilde{\varphi}(t,\xi)\|'_t \le \|\xi\|'_t$ and hence,

$$\varphi(t, B_t(E(t))) \subset B_t(F(t)) \quad \text{and} \quad \widetilde{\varphi}(t,\Delta_t(E(t))) \subset \Delta_t(F(t)). \tag{2.10}$$

In particular this implies that

$$(\xi, \widetilde{\varphi}(t,\xi)) \in \Delta_t(\mathbb{R}^k) \quad \text{for every } \xi \in \Delta_t(E(t)).$$

Given a function $\varphi \in \mathcal{X}$, for each $t > \varrho$ we consider the graph

$$\mathcal{V}_t = \{(\xi,\varphi(t,\xi)) : \xi \in B_t(E(t))\} \subset \mathbb{R}^k.$$

The following is our stable manifold theorem.

Theorem 2.2. *Assume that equation* (2.1) *admits a nonuniform exponential dichotomy and take* $f \in \mathcal{H}$. *Provided that* δ *in* (2.8) *is sufficiently small, there exists a unique* $\varphi \in \mathcal{X}$ *such that*

$$\Psi_\tau(\mathcal{V}_s) \subset \mathcal{V}_{\tau+s} \quad \textit{for every } \tau \ge 0,\ s > \varrho. \tag{2.11}$$

Moreover:

(a) $\mathcal{V}_t$ *is an analytic manifold*, $0 \in \mathcal{V}_t$, *and* $T_0\mathcal{V}_t = E(t)$ *for every* $t \in \mathbb{R}^+_0$;

(b) *for every sufficiently small* $\alpha > 0$, *there exists* $K > 0$ *such that*

$$\|\Psi_\tau(s,\xi,\varphi(s,\xi)) - \Psi_\tau(s,\bar{\xi},\varphi(s,\bar{\xi}))\| \le K e^{(\bar{a}+\alpha+\varepsilon)(t-s)+2\varepsilon s}\|\xi-\bar{\xi}\|$$

for every $t \ge s > \varrho$ *and* ξ, $\bar{\xi} \in B_s(E(s))$.

3. Proof of Theorem 2.2

3.1. Function spaces. In view of the desired property (2.11), given $s > \varrho$ and $\xi \in B_s(E(s))$, and setting $v(s) = (\xi, \varphi(s,\xi)) \in \mathcal{V}_s$, we must have

$$\Psi_{t-s}(s,\xi,\varphi(s,\xi)) = (x(t,\xi), \varphi(t, x(t,\xi))), \quad t \geq s$$

for some $x(t,\xi) \in E(t)$. Therefore, equation (1.1) can be written in the form

$$\begin{aligned} x(t,\xi) &= T(t,s)P(s)\xi + \int_s^t T(t,\tau)P(\tau)f(\tau, x(\tau,\xi), \varphi(\tau, x(\tau,\xi)))\,d\tau \\ \varphi(t, x(t,\xi)) &= T(t,s)Q(s)\varphi(s,\xi) \\ &\quad + \int_s^t T(t,\tau)Q(\tau)f(\tau, x(\tau,\xi), \varphi(\tau, x(\tau,\xi)))\,d\tau \end{aligned} \tag{3.1}$$

for $t \geq s$. Now we equip the space $\mathcal{X}$ with the norm

$$|||\varphi||| = \sup\{\|\widetilde{\varphi}(t,x)\|'_t/\|x\|'_t : t > \varrho \text{ and } x \in \Delta_t(E(t)) \setminus \{0\}\}. \tag{3.2}$$

One can easily verify that $\mathcal{X}$ is a complete metric space with this norm.

For a fixed $s > \varrho$, given $t \geq s$ we set

$$\rho(t,s) = (\overline{a} + 2\zeta)(t-s). \tag{3.3}$$

Let also $\mathcal{B}$ be the space of all continuous functions

$$x\colon \{(t,\xi) \in [s,+\infty) \times \mathbb{R}^k : \xi \in B_s(E(s))\} \to \mathbb{R}^k$$

such that:

1. $x(t, B_s(E(s))) \subset E(t)$ for each $t \geq s$, which each map $x(t,\cdot)|B_s(E(s))$ having a holomorphic extension $\widetilde{x}(t,\cdot)$ to the interior of $\Delta_s(E(s))$ which is continuous on $\Delta_s(E(s))$;
2. $x(s,\xi) = \xi$ for each $\xi \in B_s(E(s))$, and

$$\|x\|' := \sup\left\{\frac{\|\widetilde{x}(t,\xi)\|'_t e^{-\rho(t,s)}}{\|\xi\|'_s} : \ t \geq s \text{ and } \xi \in \Delta_s(E(s)) \setminus \{0\}\right\} \leq 1. \tag{3.4}$$

It follows from (3.4) and (3.3) that

$$x(t, B_s(E(s))) \subset B_t(E(t)), \qquad \widetilde{x}(t, \Delta_s(E(s))) \subset \Delta_t(E(t)). \tag{3.5}$$

One can also verify that $\mathcal{B}$ is a complete metric space with the norm in (3.4).

3.2. Solution on the stable direction. Now we establish the existence of solutions of the first equation in (3.1) in the space $\mathcal{B}$.

Lemma 3.1. *Provided that δ is sufficiently small, for each $\varphi \in \mathcal{X}$ and $s > \rho$ there is a unique $x_\varphi \in \mathcal{B}$ satisfying the first equation in* (3.1) *for every $t \geq s$.*

PROOF. Given $x \in \mathcal{B}$, we define

$$(Jx)(t,\xi) = T(t,s)P(s)\xi + \int_s^t T(t,\tau)P(\tau)f(\tau, x(\tau,\xi), \varphi(\tau, x(\tau,\xi)))\,d\tau$$

for each $t \geq s$ and $\xi \in B_s(E(s))$. When $\xi \in \Delta_s(E(s))$ we have

$$(\widetilde{x}(\tau,\xi), \widetilde{\varphi}(\tau, \widetilde{x}(\tau,\xi))) \in \Delta_\tau(\mathbb{R}^k)$$

(see (2.10) and (3.5)). Thus, each map $(Jx)(t,\cdot)$ admits a holomorphic extension to the interior of $\Delta_s(E(s))$, which we continue to denote by $(Jx)(t,\cdot)$, and it is given by

$$(Jx)(t,\xi) = T(t,s)P(s)\xi + \int_s^t T(t,\tau)P(\tau)\widetilde{f}(\tau, \widetilde{x}(\tau,\xi), \widetilde{\varphi}(\tau, \widetilde{x}(\tau,\xi)))\,d\tau.$$

Furthermore, $(Jx)(s,\xi) = \xi$ and $(Jx)(t,\cdot)$ is continuous on $\Delta_s(E(s))$.

Now we show that $\|(Jx)(t,\cdot)\|'_t \leq 1$ for each $t \geq s$ (when $t = s$ this is immediate from the definitions). Setting

$$g^*(\tau,\xi) := P(\tau)\widetilde{f}(\tau, \widetilde{x}(\tau,\xi), \widetilde{\varphi}(\tau, \widetilde{x}(\tau,\xi))), \tag{3.6}$$

and using (2.6), (2.8), (2.9), and (3.4), we obtain

$$\begin{aligned}\|g^*(\tau,\xi)\| &\leq Ce^{\varepsilon\tau}\|g^*(\tau,s)\|'_\tau \\ &\leq C\delta e^{-\varepsilon\tau}\max\{\|\widetilde{x}(\tau,\xi)\|'_\tau, \|\widetilde{\varphi}(\tau,\widetilde{x}(\tau,\xi))\|'_\tau\} \\ &\leq C\delta e^{-\varepsilon\tau}\|\widetilde{x}(\tau,\xi)\|'_\tau \leq C\delta e^{-\varepsilon\tau}e^{\rho(\tau,s)}\|\xi\|'_s.\end{aligned} \tag{3.7}$$

By (2.5) and (2.2), since $T(\sigma,t)T(t,s) = T(\sigma,s)$ for each $\sigma \geq t \geq s$, we obtain

$$\begin{aligned}\|(Jx)(t,\xi)\|'_t &= \int_t^\infty \|T(\sigma,t)(Jx)(t,\xi)\|e^{-(\overline{a}+\zeta)(\sigma-t)}\,d\sigma \\ &\leq \int_t^\infty \|T(\sigma,t)T(t,s)P(s)\xi\|e^{-(\overline{a}+\zeta)(\sigma-t)}\,d\sigma \\ &\quad + \int_s^t\int_t^\infty \|T(\sigma,t)T(t,\tau)P(\tau)g^*(\tau,\xi)\|e^{-(\overline{a}+\zeta)(\sigma-t)}\,d\sigma\,d\tau \\ &\leq e^{(\overline{a}+\zeta)(t-s)}\int_t^\infty \|T(\sigma,s)P(s)\xi\|e^{-(\overline{a}+\zeta)(\sigma-s)}\,d\sigma \\ &\quad + \int_s^t\int_t^\infty \|T(\sigma,\tau)P(\tau)\|\cdot\|g^*(\tau,\xi)\|e^{-(\overline{a}+\zeta)(\sigma-t)}\,d\sigma\,d\tau \\ &\leq e^{(\overline{a}+\zeta)(t-s)}\|\xi\|'_s + C\delta D\|\xi\|'_s\int_s^t\int_t^\infty e^{\rho(\tau,s)}e^{\overline{a}(t-\tau)}e^{-\zeta(\sigma-t)}\,d\sigma\,d\tau \\ &\leq e^{(\overline{a}+\zeta)(t-s)}\|\xi\|'_s + C\delta D\|\xi\|'_s e^{\overline{a}(t-s)}\int_s^t\int_t^\infty e^{2\zeta(\tau-s)}e^{-\zeta(\sigma-t)}\,d\sigma\,d\tau \\ &\leq e^{(\overline{a}+\zeta)(t-s)}\|\xi\|'_s + C\delta D\|\xi\|'_s e^{\rho(t,s)}\int_s^t\int_t^\infty e^{2\zeta(\tau-t)}e^{-\zeta(\sigma-t)}\,d\sigma\,d\tau \\ &\leq e^{(\overline{a}+\zeta)(t-s)}\|\xi\|'_s + C\delta D\|\xi\|'_s e^{\rho(t,s)}\int_s^t\int_t^\infty e^{\zeta(\tau-\sigma)}e^{\zeta(\tau-t)}\,d\sigma\,d\tau\end{aligned} \tag{3.8}$$

$$\begin{aligned}
&\le e^{(\overline{a}+\zeta)(t-s)}\|\xi\|'_s + \frac{C\delta D}{\zeta}\|\xi\|'_s e^{\rho(t,s)}\int_s^t e^{\zeta(\tau-t)}\,d\tau \\
&\le e^{(\overline{a}+\zeta)(t-s)}\|\xi\|'_s + \frac{C\delta D}{\zeta^2}(1-e^{-\zeta(t-s)})\|\xi\|'_s e^{\rho(t,s)} \\
&= \biggl(e^{-\zeta(t-s)} + \frac{C\delta D}{\zeta^2}(1-e^{-\zeta(t-s)})\biggr)e^{\rho(t,s)}\|\xi\|'_s.
\end{aligned}$$

Setting

$$F(r) = e^{-\zeta r} + \frac{C\delta D}{\zeta^2}(1-e^{-\zeta r})$$

we obtain

$$F'(r) = \biggl(-\zeta + \frac{C\delta D}{\zeta}e^{-\zeta r}\biggr)e^{-\zeta r} < 0$$

for δ sufficiently small. Since $F(0)=1$, for $t-s>0$ we have $F(t-s)<1$, i.e.

$$\|(Jx)(t,\xi)\|'_t \le e^{\rho(t,s)}\|\xi\|'_s \quad \text{and} \quad \|Jx\|' \le 1.$$

Hence, $Jx \in \mathcal{B}$, and we obtain a well-defined operator $J\colon \mathcal{B} \to \mathcal{B}$.

Now we show that J is a contraction. Given $x, y \in \mathcal{B}$ and $\tau \ge s$, proceeding in a similar manner to that in (3.7), and setting

$$L(\tau) = P(\tau)[\widetilde{f}(\tau, \widetilde{x}(\tau,\xi), \widetilde{\varphi}(\tau, \widetilde{x}(\tau,\xi))) - \widetilde{f}(\tau, \widetilde{y}(\tau,\xi), \widetilde{\varphi}(\tau, \widetilde{y}(\tau,\xi)))],$$

we obtain $\|L(\tau)\| \le C\delta e^{-\varepsilon\tau}e^{\rho(\tau,s)}\|\xi\|'_s\|x-y\|'$, and hence,

$$\begin{aligned}
\|(Jx)(t,\xi) - (Jy)(t,\xi)\|'_t &\le \int_s^t\int_t^\infty \|T(\sigma,\tau)P(\tau)\|\cdot\|L(\tau)\|e^{-(\overline{a}+\zeta)(\sigma-t)}\,d\sigma\,d\tau \\
&\le \frac{C\delta D}{\zeta^2}(1-e^{-\zeta(t-s)})\|\xi\|'_s e^{\rho(t,s)}\|x-y\|' \le \frac{C\delta D}{\zeta^2}\|\xi\|'_s e^{\rho(t,s)}\|x-y\|'.
\end{aligned}$$

Therefore,

$$\|Jx - Jy\|' \le \frac{C\delta D}{\zeta^2}\|x-y\|'.$$

Provided that δ is sufficiently small, we have $C\delta D/\zeta^2 < 1$ and hence J is a contraction. Thus, there exists a unique $x = x_\varphi \in \mathcal{B}$ such that $Jx = x$. This completes the proof of the lemma. □

We note that in view of (3.4) each function $x_{t,\varphi}$ in Lemma 3.1 satisfies

$$\|x_\varphi(t,\xi)\|'_t \le e^{\rho(t,s)}\|\xi\|'_s \quad \text{for} \quad t \ge s. \tag{3.9}$$

3.3. Auxiliary bounds.

Lemma 3.2. *Provided that δ is sufficiently small, for each $\varphi \in \mathcal{X}$, $s > \varrho$, and $\xi, \overline{\xi} \in \Delta_s(E(s))$ we have*

$$\|\widetilde{x}_\varphi(t,\xi) - \widetilde{x}_\varphi(t,\overline{\xi})\|'_t \le 2\|\xi-\overline{\xi}\|'_s e^{\rho(t,s)}, \quad t \ge s. \tag{3.10}$$

Proof. Take $\tau \geq s$. Using the notation in (3.6) and proceeding as in (3.7), we obtain

$$\|g^*(\tau,\xi) - g^*(\tau,\overline{\xi})\| \leq C\delta e^{-\varepsilon\tau}\|\widetilde{x}(\tau,\xi) - \widetilde{x}(\tau,\overline{\xi})\|'_\tau.$$

Set

$$z(\tau) = \|\widetilde{x}_\varphi(\tau,\xi) - \widetilde{x}_\varphi(\tau,\overline{\xi})\|'_\tau \quad \text{and} \quad S(\tau) = e^{-\rho(\tau,s)}z(\tau)$$

for each $\tau \geq s$. Proceeding as in (3.8), we obtain

$$\begin{aligned}
z(t) \leq{}& \int_t^\infty \|T(\sigma,s)P(s)(\xi - \overline{\xi})\|e^{-(\overline{a}+\zeta)(\sigma-t)}\,d\sigma \\
&+ \int_s^t \int_t^\infty \|T(\sigma,\tau)P(\tau)\| \cdot \|g^*(\tau,\xi) - g^*(\tau,\overline{\xi})\|e^{-(\overline{a}+\zeta)(\sigma-t)}\,d\sigma\,d\tau \\
\leq{}& e^{(\overline{a}+\zeta)(t-s)}\|\xi - \overline{\xi}\|'_s + C\delta D\int_s^t \int_t^\infty e^{\overline{a}(t-\tau)}e^{-\zeta(\sigma-t)}z(\tau)\,d\sigma\,d\tau \\
\leq{}& e^{\rho(t,s)-\zeta(t-s)}\|\xi - \overline{\xi}\|'_s \\
&+ C\delta D e^{\overline{a}(t-s)}\int_s^t \int_t^\infty e^{\overline{a}(s-\tau)}z(\tau)e^{-\zeta(\sigma-t)}\,d\sigma\,d\tau \\
={}& e^{\rho(t,s)-\zeta(t-s)}\|\xi - \overline{\xi}\|'_s + \frac{C\delta D}{\zeta}e^{\rho(t,s)}\int_s^t e^{2\zeta(\tau-t)}S(\tau)\,d\tau \\
\leq{}& e^{\rho(t,s)}\left[e^{-\zeta(t-s)}\|\xi - \overline{\xi}\|'_s + \frac{C\delta D}{\zeta}\int_s^t e^{2\zeta(\tau-t)}S(\tau)\,d\tau\right],
\end{aligned} \tag{3.11}$$

which yields

$$\begin{aligned}
S(t) &\leq e^{-\zeta(t-s)}\|\xi - \overline{\xi}\|'_s + \frac{C\delta D}{\zeta}\int_s^t e^{2\zeta(\tau-t)}S(\tau)\,d\tau \\
&\leq \|\xi - \overline{\xi}\|'_s + \frac{C\delta D}{\zeta}\int_s^t e^{2\zeta(\tau-t)}S(\tau)\,d\tau.
\end{aligned}$$

Applying Gronwall's lemma we obtain

$$S(t) \leq \|\xi - \overline{\xi}\|'_s e^{(C\delta D/\zeta)\int_s^t e^{2\zeta(\tau-t)}\,d\tau} \leq e^{C\delta D/(2\zeta^2)}\|\xi - \overline{\xi}\|'_s,$$

and hence,

$$\|\widetilde{x}_\varphi(t,\xi) - \widetilde{x}_\varphi(t,\overline{\xi})\|'_t \leq e^{C\delta D/(2\zeta^2)}\|\xi - \overline{\xi}\|'_s e^{\rho(t,s)}.$$

Taking δ sufficiently small, this yields inequality (3.10). □

Lemma 3.3. *Provided that δ is sufficiently small, for each φ, $\psi \in \mathfrak{X}$, $s > \varrho$, and $\xi \in \Delta_s(E(s))$ we have*

$$\|\widetilde{x}_\varphi(t,\xi) - \widetilde{x}_\psi(t,\xi)\|'_t \leq \|\xi\|'_s\|\varphi - \psi\|e^{\rho(t,s)}, \quad t \geq s. \tag{3.12}$$

Proof. Take $\tau \geq s$. Proceeding as in (3.7) and using (3.2) and (3.9) we obtain

$$\begin{aligned}
a(\tau) :={}& \|\widetilde{g}(\tau,\widetilde{x}_\varphi(\tau,\xi),\widetilde{\varphi}(\tau,\widetilde{x}_\varphi(\tau,\xi))) - \widetilde{g}(\tau,\widetilde{x}_\psi(\tau,\xi),\widetilde{\psi}(\tau,\widetilde{x}_\psi(\tau,\xi)))\| \\
\leq{}& C\delta e^{-\varepsilon\tau}\|(\widetilde{x}_\varphi(\tau,\xi) - \widetilde{x}_\psi(\tau,\xi),\widetilde{\varphi}(\tau,\widetilde{x}_\varphi(\tau,\xi)) - \widetilde{\psi}(\tau,\widetilde{x}_\psi(\tau,\xi)))\|'_\tau
\end{aligned} \tag{3.13}$$

$$\le C\delta e^{-\varepsilon\tau}(\|\widetilde{x}_\varphi(\tau,\xi)\|'_\tau \cdot \|\varphi-\psi\| + 2\|\widetilde{x}_\varphi(\tau,\xi)-\widetilde{x}_\psi(\tau,\xi)\|'_\tau)$$
$$\le C\delta e^{-\varepsilon\tau}e^{\rho(\tau,s)}\|\xi\|'_s\|\varphi-\psi\| + 2C\delta e^{-\varepsilon\tau}\|\widetilde{x}_\varphi(\tau,\xi)-\widetilde{x}_\psi(\tau,\xi)\|'_\tau.$$

Set

$$\overline{\rho}(\tau) = \|\widetilde{x}_\varphi(\tau,\xi)-\widetilde{x}_\psi(\tau,\xi)\|'_\tau \quad \text{and} \quad T(\tau) = e^{-\rho(\tau,s)}\overline{\rho}(\tau)$$

for each $\tau \ge s$. Proceeding as in (3.11) (see also (3.7) and (3.8)) we obtain

$$\begin{aligned}
\overline{\rho}(t) &\le \int_s^t \int_t^\infty \|T(\sigma,\tau)P(\tau)\| a(\tau) e^{-(\overline{a}+\zeta)(\sigma-t)}\,d\sigma\,d\tau \\
&= C\delta D\|\xi\|'_s\|\varphi-\psi\| \int_s^t \int_t^\infty e^{\overline{a}(\sigma-\tau)} e^{\rho(\tau,s)} e^{-(\overline{a}+\zeta)(\sigma-t)}\,d\sigma\,d\tau \\
&\quad + 2CD\delta \int_s^t \int_t^\infty e^{\overline{a}(\sigma-\tau)}\overline{\rho}(\tau) e^{-(\overline{a}+\zeta)(\sigma-t)}\,d\sigma\,d\tau \\
&= C\delta D\|\xi\|'_s\|\varphi-\psi\| e^{\overline{a}(t-s)} \int_s^t \int_t^\infty e^{2\zeta(\tau-s)} e^{-\zeta(\sigma-t)}\,d\sigma\,d\tau \\
&\quad + 2CD\delta \int_s^t \int_t^\infty e^{\overline{a}(t-\tau)} e^{-\zeta(\sigma-t)}\overline{\rho}(\tau)\,d\sigma\,d\tau \\
&= C\delta D\|\xi\|'_s\|\varphi-\psi\| e^{\rho(t,s)} \int_s^t \int_t^\infty e^{2\zeta(\tau-t)} e^{-\zeta(\sigma-t)}\,d\sigma\,d\tau \\
&\quad + 2CD\delta e^{\rho(t,s)} \int_s^t \int_t^\infty e^{2\zeta(\tau-t)} e^{-\zeta(\sigma-t)} T(\tau)\,d\sigma\,d\tau \\
&= \frac{C\delta D}{2\zeta^2}\|\xi\|'_s\|\varphi-\psi\| e^{\rho(t,s)} + \frac{2CD\delta}{\zeta} e^{\rho(t,s)} \int_s^t e^{2\zeta(\tau-t)} T(\tau)\,d\tau,
\end{aligned}$$

and thus,

$$T(t) \le \frac{C\delta D}{2\zeta^2}\|\xi\|'_s\|\varphi-\psi\| + \frac{2CD\delta}{\zeta}\int_s^t e^{2\zeta(\tau-t)}T(\tau)\,d\tau.$$

Applying Gronwall's lemma we obtain

$$T(t) \le \frac{C\delta D}{2\zeta^2} e^{C\delta D/\zeta^2}\|\xi\|'_s\|\varphi-\psi\|,$$

and taking δ sufficiently small yields inequality (3.12). □

3.4. Existence of the graph. Now we use the former lemmas to establish the existence of a function $\varphi \in \mathfrak{X}$ satisfying the second identity in (3.1). We start with an auxiliary statement.

Lemma 3.4. *Provided that δ is sufficiently small, there exists a unique $\varphi \in \mathfrak{X}$ such that for every $s > \varrho$ and $\xi \in B_s(E(s))$ we have*

$$\varphi(s,\xi) = -\int_s^\infty T(\tau,s)^{-1}Q(\tau)f(\tau,x_\varphi(\tau,\xi),\varphi(\tau,x_\varphi(\tau,\xi)))\,d\tau. \tag{3.14}$$

Proof. We define an operator Φ in $\mathcal{X}$ by

$$(\Phi\varphi)(s,\xi) = -\int_s^\infty T(\tau,s)^{-1}Q(\tau)f(\tau,x_\varphi(\tau,\xi),\varphi(\tau,x_\varphi(\tau,\xi)))\,d\tau \tag{3.15}$$

for $s > \varrho$ and $\xi \in B_s(E(s))$, where x_φ is the unique function given by Lemma 3.1. We first show that the integral in (3.15) (or more precisely its extension to the complex domain) converges uniformly on $\Delta_s(E(s))$. Indeed, by (3.9) and (2.6), writing

$$h^*(\tau,\xi) = Q(\tau)\widetilde{f}(\tau,\widetilde{x}_\varphi(\tau,\xi),\widetilde{\varphi}(\tau,\widetilde{x}_\varphi(\tau,\xi)))$$

and proceeding as in (3.7), we obtain

$$\|h^*(\tau,\xi)\| \le C\delta e^{-\varepsilon\tau}\|x(\tau,\xi)\|'_\tau \le C\delta e^{-\varepsilon\tau}e^{\rho(\tau,s)}\|\xi\|'_s.$$

By the second inequality in (2.3), for every $r \ge s$ we have

$$\begin{aligned}\|T(\tau,s)^{-1}Q(\tau)\|\cdot\|h^*(\tau,\xi)\| &\le DC\delta\|\xi\|'_s e^{-\underline{b}(\tau-s)+\varepsilon\tau}e^{-\varepsilon\tau}e^{(\overline{a}+2\zeta)(\tau-s)}\\ &= DC\delta e^{(\overline{a}-\underline{b}+2\zeta)(\tau-s)},\end{aligned} \tag{3.16}$$

where $\overline{a}-\underline{b}+2\zeta < 0$. This shows that the integral in (3.15) converges uniformly on $\Delta_s(E(s))$, and hence the right-hand side of (3.15) defines a holomorphic extension of $(\Phi\varphi)(s,\,\cdot\,)$ to the interior of $\Delta_s(E(s))$ which is continuous on $\Delta_s(E(s))$ (we continue to denote the extension by $(\Phi\varphi)(s,\,\cdot\,)$). Since $x_\varphi(\tau,0)=0$ for every $\varphi \in \mathcal{X}$ and $\tau \ge s$ (see (3.9)), it follows from (3.15) that $(\Phi\varphi)(s,0)=0$ for every $s > \varrho$. Furthermore, also by (3.15) and since $(\partial f/\partial v)(t,0)=0$, we have $\partial(\Phi\varphi/\partial v)(s,0)=0$ for every $s>\varrho$.

By (2.6), (3.9) and (3.10), proceeding as in (3.7) we obtain

$$\begin{aligned}b(\tau) &:= \|h^*(\tau,\xi)-h^*(\tau,\overline{\xi})\|\\ &\le C\delta e^{-\varepsilon\tau}\|x(\tau,\xi)-x(\tau,\overline{\xi})\|'_\tau \le 2C\delta e^{-\varepsilon\tau}e^{\rho(\tau,s)}\|\xi-\overline{\xi}\|'_s.\end{aligned}$$

In an analogous manner to that in (3.16) we also have

$$\begin{aligned}&\|(\Phi\varphi)(s,\xi)-(\Phi\varphi)(s,\overline{\xi})\|'_s\\ &\le \int_{s-\varrho}^s\int_s^\infty \|T(\tau,\sigma)^{-1}Q(\tau)\|b(\tau)e^{-(\underline{b}+\zeta)(\sigma-s)}\,d\tau\,d\sigma\\ &\le 2CD\delta\|\xi-\overline{\xi}\|'_s\int_s^\infty\int_{s-\varrho}^s e^{-\underline{b}(\tau-\sigma)}e^{(\overline{a}+2\zeta)(\tau-s)}e^{-(\underline{b}+\zeta)(\sigma-s)}\,d\sigma\,d\tau\\ &\le 2CD\delta\|\xi-\overline{\xi}\|'_s\int_s^\infty\int_{s-\varrho}^s e^{(\overline{a}-\underline{b}+2\zeta)(\tau-s)}e^{-\zeta(\sigma-s)}\,d\sigma\,d\tau\\ &= \frac{2CD\delta(e^{\varrho\zeta}-1)}{\zeta|\overline{a}-\underline{b}+2\zeta|}\|\xi-\overline{\xi}\|'_s.\end{aligned} \tag{3.17}$$

Taking δ sufficiently small we have $\Phi(\mathcal{X}) \subset \mathcal{X}$ and the operator $\Phi\colon \mathcal{X}\to\mathcal{X}$ is well-defined.

Now we show that $\Phi\colon \mathcal{X} \to \mathcal{X}$ is a contraction with the norm in (3.2). Given $\varphi, \psi \in \mathcal{X}$ and $s > \varrho$, let x_φ and x_ψ be the unique functions given by Lemma 3.1. Proceeding as in (3.13), and using Lemma 3.3, (3.9), and (2.6), we obtain

$$\begin{aligned}c(\tau) :&= \big\|Q(\tau)[\widetilde{f}(\tau,\widetilde{x}_\varphi(\tau,\xi),\widetilde{\varphi}(\tau,\widetilde{x}_\varphi(\tau,\xi))) - \widetilde{f}(\tau,\widetilde{x}_\psi(\tau,\xi),\widetilde{\psi}(\tau,\widetilde{x}_\psi(\tau,\xi)))]\big\| \\ &\le C\delta e^{-\varepsilon\tau}[e^{\rho(\tau,s)}\|\xi\|'_s \cdot \|\!|\varphi-\psi|\!\| + 2\|x_\varphi(\tau,\xi) - x_\psi(\tau,\xi)\|'_\tau] \\ &\le 3C\delta e^{-\varepsilon\tau}e^{\rho(\tau,s)}\|\xi\|'_s \cdot \|\!|\varphi-\psi|\!\|.\end{aligned}$$

In an analogous manner to that in (3.17) we conclude that

$$\begin{aligned}\|(\Phi\varphi)(s,\xi) &- (\Phi\psi)(s,\xi)\|'_s \\ &\le \int_{s-\varrho}^{s}\int_s^\infty \|T(\tau,\sigma)^{-1}Q(\tau)\|c(\tau)e^{-(\underline{b}+\zeta)(\sigma-s)}\,d\tau\,d\sigma \\ &\le 3C\delta D\|\xi\|'_s \cdot \|\!|\varphi-\psi|\!\| \int_s^\infty \int_{s-\varrho}^{s} e^{(\overline{a}-\underline{b}+2\zeta)(\tau-s)}e^{-\zeta(\sigma-s)}\,d\sigma\,d\tau \\ &= \frac{3C\delta D(e^{\zeta\varrho}-1)}{\zeta|\overline{a}-\underline{b}+2\zeta|}\|\xi\|'_s \cdot \|\!|\varphi-\psi|\!\|.\end{aligned}$$

Thus, taking δ sufficiently small, the operator $\Phi\colon \mathcal{X} \to \mathcal{X}$ is a contraction in the complete metric space $\mathcal{X}$. Hence, there exists a unique $\varphi \in \mathcal{X}$ satisfying $\Phi\varphi = \varphi$. This completes the proof of the lemma. □

We can now establish Theorem 2.2.

Proof of Theorem 2.2. Consider the unique function $\varphi \in \mathcal{X}$ given by Lemma 3.4. Since $T(t,s)T(\tau,s)^{-1} = T(t,\tau)$, for every $\xi \in B_s(E(s))$ and $t \ge s$ it follows from (3.14) that

$$\begin{aligned}T(t,s)Q(s)\varphi(s,\xi) &+ \int_s^t T(t,\tau)Q(\tau)f(\tau,x(\tau,\xi),\varphi(\tau,x(\tau,\xi)))\,d\tau \\ &= -\int_t^\infty T(\tau,t)^{-1}Q(\tau)f(\tau,x(\tau,\xi),\varphi(\tau,x(\tau,\xi)))\,d\tau.\end{aligned} \tag{3.18}$$

We know from (3.5) that $x(t,\xi) \in B_t(E(t))$, and hence it follows from (3.14) that the right-hand side of (3.18) is equal to $\varphi(t,x(t,\xi))$. This establishes property (2.11).

It follows from the mean value theorem together with Lemmas 3.2 and 2.1, that for every $t \ge s$ and $\xi, \overline{\xi} \in B_s(E(s))$ we have (see (2.4))

$$\begin{aligned}\|\Psi_{t-s}&(s,\xi,\varphi(s,\xi)) - \Psi_{t-s}(s,\overline{\xi},\varphi(s,\overline{\xi}))\| \\ &= \|x(t,\xi) - x(t,\overline{\xi})\| + \|\varphi(t,x(t,\xi)) - \varphi(t,x(t,\overline{\xi}))\| \\ &\le e^{\varepsilon t}[C\|x(t,\xi) - x(t,\overline{\xi})\|'_t + C\|\varphi(t,x(t,\xi)) - \varphi(t,x(t,\overline{\xi}))\|'_t] \\ &\le 2Ce^{\varepsilon t}\|x(t,\xi) - x(t,\overline{\xi})\|'_t \le 4Ce^{\varepsilon t}e^{\rho(t,s)}\|\xi-\overline{\xi}\|'_s \\ &\le \frac{4CD}{\zeta}e^{\varepsilon(t+s)}e^{\rho(t,s)}\|\xi-\overline{\xi}\| \le \frac{4CD}{\zeta}e^{(\overline{a}+2\zeta+\varepsilon)(t-s)+2\varepsilon s}\|\xi-\overline{\xi}\|.\end{aligned}$$

□

4. Decay of derivatives along the stable manifold

We show in this section that the derivatives of the functions

$$\xi \mapsto \Psi_{t-s}(s, \xi, \varphi(s, \xi))$$

also exhibit an exponential decay.

THEOREM 4.1. *Under the assumptions in Theorem* 2.2, *for the unique* $\varphi \in \mathfrak{X}$ *there exists* $\kappa > 1$ *such that given* $j \in \mathbb{N}$, $t \geq s > \varrho$, *and* ξ, $\bar{\xi} \in E(s)$ *with* $\|\xi\|, \|\bar{\xi}\| \leq C^{-1}e^{-\varepsilon s}$ *we have*

$$\left\| \frac{\partial^j}{\partial \xi^j} \Psi_{t-s}(s, \xi, \varphi(s, \xi)) - \frac{\partial^j}{\partial \xi^j} \Psi_{t-s}(s, \bar{\xi}, \varphi(s, \bar{\xi})) \right\| \leq \kappa^j e^{(\bar{a}+\alpha+3\varepsilon)(t-s)+\varepsilon(j+4)s} \|\xi - \bar{\xi}\|.$$

PROOF. Given $\sigma \in (0, 1]$ we set

$$\Delta(\sigma) = \left\{ z = (z_1, \dots, z_k) \in \mathbb{C}^k : |z_i| \leq \sigma \text{ for } i = 1, \dots, k \right\}. \tag{4.1}$$

LEMMA 4.2 (see [2, Lemma 5]). *If* $f\colon \operatorname{int} \Delta(\sigma) \to \mathbb{C}$ *is holomorphic, then there exists* $d = d(k) > 0$ *such that*

$$\sup_{z \in \Delta(\sigma e^{-\rho})} \left| \frac{\partial f}{\partial z_j}(z) \right| \leq \frac{d}{\sigma \rho^{k+1}} \sup_{z \in \Delta(\sigma)} |f(z)|$$

for every $\rho \in (0, 1]$ *and* $j = 1, \dots, k$.

We consider a sequence $(\sigma_j)_{j \in \mathbb{N}_0}$ satisfying $\sigma_j = \sigma_{j-1}e^{-1/j^2}$ for $j \geq 1$, and we set

$$T(j) = \begin{cases} 1, & j = 2, \\ \displaystyle\prod_{l=1}^{j-2} \prod_{i=1}^{l} e^{1/i^2}, & j \geq 3. \end{cases}$$

We have

$$\sigma_l = \sigma_0 \prod_{r=1}^{l} e^{-1/r^2} \quad \text{and} \quad \lim_{l \to \infty} \sigma_l = \sigma_0 e^{-\pi^2/6}. \tag{4.2}$$

Now we set $\varphi^*(t, \xi) = \widetilde{\varphi}(t, \widetilde{x}(t, \xi))$. By the chain rule,

$$\left\| \left(\frac{\partial \varphi^*}{\partial \xi} \right)(t, \xi) \right\| \leq \left\| \left(\frac{\partial \widetilde{\varphi}}{\partial x} \right)(t, \widetilde{x}(t, \xi)) \right\| \cdot \left\| \left(\frac{\partial \widetilde{x}}{\partial \xi} \right)(t, \xi) \right\|.$$

For each $\widetilde{x} = \widetilde{x}(t, \xi)$, using Lemma 2.1 we obtain

$$\left\| \left(\frac{\partial \widetilde{\varphi}}{\partial x} \right)(t, \widetilde{x}) \right\| \leq \sup \left\{ \frac{\|\widetilde{\varphi}(t, x+h) - \widetilde{\varphi}(t, x)\|}{\|h\|} : x + h \in \Delta_t(E(t)), h \neq 0 \right\}$$
$$\leq C^2 e^{2\varepsilon t} \sup \left\{ \frac{\|\widetilde{\varphi}(t, x+h) - \widetilde{\varphi}(t, x)\|_t'}{\|h\|_t'} : x + h \in \Delta_t(E(t)), h \neq 0 \right\} \leq C^2 e^{2\varepsilon t}.$$

Thus, setting $\sigma_0 = C^{-1}e^{-\varepsilon s}$ (for each fixed s) and $a_1 = \sup\{\|(\partial\widetilde{x}/\partial\xi)(t,\xi)\| : \xi \in \Delta(\sigma_0)\}$, with $\mathbb{C}^k$ replaced by $E(s)$ in (4.1), we have

$$b_1 := \sup\left\{\left\|\left(\frac{\partial\varphi^*}{\partial\xi}\right)(t,\xi)\right\| : \xi \in \Delta(\sigma_0)\right\} \le C^2e^{2\varepsilon t}a_1 =: \widetilde{a}_1. \tag{4.3}$$

We claim that, for each $j \ge 2$,

$$a_j := \sup\left\{\left\|\left(\frac{\partial^j\widetilde{x}}{\partial\xi^j}\right)(t,\xi)\right\| : \xi \in \Delta(\sigma_{j-1})\right\} \le \frac{T(j)}{\sigma_0^{j-1}}\prod_{l=1}^{j-1}(de^{(k+1)/l^2})a_1, \tag{4.4}$$

and

$$b_j := \sup\left\{\left\|\left(\frac{\partial^j\varphi^*}{\partial\xi^j}\right)(t,\xi)\right\| : \xi \in \Delta(\sigma_{j-1})\right\} \le \frac{T(j)}{\sigma_0^{j-1}}\prod_{l=1}^{j-1}(de^{(k+1)/l^2})\widetilde{a}_1. \tag{4.5}$$

For $j = 2$, by Lemma 4.2 and (4.3) we have

$$a_2 \le \frac{de^{k+1}}{\sigma_0}a_1 \quad \text{and} \quad b_2 \le \frac{de^{k+1}}{\sigma_0}b_1 \le \frac{de^{k+1}}{\sigma_0}\widetilde{a}_1.$$

This proves (4.4) and (4.5) for $j = 2$. Now we assume that (4.4) and (4.5) hold for $j = l - 1$ ($l \ge 3$). Then, by Lemma 4.2,

$$a_l \le \frac{de^{(k+1)/(l-1)^2}}{\sigma_{l-2}}a_{l-1} \le \frac{T(l-1)}{\sigma_{l-2}\sigma_0^{l-2}}\prod_{j=1}^{l-1}(de^{(k+1)/j^2})a_1. \tag{4.6}$$

By (4.2) we have

$$\sigma_0\frac{T(l-1)}{\sigma_{l-2}} = \prod_{j=1}^{l-3}\prod_{i=1}^{j}e^{1/i^2}\prod_{r=1}^{l-2}e^{1/r^2} = \prod_{j=1}^{l-2}\prod_{i=1}^{j}e^{1/i^2} = T(l),$$

and we conclude from (4.6) that

$$a_l \le \frac{T(l)}{\sigma_0^{l-1}}\prod_{j=1}^{l-1}(de^{(k+1)/j^2})a_1.$$

This shows that (4.4) holds for $j = l$. One can show in a similar manner that (4.5) holds for $j = l$.

Now we establish the statement in the theorem. By Lemmas 2.1 and 3.2 we obtain

$$\begin{aligned} a_1 &\le \sup\left\{\frac{\|\widetilde{x}(t,\xi+h) - \widetilde{x}(t,\xi)\|}{\|h\|} : \xi, \xi+h \in \Delta_s(E(s)), h \ne 0\right\} \\ &\le C^2e^{\varepsilon(t+s)}\sup\left\{\frac{\|\widetilde{x}(t,\xi+h) - \widetilde{x}(t,\xi)\|'_t}{\|h\|'_s} : \xi, \xi+h \in \Delta_s(E(s)), h \ne 0\right\} \\ &\le 2C^2e^{\varepsilon(t+s)}e^{\rho(t,s)} = 2C^2e^{(\overline{a}+2\zeta+\varepsilon)(t-s)+2\varepsilon s}, \end{aligned}$$

and thus,

$$\widetilde{a}_1 \le 2C^4e^{2\varepsilon t}e^{(\overline{a}+2\zeta+\varepsilon)(t-s)+2\varepsilon s} = 2C^4e^{(\overline{a}+2\zeta+3\varepsilon)(t-s)+4\varepsilon s}.$$

By the mean value theorem, (4.4), and (4.5), for each $j \ge 1$ and $\xi, \overline{\xi} \in E(s)$ with $\|\xi\|, \|\overline{\xi}\| \le C^{-1}e^{-\varepsilon s}$ we obtain

$$\begin{aligned}
&\left\| \frac{\partial^j}{\partial \xi^j} \Psi_{t-s}(s,\xi,\varphi(s,\xi)) - \frac{\partial^j}{\partial \xi^j} \Psi_{t-s}(s,\overline{\xi},\varphi(s,\overline{\xi})) \right\| \\
&\quad = \left\| \frac{\partial^j}{\partial \xi^j} \widetilde{x}(t,\xi) - \frac{\partial^j}{\partial \xi^j} \widetilde{x}(t,\overline{\xi}) \right\| + \left\| \frac{\partial^j}{\partial \xi^j} \varphi^*(t,\xi) - \frac{\partial^j}{\partial \xi^j} \varphi^*(t,\overline{\xi}) \right\| \\
&\quad \le (a_{j+1} + b_{j+1}) \|\xi - \overline{\xi}\| \\
&\quad \le 2C^{4+j} e^{(\overline{a}+2\zeta+3\varepsilon)(t-s)+\varepsilon(j+4)s} T(j+1) \prod_{l=1}^{j} (de^{(k+1)/l^2}) \|\xi - \overline{\xi}\| \\
&\quad \le 2C^4 (dC)^j e^{(k+1)\pi^2/6} e^{(\overline{a}+2\zeta+3\varepsilon)(t-s)+\varepsilon(j+4)s} T(j+1) \|\xi - \overline{\xi}\|.
\end{aligned}$$

Noticing that

$$\log T(j+1) = \sum_{l=1}^{j-1} \sum_{i=1}^{l} \frac{1}{i^2} = \sum_{i=1}^{j-1} \frac{j-i}{i^2} \le j \frac{\pi^2}{6},$$

we obtain the desired statement taking $\kappa > dCe^{\pi^2/6}$. □

REFERENCES

[1] L. BARREIRA AND YA. PESIN, *Nonuniform Hyperbolicity*, Encyclopedia Math. Appl. **115**, Cambridge Univ. Press, 2007.

[2] L. BARREIRA AND C. VALLS, *Analytic invariant manifolds for sequences of diffeomorphisms*, J. Differential Equations **245** (2008), 80–101.

[3] E. BEDFORD, M. LYUBICH AND J. SMILLIE, *Polynomial diffeomorphisms of* $\mathbb{C}^2$. IV: *the measure of maximal entropy and laminar currents*, Invent. Math. **112** (1993), 77–125.

[4] E. BEDFORD AND J. SMILLIE, *Polynomial diffeomorphisms of* $\mathbb{C}^2$ *: currents, equilibrium measure and hyperbolicity*, Invent. Math. **103** (1991), 69–99.

[5] M. JONSSON AND D. VAROLIN, *Stable manifolds of holomorphic diffeomorphisms*, Invent. Math. **149** (2002), 409–430.

[6] YA. PESIN, *Families of invariant manifolds corresponding to nonzero characteristic exponents*, Math. USSR Izv. **10** (1976), 1261–1305.

[7] H. WU, *Complex stable manifolds of holomorphic diffeomorphisms*, Indiana Univ. Math. J. **42** (1993), 1349–1358.

Manuscript received July 28, 2012

LUIS BARREIRA AND CLAUDIA VALLS
Departamento de Matemática
Instituto Superior Técnico
Universidade de Lisboa
1049-001 Lisboa, PORTUGAL

E-mail address: barreira@math.ist.utl.pt, cvalls@math.ist.utl.pt