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COEXISTENCE STATES OF DIFFUSIVE PREDATOR-PREY SYSTEMS WITH PREYS COMPETITION AND PREDATOR SATURATION

Jun Zhou

Abstract. In this paper, we study the existence, stability, permanence, and global attractor of coexistence states (i.e. the densities of all the species are positive in Ω) to the following diffusive two-competing-prey and one-predator systems with preys competition and predator saturation:

$$\begin{cases} -\Delta u = u\bigg(a_1 - u - b_{12}v - \dfrac{c_1 w}{(1+\alpha_1 u)(1+\beta_1 w)}\bigg) & \text{in } \Omega, \\ -\Delta v = v\bigg(a_2 - b_{21}u - v - \dfrac{c_2 w}{(1+\alpha_2 v)(1+\beta_2 w)}\bigg) & \text{in } \Omega, \\ -\Delta w = w\bigg(\dfrac{e_1 u}{(1+\alpha_1 u)(1+\beta_1 w)} + \dfrac{e_2 v}{(1+\alpha_2 v)(1+\beta_2 w)} - d\bigg) & \text{in } \Omega, \\ k_1\partial_\nu u + u = k_2\partial_\nu v + v = k_3\partial_\nu w + w = 0 & \text{on } \partial\Omega, \end{cases}$$

where $k_i \geq 0$ ($i = 1,2,3$) and all the other parameters are positive, ν is the outward unit rector on $\partial\Omega$, u and v are densities of the competing preys, w is the density of the predator.

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1. Introduction and main results

Continuous models, usually in the form of differential equations, have formed a large part of the traditional mathematical ecology literature. In these models, the key terms specifying the outcome of predator-prey interactions are the functional and numerical response, which reflect the relationship between predators and their prey. In general, a predator-prey system has the following form

$$\begin{cases} u_t = u(a - bu) - g(u)v, \\ v_t = -dv + \epsilon g(u)v, \end{cases}$$

where a, b, d, ϵ are positive constants and $g(u)$ is the so-called prey-dependent functional response. And ones usually consider the following two cases.

(I) For the case of monotonic response, the functional response $g(u)$ is taken as follows (see [10], [12], [38] and the references therein).

(i) $g(u) = \dfrac{mu}{e+u}$,

(ii) $g(u) = \dfrac{mu^2}{e+\varepsilon u+u^2}$,

(iii) $g(u) = \dfrac{mu^2}{e+u^2}$,

where m, e and ε are positive constants, m denotes the maximal growth rate of the species and e is the half-saturation constant. The model (i) is called the Michaelis–Menten or Holling type-II function, (ii) is called the sigmoidal response function, and (iii) is called the Holling type-III function.

(II) For the case of non-monotonic response, the functional response $g(u)$ is taken as follows (see [4] and the references therein).

$$g(u) = \frac{mu}{e+\varepsilon u+u^2}, \tag{1.1}$$

which is called the Monod–Haldane function. Collings [20] utilized (1.1) to study the effects of functional response on the bifurcation behavior of a mite predator-prey interaction model, and called it a Holling type-IV function. The experiments of Edwards [33] supported the use of the function (1.1) to describe the dependence of the growth rate on an inhibitory substrate. In experiments on the uptake of phenol by pure culture of pseudomonas putida, Sokol and Howell [72] suggested a simplified version of (1.1), namely,

$$g(u) = \frac{mu}{e+u^2}. \tag{1.2}$$

For details of the background of the response functions (1.1) and (1.2), we refer to Ruan and Xiao [68].

When predators have to search, share and compete for food, a more suitable general predator-prey model should be based on the so-called ratio-dependent theory, which asserts that the per capita predator growth rate should be a function of the ratio of prey to predator abundance (see [5]–[8], [20], [36], [37] and the references therein). The following models was proposed

$$\begin{cases} u_t = u(a - bu) - vf(u,v), \\ v_t = -dv + \epsilon v f(u,v)), \end{cases} \tag{1.3}$$

with functional response

$$f(u,v) = \frac{mu}{u + ev},$$

where a, b, d, ϵ, m, e are positive constants.

In order to study the destabilizing force of predator saturation and the stabilizing force of competition for prey, Bazykin [9] proposed the following functional response in model (1.3):

$$f(u,v) = \frac{mu}{(1 + \alpha u)(1 + \beta v)}, \tag{1.4}$$

where a, b, d, ϵ, m, α, β are positive constants.

The qualitative properties about the above ODE models have been studied extensively in recent years (see [1], [2], [11], [39]–[41], [48], [73], [83] and the references therein).

The role of diffusion in the modelling of many physical, chemical and biological processes has been extensively studied. Starting with Turing's seminal 1952 paper [74], diffusion and cross-diffusion have been observed as causes of the spontaneous emergence of ordered structures, called patterns, in a variety of non-equilibrium situations. These include the Gierer–Meinhardt model [35], [43], [75], [80], [82], the Sel'kov model [28], [76], the Noyes–Field model for Belousov–Zhabotinskiĭ reaction [66], the chemotactic diffusion model [54], [79], the competition model [19], [27], [56], [57], [58], the predator-prey model with the above cited functional response [31], [32], [42], [44], [45], [63], [64], [77], as well as models of semiconductors, plasmas, chemical waves, combustion systems, embryogenesis, etc. see e.g. [15], [18], [22] and references therein. Diffusion-driven instability, also called Turing instability, has also been verified empirically [17], [60].

In this paper we are interested in a diffusive two-competing-prey and one-predator system in spatially inhomogeneous environment, where competing prey species are in Lotka–Volterra interaction but predator's functional response is the type of (1.4). Let u and v be the population densities of two competing preys, and w be the density of the predator, then the mathematical model is given by

$$
(1.5)\quad \begin{cases} u_t - \Delta u = u\left(a_1 - u - b_{12}v - \dfrac{c_1 w}{(1+\alpha_1 u)(1+\beta_1 w)}\right) & \text{in } \Omega\times(0,\infty),\\ v_t - \Delta v = v\left(a_2 - b_{21}u - v - \dfrac{c_2 w}{(1+\alpha_2 v)(1+\beta_2 w)}\right) & \text{in } \Omega\times(0,\infty),\\ w_t - \Delta w = w\left(\dfrac{e_1 u}{(1+\alpha_1 u)(1+\beta_1 w)} + \dfrac{e_2 v}{(1+\alpha_2 v)(1+\beta_2 w)} - d\right) & \\ & \text{in } \Omega\times(0,\infty),\\ k_1\partial_\nu u + u = k_2\partial_\nu v + v = k_3\partial_\nu w + w = 0 & \text{on } \partial\Omega\times(0,\infty), \end{cases}
$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, the given coefficients a_i, c_i, e_i, α_i, β_i $(i = 1, 2)$, d, b_{12} and b_{21} are all positive constants, k_i $(i = 1, 2, 3)$ is nonnegative constant, ν is the outward unit rector on $\partial\Omega$.

In order to study the dynamics of (1.5), we are mainly interested in the steady-state system, that is, we investigate the existence, uniqueness and asymptotic stability of positive steady-state solutions of the following elliptic system corresponding to (1.5):

$$
(1.6)\quad \begin{cases} -\Delta u = u\left(a_1 - u - b_{12}v - \dfrac{c_1 w}{(1+\alpha_1 u)(1+\beta_1 w)}\right) & \text{in } \Omega,\\ -\Delta v = v\left(a_2 - b_{21}u - v - \dfrac{c_2 w}{(1+\alpha_2 v)(1+\beta_2 w)}\right) & \text{in } \Omega,\\ -\Delta w = w\left(\dfrac{e_1 u}{(1+\alpha_1 u)(1+\beta_1 w)} + \dfrac{e_2 v}{(1+\alpha_2 v)(1+\beta_2 w)} - d\right) & \text{in } \Omega,\\ k_1\partial_\nu u + u = k_2\partial_\nu v + v = k_3\partial_\nu w + w = 0 & \text{on } \partial\Omega. \end{cases}
$$

The study of positive solutions of elliptic system corresponding to prey-predator models has attracted many people in recent years, and many good works have been done (see [16], [29], [30], [46], [50], [52], [55], [61], [66], [67], [70] and references therein).

Recently, Wang and Wu [78] considered the existence, multiplicity, bifurcation and stability of positive solution to the following prey-predator model with functional response of type (1.4):

$$
\begin{cases} -\Delta u = u\left(a - u - b\,\dfrac{v}{(1+\alpha u)(1+\beta v)}\right) & \text{in } \Omega,\\ -\Delta v = v\left(c - v + d\,\dfrac{u}{(1+\alpha u)(1+\beta v)}\right) & \text{in } \Omega,\\ u = v = 0 & \text{on } \partial\Omega, \end{cases}
$$

where the parameters are all positive constant, and u and v are the densities of the prey and predator, respectively.

However, the dynamics to three species interaction systems are more complicate than those of two species (see [26], [34], [47], [49], [51], [53], [55]). For this reason, the reaction-diffusion systems among three species have not been well understood. In [26] and [55], using their own developed degree theory, the existences of positive steady-state solutions of Lotka–Volterra type systems were investigated.

Motivated by above papers, in the present paper, we study the existence of the positive solution of (1.6). In order to state our results, let us introduce some notations. For each $q \in C^{\alpha}(\Omega)$ $(0 < \alpha < 1)$ and $k \geq 0$, denote the principle eigenvalue of

$$\begin{cases} -\Delta u + q(x)u = \lambda u & \text{in } \Omega, \\ k\partial_\nu u + u = 0 & \text{on } \partial\Omega, \end{cases}$$

by $\lambda_{1,k}(q)$ and simply denote $\lambda_{1,k}(0)$ by $\lambda_{1,k}$.

It is well known that $\lambda_{1,k}(q(x))$ is strictly increasing in the sense that $q_1(x) \leq q_2(x)$ and $q_1(x) \not\equiv q_2(x)$ implies

$$\lambda_{1,k}(q_1(x)) < \lambda_{1,k}(q_2(x))$$

(see Proposition 1.1 of [81]).

If $\lambda_{1,k}(-\rho(x)) < 0$ and $\rho(x) \in C^{\alpha}(\overline{\Omega})$ $(0 < \alpha < 1)$ is a positive function, let $\Theta_k[\rho(x)]$ be the unique positive solution of the following equation

$$\begin{cases} -\Delta \phi = \phi(\rho(x) - \phi) & \text{in } \Omega, \\ k\partial_\nu \phi + \phi = 0 & \text{on } \partial\Omega, \end{cases}$$

and it is easy to see $\Theta_k[\rho(x)] \leq \max\limits_{x \in \overline{\Omega}} \rho(x)$. The existence of $\Theta_k[\rho(x)]$ follows from Theorem 2.3 in Section 2 below.

Throughout this paper, a solution (u, v, w) of (1.6) is called a coexistence state if $u(x) > 0,\ v(x) > 0,\ w(x) > 0$ for all $x \in \Omega$.

Now, we state firstly a result on a priori bound for coexistence states of (1.6).

Theorem 1.1. *Any coexistence state* (u, v, w) *of* (1.6) *has an a priori bounds*

$$u(x) \leq Q_1, \quad v(x) \leq Q_2, \quad \text{and} \quad w(x) \leq Q_3,$$

where $Q_1 = a_1$, $Q_2 = a_2$ *and* Q_3 *is the positive root of the following equation with respect to* w

$$\frac{e_1 a_1}{(1 + \alpha_1 a_1)(1 + \beta_1 w)} + \frac{e_2 a_2}{(1 + \alpha_2 a_2)(1 + \beta_2 w)} - d = 0.$$

Secondly, by using the fixed point index theory (see [26], [55], [81]), we will get some sufficient conditions for coexistence state of (1.6). To this, we first introduce the following notations:

$$\Gamma_1^{(1)} = -\frac{e_1\Theta_{k_1}[a_1]}{1+\alpha_1\Theta_{k_1}[a_1]}, \qquad \Gamma_1^{(2)} = -\frac{e_2\Theta_{k_2}[a_2]}{1+\alpha_2\Theta_{k_2}[a_2]},$$

$$\Gamma_2^{(1)} = -\frac{e_1\Theta_{k_1}[a_1-b_{12}\Theta_{k_2}[a_2]]}{1+\alpha_1\Theta_{k_1}[a_1-b_{12}\Theta_{k_2}[a_2]]}, \qquad \Gamma_2^{(2)} = -\frac{e_2\Theta_{k_2}[a_2-b_{21}\Theta_{k_1}[a_1]]}{1+\alpha_2\Theta_{k_2}[a_2-b_{21}\Theta_{k_1}[a_1]]},$$

$$\Gamma_3^{(1)} = b_{12}\Theta_{k_2}\left[a_2-\frac{c_2}{\beta_2}\right]+\frac{c_1w_*^{(1)}}{1+\beta_1w_*^{(1)}}, \qquad \Gamma_3^{(2)} = b_{21}\Theta_{k_1}\left[a_1-\frac{c_1}{\beta_1}\right]+\frac{c_2w_*^{(2)}}{1+\beta_2w^{(2)}},$$

$$\Gamma_4^{(1)} = -\frac{e_1\Theta_{k_1}[a_1-c_1/\beta_1]}{1+\alpha_1\Theta_{k_1}[a_1-c_1/\beta_1]}, \qquad \Gamma_4^{(2)} = -\frac{e_2\Theta_{k_2}[a_2-c_2/\beta_2]}{1+\alpha_2\Theta_{k_2}[a_2-c_2/\beta_2]},$$

where $w_*^{(1)}$, $w_*^{(2)}$ are the unique positive solution of the following two problem respectively,

$$\begin{cases} -\Delta w = w\left(\dfrac{e_2\Theta_{k_2}[a_2-c_2/\beta_2]}{(1+\alpha_2\Theta_{k_2}[a_2-c_2/\beta_2])(1+\beta_2w)}-d\right) & \text{in } \Omega,\\ k_3\partial_\nu w+w=0 & \text{on } \partial\Omega, \end{cases} \tag{1.7}$$

$$\begin{cases} -\Delta w = w\left(\dfrac{e_1\Theta_{k_1}[a_1-c_1/\beta_1]}{(1+\alpha_1\Theta_{k_1}[a_1-c_1/\beta_1])(1+\beta_1w)}-d\right) & \text{in } \Omega,\\ k_3\partial_\nu w+w=0 & \text{on } \partial\Omega. \end{cases} \tag{1.8}$$

REMARK 1.2. From (c) of Theorem 2.3 in Section 2 below, we see that if $-d > \lambda_{1,k_3}(\Gamma_4^{(2)})$, then there exists a unique positive solution $w_*^{(1)}$ of (1.7). Similarly, if $-d > \lambda_{1,k_3}(\Gamma_4^{(1)})$, then there exists a unique positive solution $w_*^{(2)}$ of (1.8).

THEOREM 1.3. *Assume that $a_i > \lambda_{1,k_i}$ for $i = 1, 2$. If one of the following conditions holds, then* (1.6) *has least one coexistence state*

(a) $\begin{cases} a_1-\dfrac{c_1}{\beta_1} > \lambda_{1,k_1}(b_{12}\Theta_{k_2}[a_2]), \quad a_2-\dfrac{c_2}{\beta_2} > \lambda_{1,k_2}(b_{21}\Theta_{k_1}[a_1]),\\ -d > \max\{\lambda_{1,k_3}(\Gamma_1^{(1)}),\ \lambda_{1,k_3}(\Gamma_1^{(2)}),\ \lambda_{1,k_3}(\Gamma_2^{(1)}+\Gamma_2^{(2)})\}; \end{cases}$

(b) $\begin{cases} a_1 > \lambda_{1,k_1}(b_{12}\Theta_{k_2}[a_2]), \quad a_2-\dfrac{c_2}{\beta_2} > \lambda_{1,k_2}(b_{21}\Theta_{k_1}[a_1]),\\ \lambda_{1,k_3}(\Gamma_1^{(2)}) > -d > \max\{\lambda_{1,k_3}(\Gamma_1^{(1)}),\ \lambda_{1,k_3}(\Gamma_2^{(1)}+\Gamma_2^{(2)})\}; \end{cases}$

(c) $\begin{cases} a_1-\dfrac{c_1}{\beta_1} > \lambda_{1,k_1}(b_{12}\Theta_{k_2}[a_2]), \quad a_2 > \lambda_{1,k_2}(b_{21}\Theta_{k_1}[a_1]),\\ \lambda_{1,k_3}(\Gamma_1^{(1)}) > -d > \max\{\lambda_{1,k_3}(\Gamma_1^{(2)}),\ \lambda_{1,k_3}(\Gamma_2^{(1)}+\Gamma_2^{(2)})\}; \end{cases}$

(d) $\begin{cases} a_1 > \lambda_{1,k_1}(b_{12}\Theta_{k_2}[a_2]), \quad a_2 > \lambda_{1,k_2}(b_{21}\Theta_{k_1}[a_1]),\\ \min\{\lambda_{1,k_3}(\Gamma_1^{(1)}),\ \lambda_{1,k_3}(\Gamma_1^{(2)})\} > -d > \lambda_{1,k_3}(\Gamma_2^{(1)}+\Gamma_2^{(2)}); \end{cases}$

$$\text{(e)}\quad \begin{cases} \lambda_{1,k_1}(\Gamma_3^{(1)}) > a_1 > \max\left\{\lambda_{1,k_1}(b_{12}\Theta_{k_2}[a_2]),\ \lambda_{1,k_1} + c_1/\beta_1\right\}, \\ \lambda_{1,k_2}(\Gamma_3^{(2)}) > a_2 > \max\left\{\lambda_{1,k_2}(b_{21}\Theta_{k_1}[a_1]),\ \lambda_{1,k_2} + c_2/\beta_2\right\}, \\ -d > \max\left\{\lambda_{1,k_3}(\Gamma_4^{(1)}),\ \lambda_{1,k_3}(\Gamma_4^{(2)}),\ \lambda_{1,k_3}(\Gamma_2^{(1)} + \Gamma_2^{(2)})\right\}; \end{cases}$$

$$\text{(f)}\quad \begin{cases} \min\left\{\lambda_{1,k_1}(\Gamma_3^{(1)}),\ \lambda_{1,k_1}(b_{12}\Theta_{k_2}[a_2])\right\} > a_1 > \lambda_{1,k_1} + c_1/\beta_1, \\ \lambda_{1,k_2}(\Gamma_3^{(2)}) > a_2 > \max\left\{\lambda_{1,k_2}(b_{21}\Theta_{k_1}[a_1]),\ \lambda_{1,k_2} + c_2/\beta_2\right\}, \\ -d > \max\left\{\lambda_{1,k_3}(\Gamma_4^{(1)}),\ \lambda_{1,k_3}(\Gamma_4^{(2)}),\ \lambda_{1,k_3}(\Gamma_2^{(2)})\right\}; \end{cases}$$

$$\text{(g)}\quad \begin{cases} \min\left\{\lambda_{1,k_2}(\Gamma_3^{(2)}),\ \lambda_{1,k_2}(b_{21}\Theta_{k_1}[a_1])\right\} > a_2 > \lambda_{1,k_2} + c_2/\beta_2, \\ \lambda_{1,k_1}(\Gamma_3^{(1)}) > a_1 > \max\left\{\lambda_{1,k_1}(b_{12}\Theta_{k_2}[a_2]),\ \lambda_{1,k_1} + c_1/\beta_1\right\}, \\ -d > \max\left\{\lambda_{1,k_3}(\Gamma_4^{(1)}),\ \lambda_{1,k_3}(\Gamma_4^{(2)}),\ \lambda_{1,k_3}(\Gamma_2^{(1)})\right\}; \end{cases}$$

$$\text{(h)}\quad \begin{cases} \lambda_{1,k_1}(\Gamma_3^{(1)}) > a_1 > \max\left\{\lambda_{1,k_1}(b_{12}\Theta_{k_2}[a_2]),\ \lambda_{1,k_1} + c_1/\beta_1\right\}, \\ \min\left\{\lambda_{1,k_2}(\Gamma_3^{(2)}), \lambda_{1,k_2}(b_{21}\Theta_{k_1}[a_1 - b_{12}\Theta_{k_2}[a_2]])\right\} \\ \qquad\qquad\qquad\qquad > a_2 > \lambda_{1,k_2} + c_2/\beta_2, \\ -d > \max\left\{\lambda_{1,k_3}(\Gamma_4^{(1)}),\ \lambda_{1,k_3}(\Gamma_4^{(2)})\right\}; \end{cases}$$

$$\text{(i)}\quad \begin{cases} \lambda_{1,k_2}(\Gamma_3^{(2)}) > a_2 > \max\left\{\lambda_{1,k_2}(b_{21}\Theta_{k_1}[a_1]),\ \lambda_{1,k_2} + c_2/\beta_2\right\}, \\ \min\left\{\lambda_{1,k_1}(\Gamma_3^{(1)}),\ \lambda_{1,k_1}(b_{12}\Theta_{k_2}[a_2 - b_{21}\Theta_{k_1}[a_1]])\right\} \\ \qquad\qquad\qquad\qquad > a_1 > \lambda_{1,k_1} + c_1/\beta_1, \\ -d > \max\left\{\lambda_{1,k_3}(\Gamma_4^{(1)}),\ \lambda_{1,k_3}(\Gamma_4^{(2)})\right\}. \end{cases}$$

Thirdly, by using comparison principle, we obtain some sufficient conditions for non-existence of coexistence states of (1.6).

THEOREM 1.4. *If any one of the following conditions holds, then* (1.6) *has no coexistence states:*

(a) $a_i \le \lambda_{1,k_i}$ *for* $i = 1$ *or* 2*;*

(b) $a_i > \lambda_{1,k_i}$ *for* $i = 1, 2$ *and*

$$-d \le \lambda_{1,k_3}\left(-\frac{e_1\Theta_{k_1}[a_1]}{1 + \alpha_1\Theta_{k_1}[a_1]} - \frac{e_2\Theta_{k_2}[a_2]}{1 + \alpha_2\Theta_{k_2}[a_2]}\right);$$

(c) $a_2 > \lambda_{1,k_2}$, $a_1 - c_1/\beta_1 > \lambda_{1,k_1}(b_{12}\Theta_{k_2}[a_2])$ *and* $a_2 \le \lambda_{1,k_2}(b_{21}\Theta_{k_1}[a_1 - b_{12}\Theta_{k_2}[a_2] - c_1/\beta_1])$;

(d) $a_1 > \lambda_{1,k_1}$, $a_2 - c_2/\beta_2 > \lambda_{1,k_2}(b_{21}\Theta_{k_1}[a_1])$ *and* $a_1 \le \lambda_{1,k_1}(b_{12}\Theta_{k_2}[a_2 - b_{21}\Theta_{k_1}[a_1] - c_2/\beta_2])$.

Next, the results for the uniqueness and stability of coexistence states of (1.6) are stated as follows.

THEOREM 1.5.

(a) *If one of the conditions* (a)–(d) *in Theorem* 1.3 *and the following condition hold*:

$$(b_{21}^2 + b_{12}^2)\ell + 2b_{12}b_{21} < 4, \tag{1.9}$$

where

$$\ell = \max\left\{\max_{x\in\overline{\Omega}} \frac{\Theta_{k_2}[a_2]}{\Theta_{k_1}[a_1 - b_{12}\Theta_{k_2}[a_2]]},\ \max_{x\in\overline{\Omega}} \frac{\Theta_{k_1}[a_1]}{\Theta_{k_2}[a_2 - b_{21}\Theta_{k_1}[a_1]]}\right\},$$

then there exists a positive constant $\widetilde{C} = \widetilde{C}(\alpha_1, \alpha_2, \beta_1, \beta_2)$ *such that for* $c_1, c_2 \le \widetilde{C}$, (1.6) *has exactly one coexistence state which is nondegenerate and linearly stable.*

(b) *Assume that* $a_1 - c_1/\beta_1 > \lambda_{1,k_1}(b_{12}\Theta_{k_2}[a_2])$,

$$-d > \max\left\{\lambda_{1,k_3}\left(-\frac{e_1\Theta_{k_1}[a_1 - b_{12}\Theta_{k_2}[a_2] - c_1/\beta_1]}{1 + \alpha_1\Theta_{k_1}[a_1 - b_{12}\Theta_{k_2}[a_2] - c_1/\beta_1]}\right),\right.$$
$$\left.\lambda_{1,k_3}\left(-\frac{e_2\Theta_{k_2}[a_2]}{1 + \alpha_2\Theta_{k_2}[a_2]}\right)\right\}$$

and $a_2 - c_2/\beta_2 > \lambda_{1,k_2}(b_{21}\Theta_{k_1}[a_1])$. *In addition, if* (1.9) *holds, then there exists a positive constant* $\widetilde{C} = \widetilde{C}(c_2, \alpha_1, \alpha_2, \beta_1, e_2)$ *such that for* $1/\beta_2, c_1 \le \widetilde{C}$, (1.6) *has exactly one coexistence state which is nondegenerate and linearly stable.*

(c) *Assume that* $a_2 - c_2/\beta_2 > \lambda_{1,k_2}(b_{21}\Theta_{k_1}[a_1])$,

$$-d > \max\left\{\lambda_{1,k_3}\left(-\frac{e_2\Theta_{k_2}[a_2 - b_{21}\Theta_{k_1}[a_1] - c_2/\beta_2]}{1 + \alpha_2\Theta_{k_2}[a_2 - b_{21}\Theta_{k_1}[a_1] - c_2/\beta_2]}\right),\right.$$
$$\left.\lambda_{1,k_3}\left(-\frac{e_1\Theta_{k_1}[a_1]}{1 + \alpha_1\Theta_{k_1}[a_1]}\right)\right\}$$

and $a_1 - c_1/\beta_1 > \lambda_{1,k_1}(b_{12}\Theta_{k_2}[a_2])$. *In addition, if* (1.9) *holds, then there exists a positive constant* $\widetilde{C} = \widetilde{C}(c_1, \alpha_1, \alpha_2, \beta_2, e_1)$ *such that for* $1/\beta_1, c_2 \le \widetilde{C}$, (1.6) *has exactly one coexistence state which is nondegenerate and linearly stable.*

Finally, we give some sufficient conditions on the global asymptotic stability of the trivial and semi-trivial solutions and on global attractor of coexistence states of time-dependent system (1.5).

THEOREM 1.6. *Let* (u, v, w) *be a positive solution of* (1.5), *then we have*

(a) *If* $a_i \le \lambda_{1,k_i}$ *for* $i = 1, 2$, *then* $(u, v, w) \to (0, 0, 0)$ *as* $t \to \infty$.

(b) *If* $a_1 > \lambda_{1,k_1}$, $a_2 \le \lambda_{1,k_2}$ *and* $-d < \lambda_{1,k_3}(-e_1\Theta_{k_1}[a_1]/(1 + \alpha_1\Theta_{k_1}[a_1]))$, *then* $(u, v, w) \to (\Theta_{k_1}[a_1], 0, 0)$ *as* $t \to \infty$.

(c) *If* $a_2 > \lambda_{1,k_2}$, $a_1 \le \lambda_{1,k_1}$ *and* $-d < \lambda_{1,k_3}(-e_2\Theta_{k_2}[a_2]/(1 + \alpha_2\Theta_{k_2}[a_2]))$, *then* $(u, v, w) \to (0, \Theta_{k_2}[a_2], 0)$ *as* $t \to \infty$.

Remark 1.7. Please see Theorem 5.2 in Section 5 for the dynamics of other semi-trivial solutions of (1.5), i.e. the solutions with the density of one specie is zero and the densities of other two species are positive in Ω.

Theorem 1.8. *If*

$$\begin{cases} a_1 - \dfrac{c_1}{\beta_1} > \lambda_{1,k_1}(b_{12}\Theta_{k_2}[a_2]), \\ a_2 - \dfrac{c_2}{\beta_2} > \lambda_{1,k_2}(b_{21}\Theta_{k_1}[a_1]), \\ -d > \lambda_{1,k_3}\bigg(- \dfrac{e_1 u^*}{1+\alpha_1 u^*} - \dfrac{e_2 v^*}{1+\alpha_2 v^*}\bigg), \end{cases} \tag{1.10}$$

where $u^* = \Theta_{k_1}[a_1 - b_{12}\Theta_{k_2}[a_2] - c_1/\beta_1]$ *and* $v* = \Theta_{k_2}[a_2 - b_{21}\Theta_{k_1}[a_1] - c_2/\beta_2]$. *Then there exists a pair of functions* $(\widetilde{u},\widetilde{v},\widetilde{w})$ *and* $(\widehat{u},\widehat{v},\widehat{w})$ *in* $C^2(\Omega)\cap C^1(\overline{\Omega})$ *such that*

$$\begin{cases} -\Delta\widetilde{u} = \widetilde{u}\bigg(a_1 - \widetilde{u} - b_{12}\widehat{v} - \dfrac{c_1\widehat{w}}{(1+\alpha_1\widetilde{u})(1+\beta_1\widehat{w})}\bigg) & \text{in } \Omega, \\ -\Delta\widehat{u} = \widehat{u}\bigg(a_1 - \widehat{u} - b_{12}\widetilde{v} - \dfrac{c_1\widetilde{w}}{(1+\alpha_1\widehat{u})(1+\beta_1\widetilde{w})}\bigg) & \text{in } \Omega, \\ -\Delta\widetilde{v} = \widetilde{v}\bigg(a_2 - b_{21}\widehat{u} - \widetilde{v} - \dfrac{c_2\widehat{w}}{(1+\alpha_2\widetilde{v})(1+\beta_2\widehat{w})}\bigg) & \text{in } \Omega, \\ -\Delta\widehat{v} = \widehat{v}\bigg(a_2 - b_{21}\widetilde{u} - \widehat{v} - \dfrac{c_2\widetilde{w}}{(1+\alpha_2\widehat{v})(1+\beta_2\widetilde{w})}\bigg) & \text{in } \Omega, \\ -\Delta\widetilde{w} = \widetilde{w}\bigg(\dfrac{e_1\widetilde{u}}{(1+\alpha_1\widetilde{u})(1+\beta_1\widetilde{w})} + \dfrac{e_2\widetilde{v}}{(1+\alpha_2\widetilde{v})(1+\beta_2\widetilde{w})} - d\bigg) & \text{in } \Omega, \\ -\Delta\widehat{w} = \widehat{w}\bigg(\dfrac{e_1\widehat{u}}{(1+\alpha_1\widehat{u})(1+\beta_1\widehat{w})} + \dfrac{e_2\widehat{v}}{(1+\alpha_2\widehat{v})(1+\beta_2\widehat{w})} - d\bigg) & \text{in } \Omega, \\ k_1\partial_\nu\widetilde{u} + \widetilde{u} = 0 = k_1\partial_\nu\widehat{u} + \widehat{u} & \text{on } \partial\Omega, \\ k_2\partial_\nu\widetilde{v} + \widetilde{v} = 0 = k_2\partial_\nu\widehat{v} + \widehat{v} & \text{on } \partial\Omega, \\ k_3\partial_\nu\widetilde{w} + \widetilde{w} = 0 = k_3\partial_\nu\widehat{w} + \widehat{w} & \text{on } \partial\Omega, \end{cases}$$

and satisfying the following estimates

$$u^* \le \widehat{u} \le \widetilde{u} \le \Theta_{k_1}[a_1], \qquad v^* \le \widehat{v} \le \widetilde{v} \le \Theta_{k_1}[a_2],$$
$$w_{(u^*,v^*)} \le \widehat{w} \le \widetilde{w} \le w_{(\Theta_{k_1}[a_1],\Theta_{k_1}[a_2])},$$

where $w_{(u,v)}$ *is the unique positive solution of the equation*

$$\begin{cases} -\Delta w = w\bigg(\dfrac{e_1 u}{(1+\alpha_1 u)(1+\beta_1 w)} + \dfrac{e_2 v}{(1+\alpha_2 v)(1+\beta_2 w)} - d\bigg) & \text{in } \Omega, \\ k_3\partial_\nu w + w = 0 & \text{on } \partial\Omega. \end{cases}$$

Furthermore, $[\widehat{u},\widetilde{u}]\times[\widehat{v},\widetilde{v}]\times[\widehat{w},\widetilde{w}]$ *is a positive global attractor of* (1.5).

Remark 1.9. (a) We point out such functions $(\widetilde{u},\widetilde{v},\widetilde{w})$ and $(\widehat{u},\widehat{v},\widehat{w})$ are called quasi-solutions of (1.6) in this paper.

(b) Since

$$
\begin{aligned}
-d &> \lambda_{1,k_3}\Big(- \frac{e_1 u^*}{1+\alpha_1 u^*} - \frac{e_2 v^*}{1+\alpha_2 v^*}\Big) \\
&> \lambda_{1,k_3}\Big(- \frac{e_1 \Theta_{k_1}[a_1]}{1+\alpha_1 e_1 \Theta_{k_1}[a_1]} - \frac{e_2 \Theta_{k_2}[a_2]}{1+\alpha_2 e_2 \Theta_{k_2}[a_2]}\Big)
\end{aligned}
$$

by comparison property of principle eigenvalue, the existence of $w_{(u^*,v^*)}$ and $w_{(\Theta_{k_1}[a_1],\Theta_{k_1}[a_2])}$ follows by (c) of Theorem 2.3 in Section 2.

(c) For more results about the positive global attractor of (1.5), please see Theorem 5.2 in Section 5.

This paper is organized as follows: In Section 2, we give some fundamental theorems, which play an important role in this paper. In Section 3, the necessary and sufficient conditions for coexistence states of (1.6) are investigated, and Theorems 1.1, 1.3, 1.4 are proved. In Section 4, we show that the stability and uniqueness of coexistence states of (1.6) depend on some parameters, and prove Theorem 1.5. Finally, the proofs of Theorems 1.6 and 1.8 are given in Section 5.

2. Preliminaries

In this section, we give some fundamental theorems, especially some degree theorems, which play an important role in this paper. The following theorem follows from Proposition 1.4 of [81]) (see also Proposition 1 of [23] and Lemmas 2.1 and 2.3 of [52]).

THEOREM 2.1. *For $q \in C^\alpha(\overline{\Omega})$ $(0 < \alpha < 1)$ and P be a sufficiently large number such that $P > q(x)$ for all $x \in \overline{\Omega}$, define an positive compact operator $\mathcal{L} := (-\Delta + P)^{-1}(P - q(x))\colon C^1_k(\overline{\Omega}) \to C^1_k(\overline{\Omega}) = \{u \in C^1(\overline{\Omega}) : k\partial_\nu u + u = 0 \text{ on } \partial\Omega\}$ for $k \geq 0$ a constant. Denote the spectral radius of $\mathcal{L}$ by $r_k(\mathcal{L})$. Then we have*:

(a) $\lambda_{1,k}(q) > 0 \Leftrightarrow r_k(\mathcal{L}) < 1$;
(b) $\lambda_{1,k}(q) < 0 \Leftrightarrow r_k(\mathcal{L}) > 1$;
(c) $\lambda_{1,k}(q) = 0 \Leftrightarrow r_k(\mathcal{L}) = 1$.

From Theorem 2.1, we see that it is crucial to determine the eigenvalue $\lambda_{1,k}(q)$. The following theorem is stated in Theorem 2.4 of [3] and Theorem 11.10 of [71] (see also [13], [14], [23], [24], [61]):

THEOREM 2.2. *Let $q(x) \in L^\infty(\Omega)$ and $\phi \geq 0$, $\phi \not\equiv 0$ in Ω with $k\partial_\nu \phi + \phi = 0$ on $\partial\Omega$ for $k \geq 0$ a constant. Then we have*:

(a) *If* $0 \not\equiv -\Delta\phi + q(x)\phi \leq 0$, *then*$\lambda_{1,k}(q) < 0$;
(b) *If* $0 \not\equiv -\Delta\phi + q(x)\phi \geq 0$, *then* $\lambda_{1,k}(q) > 0$;
(c) *If* $-\Delta\phi + q(x)\phi \equiv 0$, *then* $\lambda_{1,k}(q) = 0$.

Consider the following single equation:

$$(2.1)\qquad \begin{cases} -\Delta u = uf(x,u) & \text{in } \Omega, \\ k\partial_\nu u + u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, k is nonnegative constant, ν is the outward unit rector on $\partial\Omega$. Assume that the function $f(x,u)\colon \overline{\Omega}\times[0,\infty)\to\mathbb{R}$ satisfies the following hypotheses:

(H1) $f(x,u)$ is C^α-function in x, where $0<\alpha<1$.
(H2) $f(x,u)$ is C^1-function in u with $f_u(x,u)<0$ for all $(x,u)\in\overline{\Omega}\times[0,\infty)$.
(H3) $f(x,u)\le 0$ in $\overline{\Omega}\times[C,\infty)$ for some positive constant C.

Theorem 2.3 (see [13], [61]).

(a) *The nonnegative solution $u(x)$ of* (2.1) *satisfies $u(x)\le C$ for all $x\in\overline{\Omega}$.*
(b) *If $\lambda_{1,k}(-f(x,0))\ge 0$, then* (2.1) *has no positive solutions. Moreover, the trivial solution $u(x)=0$ is globally asymptotically stable.*
(c) *If $\lambda_{1,k}(-f(x,0))<0$, then* (2.1) *has a unique positive solution which is globally asymptotically stable. In this case, the trivial solution $u(x)=0$ is unstable.*

Now, we state the fixed point index theory, which is a fundamental tool in our proofs.

Let E be a Banach space and $\mathcal{W}\subset E$ is a closed convex set. $\mathcal{W}$ is called a total wedge if $\gamma\mathcal{W}\subset\mathcal{W}$ for all $\gamma\ge 0$ and $\overline{\mathcal{W}-\mathcal{W}}=E$. For $y\in\mathcal{W}$, define

$$\begin{aligned} \mathcal{W}_y &= \{x\in E : y+\gamma x\in\mathcal{W} \text{ for some } \gamma>0\}, \\ S_y &= \{x\in\overline{\mathcal{W}}_y : -x\in\overline{\mathcal{W}}_y\}. \end{aligned}$$

Then $\overline{\mathcal{W}}_y$ is a wedge containing $\mathcal{W}$, y, $-y$, while S_y is a closed subset of E containing y. Let T be a compact linear operator on E which satisfies $T(\overline{\mathcal{W}}_y)\subset\overline{\mathcal{W}}_y$. We say that T has property α on $\overline{\mathcal{W}}_y$ if there is a $t\in(0,1)$ and a $\omega\in\overline{\mathcal{W}}_y\backslash S_y$ such that $(I-tT)\omega\in S_y$. Let $A\colon\mathcal{W}\to\mathcal{W}$ is a compact operator with a fixed point $y\in\mathcal{W}$ and A is Fréchet differentiable at y. Let $\mathcal{L}=A'(y)$ be the Fréchet derivative of A at y. Then $\mathcal{L}$ maps $\overline{\mathcal{W}}_y$ into itself. We denote by $\deg_{\mathcal{W}}(I-A,D)$ the degree of $I-A$ in D relative to $\mathcal{W}$, $\operatorname{index}_{\mathcal{W}}(A,y)$ the fixed point index of A at y relative to $\mathcal{W}$ and

$$\deg_{\mathcal{W}}(I-A,\Phi)=\sum_{y\in\Phi}\operatorname{index}_{\mathcal{W}}(A,y),$$

where $\operatorname{index}_{\mathcal{W}}(A,y)$ the fixed point index of A at y relative to $\mathcal{W}$ and Φ only contains discrete point.

Theorem 2.4 (see [25], [52], [69]). *Assume that $I-\mathcal{L}$ is invertible on $\overline{\mathcal{W}}_y$. Then, we have*:

(a) *If $\mathcal{L}$ has property α on $\overline{\mathcal{W}}_y$, then* $\mathrm{index}_{\mathcal{W}}(A, y) = 0$.
(b) *If $\mathcal{L}$ does not have property α on $\overline{\mathcal{W}}_y$, then* $\mathrm{index}_{\mathcal{W}}(A, y) = (-1)^\sigma$, *where σ is the sum of multiplicities of all eigenvalues of $\mathcal{L}$ which is greater than* 1.

3. Existence and no-existence of coexistence states

3.1. Existence of coexistence states. To give some sufficient conditions for the existence of positive steady-state solutions of (1.6) by using fixed point index theory, we need an a priori estimates for coexistence states of (1.6). So, we first give the proof of Theorem 1.1.

PROOF OF THEOREM 1.1. Since u satisfies

$$\begin{cases} -\Delta u = u\left(a_1 - u - b_{12}v - \dfrac{c_1 w}{(1+\alpha_1 u)(1+\beta_1 w)}\right) \le u(a_1 - u) & \text{in } \Omega, \\ k_1 \partial_\nu u + u = 0 & \text{on } \partial\Omega, \end{cases}$$

we get $u(x) \le a_1$ by maximum principle. Similarly, we get $v(x) \le a_2$. Since $e_1 u/((1+\alpha_1 u)(1+\beta_1 w))$ is increasing in u and $e_2 v/((1+\alpha_2 v)(1+\beta_2 w))$ is increasing in v, then w satisfies

$$\begin{cases} -\Delta w = w\left(\dfrac{e_1 u}{(1+\alpha_1 u)(1+\beta_1 w)} + \dfrac{e_2 v}{(1+\alpha_2 v)(1+\beta_2 w)} - d\right) & \\ \qquad \le w\left(\dfrac{e_1 a_1}{(1+\alpha_1 a_1)(1+\beta_1 w)} + \dfrac{e_2 a_2}{(1+\alpha_2 a_2)(1+\beta_2 w)} - d\right) & \text{in } \Omega, \\ k_3 \partial_\nu w + w = 0 & \text{on } \partial\Omega, \end{cases}$$

we get $w(x) \le Q_3$ by maximum principle. □

We introduce the following notations:

- $E = C^1_{k_1}(\overline{\Omega}) \times C^1_{k_2}(\overline{\Omega}) \times C^1_{k_3}(\overline{\Omega})$,
  where $C^1_{k_i}(\overline{\Omega}) = \{\phi \in C^1(\overline{\Omega}) : k_i \partial_\nu \phi + \phi = 0, \text{ on } \partial\Omega\}$, $i = 1, 2, 3$,
- $\mathcal{W} = K_1 \times K_2 \times K_3$,
  where $K_i = \{\phi \in C^1_{k_i}(\overline{\Omega}) : \phi \ge 0 \text{ on } \overline{\Omega}\}$, $i = 1, 2, 3$,
- $D = \{(u, v, w) \in \mathcal{W} : u < Q_1 + 1,\ v < Q_2 + 1,\ w < Q_3 + 1\}$,
  where Q_1, Q_2, Q_3 are defined in Theorem 1.1.

From Theorem 1.1, we see that the nonnegative solutions of (1.6) must be in D. Take P sufficiently large positive constant with

$$P > \max\{a_1 + b_{12}a_2 + c_1/\beta_1,\ a_2 + b_{21}a_1 + c_2/\beta_2,\ d\}$$

such that

$$u\left(a_1 - u - b_{12}v - \frac{c_1 w}{(1+\alpha_1 u)(1+\beta_1 w)}\right) + Pu,$$
$$v\left(a_2 - b_{21}u - v - \frac{c_2 w}{(1+\alpha_2 v)(1+\beta_2 w)}\right) + Pv,$$

and

$$w\left(\frac{e_1u}{(1+\alpha_1u)(1+\beta_1w)}+\frac{e_2v}{(1+\alpha_2v)(1+\beta_2w)}-d\right)+Pw$$

are respectively monotone increasing with respect to u, v and w for all $(u,v,w)\in[0,Q_1]\times[0,Q_2]\times[0,Q_3]$.

Define a positive compact operator $A\colon D\to\mathcal{W}$ by

$$A(u,v,w)$$
$$=(-\Delta+P)^{-1}\begin{pmatrix} u\left(a_1-u-b_{12}v-\dfrac{c_1w}{(1+\alpha_1u)(1+\beta_1w)}\right)+Pu \\ v\left(a_2-b_{21}u-v-\dfrac{c_2w}{(1+\alpha_2v)(1+\beta_2w)}\right)+Pv \\ w\left(\dfrac{e_1u}{(1+\alpha_1u)(1+\beta_1w)}+\dfrac{e_2v}{(1+\alpha_2v)(1+\beta_2w)}-d\right)+Pw \end{pmatrix}.$$

Remark 3.1. Note that (1.6) is equivalent to $(u,v,w)=A(u,v,w)$, and therefore it is sufficient to prove A has a positive fixed point in D to show that (1.6) has a positive solution.

The following lemma give the degree of $I-A$ in D relative to $\mathcal{W}$ and the fixed point index of A at the trivial solution $(0,0,0)$ of (1.6) relative to $\mathcal{W}$.

Lemma 3.2. *Assume that $a_i>\lambda_{1,k_i}$ for $i=1,2$, then we have*:

(a) $\deg_{\mathcal{W}}(I-A,D)=1$,
(b) $\operatorname{index}_{\mathcal{W}}(A,(0,0,0))=0$.

Proof. (a) It is easy to see that A has no fixed point on ∂D, so the $\deg_{\mathcal{W}}(I-A,D)$ is well defined. For $\theta\in[0,1]$, we define a positive and compact operator $A_\theta\colon E\to E$ by

$$A_\theta(u,v,w)$$
$$=(-\Delta+P)^{-1}\begin{pmatrix} \theta u\left(a_1-u-b_{12}v-\dfrac{c_1w}{(1+\alpha_1u)(1+\beta_1w)}\right)+Pu \\ \theta v\left(a_2-b_{21}u-v-\dfrac{c_2w}{(1+\alpha_2v)(1+\beta_2w)}\right)+Pv \\ \theta w\left(\dfrac{e_1u}{(1+\alpha_1u)(1+\beta_1w)}+\dfrac{e_2v}{(1+\alpha_2v)(1+\beta_2w)}-d\right)+Pw \end{pmatrix},$$

then $A_1=A$.

For each θ, a fixed point of A_θ is a solution of the following problem:

$$\begin{cases} -\Delta u = \theta u\left(a_1 - u - b_{12}v - \dfrac{c_1 w}{(1+\alpha_1 u)(1+\beta_1 w)}\right) & \text{in } \Omega, \\ -\Delta v = \theta v\left(a_2 - b_{21}u - v - \dfrac{c_2 w}{(1+\alpha_2 v)(1+\beta_2 w)}\right) & \text{in } \Omega, \\ -\Delta w = \theta w\left(\dfrac{e_1 u}{(1+\alpha_1 u)(1+\beta_1 w)} + \dfrac{e_2 v}{(1+\alpha_2 v)(1+\beta_2 w)} - d\right) & \text{in } \Omega, \\ k_1\partial_\nu u + u = k_2\partial_\nu v + v = k_3\partial_\nu w + w = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.1}$$

As in Theorem 1.1, we see that the fixed point of A_θ satisfies $u(x) \le Q_1$, $v(x) \le Q_2$, $w(x) \le Q_3$ for each $\theta \in [0,1]$. So A_θ has no fixed point on ∂D, the $\deg_{\mathcal{W}}(I - A_\theta, D)$ is well defined and $\deg_{\mathcal{W}}(I - A_\theta, D)$ is independent of θ. Therefore

$$\deg_{\mathcal{W}}(I - A, D) = \deg_{\mathcal{W}}(I - A_1, D) = \deg_{\mathcal{W}}(I - A_0, D).$$

Note that (3.1) has only the trivial solution $(0,0,0)$ when $\theta = 0$. Set

$$\mathcal{L} = A_0'(0,0,0) = (-\Delta + P)^{-1}\begin{pmatrix} P & 0 & 0 \\ 0 & P & 0 \\ 0 & 0 & P \end{pmatrix}.$$

Assume that $\mathcal{L}(\xi_1,\xi_2,\xi_3) = (\xi_1,\xi_2,\xi_3)$ for some $(\xi_1,\xi_2,\xi_3) \in \overline{\mathcal{W}}_{(0,0,0)} = K_1 \times K_2 \times K_3$. It is easy to see $(\xi_1,\xi_2,\xi_3) = (0,0,0)$. Thus $I - \mathcal{L}$ is invertible on $\overline{\mathcal{W}}_{(0,0,0)}$. Since $\lambda_{1,k_i} > 0$, we see that $r_{k_i}((-\Delta + P)^{-1}(P)) < 1$ for $i = 1,2,3$ by Theorem 2.1. This implies that $\mathcal{L}$ does not have property α. So, by Theorem 2.4, we get

$$\deg_{\mathcal{W}}(I - A, D) = \deg_{\mathcal{W}}(I - A_0, D) = \operatorname{index}_{\mathcal{W}}(A_0, (0,0,0)) = 1.$$

(b) Observe that $A(0,0,0) = (0,0,0)$. Let $\mathcal{L} = A'(0,0,0)$, then

$$\mathcal{L} = A'(0,0,0) = (-\Delta + P)^{-1}\begin{pmatrix} a_1 + P & 0 & 0 \\ 0 & a_2 + P & 0 \\ 0 & 0 & -d + P \end{pmatrix}.$$

Assume that $\mathcal{L}(\xi_1,\xi_2,\xi_3) = (\xi_1,\xi_2,\xi_3)$ for some $(\xi_1,\xi_2,\xi_3) \in \overline{\mathcal{W}}_{(0,0,0)}$, then

$$\begin{cases} -\Delta\xi_1 = a_1\xi_1 & \text{in } \Omega, \\ k_1\partial_\nu\xi_1 + \xi_1 = 0 & \text{on } \partial\Omega, \end{cases} \qquad \begin{cases} -\Delta\xi_2 = a_2\xi_2 & \text{in } \Omega, \\ k_2\partial_\nu\xi_2 + \xi_2 = 0 & \text{on } \partial\Omega, \end{cases}$$

$$\begin{cases} -\Delta\xi_3 = -d\xi_3 & \text{in } \Omega, \\ k_3\partial_\nu\xi_3 + \xi_3 = 0 & \text{on } \partial\Omega. \end{cases}$$

Since $d > 0$, we see $\xi_3 = 0$. If $\xi_i \ge 0$, $\xi_i \not\equiv 0$, then $a_i = \lambda_{1,k_i}$ for $i = 1,2$, which is contradicts to $\lambda_{1,k_i} < a_i$. So $(\xi_1,\xi_2,\xi_3) = (0,0,0)$. Thus $I - \mathcal{L}$ is invertible on $\overline{\mathcal{W}}_{(0,0,0)}$.

Since $a_1 > \lambda_{1,k_1}$, by Theorem 2.1, we see that $r = r_{k_1}((-\Delta+P)^{-1}(a_1+P)) > 1$ and r is the principle eigenvalue of $(-\Delta + P)^{-1}(a_1 + P)$ with a corresponding eigenfunction $\phi > 0$.

Since $S_{(0,0,0)} = \{(0,0,0)\}$, we see that $(\phi, 0, 0) \in \overline{\mathcal{W}}_{(0,0,0)} \setminus S_{(0,0,0)}$. Set $t = 1/r \in (0,1)$, then $(I - t\mathcal{L})(\phi, 0, 0) = (0,0,0) \in S_{(0,0,0)}$. This shows that $\mathcal{L}$ has property α. Thus $\mathrm{index}_{\mathcal{W}}(A, (0,0,0)) = 0$ by Theorem 2.4. $\square$

Next two lemmas give the index of the semi-trivial solutions $(\Theta_{k_1}[a_1], 0, 0)$ and $(0, \Theta_{k_2}[a_2], 0)$ of (1.6), respectively.

LEMMA 3.3. *Let* $a_1 > \lambda_{1,k_1}$, $a_2 \neq \lambda_{1,k_2}(b_{21}\Theta_{k_1}[a_1])$ *and*

$$-d \neq \lambda_{1,k_3}\left(- \frac{e_1 \Theta_{k_1}[a_1]}{1 + \alpha_1 \Theta_{k_1}[a_1]} \right).$$

Then we have:

(a) *If* $a_2 > \lambda_{1,k_2}(b_{21}\Theta_{k_1}[a_1])$ *or*

$$-d > \lambda_{1,k_3}\left(- \frac{e_1 \Theta_{k_1}[a_1]}{1 + \alpha_1 \Theta_{k_1}[a_1]} \right),$$

then, $\mathrm{index}_{\mathcal{W}}(A, (\Theta_{k_1}[a_1], 0, 0)) = 0$.

(b) *If* $a_2 < \lambda_{1,k_2}(b_{21}\Theta_{k_1}[a_1])$ *and*

$$-d < \lambda_{1,k_3}\left(- \frac{e_1 \Theta_{k_1}[a_1]}{1 + \alpha_1 \Theta_{k_1}[a_1]} \right),$$

then, $\mathrm{index}_{\mathcal{W}}(A, (\Theta_{k_1}[a_1], 0, 0)) = 1$.

PROOF. Observe $A(\Theta_{k_1}[a_1], 0, 0) = (\Theta_{k_1}[a_1], 0, 0)$. Let $\mathcal{L} = A'(\Theta_{k_1}[a_1], 0, 0)$, then

$$\begin{aligned}\mathcal{L} &= A'(\Theta_{k_1}[a_1], 0, 0)\\ &= (-\Delta+P)^{-1} \begin{pmatrix} a_1 + P - 2\Theta_{k_1}[a_1] & -b_{12}\Theta_{k_1}[a_1] & -\dfrac{c_1 \Theta_{k_1}[a_1]}{1 + \alpha_1 \Theta_{k_1}[a_1]} \\ 0 & a_2 + P - b_{21}\Theta_{k_1}[a_1] & 0 \\ 0 & 0 & -d + P + \dfrac{e_1 \Theta_{k_1}[a_1]}{1 + \alpha_1 \Theta_{k_1}[a_1]} \end{pmatrix}.\end{aligned}$$

Let $(\xi_1, \xi_2, \xi_3) = \mathcal{L}(\xi_1, \xi_2, \xi_3)$ for some $(\xi_1, \xi_2, \xi_3) \in \overline{\mathcal{W}}_{(\Theta_{k_1}[a_1],0,0)} = C^1_{k_1}(\overline{\Omega}) \times K_2 \times K_3$. Then

$$(3.2)\qquad \begin{cases} -\Delta \xi_1 + (2\Theta_{k_1}[a_1] - a_1)\xi_1 \\ \qquad\qquad = -b_{12}\Theta_{k_1}[a_1]\xi_2 - \dfrac{c_1 \Theta_{k_1}[a_1]}{1 + \alpha_1 \Theta_{k_1}[a_1]}\xi_3 & \text{in } \Omega, \\ -\Delta \xi_2 + (b_{21}\Theta_{k_1}[a_1] - a_2)\xi_2 = 0 & \text{in } \Omega, \\ -\Delta \xi_3 + \left(d - \dfrac{e_1 \Theta_{k_1}[a_1]}{1 + \alpha_1 \Theta_{k_1}[a_1]} \right)\xi_3 = 0 & \text{in } \Omega, \\ k_1 \partial_\nu \xi_1 + \xi_1 = k_2 \partial_\nu \xi_2 + \xi_2 = k_3 \partial_\nu \xi_3 + \xi_3 = 0 & \text{on } \partial\Omega. \end{cases}$$

Since $\xi_2 \in K_2$, $\xi_3 \in K_3$, $a_2 \neq \lambda_{1,k_2}(b_{21}\Theta_{k_1}[a_1])$ and

$$-d \neq \lambda_{1,k_3}\Big(- \frac{e_1\Theta_{k_1}[a_1]}{1+\alpha_1\Theta_{k_1}[a_1]}\Big),$$

we see $\xi_2 = \xi_3 = 0$. So, we get from the first equation of (3.2) that

$$\begin{cases} -\Delta\xi_1 + (2\Theta_{k_1}[a_1] - a_1)\xi_1 = 0 & \text{in } \Omega, \\ k_1\partial_\nu\xi_1 + \xi_1 = 0 & \text{on } \partial\Omega. \end{cases}$$

If $\xi \not\equiv 0$, then $\lambda_{1,k_1}(2\Theta_{k_1}[a_1] - a_1) = 0$, by Theorem 2.2. On the other hand, $\lambda_{1,k_1}(2\Theta_{k_1}[a_1]-a_1) > \lambda_{1,k_1}(\Theta_{k_1}[a_1]-a_1) = 0$, we get a contradiction. Therefore, $(\xi_1,\xi_2,\xi_3) = (0,0,0)$, i.e. $I - \mathcal{L}$ is invertible on $\overline{\mathcal{W}}_{(\Theta_{k_1}[a_1],0,0)}$. Let us first prove (a).

Case 1. $a_2 > \lambda_{1,k_2}(b_{21}\Theta_{k_1}[a_1])$. Since $a_2 > \lambda_{1,k_2}(b_{21}\Theta_{k_1}[a_1])$, by Theorem 2.1, we have $r = r_{k_2}((-\Delta+P)^{-1}(P + a_2 - b_{21}\Theta_{k_1}[a_1])) > 1$ is an eigenvalue of $(-\Delta+P)^{-1}(P+a_2-b_{21}\Theta_{k_1}[a_1])$ with a corresponding eigenfunction $\phi > 0$. Since$S_{(\Theta_{k_1}[a_1],0,0)} = C^1_{k_1}(\overline{\Omega}) \times \{0\} \times \{0\}$, we see $(0,\phi,0) \in \overline{\mathcal{W}}_{(\Theta_{k_1}[a_1],0,0)} \setminus S_{(\Theta_{k_1}[a_1],0,0)}$. Set $t = 1/r \in (0,1)$, then

$$\begin{aligned}(I-tL)\begin{pmatrix} 0 \\ \phi \\ 0 \end{pmatrix} &= \begin{pmatrix} 0 \\ \phi \\ 0 \end{pmatrix} - t(-\Delta+P)^{-1}\begin{pmatrix} -b_{12}\Theta_{k_1}[a_1]\phi \\ (P+a_2-b_{21}\Theta_{k_1}[a_1])\phi \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} (-\Delta+P)^{-1}tb_{12}\Theta_{k_1}[a_1]\phi \\ 0 \\ 0 \end{pmatrix} \in S_{(\Theta_{k_1}[a_1],0,0)}.\end{aligned}$$

So, $\mathcal{L}$ has property α on $\overline{\mathcal{W}}_{(\Theta_{k_1}[a_1],0,0)}$. Therefore, $\mathrm{index}_{\mathcal{W}}(A,(\Theta_{k_1}[a_1],0,0)) = 0$ according to Theorem 2.4.

Case 2. $-d > \lambda_{1,k_3}(-e_1\Theta_{k_1}[a_1]/(1+\alpha_1\Theta_{k_1}[a_1]))$.

Similar to the proof of Case 1, we can see $\mathcal{L}$ has property α on $\overline{\mathcal{W}}_{(\Theta_{k_1}[a_1],0,0)}$ since $-d > \lambda_{1,k_3}(-e_1\Theta_{k_1}[a_1]/(1+\alpha_1\Theta_{k_1}[a_1]))$. Therefore,

$$\mathrm{index}_{\mathcal{W}}(A,(\Theta_{k_1}[a_1],0,0)) = 0$$

according to Theorem 2.4.

Next, we prove (b). First, we prove that $\mathcal{L}$ does not have property α on $\overline{\mathcal{W}}_{(\Theta_{k_1}[a_1],0,0)}$. Since $a_2 < \lambda_{1,k_2}(b_{21}\Theta_{k_1}[a_1])$, from Theorem 2.1, we have

$$r_{k_2}((-\Delta+P)^{-1}(P+a_2-b_{21}\Theta_{k_1}[a_1])) < 1.$$

On the contrary, we suppose that $\mathcal{L}$ has property α on $\overline{\mathcal{W}}_{(\Theta_{k_1}[a_1],0,0)}$. Then there exist $t \in (0,1)$ and $(\phi_1,\phi_2,\phi_3) \in \overline{\mathcal{W}}_{(\Theta_{k_1}[a_1],0,0)} \setminus S_{(\Theta_{k_1}[a_1],0,0)}$ such that

$$(I - t\mathcal{L})(\phi_1,\phi_2,\phi_3) \in S_{(\Theta_{k_1}[a_1],0,0)}.$$

So,

$$(-\Delta + P)^{-1}(P + a_2 - b_{21}\Theta_{k_1}[a_1])\phi_2 = \frac{1}{t}\,\phi_2.$$

Since $\phi_2 \in K_2 \diagdown \{0\}$, it follows that $1/t > 1$ is an eigenvalue of the operator $(-\Delta+P)^{-1}(P+a_2-b_{21}\Theta_{k_1}[a_1])$, which is contradiction to $r_{k_2}((-\Delta+P)^{-1}(P+a_2-b_{21}\Theta_{k_1}[a_1])) < 1$. So, $\mathcal{L}$ does not have property α on $\overline{\mathcal{W}}_{(\Theta_{k_1}[a_1],0,0)}$. By Theorem 2.4, we have

$$\mathrm{index}_{\mathcal{W}}(A, (\Theta_{k_1}[a_1], 0, 0)) = (-1)^{\sigma},$$

where σ is the sum of the multiplicities of all real eigenvalues of $\mathcal{L}$ which are greater that 1.

Next we will prove $\sigma = 0$. Suppose $1/\rho > 1$ is an eigenvalue of $\mathcal{L}$ with corresponding eigenvalue function (ξ_1, ξ_2, ξ_3), then

$$(-\Delta + P)^{-1}\begin{pmatrix} a_1+P-2\Theta_{k_1}[a_1] & -b_{12}\Theta_{k_1}[a_1] & -\dfrac{c_1\Theta_{k_1}[a_1]}{1+\alpha_1\Theta_{k_1}[a_1]} \\ 0 & a_2+P-b_{21}\Theta_{k_1}[a_1] & 0 \\ 0 & 0 & -d+P+\dfrac{e_1\Theta_{k_1}[a_1]}{1+\alpha_1\Theta_{k_1}[a_1]} \end{pmatrix}$$
$$\cdot \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \frac{1}{\rho}\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix},$$

equivalently,

$$\tag{3.3} \begin{cases} -\Delta\xi_1 + P\xi_1 = \rho\Big((a_1 + P - 2\Theta_{k_1}[a_1])\xi_1 \\ \qquad\qquad -b_{12}\Theta_{k_1}[a_1]\xi_2 - \dfrac{c_1\Theta_{k_1}[a_1]}{1+\alpha_1\Theta_{k_1}[a_1]}\,\xi_3\Big) & \text{in } \Omega, \\ -\Delta\xi_2 + P\xi_2 = \rho(a_2 + P - b_{21}\Theta_{k_1}[a_1])\xi_2 & \text{in } \Omega, \\ -\Delta\xi_3 + P\xi_3 = \rho\Big(-d + P + \dfrac{e_1\Theta_{k_1}[a_1]}{1+\alpha_1\Theta_{k_1}[a_1]}\Big)\xi_3 & \text{in } \Omega, \\ k_1\partial_\nu\xi_1 + \xi_1 = k_2\partial_\nu\xi_2 + \xi_2 = k_3\partial_\nu\xi_3 + \xi_3 = 0 & \text{on} \partial\Omega. \end{cases}$$

If $\xi_2 \not\equiv 0$, it follows from the second equation of (3.3) and Theorem 2.2 that

$$\begin{aligned} 0 &= \lambda_{1,k_2}(P(1-\rho) - \rho(a_2 - b_{21}\Theta_{k_1}[a_1])) \\ &> \lambda_{1,k_2}(b_{21}\Theta_{k_1}[a_1] - a_2) = \lambda_{1,k_2}(b_{21}\Theta_{k_1}[a_1]) - a_2, \end{aligned}$$

which is contradict to $a_2 < \lambda_{1,k_2}(b_{21}\Theta_{k_1}[a_1])$, and so $\xi_2 = 0$. Similarly, we can prove $\xi_3 = 0$ since $-d < \lambda_{1,k_3}(-e_1\Theta_{k_1}[a_1]/(1+\alpha_1\Theta_{k_1}[a_1]))$. If $\xi_1 \not\equiv 0$, it follows from the first equation of (3.3) that

$$\begin{aligned} 0 &= \lambda_{1,k_1}(P(1-\rho) - \rho(a_1 - 2\Theta_{k_1}[a_1])) \\ &\geq \lambda_{1,k_1}(2\Theta_{k_1}[a_1] - a_1) > \lambda_{1,k_1}(\Theta_{k_1}[a_1] - a_1) = 0. \end{aligned}$$

This contradiction shows that $\xi_1 = 0$. So, $(\xi_1, \xi_2, \xi_3) = (0, 0, 0)$, which implies that $\mathcal{L}$ has no eigenvalues being greater than 1. Consequently, $\sigma = 0$. Hence, $\mathrm{index}_{\mathcal{W}}(A, (\Theta_{k_1}[a_1], 0, 0)) = 1$. $\square$

Similar to the proof of the above lemma, we have the following lemma about the index of the semi-trivial solution $(0, \Theta_{k_2}[a_2], 0)$ of (1.6):

LEMMA 3.4. *Let* $a_2 > \lambda_{1,k_2}$, $a_1 \neq \lambda_{1,k_1}(b_{12}\Theta_{k_2}[a_2])$ *and*

$$-d \neq \lambda_{1,k_3}\left(-\frac{e_2\Theta_{k_2}[a_2]}{1+\alpha_2\Theta_{k_2}[a_2]}\right).$$

Then we have:

(a) *If* $a_1 > \lambda_{1,k_1}(b_{12}\Theta_{k_2}[a_2])$ *or*

$$-d > \lambda_{1,k_3}\left(-\frac{e_2\Theta_{k_2}[a_2]}{1+\alpha_2\Theta_{k_2}[a_2]}\right),$$

then, $\mathrm{index}_{\mathcal{W}}(A, (0, \Theta_{k_2}[a_2], 0)) = 0$.

(b) *If* $a_1 < \lambda_{1,k_1}(b_{12}\Theta_{k_2}[a_2])$ *and*

$$-d < \lambda_{1,k_3}\left(-\frac{e_2\Theta_{k_2}[a_2]}{1+\alpha_2\Theta_{k_2}[a_2]}\right),$$

then $\mathrm{index}_{\mathcal{W}}(A, (0, \Theta_{k_2}[a_2], 0)) = 1$.

In order to study the other semi-trivial solutions of (1.6), let us consider the following three sub-systems:

$$\begin{cases} -\Delta v = v\left(a_2 - v - \dfrac{c_2 w}{(1+\alpha_2 v)(1+\beta_2 w)}\right) & \text{in } \Omega, \\ -\Delta w = w\left(\dfrac{e_2 v}{(1+\alpha_2 v)(1+\beta_2 w)} - d\right) & \text{in } \Omega, \\ k_2\partial_\nu v + v = k_3\partial_\nu w + w = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.4}$$

$$\begin{cases} -\Delta u = u\left(a_1 - u - \dfrac{c_1 w}{(1+\alpha_1 u)(1+\beta_1 w)}\right) & \text{in } \Omega, \\ -\Delta w = w\left(\dfrac{e_1 u}{(1+\alpha_1 u)(1+\beta_1 w)} - d\right) & \text{in } \Omega, \\ k_1\partial_\nu u + u = k_3\partial_\nu w + w = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.5}$$

$$\begin{cases} -\Delta u = u(a_1 - u - b_{12}v) & \text{in } \Omega, \\ -\Delta v = v(a_2 - b_{21}u - v) & \text{in } \Omega, \\ k_1\partial_\nu u + u = k_2\partial_\nu v + v = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.6}$$

Theorem 3.5.

(a) (3.4) *has a coexistence state* $(v^{(1)}, w^{(1)})$ *with* $v^{(1)} \le \Theta_{k_2}[a_2]$ *if and only if*

$$a_2 > \lambda_{1,k_2} \quad \textit{and} \quad -d > \lambda_{1,k_3}\Big(-\frac{e_2\Theta_{k_2}[a_2]}{1+\alpha_2\Theta_{k_2}[a_2]}\Big).$$

Furthermore, if

$$a_2 - c_2/\beta_2 > \lambda_{1,k_2} \quad \textit{and} \quad -d > \lambda_{1,k_3}\Big(-\frac{e_2\Theta_{k_2}[a_2 - c_2/\beta_2]}{1+\alpha_2\Theta_{k_2}[a_2 - c_2/\beta_2]}\Big),$$

then the coexistence state $(v^{(1)}, w^{(1)})$ *satisfies*

$$\Theta_{k_2}[a_2 - c_2/\beta_2] \le v^{(1)} \quad \textit{and} \quad w_*^{(1)} \le w^{(1)},$$

where $w_*^{(1)}$ *is defined in Theorem* 1.3.

Denote $\Phi_1 = \{(v, w) : (v, w)$ *is the coexistence state of* $(3.4)\}$, *then if* $a_2 > \lambda_{1,k_2}$ *and* $-d > \lambda_{1,k_3}(-e_2\Theta_{k_2}[a_2]/(1 + \alpha_2\Theta_{k_2}[a_2]))$, *we have*

$$\deg_{K_2\times K_3}(I - A|_{K_2\times K_3}, \Phi_1) = 1.$$

(b) (3.5) *has a coexistence state* $(u^{(2)}, w^{(2)})$ *with* $u^{(2)} \le \Theta_{k_1}[a_1]$ *if and only if*

$$a_1 > \lambda_{1,k_1} \quad \textit{and} \quad -d > \lambda_{1,k_3}\Big(-\frac{e_1\Theta_{k_1}[a_1]}{1+\alpha_1\Theta_{k_1}[a_1]}\Big).$$

Furthermore, if

$$a_1 - c_1/\beta_1 > \lambda_{1,k_1} \quad \textit{and} \quad -d > \lambda_{1,k_3}\Big(-\frac{e_1\Theta_{k_1}[a_1 - c_1/\beta_1]}{1+\alpha_1\Theta_{k_1}[a_1 - c_1/\beta_1]}\Big),$$

then the coexistence state $(u^{(2)}, w^{(2)})$ *satisfies*

$$\Theta_{k_1}[a_1 - c_1/\beta_1] \le u^{(2)} \quad \textit{and} \quad w_*^{(2)} \le w^{(2)},$$

where $w_*^{(2)}$ *is defined in Theorem* 1.3.

Denote $\Phi_2 = \{(u, w) : (u, w)$ *is the coexistence state of* $(3.5)\}$, *then if* $a_1 > \lambda_{1,k_1}$ *and*$-d > \lambda_{1,k_3}(-e_1\Theta_{k_1}[a_1]/(1 + \alpha_1\Theta_{k_1}[a_1]))$, *we have*

$$\deg_{K_1\times K_3}(I - A|_{K_1\times K_3}, \Phi_2) = 1.$$

(c) *If* $a_1 > \lambda_{1,k_1}(b_{12}\Theta_{k_2}[a_2])$ *and* $a_2 > \lambda_{1,k_2}(b_{21}\Theta_{k_1}[a_1])$, *then* (3.6) *has a coexistence state* $(u^{(3)}, v^{(3)})$. *Furthermore, we have the following estimates:*

$$\Theta_{k_1}[a_1 - b_{12}\Theta_{k_2}[a_2]] \le u^{(3)} \le \Theta_{k_1}[a_1],$$
$$\Theta_{k_2}[a_2 - b_{21}\Theta_{k_1}[a_1]] \le v^{(3)} \le \Theta_{k_2}[a_2].$$

Denote $\Phi_3 = \{(u, v) : (u, v)$ *is the coexistence state of (3.6)*$\}$, *then if* $a_1 > \lambda_{1,k_1}(b_{12}\Theta_{k_2}[a_2])$ *and* $a_2 > \lambda_{1,k_2}(b_{21}\Theta_{k_1}[a_1])$, *we have*

$$\deg_{K_1\times K_2}(I - A|_{K_1\times K_2}, \Phi_3) = 1.$$

Proof. We only give the proof of (a) since the proofs of (b) and (c) are similar. We first prove the sufficient condition:

$$a_2 > \lambda_{1,k_2} \quad \text{and} \quad -d > \lambda_{1,k_3}\bigg(-\frac{e_2\Theta_{k_2}[a_2]}{1+\alpha_2\Theta_{k_2}[a_2]}\bigg) \Rightarrow \Phi_1 \neq \emptyset.$$

Similarly to the proof of Theorem 1.1, we see any solution (v,w) of (3.4) satisfying $v \le Q_2$ and $w \le Q_3$, where Q_2 and Q_3 are defined in Theorem 1.1.

Denote

- $E_{23} = C^1_{k_2}(\overline{\Omega}) \times C^1_{k_3}(\overline{\Omega})$,
  where $C^1_{k_i}(\overline{\Omega}) = \{\phi \in C^1(\overline{\Omega}) : k_i\partial_\nu\phi + \phi = 0, \text{ on } \partial\Omega\}$, $i = 2,3$,
- $\mathcal{W}_{23} = K_2 \times K_3$, where $K_i = \{\phi \in C^1_{k_i}(\overline{\Omega}) : \phi \ge 0 \text{ on } \overline{\Omega}\}$, $i = 2,3$,
- $D_{23} = \{(v,w) \in \mathcal{W}_{23} : v < Q_2 + 1,\ w < Q_3 + 1\}$.

Define a positive compact operator $A_{23}\colon D_{23} \to \mathcal{W}_{23}$ by

$$A_{23}(v,w) = (-\Delta + P)^{-1}\begin{pmatrix} v\bigg(a_2 - v - \dfrac{c_2 w}{(1+\alpha_2 v)(1+\beta_2 w)}\bigg) + Pv \\ w\bigg(\dfrac{e_2 v}{(1+\alpha_2 v)(1+\beta_2 w)} - d\bigg) + Pw \end{pmatrix} = A|_{K_2\times K_3}.$$

Note that (3.4) is equivalent to $(v,w) = A_{23}(v,w)$, and therefore it suffices to prove A_{23} has a positive fixed point in D_{23} to show that (3.4) has a positive solution.

It is easy to see that (3.4) has a trivial solution $(0,0)$ and a semi-trivial solution $(\Theta_{k_2}(a_2),0)$, similar to the proof of Lemma 3.2 and (a) of Lemma 3.4 (see also [23], [24], [25], [78]), we have

$$\begin{aligned} &\deg_{\mathcal{W}_{23}}(I - A_{23}, D_{23}) = 1, \qquad \mathrm{index}_{\mathcal{W}_{23}}(A_{23},(0,0)) = 0, \\ &\mathrm{index}_{\mathcal{W}_{23}}(A_{23},(\Theta_{k_2}(a_2),0)) = 0 \end{aligned}$$

under the condition $a_2 > \lambda_{1,k_2}$ and $-d > \lambda_{1,k_3}(-e_2\Theta_{k_2}[a_2]/(1+\alpha_2\Theta_{k_2}[a_2]))$. So, it follows from the theory of Leray–Schauder degree [59] that

$$\begin{aligned} &\deg_{K_2\times K_3}(I - A|_{K_2\times K_3}, \Phi_1) = \deg_{\mathcal{W}_{23}}(I - A_{23}, \Phi_1) \\ &= \deg_{\mathcal{W}_{23}}(I - A_{23}, D_{23}) - \mathrm{index}_{\mathcal{W}_{23}}(A_{23},(0,0)) - \mathrm{index}_{\mathcal{W}_{23}}(A_{23},(0,0)) = 1. \end{aligned}$$

So $\Phi_1 \neq \emptyset$, i.e. there exist coexistence state $(v^{(1)}, w^{(1)})$ of (3.4).

We next prove the necessary condition:

$$\Phi_1 \neq \emptyset \Rightarrow a_2 > \lambda_{1,k_2} \quad \text{and} \quad -d > \lambda_{1,k_3}\bigg(-\frac{e_2\Theta_{k_2}[a_2]}{1+\alpha_2\Theta_{k_2}[a_2]}\bigg).$$

Suppose the result does not hold, i.e. $a_2 \le \lambda_{1,k_2}$ or

$$-d < \lambda_{1,k_3}\bigg(-\frac{e_2\Theta_{k_2}[a_2]}{1+\alpha_2\Theta_{k_2}[a_2]}\bigg).$$

Without loss of generality, we assume $a_2 \le \lambda_{1,k_2}$, then from the first equation of (3.4), we get

$$\begin{cases} -\Delta v = v\left(a_2 - v - \dfrac{c_2 w}{(1+\alpha_2 v)(1+\beta_2 w)}\right) \le v(a_2 - v) & \text{in } \Omega, \\ k_2 \partial_\nu v + v = 0 & \text{on } \partial\Omega, \end{cases}$$

then we get $v(x) = 0$ from (b) of Theorem 2.3 since $a_2 \le \lambda_{1,k_2}$, which is contradict to $\Phi_1 \ne \emptyset$. So, (3.4) has a coexistence state $(v^{(1)}, w^{(1)})$ with $v^{(1)} \le \Theta_{k_2}[a_2]$ if and only if $a_2 > \lambda_{1,k_2}$ and $-d > \lambda_{1,k_3}(-e_2\Theta_{k_2}[a_2]/(1+\alpha_2\Theta_{k_2}[a_2]))$. The rest proof is similar to the proof Theorem 1.1 by comparison principle [61], so we omit it. The proof is complete. □

Denote

$$\begin{aligned} \Psi_1 &= \{(0, v^{(1)}, w^{(1)}) : \text{where}(v^{(1)}, w^{(1)}) \in \Phi_1\}, \\ \Psi_2 &= \{(u^{(2)}, 0, w^{(2)}) : \text{where}(u^{(2)}, w^{(2)}) \in \Phi_2\}, \\ \Psi_3 &= \{(u^{(3)}, v^{(3)}, 0) : \ \text{where}(u^{(3)}, v^{(3)}) \in \Phi_3\}. \end{aligned}$$

It is easy to see that (1.6) has semi-trivial solutions

$$(0, v^{(1)}, w^{(1)}) \in \Psi_1, \quad (u^{(2)}, 0, w^{(2)}) \in \Psi_2 \quad \text{and} \quad (u^{(3)}, v^{(3)}, 0) \in \Psi_3$$

from Theorem 3.5. Next, we give the degree of $I - A$ in Ψ_1, Ψ_2, Ψ_3 relative to $\mathcal{W}$, respectively.

Lemma 3.6.

(a) *If* $\Phi_1 \ne \emptyset$ *and*

$$a_1 > \lambda_{1,k_1}\left(b_{12} v^{(1)} + \frac{c_1 w^{(1)}}{1+\beta_1 w^{(1)}}\right) \quad \textit{for any } (0, v^{(1)}, w^{(1)}) \in \Psi_1,$$

then $\deg_{\mathcal{W}}(I - A, \Psi_1) = 0$.

(b) *If* $\Phi_1 \ne \emptyset$ *and*

$$a_1 < \lambda_{1,k_1}\left(b_{12} v^{(1)} + \frac{c_1 w^{(1)}}{1+\beta_1 w^{(1)}}\right) \quad \textit{for any } (0, v^{(1)}, w^{(1)}) \in \Psi_1,$$

then $\deg_{\mathcal{W}}(I - A, \Psi_1) = \deg_{K_2\times K_3}(I - A|_{K_2\times K_3}, \Phi_1) = 1$.

(c) *If* $\Phi_2 \ne \emptyset$ *and*

$$a_2 > \lambda_{1,k_2}\left(b_{21} u^{(2)} + \frac{c_2 w^{(2)}}{1+\beta_2 w^{(2)}}\right) \quad \textit{for any } (u^{(2)}, 0, w^{(2)}) \in \Psi_2,$$

then $\deg_{\mathcal{W}}(I - A, \Psi_2) = 0$.

(d) *If* $\Phi_2 \ne \emptyset$ *and*

$$a_2 < \lambda_{1,k_2}\left(b_{21} u^{(2)} + \frac{c_2 w^{(2)}}{1+\beta_2 w^{(2)}}\right) \quad \textit{for any } (u^{(2)}, 0, w^{(2)}) \in \Psi_2,$$

then $\deg_{\mathcal{W}}(I - A, \Psi_2) = \deg_{K_1\times K_3}(I - A|_{K_1\times K_3}, \Phi_2) = 1$.

(e) *If* $\Phi_3 \neq \emptyset$ *and*

$$-d > \lambda_{1,k_3}\bigg(-\frac{e_1 u^{(3)}}{1+\alpha_1 u^{(3)}} - \frac{e_2 v^{(3)}}{1+\alpha_2 v^{(3)}}\bigg) \quad \textit{for any } (u^{(3)}, v^{(3)}, 0) \in \Psi_3,$$

then $\deg_{\mathcal{W}}(I-A, \Psi_3) = 0$.

(f) *If* $a_1 > \lambda_{1,k_1}(b_{12}\Theta_{k_2}[a_2])$, $a_2 > \lambda_{1,k_2}(b_{21}\Theta_{k_1}[a_1])$ *and*

$$-d < \lambda_{1,k_3}\bigg(-\frac{e_1 u^{(3)}}{1+\alpha_1 u^{(3)}} - \frac{e_2 v^{(3)}}{1+\alpha_2 v^{(3)}}\bigg) \quad \textit{for any } (u^{(3)}, v^{(3)}, 0) \in \Psi_3,$$

then $\deg_{\mathcal{W}}(I-A, \Psi_3) = \deg_{K_1\times K_2}(I - A|_{K_1\times K_2}, \Phi_3) = 1$.

REMARK 3.7. (a) If $\Phi_i = \emptyset$ $(i = 1, 2, 3)$, it is easy to see $\deg_{\mathcal{W}}(I-A, \Psi_i) = 0$ from Leray–Schauder degree theory [59].

(b) From Theorem 3.5, we know that

$$\Phi_1 \neq \emptyset \Leftrightarrow a_2 > \lambda_{1,k_2} \quad \text{and} \quad -d > \lambda_{1,k_3}\bigg(-\frac{e_2\Theta_{k_2}[a_2]}{1+\alpha_2\Theta_{k_2}[a_2]}\bigg).$$

So, we have $\Phi_1 \neq \emptyset \Rightarrow \deg_{K_2\times K_3}(I-A|_{K_2\times K_3}, \Phi_1) = 1$ in case (b) of Lemma 3.6 by (a) of Theorem 3.5. The same argument holds for case (d) of Lemma 3.6. However, this is not true for (f) of Lemma 3.6 since

$$\Phi_3 \neq \emptyset \quad \begin{matrix} \not\Rightarrow \\ \Leftarrow \end{matrix} \quad a_1 > \lambda_{1,k_1}(b_{12}\Theta_{k_2}[a_2]) \quad \text{and} \quad a_2 > \lambda_{1,k_2}(b_{21}\Theta_{k_1}[a_1])$$

by (c) of Theorem 3.5. In fact, similar to [23], [24], [52], we have

$$\begin{aligned}
&\deg_{K_1\times K_2}(I - A|_{K_1\times K_2}, \Phi_3) \\
&\quad = \begin{cases} -1 & \text{if} a_1 < \lambda_{1,k_1}(b_{12}\Theta_{k_2}[a_2]), \text{ and } a_2 < \lambda_{1,k_2}(b_{21}\Theta_{k_1}[a_1]), \\ \ \ 1 & \text{if } a_1 > \lambda_{1,k_1}(b_{12}\Theta_{k_2}[a_2]), \text{ and } a_2 > \lambda_{1,k_2}(b_{21}\Theta_{k_1}[a_1]), \\ \ \ 0 & \text{if } (a_1 - \lambda_{1,k_1}(b_{12}\Theta_{k_2}[a_2]))(a_2 - \lambda_{1,k_2}(b_{21}\Theta_{k_1}[a_1])) < 0. \end{cases}
\end{aligned}$$

PROOF. We only prove (a) and (b) since the proofs of (c) and (e) are similar to (a) and the proof of (d) and (d) are similar to (b).

(a) For $\theta \geq 0$, we define a homotopy A_θ by

$$\begin{aligned}
&A_\theta(u, v, w) \\
&= (-\Delta+P)^{-1}\left(\begin{matrix} u\Big(\theta + a_1 - u - b_{12}v - \dfrac{c_1 w}{(1+\alpha_1 u)(1+\beta_1 w)}\Big) + Pu \\ v\Big(a_2 - b_{21}u - v - \dfrac{c_2 w}{(1+\alpha_2 v)(1+\beta_2 w)}\Big) + Pv \\ w\Big(\dfrac{e_1 u}{(1+\alpha_1 u)(1+\beta_1 w)} + \dfrac{e_2 v}{(1+\alpha_2 v)(1+\beta_2 w)} - d\Big) + Pw \end{matrix}\right).
\end{aligned}$$

Next, we prove the following two claims:

Claim 1. For any $\theta \geq 0$, there is a sufficiently small open neighbourhood $N_\delta(0, v^{(1)}, w^{(1)})$ for any $(0, v^{(1)}, w^{(1)}) \in \Psi_1$ such that A_θ has no fixed point on $\partial N_\delta(0, v^{(1)}, w^{(1)})$, i.e. $\deg_{\mathcal{W}}(I - A_\theta, N_\delta(0, v^{(1)}, w^{(1)}))$ is well defined.

Claim 2. A_θ has no fixed point in $N_\delta(0, v^{(1)}, w^{(1)})$ for θ sufficiently large, i.e. $\deg_{\mathcal{W}}(I - A_\theta, N_\delta(0, v^{(1)}, w^{(1)})) = 0$ for θ sufficiently large.

Proof of Claim 1. Suppose the result does not hold, then there exist $\delta_n \to 0$, $\theta_n \geq 0$ and $(u_n, v_n, w_n) \in \partial N_{\delta_n}(0, v^{(1)}, w^{(1)})$ with $A_\theta(u_n, v_n, w_n) = (u_n, v_n, w_n)$. Since $\delta_n \to 0$, we see that $u_n \to 0$, $v_n \to v^{(1)}$, $w_n \to w^{(1)}$. If $u_n \equiv 0$, then $\partial N_{\delta_n}(0, v^{(1)}, w^{(1)}) \ni (0, v_n, w_n) = (0, v^{(1)}, w^{(1)}) \notin \partial N_{\delta_n}(0, v^{(1)}, w^{(1)})$, a contradiction. Therefore, $u_n \not\equiv 0$ in Ω for all n. Since (u_n, v_n, w_n) is a positive fixed point of A_θ, we have

$$\theta_n + a_1 = \lambda_{1,k_1}\left(u_n + b_{12}v_n + \frac{c_1 w_n}{(1 + \alpha_1 u_n)(1 + \beta_1 w_n)}\right)$$

by Theorem 2.2. Thus,

$$\begin{aligned} a_1 &\leq \lambda_{1,k_1}\left(u_n + b_{12}v_n + \frac{c_1 w_n}{(1 + \alpha_1 u_n)(1 + \beta_1 w_n)}\right) \\ &\to \lambda_{1,k_1}\left(b_{12}v^{(1)} + \frac{c_1 w^{(1)}}{1 + \beta_1 w^{(1)}}\right) \quad (\text{as} \quad \delta_n \to 0) \\ &< a_1, \end{aligned}$$

a contradiction. So we get Claim 1.

Proof of Claim 2. A contradiction argument will be used again by assuming there exist $\theta_n \to \infty$ and $(u_n, v_n, w_n) \in N_\delta(0, v^{(1)}, w^{(1)})$ such that $A_\theta(u_n, v_n, w_n) = (u_n, v_n, w_n)$. Then, by Theorem 2.2,

$$\theta_n + a_1 = \lambda_{1,k_1}\left(u_n + b_{12}v_n + \frac{c_1 w_n}{(1 + \alpha_1 u_n)(1 + \beta_1 w_n)}\right).$$

Moreover, since $(u_n, v_n, w_n) \in N_\delta(0, v^{(1)}, w^{(1)})$, we have

$$\max_{x \in \overline{\Omega}} u_n(x) \leq \delta, \quad \max_{x \in \overline{\Omega}} v_n(x) \leq \delta + \max_{x \in \overline{\Omega}} v^{(1)}(x),$$

and thus

$$\theta_n + a_1 \leq \lambda_{1,k_1}\left(\delta + b_{12}\delta + b_{12}\max_{x \in \overline{\Omega}} v^{(1)}(x) + c_1/\beta_1\right)$$

by the comparison property of principle eigenvalues. This derives a contradiction since $\theta_n \to \infty$. From Claims 1, 2 and the Leray–Schauder degree theory [59], we have

$$\begin{aligned} \deg_{\mathcal{W}}(I - A, \Psi_1) &= \sum_{(0, v^{(1)}, w^{(1)}) \in \Psi_1} \operatorname{index}_{\mathcal{W}}(A, (0, v^{(1)}, w^{(1)})) \\ &= \sum_{(0, v^{(1)}, w^{(1)}) \in \Psi_1} \deg_{\mathcal{W}}(I - A_\theta, N_\delta(0, v^{(1)}, w^{(1)})) = 0. \end{aligned}$$

(b) For $\theta \in [0,1]$, we define a homotopy A_θ by

$$A_\theta(u,v,w) = (-\Delta+P)^{-1}\begin{pmatrix} \theta\Big(u\Big(a_1 - u - b_{12}v - \dfrac{c_1 w}{(1+\alpha_1 u)(1+\beta_1 w)}\Big) + Pu\Big) \\ v\Big(a_2 - b_{21}u - v - \dfrac{c_2 w}{(1+\alpha_2 v)(1+\beta_2 w)}\Big) + Pv \\ w\Big(\dfrac{e_1 u}{(1+\alpha_1 u)(1+\beta_1 w)} + \dfrac{e_2 v}{(1+\alpha_2 v)(1+\beta_2 w)} - d\Big) + Pw \end{pmatrix}.$$

Next, we prove the following claim:

CLAIM 3. For any $\theta \in [0,1]$, there is a sufficiently small open neighbourhood $N_\delta(0, v^{(1)}, w^{(1)})$ for any $(0, v^{(1)}, w^{(1)}) \in \Psi_1$ such that A_θ has no fixed point on $\partial N_\delta(0, v^{(1)}, w^{(1)})$, i.e. $\deg_{\mathcal{W}}(I - A_\theta, N_\delta(0, v^{(1)}, w^{(1)}))$ is well defined.

PROOF OF THE CLAIM 3. Suppose by contradiction that there is $\delta_n \to 0$, $\theta_n \in [0,1]$ and $(u_n, v_n, w_n) \in \partial N_{\delta_n}(0, v^{(1)}, w^{(1)})$ such that $A_{\theta_n}(u_n, v_n, w_n) = (u_n, v_n, w_n)$. As the proof of (a), we can easily see that $u_n \to 0$, $v_n \to v^{(1)}$, $w_n \to w^{(1)}$ as $n \to \infty$ with $u_n \not\equiv 0$. Since (u_n, v_n, w_n) is a positive fixed point of A_{θ_n}, we have

$$\begin{cases} -\Delta u_n = \theta_n u_n\Big(a_1 - u_n - b_{12}v_n - \dfrac{c_1 w_n}{(1+\alpha_1 u_n)(1+\beta_1 w_n)} + P\Big) - Pu_n & \text{in } \Omega, \\ k_1 \partial_\nu u_n + u_n = 0 & \text{on } \partial\Omega. \end{cases}$$

Therefore,

$$\theta_n a_1 = \lambda_{1,k_1}\Big(\theta_n\Big(u_n + b_{12}v_n + \frac{c_1 w_n}{(1+\alpha_1 u_n)(1+\beta_1 w_n)} - P\Big) + P\Big)$$

by Theorem 2.2. Since

$$\theta u\Big(a_1 - u - b_{12}v - \frac{c_1 w_n}{(1+\alpha_1 u_n)(1+\beta_1 w_n)} + P\Big) > 0$$

for all $(u,v,w) \in D$ and $\theta \in [0,1]$ from the definition of P, we can derive the following contradiction as $\delta_n \to 0$:

$$\begin{aligned} a_1 \geq \theta_n a_1 &= \lambda_{1,k_1}\Big(\theta_n\Big(u_n + b_{12}v_n + \frac{c_1 w_n}{(1+\alpha_1 u_n)(1+\beta_1 w_n)} - P\Big) + P\Big) \\ &\geq \lambda_{1,k_1}\Big(u_n + b_{12}v_n + \frac{c_1 w_n}{(1+\alpha_1 u_n)(1+\beta_1 w_n)}\Big) \\ &\to \lambda_{1,k_1}\Big(b_{12}v^{(1)} + \frac{c_1 w^{(1)}}{1+\beta_1 w^{(1)}}\Big) > a_1. \end{aligned}$$

Since $\deg_{\mathcal{W}}(I - A_\theta, N_\delta(0, v^{(1)}, w^{(1)}))$ is independent of θ [59], we see that

$$\deg_{\mathcal{W}}(I - A, \Psi_1) = \deg_{\mathcal{W}}(I - A_0, \Psi_1)$$

$$
\begin{aligned}
&= \sum_{(0,v^{(1)},w^{(1)})\in\Psi_1} \deg_{\mathcal{W}}(I-A_0, N_\delta(0,v^{(1)},w^{(1)})) \\
&= \sum_{(v^{(1)},w^{(1)})\in\Phi_1} \deg_{K_2\times K_3}(I-A_0|_{K_2\times K_3}, N_\delta(v^{(1)},w^{(1)}))\cdot \deg_{K_1}(I-A_0|_{K_1}, N_\delta(0)) \\
&= \sum_{(v^{(1)},w^{(1)})\in\Phi_1} \deg_{K_2\times K_3}(I-A|_{K_2\times K_3}, N_\delta(v^{(1)},w^{(1)})) \\
&= \deg_{K_2\times K_3}(I-A|_{K_2\times K_3}, \Phi_1) = 1
\end{aligned}
$$

by theory of Leray–Schauder degree [59] and (a) of Theorem 3.5. □

Now, we are ready to discuss the existence of positive steady-state solution of (1.5), that is, give the proof of Theorem 1.3.

PROOF OF THEOREM 1.3. We prove (a) in detail only since the other cases can be obtained by using a similar argument. Since $a_i > \lambda_{1,k_i}$ for $i=1,2$, we obtain $\deg_{\mathcal{W}}(I-A,D)=1$ and $\mathrm{index}_{\mathcal{W}}(A,(0,0,0))=0$ from Lemma 3.2. Thus, it suffices to prove that

$$
\mathrm{index}_{\mathcal{W}}(A,(\Theta_{k_1}[a_1],0,0)+\mathrm{index}_{\mathcal{W}}(A,(0,\Theta_{k_2}[a_2],0))+\sum_{i=1}^{3}\deg_{\mathcal{W}}(I-A,\Psi_i)\neq 1.
$$

(a) Since $a_2 > \lambda_{1,k_2}(b_{21}\Theta_{k_1}[a_1])$ and $a_1 > \lambda_{1,k_1}(b_{12}\Theta_{k_2}[a_2])$, we have

$$
\mathrm{index}_{\mathcal{W}}(A,(\Theta_{k_1}[a_1],0,0)=\mathrm{index}_{\mathcal{W}}(A,(0,\Theta_{k_2}[a_2],0))=0
$$

from (a) of Lemma 3.3 and (a) of Lemma 3.4, respectively.

Moreover, Theorem 3.5 implies $\Psi_i\neq\emptyset$ for all $i=1,2,3$. Let $(0,v^{(1)},w^{(1)})\in\Psi_1$, $(u^{(2)},0,w^{(2)})\in\Psi_2$ and $(u^{(3)},v^{(3)},0)\in\Psi_3$. Then, since $v^{(1)}\leq\Theta_{k_2}[a_2]$, $u^{(2)}\leq\Theta_{k_1}[a_1]$, $u^{(3)}\geq\Theta_{k_1}[a_1-b_{12}\Theta_{k_2}[a_2]]$, $v^{(3)}\geq\Theta_{k_2}[a_2-b_{21}\Theta_{k_1}[a_1]]$, we have

$$
\begin{aligned}
a_1 &> \lambda_{1,k_1}\left(b_{12}\Theta_{k_2}[a_2]+\frac{c_1}{\beta_1}\right)\geq\lambda_{1,k_1}\left(b_{12}v^{(1)}+\frac{c_1w^{(1)}}{1+\beta_1w^{(1)}}\right),\\
a_2 &> \lambda_{1,k_2}\left(b_{21}\Theta_{k_1}[a_1]+\frac{c_2}{\beta_2}\right)\geq\lambda_{1,k_2}\left(b_{21}u^{(2)}+\frac{c_2w^{(2)}}{1+\beta_2w^{(2)}}\right),\\
-d &> \lambda_{1,k_3}\left(\Gamma_2^{(1)}+\Gamma_2^{(2)}\right)\geq\lambda_{1,k_3}\left(-\frac{e_1u^{(3)}}{1+\alpha_1u^{(3)}}-\frac{e_2v^{(3)}}{1+\alpha_2v^{(3)}}\right)
\end{aligned}
$$

by the comparison property of principle eigenvalues, thus $\deg_{\mathcal{W}}(I-A,\Psi_i)=0$ for $i=1,2,3$ by Lemma 3.6. Synthetically, we have

$$
\mathrm{index}_{\mathcal{W}}(A,(\Theta_{k_1}[a_1],0,0)+\mathrm{index}_{\mathcal{W}}(A,(0,\Theta_{k_2}[a_2],0))+\sum_{i=1}^{3}\deg_{\mathcal{W}}(I-A,\Psi_i)=0.
$$

(b) $\Psi_1=\varnothing$ from (a) of Theorem 3.5 since $-d<\lambda_{1,k_3}(\Gamma_1^{(2)})$, and therefore $\deg_{\mathcal{W}}(I-A,\Psi_1)=0$. The remaining proofs are similar to (a).

(c) The proof is similar to (b).

(d) Since $-d < \min\{\lambda_{1,k_3}(\Gamma_1^{(1)}), \lambda_{1,k_3}(\Gamma_1^{(2)})\}$, $\Psi_1 = \Psi_2 = \emptyset$ by (a) and (b) of Theorem 3.5, and thus one can similarly show that

$$\mathrm{index}_{\mathcal{W}}(A, (\Theta_{k_1}[a_1], 0, 0) + \mathrm{index}_{\mathcal{W}}(A, (0, \Theta_{k_2}[a_2], 0)) + \sum_{i=1}^{3} \deg_{\mathcal{W}}(I - A, \Psi_i) = 0.$$

(e) We have

$$\mathrm{index}_{\mathcal{W}}(A, (\Theta_{k_1}[a_1], 0, 0) = \mathrm{index}_{\mathcal{W}}(A, (0, \Theta_{k_2}[a_2], 0)) = \deg_{\mathcal{W}}(I - A, \Psi_3) = 0,$$

but $\deg_{\mathcal{W}}(I - A, \Psi_1) = \deg_{\mathcal{W}}(I - A, \Psi_2) = 1$. Therefore,

$$\mathrm{index}_{\mathcal{W}}(A, (\Theta_{k_1}[a_1], 0, 0) + \mathrm{index}_{\mathcal{W}}(A, (0, \Theta_{k_2}[a_2], 0)) + \sum_{i=1}^{3} \deg_{\mathcal{W}}(I - A, \Psi_i) = 2.$$

(f) Observe that $a_1 < \lambda_{1,k_1}(b_{12}\Theta_{k_2}[a_2])$ and $a_2 > \lambda_{1,k_2}(b_{21}\Theta_{k_1}[a_1])$ do not imply $\Psi_3 = \emptyset$ or $\Psi_3 \neq \emptyset$. If $\Psi_3 = \emptyset$, then clearly $\deg_{\mathcal{W}}(I - A, \Psi_3) = 0$ holds. Even if $\Psi_3 \neq \emptyset$ are assumed, we have $\deg_{\mathcal{W}}(I - A, \Psi_3) = 0$ by (e) of Lemma 3.6 since

$$-d > \lambda_{1,k_3}(\Gamma_2^{(2)}) \geq \lambda_{1,k_3}\bigg(- \frac{e_1 u^{(3)}}{1 + \alpha_1 u^{(3)}} - \frac{e_2 v^{(3)}}{1 + \alpha_2 v^3} \bigg).$$

The other degrees re the same as (e), and thus

$$\mathrm{index}_{\mathcal{W}}(A, (\Theta_{k_1}[a_1], 0, 0) + \mathrm{index}_{\mathcal{W}}(A, (0, \Theta_{k_2}[a_2], 0)) + \sum_{i=1}^{3} \deg_{\mathcal{W}}(I - A, \Psi_i) = 2.$$

(g) The proof is similar to (f).

(h) Since $a_2 < \lambda_{1,k_2}(b_{21}\Theta_{k_1}[a_1 - b_{12}\Theta_{k_2}[a_2]])$, $\Psi_3 = \emptyset$. More precisely, if there is $(u^{(3)}, v^{(3)}, 0) \in \Psi_3$, then we have

$$a_2 = \lambda_{1,k_2}(b_{21}u^{(3)} + v^{(3)}) > \lambda_{1,k_2}(b_{21}\Theta_{k_1}[a_1 - b_{12}\Theta_{k_2}[a_2]]) > a_2$$

by Theorem 2.2 and the comparison property of principle eigenvalues, a contradiction. Then as in (f), we have

$$\mathrm{index}_{\mathcal{W}}(A, (\Theta_{k_1}[a_1], 0, 0) + \mathrm{index}_{\mathcal{W}}(A, (0, \Theta_{k_2}[a_2], 0)) + \sum_{i=1}^{3} \deg_{\mathcal{W}}(I - A, \Psi_i) = 2.$$

(i) The proof is similar to (h). □

We discuss the biology meaning of Theorem 1.3 (b)–(d), (h) and (i).

REMARK 3.8. (a) Theorem 1.3(b)–(c) imply that, even if the subsystem (3.4) or (3.5) does not have a coexistence state ($\Psi_1 = \emptyset$ or $\Psi_2 = \emptyset$), we may expect that (1.6) has at leat one coexistence state by introducing another prey species or by putting them in the same circumstance with weak competition interaction between two prey species u and v (i.e. $a_1 > \lambda_{1,k_1}(b_{12}\Theta_{k_2}[a_2])$ and $a_2 > \lambda_{1,k_2}(b_{21}\Theta_{k_1}[a_1])$).

(b) In Theorem 1.3(h), the assumption

$$a_1 > \lambda_{1,k_1}(b_{12}\Theta_{k_2}[a_2]) \quad \text{and} \quad \lambda_{1,k_2}(b_{21}\Theta_{k_1}[a_1 - b_{12}\Theta_{k_2}[a_2]])$$

can be observed that the first species u dominates the other species v in the competing subsystem (3.6), and therefore u may wipe out v. In this situation, if some predator species w is introduced properly in the system, then we may expect that they coexist.

(c) The remark on Theorem 1.3(i) is similar to remark (b).

3.2. No-existence of coexistence states. Before closing this section, we consider the non-existence of coexistence states of (1.6) and give the proof of Theorem 1.4.

Proof of Theorem 1.4. Assume (1.6) has a coexistence state (u, v, w), then u, v, w satisfy the following three equations respectively,

$$\begin{cases} -\Delta u = u\Big(a_1 - u - b_{12}v - \dfrac{c_1 w}{(1+\alpha_1 u)(1+\beta_1 w)}\Big) & \text{in } \Omega, \\ k_1\partial_\nu u + u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.7}$$

$$\begin{cases} -\Delta v = v\Big(a_2 - b_{21}u - v - \dfrac{c_2 w}{(1+\alpha_2 v)(1+\beta_2 w)}\Big) & \text{in } \Omega, \\ k_2\partial_\nu v + v = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.8}$$

$$\begin{cases} -\Delta w = w\Big(\dfrac{e_1 u}{(1+\alpha_1 u)(1+\beta_1 w)} + \dfrac{e_2 v}{(1+\alpha_2 v)(1+\beta_2 w)} - d\Big) & \text{in } \Omega, \\ k_3\partial_\nu w + w = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.9}$$

(a) Without loss of generality, we only consider the case $a_1 \le \lambda_{1,k_1}$. Since $u > 0$ in Ω, we get from (3.7) that

$$a_1 = \lambda_{1,k_1}\Big(u + b_{12}v + \frac{c_1 w}{(1+\alpha_1 u)(1+\beta_1 w)}\Big) > \lambda_{1,k_1}$$

by Theorem 2.2 and comparison property of principle eigenvalue, this is contradict to $a_1 \le \lambda_{1,k_1}$.

(b) Since $w > 0$ in Ω, $u \le \Theta_{k_1}[a_1]$ and $v \le \Theta_{k_2}[a_2]$, we get from (3.9) that

$$\begin{aligned} -d &= \lambda_{1,k_3}\Big(- \frac{e_1 u}{(1+\alpha_1 u)(1+\beta_1 w)} - \frac{e_2 v}{(1+\alpha_2 v)(1+\beta_2 w)}\Big) \\ &> \lambda_{1,k_3}\Big(- \frac{e_1\Theta_{k_1}[a_1]}{1+\alpha_1\Theta_{k_1}[a_1]} - \frac{e_2\Theta_{k_2}[a_2]}{1+\alpha_2\Theta_{k_2}[a_2]}\Big) \end{aligned}$$

by Theorem 2.2 and comparison property of principle eigenvalue, this is contradict to

$$-d \le \lambda_{1,k_3}\Big(- \frac{e_1\Theta_{k_1}[a_1]}{1+\alpha_1\Theta_{k_1}[a_1]} - \frac{e_2\Theta_{k_2}[a_2]}{1+\alpha_2\Theta_{k_2}[a_2]}\Big).$$

(c) Since $v \leq \Theta_{k_2}[a_2]$ and $c_1 w/((1+\alpha_1 u)(1+\beta_1 w)) \leq c_1/\beta_1$, we get from (3.7) that

$$\begin{cases} -\Delta u \geq u(a_1 - u - b_{12}\Theta_{k_2}[a_2] - c_1/\beta_1) & \text{in } \Omega, \\ k_1 \partial_\nu u + u = 0 & \text{on } \partial\Omega, \end{cases}$$

then, we have $u \geq \Theta_{k_1}[a_1 - b_{12}\Theta_{k_2}[a_2] - c_1/\beta_1]$ by comparison principle. Since $v > 0$ in Ω, we get from (3.8) that

$$a_2 = \lambda_{1,k_2}\left(b_{21}u + v + \frac{c_2 w}{(1+\alpha_2 v)(1+\beta_2 w)}\right) > \lambda_{1,k_2}(b_{21}\Theta_{k_1}[a_1 - b_{12}\Theta_{k_2}[a_2] - c_1/\beta_1])$$

by Theorem 2.2 and comparison property of principle eigenvalue, this is contradict to $a_2 \leq \lambda_{1,k_2}(b_{21}\Theta_{k_1}[a_1 - b_{12}\Theta_{k_2}[a_2] - c_1/\beta_1])$.

(d) The proof is similar to (c). □

4. Uniqueness and stability of coexistence states

In this section, in order to investigate the stability and uniqueness of coexistence states of (1.6), we first present a lemma that mainly show the non-degenerateness and linear stability of coexistence states for (1.6) under some restricted assumptions.

Lemma 4.1. *Assume that* $a_1 > \lambda_{1,k_1}(b_{12}\Theta_{k_2}[a_2])$, $a_2 > \lambda_{1,k_2}(b_{21}\Theta_{k_1}[a_1])$ *and* (1.9) *holds. Then we have*:

(a) (3.6) *has a unique coexistence state* $(u^{(3)}, v^{(3)})$.

(b) *Let* $w^{(3)}$ *be the unique positive solution of the following equation*

$$\begin{cases} -\Delta w = w\left(\dfrac{e_1 u^{(3)}}{(1+\alpha_1 u^{(3)})(1+\beta_1 w)} + \dfrac{e_2 v^{(3)}}{(1+\alpha_2 v^{(3)})(1+\beta_2 w)} - d\right) & \text{in } \Omega, \\ k_3 \partial_\nu w + w = 0 & \text{on } \partial\Omega \end{cases}$$

for

$$-d > \lambda_{1,k_3}\left(-\frac{e_1 u^{(3)}}{1+\alpha_1 u^{(3)}} - \frac{e_2 v^{(3)}}{1+\alpha_2 v^{(3)}}\right)$$

and $w^{(3)} \equiv 0$ *for the other cases. Then the coexistence states of* (1.6) (*if they exist*) *converge to* $(u^{(3)}, v^{(3)}, w^{(3)})$ *as* $c_i \to 0$ *for* $i = 1, 2$.

(c) *There exists a positive constant* $\widetilde{C} = \widetilde{C}(\alpha_1, \alpha_2, \beta_1, \beta_2)$ *such that for* $c_1, c_2 \leq \widetilde{C}$, *any coexistence state of* (1.6) *is non-degenerate and linearly stable.*

(d) *If* $a_1 - c_1/\beta_1 > \lambda_{1,k_1}(b_{12}\Theta_{k_2}[a_2])$ *and*

$$-d > \lambda_{1,k_3}\left(-\frac{e_1\Theta_{k_1}[a_1 - b_{12}\Theta_{k_2}[a_2] - c_1/\beta_1]}{1+\alpha_1\Theta_{k_1}[a_1 - b_{12}\Theta_{k_2}[a_2] - c_1/\beta_1]}\right),$$

then there exists a positive constant $\widetilde{C} = \widetilde{C}(c_2, \alpha_1, \alpha_2, \beta_1, e_2)$ *such that for* $1/\beta_2, c_1 \leq \widetilde{C}$ *any coexistence state of* (1.6) (*if it exists*) *is non-degenerate and linearly stable.*

(e) *If* $a_2 - c_2/\beta_2 > \lambda_{1,k_2}(b_{21}\Theta_{k_1}[a_1])$ *and*

$$-d > \lambda_{1,k_3}\Big(- \frac{e_2\Theta_{k_2}[a_2 - b_{21}\Theta_{k_1}[a_1] - c_2/\beta_2]}{1+\alpha_2\Theta_{k_2}[a_2 - b_{21}\Theta_{k_1}[a_1] - c_2/\beta_2]}\Big),$$

then there exists a positive constant $\widetilde{C} = \widetilde{C}(c_1, \alpha_1, \alpha_2, \beta_2, e_1)$ *such that for* $1/\beta_1, c_2 \leq \widetilde{C}$ *any coexistence state of* (1.6) (*if it exists*) *is non-degenerate and linearly stable.*

PROOF. (a) The existence of a coexistence state of (3.6) follows from (c) of Theorem 3.5. Using the Rayleigh's formula as in [46] or [49] for principle eigenvalue, the unique can be shown similarly. So that its proof is omitted.

(b) It is easy to see that the compact operator $A(u, v, w)$ defined in Section 3 converges to the operator

$$\widetilde{A}(u,v,w) = (-\Delta + P)^{-1} \cdot \begin{pmatrix} u(a_1 - u - b_{12}v) + Pu \\ v(a_2 - b_{21}u - v) + Pv \\ w\Big(\dfrac{e_1 u}{(1+\alpha_1 u)(1+\beta_1 w)} + \dfrac{e_2 v}{(1+\alpha_2 v)(1+\beta_2 w)} - d\Big) + Pw \end{pmatrix},$$

as $c_i \to 0$ for $i = 1, 2$, and coexistence state of (1.6) converge to the fixed points of $\widetilde{A}(u, v, w)$ as $c_i \to 0$ for $i = 1, 2$ since the only fixed point of $\widetilde{A}(u, v, w)$ is $(u^{(3)}, v^{(3)}, w^{(3)})$.

(c) In view of Theorem 11.20 of [71], it suffices to show that the corresponding linearized problem of (1.6) has no eigenvalue μ with $\mathrm{Re}(\mu) \leq 0$. To do this, a contradiction argument will be used by assuming that (1.6) has a coexistence state (u_i, v_i, w_i) which is either degenerate or linearly unstable for sequence $\{c_{1,i}\}$ and $\{c_{2,i}\}$ with $c_{1,i},\ c_{2,i} \to 0$ where $i \geq 1$. Thus there exist μ_i with $\mathrm{Re}(\mu_i) \leq 0$ and $(\xi_i, \eta_i, \zeta_i) \neq (0, 0, 0)$ such that

$$(4.1)\quad \begin{cases} -\Delta\xi_i - \Big(a_1 - 2u_i - b_{12}v_i - \dfrac{c_{1,i}w_i}{(1+\alpha_1 u_i)^2(1+\beta_1 w_i)}\Big)\xi_i \\ \quad + b_{12}u_i\eta_i + \dfrac{c_{1,i}u_i}{(1+\alpha_1 u_i)(1+\beta_1 w_i)^2}\zeta_i = \mu_i\xi_i & \text{in } \Omega, \\ -\Delta\eta_i + b_{21}v_i\xi_i - \Big(a_2 - b_{21}u_i - 2v_i - \dfrac{c_{2,i}w_i}{(1+\alpha_2 v_i)^2(1+\beta_2 w_i)}\Big)\eta_i \\ \quad + \dfrac{c_{2,i}v_i}{(1+\alpha_2 v_i)(1+\beta_2 w_i)^2}\zeta_i = \mu_i\eta_i & \text{in } \Omega, \\ -\Delta\zeta_i - \dfrac{e_1 w_i}{(1+\alpha_1 u_i)^2(1+\beta_1 w_i)}\xi_i - \dfrac{e_2 w_i}{(1+\alpha_2 v_i)^2(1+\beta_2 w_i)}\eta_i \\ \quad - \Big(\dfrac{e_1 u_i}{(1+\alpha_1 u_i)(1+\beta_1 w_i)^2} + \dfrac{e_2 v_i}{(1+\alpha_2 v_i)(1+\beta_2 w_i)^2} - d\Big)\zeta_i = \mu_i\zeta_i & \text{in } \Omega, \\ k_1\partial_\nu\xi_i + \xi_i = k_2\partial_\nu\eta_i + \eta_i = k_3\partial_\nu\zeta_i + \zeta_i = 0 & \text{on } \partial\Omega. \end{cases}$$

There are two cases for our considerations.

CASE 1.

$$-d > \lambda_{1,k_3}\Big(-\frac{e_1u^{(3)}}{1+\alpha_1u^{(3)}}-\frac{e_2v^{(3)}}{1+\alpha_2v^3}\Big).$$

Assume that

$$\|\xi_i\|^2_{L^2}+\|\eta_i\|^2_{L^2}=\|\zeta_i\|^2_{L^2}=1$$

and observe that

$$(u_i,v_i,w_i)\to(u^{(3)},v^{(3)},w^{(3)})$$

with $w^{(3)}>0$ in Ω as $c_{1,i},\ c_{2,i}\to 0$ by the previous result (a). Then from (4.1), we have

$$\begin{aligned}
\mu_i = {} & \int_\Omega|\nabla\xi_i|^2-\int_\Omega\Big(a_1-2u_i-b_{12}v_i-\frac{c_{1,i}w_i}{(1+\alpha_1u_i)^2(1+\beta_1w_i)}\Big)|\xi_i|^2\\
&+\int_\Omega b_{12}u_i\eta_i\overline{\xi}_i+\int_\Omega\frac{c_{1,i}u_i}{(1+\alpha_1u_i)(1+\beta_1w_i)^2}\,\zeta_i\overline{\xi}_i+\tau_1\int_{\partial\Omega}|\xi_i|^2\\
&+\int_\Omega|\nabla\eta_i|^2-\int_\Omega\Big(a_2-b_{21}u_i-2v_i-\frac{c_{2,i}w_i}{(1+\alpha_2v_i)^2(1+\beta_2w_i)}\Big)|\eta_i|^2\\
&+\int_\Omega b_{21}v_i\xi_i\overline{\eta}_i+\int_\Omega\frac{c_{2,i}v_i}{(1+\alpha_2v_i)(1+\beta_2w_i)^2}\,\zeta_i\overline{\eta}_i+\tau_2\int_{\partial\Omega}|\eta_i|^2\\
&+\int_\Omega|\nabla\zeta_i|^2-\int_\Omega\frac{e_1w_i}{(1+\alpha_1u_i)^2(1+\beta_1w_i)}\,\xi_i\overline{\zeta}_i\\
&-\int_\Omega\frac{e_2w_i}{(1+\alpha_2v_i)^2(1+\beta_2w_i)}\,\eta_i\overline{\zeta}_i\\
&-\int_\Omega\Big(\frac{e_1u_i}{(1+\alpha_1u_i)(1+\beta_1w_i)^2}+\frac{e_2v_i}{(1+\alpha_2v_i)(1+\beta_2w_i)^2}-d\Big)|\zeta_i|^2+\tau_3\int_{\partial\Omega}|\zeta_i|^2,
\end{aligned}$$

where $\overline{\xi}_i$, $\overline{\eta}_i$ and $\overline{\zeta}_i$ are the respective complex conjugates of ξ_i, η_i and ζ_i and

$$\tau_i=\begin{cases}0 & \text{if } k_i=0,\\ \dfrac{1}{k_i} & \text{if } k_i>0\end{cases}$$

for $i=1,2,3$.

From the above the above identity, we can see that $\{\mathrm{Im}(\mu_i)\}$ and $\{\mathrm{Re}(\mu_i)\}$ are bounded. Therefore, $\{\mu_i\}$ is bounded.

Without loss of generality, we assume that $\mu_i\to\mu$, then $\mathrm{Re}(\mu)\le 0$ follows from $\mathrm{Re}(\mu_i)\le 0$.

We also assume that $\xi_i\to\xi$, $\eta_i\to\eta$ and $\zeta_i\to\zeta$ since $\{\xi_i\}$, $\{\eta_i\}$ and $\{\zeta_i\}$ are bounded.

Taking the limit in (4.1), we obtain

$$
\begin{cases}
-\Delta\xi - (a_1 - 2u^{(3)} - b_{12}v^{(3)})\xi + b_{12}u^{(3)}\eta = \mu\xi & \text{in } \Omega,\\
-\Delta\eta + b_{21}v^{(3)}\xi - (a_2 - b_{21}u^{(3)} - 2v^{(3)})\eta = \mu\eta & \text{in } \Omega,\\
-\Delta\zeta - \dfrac{e_1 w^{(3)}}{(1+\alpha_1 u^{(3)})^2(1+\beta_1 w^{(3)})}\,\xi & \\
\qquad -\dfrac{e_2 w^{(3)}}{(1+\alpha_2 v^{(3)})^2(1+\beta_2 w^{(3)})}\,\eta & \\
\qquad -\Big(\dfrac{e_1 u^{(3)}}{(1+\alpha_1 u^{(3)})(1+\beta_1 w^{(3)})^2} & \\
\qquad +\dfrac{e_2 v^{(3)}}{(1+\alpha_2 v^{(3)})(1+\beta_2 w^{(3)})^2} - d\Big)\zeta = \mu\zeta & \text{in } \Omega,\\
k_1\partial_\nu\xi + \xi = k_2\partial_\nu\eta + \eta = k_3\partial_\nu\zeta + \zeta = 0 & \text{on } \partial\Omega.
\end{cases}
\tag{4.2}
$$

From the first and the second equations of (4.2), we obtain

$$
\mu\int_\Omega(|\xi|^2+|\eta|^2) = \int_\Omega|\nabla\xi|^2 - \int_\Omega(a_1-2u^{(3)}-b_{12}v^{(3)})|\xi|^2 + \int_\Omega b_{12}u^{(3)}\eta\overline{\xi}
$$
$$
+\tau_1\int_{\partial\Omega}|\xi|^2 + \int_\Omega|\nabla\eta|^2 + \int_\Omega b_{21}v^{(3)}\xi\overline{\eta} - \int_\Omega(a_2-b_{21}u^{(3)}-2v^{(3)})|\eta|^2 + \tau_2\int_{\partial\Omega}|\eta|^2,
$$

then we have

$$
\begin{aligned}
0 \ge\;& \mathrm{Re}(\mu)\int_\Omega(|\xi|^2+|\eta|^2)\\
=\;& \int_\Omega|\nabla\mathrm{Re}(\xi)|^2 - \int_\Omega(a_1-u^{(3)}-b_{12}v^{(3)})(\mathrm{Re}(\xi))^2 + \tau_1\int_{\partial\Omega}(\mathrm{Re}(\xi))^2\\
&+\int_\Omega|\nabla\mathrm{Im}(\xi)|^2 - \int_\Omega(a_1-u^{(3)}-b_{12}v^{(3)})(\mathrm{Im}(\xi))^2 + \tau_1\int_{\partial\Omega}(\mathrm{Im}(\xi))^2\\
&+\int_\Omega|\nabla\mathrm{Re}(\eta)|^2 - \int_\Omega(a_2-b_{21}u^{(3)}-v^{(3)})(\mathrm{Re}(\eta))^2 + \tau_2\int_{\partial\Omega}(\mathrm{Re}(\eta))^2\\
&+\int_\Omega|\nabla\mathrm{Im}(\eta)|^2 - \int_\Omega(a_2-b_{21}u^{(3)}-v^{(3)})(\mathrm{Im}(\eta))^2 + \tau_2\int_{\partial\Omega}(\mathrm{Im}(\eta))^2\\
&+\int_\Omega[u^{(3)}((\mathrm{Re}(\xi))^2+(\mathrm{Im}(\xi))^2)\\
&+(b_{21}v^{(3)}+b_{12}u^{(3)})(\mathrm{Re}(\xi)\mathrm{Re}(\eta)+\mathrm{Im}(\xi)\mathrm{Im}(\eta))+v^{(3)}((\mathrm{Re}(\xi))^2+(\mathrm{Im}(\eta))^2)]\\
\ge\;& \int_\Omega[u^{(3)}(\mathrm{Re}(\xi))^2+(b_{21}v^{(3)}+b_{12}u^{(3)})\mathrm{Re}(\xi)\mathrm{Re}(\eta)+v^{(3)}(\mathrm{Re}(\eta))^2] && (\aleph_1)\\
&+\int_\Omega\Big[u^{(3)}(\mathrm{Im}(\xi))^2+(b_{21}v^{(3)}+b_{12}u^{(3)})\mathrm{Im}(\xi)\mathrm{Im}(\eta)+v^{(3)}(\mathrm{Im}(\eta))^2\Big] && (\aleph_2)\\
\ge\;& 0.
\end{aligned}
$$

The last inequality follows from the fact that the integrand in ($\aleph_1$) and ($\aleph_2$) are positive defined since $(b_{21}v^{(3)}+b_{12}u^{(3)})^2-4u^{(3)}v^{(3)} < 0$ under the assumption (1.9) (Note that $\Theta_{k_1}[a_1 - b_{12}\Theta_{k_2}[a_2]] \le u^{(3)} \le \Theta_{k_1}[a_1]$ and $\Theta_{k_2}[a_2 - b_{21}\Theta_{k_1}[a_1]] \le v^{(3)} \le \Theta_{k_2}[a_2]$ follows from (c) of Theorem 3.5). This implies that $\mathrm{Re}(\xi) = \mathrm{Im}(\xi) = \mathrm{Re}(\eta) = \mathrm{Im}(\eta) = 0$, i.e. $\xi = \eta = 0$. Thus, ζ must not be identical zero to avoid a contradiction. Since $\xi = \eta = 0$, the third equation of (4.2) becomes

$$\begin{cases} -\Delta\zeta - \Bigg(\dfrac{e_1 u^{(3)}}{(1+\alpha_1 u^{(3)})(1+\beta_1 w^{(3)})^2} \\ \qquad + \dfrac{e_2 v^{(3)}}{(1+\alpha_2 v^{(3)})(1+\beta_2 w^{(3)})^2} - d\Bigg)\zeta = \mu\zeta & \text{in } \Omega, \\ k_3\partial_\nu \zeta + \zeta = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.3}$$

and μ must be a non-positive real number. Furthermore, since $\zeta \not\equiv 0$, μ can be consider as an eigenvalue of the problem (4.3), and therefore we have

$$\begin{aligned} \mu &= \lambda_{1,k_3}\Bigg(- \frac{e_1 u^{(3)}}{(1+\alpha_1 u^{(3)})(1+\beta_1 w^{(3)})^2} - \frac{e_2 v^{(3)}}{(1+\alpha_2 v^{(3)})(1+\beta_2 w^{(3)})^2} + d\Bigg) \\ &> \lambda_{1,k_3}\Bigg(- \frac{e_1 u^{(3)}}{1+\alpha_1 u^{(3)}} - \frac{e_2 v^{(3)}}{1+\alpha_2 v^{(3)}} + d\Bigg) \ge 0 \end{aligned}$$

by the comparison property of principle eigenvalue and Theorem 2.2, which is contradict to $\mu \le 0$.

CASE 2.

$$-d \le \lambda_{1,k_3}\Bigg(- \frac{e_1 u^{(3)}}{1+\alpha_1 u^{(3)}} - \frac{e_2 v^{(3)}}{1+\alpha_2 v^3}\Bigg).$$

Even if $w^{(3)} \equiv 0$ in this case, a contradiction can be derived similarly as the previous case.

(d) From Theorem 2.3, we see the assumptions $a_1 - c_1/\beta_1 > \lambda_{1,k_1}(b_{12}\Theta_{k_2}[a_2])$ and

$$-d > \lambda_{1,k_3}\Bigg(- \frac{e_1\Theta_{k_1}[a_1 - b_{12}\Theta_{k_2}[a_2] - c_1/\beta_1]}{1+\alpha_1\Theta_{k_1}[a_1 - b_{12}\Theta_{k_2}[a_2] - c_1/\beta_1]}\Bigg),$$

guarantee the existence of $\Theta_{k_1}[a_1 - b_{12}\Theta_{k_2}[a_2] - c_1/\beta_1]$ and w_*, respectively, where w_* is the unique positive solution of the following equation

$$\begin{cases} -\Delta w = w\Bigg(\dfrac{e_1\Theta_{k_1}[a_1 - b_{12}\Theta_{k_2}[a_2] - c_1/\beta_1]}{(1+\alpha_1\Theta_{k_1}[a_1 - b_{12}\Theta_{k_2}[a_2] - c_1/\beta_1])(1+\beta_1 w)} - d\Bigg) & \text{in } \Omega, \\ k_3\partial_\nu w + w = 0 & \text{on } \partial\Omega. \end{cases}$$

Furthermore, $\Theta_{k_1}[a_1 - b_{12}\Theta_{k_2}[a_2] - c_1/\beta_1] \le u$ and $w_* \le w$ follows by comparison principle for any positive solution (u, v, w) of (1.6). Since w_* is a lower solution of w which does not dependent on β_2, we see that $\beta_2 w \to \infty$ as $\beta_2 \to \infty$, equivalently as $1/\beta_2 \to 0$. Therefore, replacing the role of the sequence $\{c_{2,i}\}$ by $\{\beta_{2,i}\}$ in the previous proof, the desired result can be obtained similarly.

(e) The proof is similar to (d). $\square$

By using of Lemma 4.1, we can give the proof of Theorem 1.5.

Proof of Theorem 1.5. We only prove (b) since the proofs for others are similar. It suffices to show the uniqueness. By a compact argument, A has at most finitely many positive fixed points in the region D defined in Section 3. Denote them by (u_i, v_i, w_i) for $i = 1, \dots, k$. For sufficiently small $1/\beta_2$, c_1, it is easy to show that $I - A'(u_i, v_i, w_i)$ is invertible under the condition (1.9) and $A'(u_i, v_i, w_i)$ does not have property α on $\overline{\mathcal{W}}_{(u_i,v_i,w_i)}$ by (d) of Lemma 4.1. In addition, $A'(u_i, v_i, w_i)$ does not have a real eigenvalue which is greater than or equal to one. Then we get $\mathrm{index}_{\mathcal{W}}(A, (u_i, v_i, w_i)) = (-1)^0 = 1$ for $i = 1, \dots, k$ by (b) of Theorem 2.4. Using the additivity property of degree [59] and

$$-d > \lambda_{1,k_3}\left(-\frac{e_2\Theta_{k_2}[a_2]}{1+\alpha_2\Theta_{k_2}[a_2]} \right),$$

we have

$$\begin{aligned} k = {} & \sum_{i=1}^{k} \mathrm{index}_{\mathcal{W}}(A, (u_i, v_i, w_i)) \\ = {} & \deg_{\mathcal{W}}(I - A, D) - \mathrm{index}_{\mathcal{W}}(A, (0,0,0)) - \mathrm{index}_{\mathcal{W}}(A, (\Theta_{k_1}[a_1], 0, 0)) \\ & - \mathrm{index}_{\mathcal{W}}(A, (0, \Theta_{k_2}[a_2], 0)) - \deg_{\mathcal{W}}(I - A, \Psi_1) \\ & - \deg_{\mathcal{W}}(I - A, \Psi_2) - \deg_{\mathcal{W}}(I - A, \Psi_3) \\ = {} & 1 - 0 - 0 - 0 - 0 - 0 - 0 = 1. \end{aligned}$$

So, the uniqueness holds, the proof is complete. □

5. Asymptotic behavior

In this section, the asymptotic behavior of the time-dependent solutions of (1.5) is considered, that is, sufficient conditions for the extinction and permanence to the time-dependent system (1.5) are investigated. We first give the proof of Theorem 1.6.

Proof of Theorem 1.6. (a) First, consider any time-dependent positive solution (u, v, w) of (1.5) satisfies

$$\begin{cases} u_t - \Delta u = u\left(a_1 - u - b_{12}v - \dfrac{c_1 w}{(1+\alpha_1 u)(1+\beta_1 w)} \right) & \\ \qquad\quad \le u(a_1 - u) & \text{in } \Omega \times (0,\infty), \\ v_t - \Delta v = v\left(a_2 - b_{21}u - v - \dfrac{c_2 w}{(1+\alpha_2 v)(1+\beta_2 w)} \right) & \\ \qquad\quad \le v(a_2 - v) & \text{in } \Omega \times (0,\infty), \\ k_1\partial_\nu u + u = k_2\partial_\nu v + v = 0 & \text{on } \partial\Omega \times (0,\infty). \end{cases}$$

Thus, from (b) of Theorem 2.3, we see that $u(x,t),\ v(x,t) \to 0$ uniformly as $t \to \infty$ by using comparison argument for elliptic problems.

Let ε be a positive constant with $\varepsilon \le d/(e_1+e_2)$, then there exists a $T_\varepsilon \ge 0$ such that $u(x,t),\ v(x,t) \le \varepsilon$ for all $t > T_\varepsilon$. Therefore, we have

$$\begin{cases} w_t - \Delta w \le w((e_1+e_2)\varepsilon - d) \le 0 & \text{in } \Omega \times (T_\varepsilon, \infty), \\ k_3 \partial_\nu w + w = 0 & \text{on } \partial\Omega \times (T_\varepsilon, \infty), \end{cases}$$

that concludes $w(x,t) \to 0$ uniformly as $t \to \infty$ by (b) of Theorem 2.3.

(b) Similar to (a), we see that $v(x,t) \to 0$ uniformly as $t \to \infty$. Let ε' be a positive constant with

$$\varepsilon' < \frac{1}{e_2}\left(d + \lambda_{1,k_3}\left(-\frac{e_1\Theta_{k_1}[a_1]}{1+\alpha_1\Theta_{k_1}[a_1]}\right)\right),$$

then there exists a $T_{\varepsilon'} \ge 0$ such that $v(x,t) \le \varepsilon'$ for all $t > T_{\varepsilon'}$. Therefore, we have

$$\begin{cases} w_t - \Delta w \le \left(\dfrac{e_1\Theta_{k_1}[a_1]}{1+\alpha_1\Theta_{k_1}[a_1]} + e_2\varepsilon' - d\right) & \text{in } \Omega \times (T_{\varepsilon'}, \infty), \\ k_3 \partial_\nu w + w = 0 & \text{on } \partial\Omega \times (T_{\varepsilon'}, \infty), \end{cases}$$

that concludes $w(x,t) \to 0$ uniformly as $t \to \infty$ by (b) of Theorem 2.3.

Let ε be a positive constant with $\varepsilon < (a_1 - \lambda_{1,k_1})/(b_{12}+c_1)$, then there exists a $T_\varepsilon \ge 0$ such that $u(x,t),\ v(x,t) \le \varepsilon$ for all $t > T_\varepsilon$. Therefore, we have

$$\begin{cases} u_t - \Delta u \ge u(a_1 - u - b_{12}\varepsilon - c_1\varepsilon) & \text{in } \Omega \times (T_\varepsilon, \infty), \\ k_1 \partial_\nu u + u = 0 & \text{on } \partial\Omega \times (T_\varepsilon, \infty), \end{cases}$$

so

$$\liminf_{t\to\infty} u(x,t) \ge \Theta_{k_1}[a_1 - b_{12}\varepsilon - c_1\varepsilon] - \varepsilon.$$

On other hand,

$$\begin{cases} u_t - \Delta u \le u(a_1 - u) & \text{in } \Omega \times (0, \infty), \\ k_1 \partial_\nu u + u = 0 & \text{on } \partial\Omega \times (0, \infty), \end{cases}$$

so,

$$\limsup_{t\to\infty} u(x,t) \le \Theta_{k_1}[a_1] + \varepsilon.$$

Therefore

$$\Theta_{k_1}[a_1 - b_{12}\varepsilon - c_1\varepsilon] - \varepsilon \le \liminf_{t\to\infty} u(x,t) \le \limsup_{t\to\infty} u(x,t) \le \Theta_{k_1}[a_1] + \varepsilon,$$

letting $\varepsilon \to 0$, we see that $u(x,t) \to \Theta_{k_1}[a_1]$ as $t \to \infty$.

(c) The proof is similar to (b), we complete the proof. $\square$

In order to prove of Theorem 1.6, we first give a definition of upper and lower solutions of (1.6).

Definition 5.1 (see [61]). A pair of functions $(\overline{u},\overline{v},\overline{w})$ and $(\underline{u},\underline{v},\underline{w})$ in $C^2(\Omega)\cap C^1(\overline{\Omega})$ are called ordered upper and lower solutions of (1.6) if they satisfy the relations:

$$\overline{u}\geq\underline{u},\qquad \overline{v}\geq\underline{v},\qquad \overline{w}\geq\underline{w}$$

and the following inequalities

$$\begin{cases}
-\Delta\overline{u}\geq\overline{u}\left(a_1-\overline{u}-b_{12}\underline{v}-\dfrac{c_1\underline{w}}{(1+\alpha_1\overline{u})(1+\beta_1\underline{w})}\right) & \text{in } \Omega,\\
-\Delta\underline{u}\leq\underline{u}\left(a_1-\underline{u}-b_{12}\overline{v}-\dfrac{c_1\overline{w}}{(1+\alpha_1\underline{u})(1+\beta_1\overline{w})}\right) & \text{in } \Omega,\\
-\Delta\overline{v}\geq\overline{v}\left(a_2-b_{21}\underline{u}-\overline{v}-\dfrac{c_2\underline{w}}{(1+\alpha_2\overline{v})(1+\beta_2\underline{w})}\right) & \text{in } \Omega,\\
-\Delta\underline{v}\leq\underline{v}\left(a_2-b_{21}\overline{u}-\underline{v}-\dfrac{c_2\overline{w}}{(1+\alpha_2\underline{v})(1+\beta_2\overline{w})}\right) & \text{in } \Omega,\\
-\Delta\overline{w}\geq\overline{w}\left(\dfrac{e_1\overline{u}}{(1+\alpha_1\overline{u})(1+\beta_1\overline{w})}+\dfrac{e_2\overline{v}}{(1+\alpha_2\overline{v})(1+\beta_2\overline{w})}-d\right) & \text{in } \Omega,\\
-\Delta\underline{w}\leq\underline{w}\left(\dfrac{e_1\underline{u}}{(1+\alpha_1\underline{u})(1+\beta_1\underline{w})}+\dfrac{e_2\underline{v}}{(1+\alpha_2\underline{v})(1+\beta_2\underline{w})}-d\right) & \text{in } \Omega,\\
k_1\partial_\nu\overline{u}+\overline{u}\geq 0\geq k_1\partial_\nu\underline{u}+\underline{u} & \text{on } \partial\Omega,\\
k_2\partial_\nu\overline{v}+\overline{v}\geq 0\geq k_2\partial_\nu\underline{v}+\underline{v} & \text{on } \partial\Omega,\\
k_3\partial_\nu\overline{w}+\overline{w}\geq 0\geq k_3\partial_\nu\underline{w}+\underline{w} & \text{on } \partial\Omega.
\end{cases}$$

Proof of Theorem 1.8. It is easy to see that

$$(\Theta_{k_1}[a_1],\Theta_{k_2}[a_2],w_{(\Theta_{k_1}[a_1],\Theta_{k_2}[a_2])})\quad\text{and}\quad(u^*,v^*,w_{(u^*,v^*)})$$

are a pair of ordered positive upper and lower solutions of (1.6) under assumption (1.10). Using the monotone iteration scheme technique in [61], the existence of a pair of functions $(\widetilde{u},\widetilde{v},\widetilde{w})$ and $(\widehat{u},\widehat{v},\widehat{w})$ can be show easily.

Next, we show that $[\widehat{u},\widetilde{u}]\times[\widehat{v},\widetilde{v}]\times[\widehat{w},\widetilde{w}]$ is a positive global attractor of (1.5). Since $\widehat{u}$, $\widehat{v}$, $\widehat{w}>0$, the positivity follows easily. So it is sufficient to show that $[\widehat{u},\widetilde{u}]\times[\widehat{v},\widetilde{v}]\times[\widehat{w},\widetilde{w}]$ is a global attractor.

Let ε be a sufficiently small constant such that

(i) $$\varepsilon<\frac{-\lambda_{1,k_3}\left(-\dfrac{e_1\Theta_{k_1}[a_1]}{1+\alpha_1\Theta_{k_1}[a_1]}-\dfrac{e_2\Theta_{k_2}[a_2]}{1+\alpha_2\Theta_{k_2}[a_2]}\right)-d}{e_1+e_2},$$

(ii) $$\max\{b_{12},b_{21}\}\varepsilon<\min\left\{a_1-\frac{c_1}{\beta_1}-\lambda_{1,k_1}(b_{12}\Theta_{k_2}[a_2]),\right.\\ \left.a_2-\frac{c_2}{\beta_2}-\lambda_{1,k_2}(b_{21}\Theta_{k_1}[a_1])\right\},$$

(iii) $$\varepsilon<\frac{-\lambda_{1,k_3}\left(-\dfrac{e_1u^*}{1+\alpha_1u^*}-\dfrac{e_2v^*}{1+\alpha_2v^*}\right)-d}{e_1+e_2}.$$

Since

$$\begin{cases} u_t - \Delta u = u\left(a_1 - u - b_{12}v - \dfrac{c_1 w}{(1+\alpha_1 u)(1+\beta_1 w)}\right) \\ \qquad\qquad \le u(a_1 - u) & \text{in } \Omega \times (0,\infty), \\ v_t - \Delta v = v\left(a_2 - b_{21}u - v - \dfrac{c_2 w}{(1+\alpha_2 v)(1+\beta_2 w)}\right) \\ \qquad\qquad \le v(a_2 - v) & \text{in } \Omega \times (0,\infty), \\ k_1 \partial_\nu u + u = k_2 \partial_\nu v + v = 0 & \text{on } \partial\Omega \times (0,\infty), \end{cases}$$

then by Theorem 2.3 and comparison principle, it is obvious that

$$\limsup_{t\to\infty} u(x,t) \le \Theta_{k_1}[a_1] \quad \text{and} \quad \limsup_{t\to\infty} v(x,t) \le \Theta_{k_2}[a_2]. \tag{5.1}$$

This implies that there exists a $T_\varepsilon \ge 0$ such that

$$u(x,t) \le \Theta_{k_1}[a_1] + \varepsilon \quad \text{and} \quad v(x,t) \le \Theta_{k_2}[a_2] + \varepsilon \quad \text{for all } t > T_\varepsilon. \tag{5.2}$$

Thus,

$$\begin{cases} w_t - \Delta w = w\left(\dfrac{e_1 u}{(1+\alpha_1 u)(1+\beta_1 w)} + \dfrac{e_2 v}{(1+\alpha_2 v)(1+\beta_2 w)} - d\right) \\ \qquad \le w\Bigg(\dfrac{e_1(\Theta_{k_1}[a_1]+\varepsilon)}{(1+\alpha_1(\Theta_{k_1}[a_1]+\varepsilon))(1+\beta_1 w)} \\ \qquad\qquad + \dfrac{e_2(\Theta_{k_2}[a_2]+\varepsilon)}{1+\alpha_2(\Theta_{k_2}[a_2]+\varepsilon)(1+\beta_2 w)} - d\Bigg) \\ \qquad \le w\Bigg(\dfrac{e_1\Theta_{k_1}[a_1]}{(1+\alpha_1\Theta_{k_1}[a_1])(1+\beta_1 w)} \\ \qquad\qquad + \dfrac{e_2\Theta_{k_2}[a_2]}{(1+\alpha_2\Theta_{k_2}[a_2])(1+\beta_2 w)} + \varepsilon(e_1+e_2) - d\Bigg) & \text{in } \Omega \times (T_\varepsilon,\infty), \\ k_3 \partial_\nu w + w = 0 & \text{on } \partial\Omega \times (T_\varepsilon,\infty). \end{cases}$$

Since ε satisfies assumption (i), we get from Theorem 2.3 that

$$\limsup_{t\to\infty} w(x,t) \le w_{(\Theta_{k_1}[a_1],\Theta_{k_2}[a_2])}. \tag{5.3}$$

On the other hand, it follows from (5.2) that

$$\begin{cases} u_t - \Delta u = u\left(a_1 - u - b_{12}v - \dfrac{c_1 w}{(1+\alpha_1 u)(1+\beta_1 w)}\right) \\ \qquad\qquad \ge u\left(a_1 - \dfrac{c_1}{\beta_1} - b_{12}(\Theta_{k_2}[a_2]+\varepsilon) - u\right) & \text{in } \Omega \times (T_\varepsilon,\infty), \\ v_t - \Delta v = v\left(a_2 - b_{21}u - v - \dfrac{c_2 w}{(1+\alpha_2 v)(1+\beta_2 w)}\right) \\ \qquad\qquad \ge v\left(a_2 - \dfrac{c_2}{\beta_2} - b_{21}(\Theta_{k_1}[a_1]+\varepsilon) - v\right) & \text{in } \Omega \times (T_\varepsilon,\infty), \\ k_1 \partial_\nu u + u = k_2 \partial_\nu v + v = 0 & \text{on } \partial\Omega \times (T_\varepsilon,\infty). \end{cases}$$

Since ε satisfies assumption (ii), we get from Theorem 2.3 that

$$\liminf_{t\to\infty} u(x,t) \geq u^* \quad \text{and} \quad \liminf_{t\to\infty} v(x,t) \geq v^*. \tag{5.4}$$

This implies that there exists a $T'_\varepsilon \geq 0$ such that

$$u(x,t) \geq u^* - \varepsilon \quad \text{and} \quad v(x,t) \geq v^* - \varepsilon \quad \text{for all } t > T'_\varepsilon. \tag{5.5}$$

Thus,

$$\begin{cases} w_t - \Delta w = w\left(\dfrac{e_1 u}{(1+\alpha_1 u)(1+\beta_1 w)} + \dfrac{e_2 v}{(1+\alpha_2 v)(1+\beta_2 w)} - d\right) \\ \qquad \geq w\left(\dfrac{e_1(u^*-\varepsilon)}{(1+\alpha_1(u^*-\varepsilon))(1+\beta_1 w)}\right. \\ \qquad\quad \left. + \dfrac{e_2(v^*-\varepsilon)}{(1+\alpha_2(v^*-\varepsilon))(1+\beta_2 w)} - d\right) \\ \qquad \geq w\left(\dfrac{e_1 u^*}{(1+\alpha_1 u^*)(1+\beta_1 w)}\right. \\ \qquad\quad + \dfrac{e_2 v^*}{(1+\alpha_2 v^*)(1+\beta_2 w)} - \varepsilon(e_1+e_2) - d) & \text{in } \Omega \times (T'_\varepsilon, \infty), \\ k_3 \partial_\nu w + w = 0 & \text{on } \partial\Omega \times (T'_\varepsilon, \infty). \end{cases}$$

Since ε satisfies assumption (iii), we get from Theorem 2.3 that

$$\limsup_{t\to\infty} w(x,t) \geq w_{(u^*,v^*)}. \tag{5.6}$$

Finally, using (5.1) and (5.3)-(5.6), it is concluded that there exist

$$T = \max\{T_\varepsilon, T'_\varepsilon\}$$

such that for any nontrivial initial condition $(u(x,0), v(x,0), w(x,0))$, the time-dependent solution (u,v,w) of (1.5) satisfies

$$(u,v,w) \in [u^*, \Theta_{k_1}[a_1]] \times [v^*, \Theta_{k_2}[a_2]] \times [w_{(u^*,v^*)}, w_{(\Theta_{k_1}[a_1], \Theta_{k_2}[a_2])}] \quad \text{for } t > T.$$

Then, our result follows by Corollary 2.1 and Theorem 2.1 in [62]. $\square$

Similar to the proof of above theorem, we obtain the following theorem.

THEOREM 5.2.

(a) *If* $a_1 \leq \lambda_{1,k_1}$, $a_2 - c_2/\beta_2 > \lambda_{1,k_2}$ *and*

$$-d > \lambda_{1,k_3}\left(-\frac{e_2\Theta_{k_2}[a_2]}{1+\alpha_2\Theta_{k_2}[a_2]}\right),$$

then there exists a pair of quasi-solution $(\widetilde{v}, \widetilde{w})$ *and* $(\widehat{v}, \widehat{w})$ *of* (3.4) *with* $\widetilde{v} \geq \widehat{v}$ *and* $\widetilde{w} \geq \widehat{w}$. *Moreover,* $\{0\} \times [\widehat{v}, \widetilde{v}] \times [\widehat{w}, \widetilde{w}]$ *is a global attractor of* (1.5).

(b) *If* $a_1 - c_1/\beta_1 > \lambda_{1,k_1}$, $a_2 \le \lambda_{1,k_2}$ *and*
$$-d > \lambda_{1,k_3}\left(-\frac{e_1\Theta_{k_1}[a_1]}{1+\alpha_1\Theta_{k_1}[a_1]}\right),$$
then there exist a pair of quasi-solution $(\widetilde{u},\widetilde{w})$ *and* $(\widehat{u},\widehat{w})$ *of* (3.5) *with* $\widetilde{u} \ge \widehat{u}$ *and* $\widetilde{w} \ge \widehat{w}$. *Moreover,* $[\widehat{u},\widetilde{u}] \times \{0\} \times [\widehat{w},\widetilde{w}]$ *is a global attractor of* (1.5).

(c) *If* $a_1 > \lambda_{1,k_1}(b_{12}\Theta_{k_2}[a_2])$, $a_2 > \lambda_{1,k_2}(b_{21}\Theta_{k_1}[a_1])$ *and*
$$-d < \lambda_{1,k_3}\left(-\frac{e_1\Theta_{k_1}[a_1]}{1+\alpha_1\Theta_{k_1}[a_1]} - \frac{e_2\Theta_{k_2}[a_2]}{1+\alpha_2\Theta_{k_2}[a_2]}\right),$$
then there exist a pair of quasi-solutions $(\widetilde{u},\widetilde{v})$ *and* $(\widehat{u},\widehat{v})$ *of* (3.6) *with* $\widetilde{u} \ge \widehat{u}$ *and* $\widetilde{v} \ge \widehat{v}$. *Moreover,* $[\widehat{u},\widetilde{u}] \times [\widehat{v},\widetilde{v}] \times \{0\}$ *is a global attractor of* (1.5).

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JUN ZHOU
School of Mathematics and Statistics
Southwest University
Chongqing, 400715, P.R.CHINA
E-mail address: jzhouwm@gmail.com