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THE NIELSEN TYPE NUMBERS FOR MAPS ON A 3-DIMENSIONAL FLAT RIEMANNIAN MANIFOLD

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Abstract. Let $f\colon M \to M$ be a self-map on a 3-dimensional flat Riemannian M. We compute the Lefschetz number and the Nielsen number of f by using the infra-nilmanifold structure of M and the averaging formulas for the Lefschetz numbers and the Nielsen numbers of maps on infra-nilmanifolds. For each positive integer n, we provide an explicit algorithm for a complete computation of the Nielsen type numbers $N\mathrm{P}_n(f)$ and $N\Phi_n(f)$ of f^n.

1. Introduction

In dynamical systems, it is often the case that topological information can be used to study qualitative and quantitative properties of the system. For the periodic points, two Nielsen type numbers $N\mathrm{P}_n(f)$ and $N\Phi_n(f)$ are lower bounds for the number of periodic points of least period exactly n and the set of periodic points of period n, respectively, see [9]. One can find the basic definitions, notions and some developments for the Nielsen periodic point theory in the survey articles [4], [5] and the references given there. In this paper we will determine these Nielsen type numbers of all homotopy classes of maps on a 3-dimensional flat Riemannian manifold.

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In order to state our main results, let us fix some notations and terminologies. Let $f\colon X \to X$ be a continuous self-map of a topological space X. We consider the following sets:

$$\begin{aligned} \mathrm{Fix}(f) &= \{x \in X \mid f(x) = x\}, \\ P^n(f) &= \mathrm{Fix}(f^n), \\ P_n(f) &= \mathrm{Fix}(f^n) - \bigcup_{k<n} \mathrm{Fix}(f^k) \\ &= \text{ the set of periodic points } f \text{ with least period } n. \end{aligned}$$

Let $\widetilde{f}\colon \widetilde{X} \to \widetilde{X}$ be a lifting of f. We denote by $\varphi\colon \Pi \to \Pi$ the homomorphism on the deck transformation group Π induced by the lifting $\widetilde{f}$. Namely,

$$\varphi(\alpha)\widetilde{f} = \widetilde{f}\alpha, \quad \text{for all } \alpha \in \Pi.$$

For each $n = 1, 2, \ldots$, $\widetilde{f}^n$ is a lifting of f^n, and the homomorphism determined by the lifting $\widetilde{f}^n$ is $\varphi^n\colon \Pi \to \Pi$. The homomorphism φ^n defines the *Reidemeister action* of Π on Π as follows:

$$\Pi \times \Pi \to \Pi, \quad (\gamma, \alpha) \mapsto \gamma\alpha\varphi^n(\gamma)^{-1}.$$

The Reidemeister class containing α will be denoted by $[\alpha]_n$ and the set of Reidemeister classes of Π determined by φ^n will be denoted by $\mathcal{R}[\varphi^n]$. The Reidemeister number $R(\varphi^n)$ of φ^n is defined as the cardinality of $\mathcal{R}[\varphi^n]$.

The *Reidemeister number* $R(f)$ of the continuous map f is defined as the Reidemeister number $R(\varphi)$ of an induced homomorphism φ. Note that the Reidemeister number $R(f)$ does not depend on the particular choice of the lifting $\widetilde{f}$ and hence on the particular choice of the induced homomorphism φ. Also the fixed point classes do not depend on the choice of liftings, although the corresponding Reidemeister classes may.

Let $O_n(\varphi)$ be the number of irreducible, essential periodic point orbits of $\mathcal{R}[\varphi^n]$. If $[\alpha]^n$ is irreducible and essential, then so is the corresponding periodic point class $\mathbb{F}$ and its f-orbit contains at least n periodic points of least period n.

The *prime Nielsen–Jiang periodic number* of period n is defined by the formula

$$N\mathrm{P}_n(f) = n \times O_n(\varphi).$$

Take the set of all the essential orbits, of any period $m \mid n$, which do not contain any essential orbits of lower period. To each such an orbit, find the lowest period which it can be reduced to. The *full Nielsen–Jiang periodic number* of period n, denoted by $N\Phi_n(f)$, is the sum of these numbers.

Then the Nielsen type numbers $N\mathrm{P}_n(f)$ and $N\Phi_n(f)$ are homotopy invariant, non-negative integers [9, Theorem III.4.10]. Therefore,

$$N\mathrm{P}_n(f) \le \min\{|P_n(g)| \mid g \simeq f\}, \qquad N\Phi_n(f) \le \min\{|P^n(g)| \mid g \simeq f\}.$$

In this paper when the topological space X is a 3-dimensional flat Riemannian manifold, we will determine these two homotopy invariants for all maps on X. This will be the first complete computation on a 3-dimensional infra-nilmanifold.

In Section 2, we will consider 3-dimensional flat Riemannian manifolds which can be considered as 3-dimensional analogues of the 2-dimensional Klein bottle, and show that they have solvmanifold structures. In Section 3, we will consider one of such manifolds and devote ourselves to compute these homotopy invariants. This is a continuation of the work done on the Klein bottle [12]. However, the computations involved in this paper are much complicated compared to the previous ones. We will refer to [12] and the references given there for necessary preliminaries and facts.

2. 3-dimensional flat Riemannian manifolds

We have a complete classification of 3-dimensional crystallographic groups. Such a group Π has an explicit representation $\Pi \to \mathbb{R}^3 \rtimes \mathrm{GL}(3,\mathbb{Z})$ (not into $\mathbb{R}^3 \rtimes \mathrm{O}(3)$) in the book [1].

There are 3-dimensional analogues of the classical 2-dimensional Klein bottle K^2. In fact, among ten 3-dimensional Bieberbach groups, there are three 3-dimensional Bieberbach groups Π with holonomy group $\mathbb{Z}_2$. These are orientable **2/1/1/02**, and non-orientable **2/2/1/02** and **2/2/2/02**. The bold-faced numbers associated to the 3-dimensional Bieberbach groups refer to the numbering in the book [1]. Indeed, it is easy to see that these are $\mathfrak{G}_2$, $\mathfrak{B}_1$ and $\mathfrak{B}_2$ in [16, Theorems 3.5.5 and 3.5.9], respectively. Write

$$e_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}, \qquad e_2 =, \begin{bmatrix}0\\1\\0\end{bmatrix}, \qquad e_3 = \begin{bmatrix}0\\0\\1\end{bmatrix};$$

$$a_1 = \begin{bmatrix}0\\1/2\\0\end{bmatrix}, \qquad A_1 = \begin{bmatrix}-1&0&0\\0&1&0\\0&0&-1\end{bmatrix};$$

$$a_2 = \begin{bmatrix}0\\0\\1/2\end{bmatrix}, \qquad A_2 = \begin{bmatrix}1&0&0\\0&-1&0\\0&0&1\end{bmatrix};$$

$$a_3 = \begin{bmatrix}0\\0\\1/2\end{bmatrix}, \qquad A_3 = \begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}.$$

Then $\alpha_i = (a_i, A_i)$ and $t_i = (e_i, I_3)$ be elements of $\mathbb{R}^3 \rtimes \mathrm{GL}(3,\mathbb{R})$ and A_i has period 2.

Furthermore,

$$\begin{aligned}
\alpha_1^2 &= t_2, & t_1\alpha_1 t_1^{-1} &= \alpha_1 t_1^{-2}, & t_3\alpha_1 t_3^{-1} &= \alpha_1 t_3^{-2},\\
\alpha_2^2 &= t_3, & t_1\alpha_2 t_1^{-1} &= \alpha_2, & t_2\alpha_2 t_2^{-1} &= \alpha_2 t_2^{-2},\\
\alpha_3^2 &= t_3, & t_1\alpha_3 t_1^{-1} &= \alpha_3 t_1^{-1} t_2, & t_2\alpha_3 t_2^{-1} &= \alpha_3 t_1 t_2^{-1}.
\end{aligned}$$

Let Γ be the integral matrices of $\mathbb{R}^3$. Then it forms a lattice of $\mathbb{R}^3$ and $\Gamma\setminus\mathbb{R}^3$ is the 3-torus. It is easy to check that the subgroup

$$\Pi_i = \langle \Gamma, (a_i, A_i)\rangle \subset \mathbb{R}^3 \rtimes \mathrm{GL}(3,\mathbb{R})$$

generated by the lattice Γ and the element $\alpha_i = (a_i, A_i)$ is discrete and torsion free, and has Γ as a normal subgroup of index 2. Thus Π_i is a 3-dimensional Bieberbach group and the quotient space $\Pi_i\setminus\mathbb{R}^3$ is a 3-dimensional flat manifold which is orientable when $i = 1$ and non-orientable when $i = 2$ or 3. The projection $\Gamma\setminus\mathbb{R}^3 \to \Pi_i\setminus\mathbb{R}^3$ is a double covering projection. We shall denote the flat manifold $\Pi_i\setminus\mathbb{R}^3$ by K_i, and the torus $\Gamma\setminus\mathbb{R}^3$ by T. Note also that

2/1/1/02: $\Pi_1 = \langle t_1, \alpha_1, t_3 \mid t_1\alpha_1 = \alpha_1 t_1^{-1},\ t_3\alpha_1 = \alpha_1 t_3^{-1},\ [t_1, t_3] = 1\rangle$,
2/2/1/02: $\Pi_2 = \langle t_1, t_2, \alpha_2 \mid t_1\alpha_2 = \alpha_2 t_1,\ t_2\alpha_2 = \alpha_2 t_2^{-1},\ [t_1, t_2] = 1\rangle$,
2/2/2/02: $\Pi_3 = \langle t_1, t_2, \alpha_3 \mid t_1\alpha_3 = \alpha_3 t_2,\ t_2\alpha_3 = \alpha_3 t_1,\ [t_1, t_2] = 1\rangle$.

Furthermore, since we can embed $\mathrm{Aff}(3) = \mathbb{R}^3 \rtimes \mathrm{GL}(3,\mathbb{R})$ in $\mathrm{GL}(4,\mathbb{R})$ as

$$\mathrm{Aff}(3) = \left\{ \begin{bmatrix} A & \mathbf{x} \\ 0 & 1 \end{bmatrix} \;\middle|\; A \in \mathrm{GL}(3,\mathbb{R}), \mathbf{x} \in \mathbb{R}^3 \right\} \subset \mathrm{GL}(4,\mathbb{R}),$$

we can embed each Π_i in $\mathrm{GL}(4,\mathbb{R})$ so that

$$t_i = \begin{bmatrix} I_3 & e_i \\ 0 & 1 \end{bmatrix}, \quad \alpha_i = \begin{bmatrix} A_i & a_i \\ 0 & 1 \end{bmatrix} \in \mathrm{GL}(4,\mathbb{R}).$$

Observe that the Π_i are non-nilpotent, 2-step solvable groups

$$\Pi_1 = \mathbb{Z}^2 \rtimes_{\phi_1} \mathbb{Z}, \qquad \Pi_2 = \mathbb{Z}^2 \rtimes_{\phi_2} \mathbb{Z}, \qquad \Pi_3 = \mathbb{Z}^2 \rtimes_{\phi_3} \mathbb{Z}$$

where

$$\phi_1 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad \phi_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad \phi_3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Since each ϕ_i has an eigenvalue -1, the solvmanifold with fundamental group Π_i is not an $\mathcal{NR}$-solvmanifold (see [10], [15] for the definition of $\mathcal{NR}$-solvmanifolds).

Proposition 2.1. *The flat manifolds K_i are compact solvmanifolds.*

Proof. Consider the simply connected solvable Lie group $G_1 = \mathbb{C}^2 \rtimes_{\sigma_1} \mathbb{R}$ where $\sigma_1(t)$ is the rotation by $2\pi t$ on each factor of $\mathbb{C}$, namely, $\sigma_1(t)\colon (z_1, z_2) \mapsto (e^{2\pi i t} z_1, e^{2\pi i t} z_2)$. Let H_1 be the closed subgroup of G_1 given by

$$H_1 = \{(m + ix, n + iy, k/2) \in G_1 \mid m, n, k \in \mathbb{Z},\ x, y \in \mathbb{R}\}.$$

Then it is easily seen that the compact solvmanifold $H_1 \setminus G_1$ is homeomorphic to the flat manifold K_1.

Note that $\Pi_2 = \langle t_1 \rangle \times \langle t_2, \alpha_2 \rangle$ is isomorphic to the product of the infinite cyclic group $\mathbb{Z}$ and the 2-dimensional Klein bottle group $\pi_1(K)$. This yields that the flat manifold K_2 is homeomorphic to the product space $S^1 \times K^2$. Hence K_2 is a compact solvmanifold. As above, we can consider the simply connected solvable Lie group $G_2 = \mathbb{R} \times (\mathbb{C} \rtimes_{\sigma_2} \mathbb{R}) = \mathbb{R} \times \widetilde{E}_0(2)$ where $\sigma_2(t)$ is the rotation by $2\pi t$ on $\mathbb{C}$, namely, $\sigma_2(t)\colon z \mapsto e^{2\pi i t} z$. Let H_2 be the closed subgroup of G_2 given by

$$H_2 = \{(m, n + ix, k/2) \in G_2 \mid m, n, k \in \mathbb{Z},\ x \in \mathbb{R}\}.$$

Then we can see that the compact solvmanifold $H_2 \setminus G_2$ is homeomorphic to the product $S^1 \times K^2$ and hence to the flat manifold K_2.

Consider the simply connected solvable Lie group $G_3 = \mathbb{C}^3 \rtimes_{\sigma_3} \mathbb{R}$ where $\sigma_3(t)$ is given by the composition (or product) of three matrices

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos 2\pi t & -\sin 2\pi t \\ 0 & 0 & 0 & 0 & \sin 2\pi t & \cos 2\pi t \end{bmatrix},$$

$$\begin{bmatrix} \cos 4\pi t & 0 & -\sin 4\pi t & 0 & 0 & 0 \\ 0 & \cos 4\pi t & 0 & 0 & -\sin 4\pi t & 0 \\ \sin 4\pi t & 0 & \cos 4\pi t & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & \sin 4\pi t & 0 & 0 & \cos 4\pi t & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\begin{bmatrix} \cos 2\pi t & 0 & -\sin 2\pi t & 0 & 0 & 0 \\ 0 & \cos 2\pi t & 0 & -\sin 2\pi t & 0 & 0 \\ \sin 2\pi t & 0 & \cos 2\pi t & 0 & 0 & 0 \\ 0 & \sin 2\pi t & 0 & \cos 2\pi t & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos 2\pi t & -\sin 2\pi t \\ 0 & 0 & 0 & 0 & \sin 2\pi t & \cos 2\pi t \end{bmatrix}.$$

Let H_3 be the closed subgroup of G_3 given by

$$H_3 = \{(x + im, y + in, u + iv, k/4) \in G_3 \mid k, m, n \in \mathbb{Z},\ x, y, u, v \in \mathbb{R}\}.$$

Then it is easily seen that the compact solvmanifold $H_3 \setminus G_3$ is homeomorphic to the flat manifold K_3. $\square$

For a compact solvmanifold $K = H \setminus G$ where S is a connected, simply connected solvable Lie group and H is a closed uniform subgroup of G, let N be the nilradical of G; then N fits a short exact sequence

$$0 \longrightarrow N \longrightarrow G \longrightarrow G/N \cong \mathbb{R}^s \longrightarrow 0.$$

The closed subgroup H of G yields a uniform subgroup $N\cap H$ of N. Moreover, the closed subgroup H of G induces a short exact sequence $0\to N\cap H\to H\to H/N\cap H\cong H\cdot N/N\to 0$ so that the following diagram is commutative

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \longrightarrow & G & \longrightarrow & G/N & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & N\cap H & \longrightarrow & H & \longrightarrow & H\cdot N/N & \longrightarrow & 0 \end{array}$$

This gives rise to the fibration, called the *Mostow fibration*,

$$N\cap H\setminus N\longrightarrow K=H\setminus G\longrightarrow H\cdot N\setminus G$$

over a torus base $H\cdot N\setminus G$ with compact nilmanifold fiber $N\cap H\setminus N$, see [15], [2]. It is known that the Mostow fibration is orientable if and only if the solvmanifold K is a nilmanifold, see [15, Lemma 3.1].

For $G_1=\mathbb{C}^2\rtimes_{\sigma_1}\mathbb{R}$, we have that

- the nilradical of G_1 is $N_1=\mathbb{C}^2$,
- $N_1\cap H_1=\{(m+ix,n+iy,0)\in G_1\mid m,n\in\mathbb{Z},\ x,y\in\mathbb{R}\}$,
- $N_1\cdot H_1=\{(\mathbb{C}^2,k/2)\mid k\in\mathbb{Z}\}$.

Thus the Mostow fibration has the base the circle with α_1 as a generator of the fundamental group, and the fiber the torus with t_1 and t_3 as generators of the fundamental group. Notice here that K_1 is orientable as a manifold, however the Mostow fibration structure on K_1 is not orientable. In this sense, K_1 is a 3-dimensional analogue of the classical 2-dimensional Klein bottle.

It is clear that $K_2=S^1\times K^2$. The Mostow fibration is the product of the trivial bundle over S^1 with the standard fibration of the Klein bottle K^2.

For $G_3=\mathbb{C}^3\rtimes_{\sigma_3}\mathbb{R}$, we have that

- the nilradical of G_3 is $N_3=\mathbb{C}^3$,
- $N_3\cap H_3=\{(x+im,y+in,u+iv,0)\in G_3\mid m,n\in\mathbb{Z},x,y,u,v\in\mathbb{R}\}$,
- $N_3\cdot H_3=\{(\mathbb{C}^3,k/4)\mid k\in\mathbb{Z}\}$.

Thus the Mostow fibration has the base the circle with α_3 as a generator of the fundamental group, and the fiber the torus with t_1 and t_2 as generators of the fundamental group.

3. The Nielsen type numbers of maps on K

Now let us recall some of the main results in [6], [7], [8].

Definition 3.1. The map $f\colon M\to M$ is called *weakly Jiang* if either $N(f)=0$ or $N(f)=R(f)$.

Theorem 3.2 ([6, Theorem 1], [7, Theorems 1.2]). *Let $f\colon M\to M$ be a self-map of a nilmanifold or $\mathcal{NR}$ solvmanifold, or if M is an arbitrary solvmanifold* *suppose that f^n is weakly Jiang. If $N(f^n) \neq 0$, then for all $m|n$*

$$N(f^m) = \sum_{k|m} N\mathrm{P}_k(f), \qquad N\mathrm{P}_m(f) = \sum_{k|m} \mu(k) N(f^{m/k}),$$

where μ is the Möbius function.

THEOREM 3.3 ([7, Corollary 4.6]). *Let $f\colon M \to M$ be a self-map. If M is a solvmanifold, then*

$$N\Phi_n(f) = \sum_{k|n} N\mathrm{P}_k(f), \qquad N\mathrm{P}_n(f) = \sum_{k|n} \mu(k) N\Phi_{n/k}(f).$$

Since our flat manifolds K_i are solvmanifolds, according to Theorem 3.3 it is enough to find the formula for the prime Nielsen–Jiang periodic number or the full Nielsen-Jiang periodic number. Note that K_i are not $\mathcal{NR}$ solvmanifolds. When f^n is weakly Jiang and $N(f^n) \neq 0$ we can find the formula easily using Theorem 3.2. However, the remaining cases are not rare and are required a lot of efforts to work, see for example [12].

In this paper we shall consider the flat manifold K_1 only and evaluate the Nielsen type numbers for all self-maps on K_1. We believe the results of this case are worth recording once and for all. For simplicity we will use the notation K and Π for the manifold K_1 and its fundamental group Π_1.

4. Self-maps on K

Note that Π fits a short exact sequence

$$0 \longrightarrow \mathbb{Z}^2 \longrightarrow \Pi \longrightarrow \mathbb{Z} \longrightarrow 0$$

where s_1 and s_2 are generators of the normal subgroup $\mathbb{Z}^2$ and α is a generator of the quotient group $\mathbb{Z}$ so that α acts on s_i by $\alpha\colon s_i \mapsto s_i^{-1}$. Of course, we can embed Π into $\mathbb{R}^3 \rtimes \mathrm{GL}(3,\mathbb{R})$ by the assignment $s_1 \mapsto t_1$, $s_2 \mapsto t_2$ and $\alpha \mapsto \alpha_1$ where

$$t_i = (e_i, I_3), \qquad \alpha_1 = \left(\begin{bmatrix} 0 \\ 0 \\ 1/2 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right).$$

Note that the group $\langle t_1, t_2, \alpha_1 \rangle$ is conjugate to the 3-dimensional Bieberbach group **2/1/1/02** in the book [1] by

$$\left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \right) \in \mathbb{R} \rtimes \mathrm{GL}(3,\mathbb{R}).$$

Lemma 4.1. *Any homomorphism* $\varphi\colon \Pi\to\Pi$ *is given as follows*:

$$\begin{aligned}\varphi(s_1) &= s_1^{(a_{11}-(-1)^\omega a_{11})/2} s_2^{(a_{21}-(-1)^\omega a_{21})/2},\\ \varphi(s_2) &= s_1^{(a_{12}-(-1)^\omega a_{12})/2} s_2^{(a_{22}-(-1)^\omega a_{22})/2},\\ \varphi(\alpha) &= s_1^{b_1} s_2^{b_2}\alpha^\omega.\end{aligned}$$

Proof. Every element of Π can be written uniquely as $s_1^\ell s_2^m \alpha^n$. Thus

$$\varphi(s_1)=s_1^{a_{11}}s_2^{a_{21}}\alpha^{\omega_1},\qquad \varphi(s_2)=s_1^{a_{12}}s_2^{a_{22}}\alpha^{\omega_2},\qquad \varphi(\alpha)=s_1^{b_1}s_2^{b_2}\alpha^{\omega},$$

for some integers a_{ij}, b_i, ω_i and ω. Since $\alpha s_i\alpha^{-1}=s_i^{-1}$ implies $\alpha=s_i\alpha s_i$, the equations

$$\varphi(\alpha)=\varphi(s_i)\,\varphi(\alpha)\,\varphi(s_i)$$

yields that

$$\begin{aligned} s_1^{b_1}s_2^{b_2}\alpha^\omega &= (s_1^{a_{1i}}s_2^{a_{2i}}\alpha^{\omega_i})(s_1^{b_1}s_2^{b_2}\alpha^{\omega})(s_1^{a_{1i}}s_2^{a_{2i}}\alpha^{\omega_i})\\ &= s_1^{a_{1i}+(-1)^{\omega_i}b_1+(-1)^{\omega_i+\omega}a_{1i}} s_2^{a_{2i}+(-1)^{\omega_i}b_2+(-1)^{\omega_i+\omega}a_{2i}} \alpha^{2\omega_i+\omega}\end{aligned}$$

or

$$\omega_i=0\quad\text{and}\quad (1+(-1)^\omega)a_{ij}=0\quad\text{for } 1\le i,j\le 2.$$

The equation $\varphi(s_1)\,\varphi(s_2)=\varphi(s_2)\,\varphi(s_1)$ is redundant! □

Explicitly we have:

Corollary 4.2. *Any homomorphism* $\varphi:\Pi\to\Pi$ *is given as follows:*

(a) *When* ω *is odd,*

$$\varphi(s_1)=s_1^{a_{11}}s_2^{a_{21}},\qquad \varphi(s_2)=s_1^{a_{12}}s_2^{a_{22}},\qquad \varphi(\alpha)=s_1^{b_1}s_2^{b_2}\alpha^\omega.$$

(b) *When* ω *is even,*

$$\varphi(s_1)=1,\qquad \varphi(s_2)=1,\qquad \varphi(\alpha)=s_1^{b_1}s_2^{b_2}\alpha^\omega.$$

Immediately we have:

Corollary 4.3. *The group of pure translations in* Π, $\Gamma=\langle s_1,s_2,s_3\rangle$, *is a fully invariant subgroup of* Π.

Let $f\colon K\to K$ be a continuous map on the flat manifold $K=\Pi\setminus\mathbb{R}^3$ and choose a lifting $\widetilde{f}:\mathbb{R}^3\to\mathbb{R}^3$ of f. The lifting $\widetilde{f}$ induces a homomorphism $\varphi\colon\Pi\to\Pi$ which is defined by the following rule:

$$\varphi(\gamma)\circ\widetilde{f}=\widetilde{f}\circ\gamma\quad\text{for all } \gamma\in\Pi.$$

Given f, we consider another lifting of f. It is of the form $\beta\circ\widetilde{f}$ for some $\beta\in\Pi$, and the homomorphism on Π induced by $\beta\circ\widetilde{f}$ is $\tau_\beta\circ\varphi$. Indeed, for all $\gamma\in\Pi$,

$$(\beta\circ\widetilde{f})\circ\gamma=\beta\circ\varphi(\gamma)\circ\widetilde{f}=(\beta\circ\varphi(\gamma)\circ\beta^{-1})\circ(\beta\circ\widetilde{f})=(\tau_\beta\circ\varphi)(\gamma)\circ(\beta\circ\widetilde{f}).$$

Now we describe $\tau_\beta\circ\varphi$.

A homomorphism φ given as in Corollary 4.2 is called of *type* $(F, \mathbf{b}, \omega)$ where F and $\mathbf{b}$ are integer matrices

$$F = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}.$$

When ω is even, F is the zero matrix. A self-map on K is called of *type* $(F, \mathbf{b}, \omega)$ if it induces a homomorphism of type $(F, \mathbf{b}, \omega)$. Since the element β of Π is of the form $\beta = s_1^\ell s_2^m \alpha^n$ for some integers ℓ, m and n, we have:

(1) When ω is odd, then

$$\begin{aligned} \tau_\beta \circ \varphi(s_i) &= s_1^{(-1)^n a_{1i}} s_2^{(-1)^n a_{2i}}, \quad (i = 1, 2), \\ \tau_\beta \circ \varphi(\alpha) &= s_1^{2\ell + (-1)^n b_1} s_2^{2m + (-1)^n b_2} \alpha^\omega. \end{aligned}$$

Thus we can choose a lifting $\widetilde{f}$ of f so that (i) F is unique up to $\pm I$ and (ii) $b_1, b_2 \in \{0, 1\}$.

(2) When ω is even, then

$$\begin{aligned} \tau_\beta \circ \varphi(s_i) &= 1, \quad (i = 1, 2), \\ \tau_\beta \circ \varphi(\alpha) &= s_1^{(-1)^n b_1} s_2^{(-1)^n b_2} \alpha^\omega. \end{aligned}$$

Thus we can choose a lifting $\widetilde{f}$ of f so that $b_1 \geq 0$; when $b_1 = 0$ then $b_2 \geq 0$.

Consequently, we can choose and then fix a lifting $\widetilde{f}$ of f so that the induced homomorphism φ as in Corollary 4.2 satisfies the following: when ω is odd, (i) F is unique up to $\pm I$ and (ii) $b_1, b_2 \in \{0, 1\}$, and when ω is even, $b_1 \geq 0$; when $b_1 = 0$ then $b_2 \geq 0$. Such a homomorphism $\varphi \colon \Pi \to \Pi$ is called of *normalized type* $(F, \mathbf{b}, \omega)$. A self-map f on $\Pi \setminus \mathbb{R}^3$ is said to be of *normalized type* $(F, \mathbf{b}, \omega)$ if it induces a homomorphism of normalized type $(F, \mathbf{b}, \omega)$.

Suppose f and f' are homotopic maps on the flat manifold K. It is well known that the induced homomorphisms φ and φ' are conjugate by an element of Π. It follows that $\varphi = \varphi'$ as normalized type. Note that when ω is even $F = 0$.

Conversely suppose f and f' are maps on K so that the homomorphisms φ and φ' are the same (as normalized type). Since K is an aspherical manifold, it is well-known that f and f' are homotopic, i.e. such a map is unique up to homotopy.

In all, we have obtained the following homotopy classification of maps on the flat manifold K.

Theorem 4.4. *Every continuous maps on the flat manifold K is homotopic to a map of normalized type $(F, \mathbf{b}, \omega)$. Furthermore, two such maps of (normalized) type are homotopic if and only if they have the same normalized types.*

Notation 4.5. A self-map $f\colon K \to K$ on the flat manifold K of type $(F,\mathbf{b},\omega)$ will be denoted by $f_{(F,\mathbf{b},\omega)}$.

Lemma 4.6. *Every self-map $f\colon K\to K$ of type $(F,\mathbf{b},\omega)$ has an affine endomorphism $(\delta,D)\in\mathbb{R}^3\rtimes\mathrm{Endo}(\mathbb{R}^3)$ as a homotopy lifting given by*

$$(\delta,D)=\begin{cases}\left(\begin{bmatrix}\mathbf{b}/2\\ *\end{bmatrix},\begin{bmatrix}F & 0\\ 0 & \omega\end{bmatrix}\right) & \text{when } \omega \text{ is odd},\\ \left(\begin{bmatrix}*\\ *\end{bmatrix},\begin{bmatrix}0 & 2\mathbf{b}\\ 0 & \omega\end{bmatrix}\right) & \text{when } \omega \text{ is even}.\end{cases}$$

Conversely, such an affine map (δ,D) induces a map on K of type $(F,\mathbf{b},\omega)$.

Proof. Let $f\colon K\to K$ be a self-map inducing a homomorphism φ on Π of type $(F,\mathbf{b},\omega)$. Due to [14, Theorem 1.1], there exists an affine endomorphism $(\delta,D)\in\mathbb{R}^3\rtimes\mathrm{Endo}(\mathbb{R}^3)$ such that

$$\begin{aligned}&\varphi(s_i)(\delta,D)=(\delta,D)s_i \quad (i=1,2),\\ &\varphi(\alpha)(\delta,D)=(\delta,D)\alpha.\end{aligned}$$

These equalities yield the formula for (δ,D) by a simple computation. Furthermore, the above equalities imply that (δ,D) induces a map $\overline{f}$ on K, i.e. $\overline{f}$ has a lifting (δ,D). Hence f and $\overline{f}$ induce the same homomorphism φ. Since K is a $K(\pi,1)$-manifold, $\overline{f}$ is homotopic to f. $\square$

Definition 4.7. Let f be a self-map on K of type $(F,\mathbf{b},\omega)$. The linear part D of a homotopy lifting of f in Lemma 4.6 is called the *linearization* of f. We call F the *fiber* of f.

Remark 4.8. Let f be a self-map on K of type $(F,\mathbf{b},\omega)$ with a lifting $\widetilde{f}$ which induces the homomorphism φ. Then $\widetilde{f}^n$ is a lifting of f^n and the corresponding homomorphism is φ^n. It can seen easily that

$$\begin{aligned}&\varphi^n(s_i)(\delta,D)^n=(\delta,D)^n s_i \quad (i=1,2),\\ &\varphi^n(\alpha)(\delta,D)^n=(\delta,D)^n\alpha.\end{aligned}$$

Thus D^n is the linearization of f^n, and F^n is the fiber of f^n. Note also that the integer ω for f becomes ω^n for f^n.

Theorem 4.9. *Let $f\colon K\to K$ be a self-map of type $(F,\mathbf{b},\omega)$. Then for any positive integer n,*

$$N(f^n)=\frac{1}{2}|1-\omega^n|(|\det(I-F^n)|+|\det(I+F^n)|).$$

In particular when ω is even $N(f^n)=|1-\omega^n|$.

Proof. We recall the averaging formula for the Nielsen number on infra-nilmanifolds from [11, Theorem 3.5] and [13, Theorem 1.4]

$$N(f) = \frac{1}{|\Psi|}\sum_{A\in\Psi} |\det(A_* - f_*)|.$$

Thus for the case of the flat manifold K, we have $\Psi = \langle A_1\rangle \cong \mathbb{Z}_2$ and

$$\begin{aligned} N(f^n) &= \frac{1}{2}(|\det(I - D^n)| + |\det(A_1 - D^n)|) \\ &= \frac{1}{2}|1-\omega^n|(|\det(I - F^n)| + |\det(I + F^n)|). \end{aligned}$$ □

Remark that if λ_i is an eigenvalue of F then $1\pm\lambda_i$ is an eigenvalue of $I\pm F$. Hence

$$\begin{aligned} \det(I\pm F^n) &= (1\pm\lambda_1^n)(1\pm\lambda_2^n) = 1+\lambda_1^n\lambda_2^n \pm(\lambda_1^n+\lambda_2^n) \\ &= 1+\det(F^n)\pm \mathrm{tr}\,(F^n). \end{aligned}$$

5. Weakly Jiang maps on K

Let f be a self-map on K of normalized type $(F,\mathbf{b},\omega)$ with induced homomorphism φ on Π. By Lemma 4.6, f^n is of type $(F^n,\mathbf{b}_n,\omega^n)$, not necessarily normalized, where

$$\mathbf{b}_n = \begin{cases} (I+F+\ldots+F^{n-1})\mathbf{b} & \text{when } \omega \text{ is odd;} \\ \omega^{n-1}\mathbf{b} & \text{when } \omega \text{ is even.} \end{cases}$$

We will discuss the case where f^n is weakly Jiang.

Theorem 5.1. *Let f be a self-map on K of type $(F,\mathbf{b},\omega)$. Then f^n is weakly Jiang if and only if one of the following holds*:

(a) *Case* $N(f^n)=0$:
- $\det(I\pm F^n)=0$,
- $\omega=1$,
- $\omega=-1$ *and* n *even.*

(b) *Case* $N(f^n)=R(f^n)$:
- ω *even,*
- $\omega=-1$, n *odd and* $\det(I\pm F^n)\neq 0$,
- $\omega\neq\pm1$ *odd and* $\det(I\pm F^n)\neq 0$.

Proof. The first case follows directly from Theorem 4.9. For the second case, we shall recall the averaging formula for the Reidemeister coincidence number on orientable infra-nilmanifolds from [3, Theorem 6.11]. We can use this result because our infra-nilmanifold K is orientable. In fact, the fixed point version of this result is true for all infra-nilmanifolds. Namely,

$$R(f)=\frac{1}{|\Psi|}\sum_{A\in\Psi}\sigma(\det(A_*-f_*)),$$

where $\sigma\colon\mathbb{R}\to\mathbb{R}\cup\{\infty\}$ is defined by $\sigma(0)=\infty$ and $\sigma(x)=|x|$ for all $x\neq 0$. Hence

$$R(f^n)=\frac{1}{2}\left(\sigma(\det(I-D^n))+\sigma(\det(A-D^n))\right).$$

Therefore, $N(f^n)=R(f^n)$ if and only if $\det(I-D^n)\neq 0$ and $\det(A-D^n)\neq 0$ if and only if $\omega^n\neq 1$, $\det(I-F^n)\neq 0\neq\det(I+F^n)$. $\square$

Immediately we have:

Corollary 5.2. *If $N(f^n)=R(f^n)$ holds, then for any m with $m\,|\,n$, we have that $N(f^m)=R(f^m)$.*

Proof. Observe that if $m\,|\,n$ then $I-F^n=(I-F^m)(I+F^m+\ldots+F^{n-m})$ and so $\det(I-F^m)\,|\,\det(I-F^n)$; moreover, in addition, if n/m is even then $I-F^n=(I+F^m)(I-F^m+\ldots-F^{n-m})$ implies that $\det(I+F^m)\,|\,\det(I-F^n)$, and if n/m is odd then $I+F^n=(I+F^m)(I-F^m+\ldots+F^{n-m})$ implies that $\det(I+F^m)\,|\,\det(I+F^n)$. This implies our result. $\square$

Corollary 5.3. *If $\det(I\pm F^n)=0$ holds, then n must be odd and the eigenvalues of F are ± 1. Moreover, for any odd m $\det(I\pm F^m)=0$ holds and, for any even m, $\det(I-F^m)=0$ and $\det(I+F^m)=4$ and so $N(f^m)=2|1-\omega^m|$.*

Proof. Assume $\det(I\pm F^n)=0$. Let λ_1 and λ_2 be the eigenvalues of F. Then $1+\lambda_1^n\lambda_2^n-(\lambda_1^n+\lambda_2^n)=0$ and $1+\lambda_1^n\lambda_2^n+(\lambda_1^n+\lambda_2^n)=0$. These identities induce that $\lambda_2^n=-\lambda_1^n$ and $\lambda_1^{2n}=1$. If the λ_i are real then $\lambda_1=-\lambda_2=\pm 1$ and n must be odd; thus in this case for any odd m, we must have $\det(I\pm F^m)=0$. If the λ_i are complex numbers (so that $\lambda_2=\overline{\lambda}_1$) then $\lambda_2^n=-\lambda_1^n$ yields that $\lambda_2^n=\overline{\lambda}_1^n=-\lambda_1^n$ and thus $\lambda_1^n=\pm i$ and $\lambda_1^{2n}=(\pm i)^2=-1$, which is a contradiction. Now it is simply a routine to check the last assertion. $\square$

Let f be a self-map on K with induced homomorphism φ on Π of type $(F,\mathbf{b},\omega)$. Assuming $N(f^n)=R(f^n)$ so that f^n is weakly Jiang, we will find a complete set of representatives for the Reidemeister set $\mathcal{R}[\varphi^n]$.

Notation 5.4. Note that for any k, φ^k acts on the integral lattice spanned by s_1 and s_2 by matrix multiplication by F^k. That is, the exponents of $\varphi^k(s_1^{i_1}s_2^{i_2})$ are the entries of F^k multiplied by the column vector $\mathbf{i}$ of (i_1,i_2). For simplicity, we will use the following notations:

$$s^{\mathbf{i}}:=s_1^{i_1}s_2^{i_2},\qquad s^{F^k\mathbf{i}}:=\varphi^k(s^{\mathbf{i}}).$$

Similarly, we will use $\bar{\mathbf{i}}$ and $F^k\bar{\mathbf{i}}$ for the elements in a quotient group of $\mathbb{Z}^2$ represented by $\mathbf{i}$ and $F^k\mathbf{i}$, respectively.

Notation 5.5. As we are dealing with only integral matrices M of order 2, when we say the image of M, $\mathrm{Im}(M)$, we shall mean that

$$\mathrm{Im}(M) = \mathrm{image}\{M : \mathbb{Z}^2 \to \mathbb{Z}^2\}.$$

Remark 5.6 (ω even). Let f be a self-map on K with induced homomorphism φ on Π of type $(F, \mathbf{b}, \omega)$ where ω is even and so $F = 0$. In this case $R(\varphi^n) = N(f^n) = |1-\omega^n| \neq 0$. Since

$$\varphi^n(s^{\mathbf{x}}\alpha^y) = s^{y\omega^{n-1}\mathbf{b}}\alpha^{y\omega^n}, \qquad \varphi^n(s^{\mathbf{x}}\alpha^y)^{-1} = s^{-y\omega^{n-1}\mathbf{b}}\alpha^{-y\omega^n},$$

we have for each $\mathbf{i} \in \mathbb{Z}^2$ and $j \in \mathbb{Z}$

$$(s^{\mathbf{x}}\alpha^y)(s^{\mathbf{i}}\alpha^j)\varphi^n(s^{\mathbf{x}}\alpha^y)^{-1} = \; s^{\mathbf{x}+(-1)^y\mathbf{i}-(-1)^{y+j}y\omega^{n-1}\mathbf{b}}\alpha^{y(1-\omega^n)+j}.$$

It follows that $\mathcal{R}[\varphi^n] = \{[\alpha]_n, [\alpha^2]_n, \dots, [\alpha^{|1-\omega^n|}]_n\}$.

In fact, $\varphi\colon \Pi \to \Pi$ induces the following commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & \mathbb{Z}^2 \arrow[r] \arrow[d, "0"] & \pi \arrow[r] \arrow[d, "\varphi"] & \mathbb{Z} \arrow[r] \arrow[d, "\overline{\varphi}"] & 1 \\
1 \arrow[r] & \mathbb{Z}^2 \arrow[r] & \pi \arrow[r] & \mathbb{Z} \arrow[r] & 1
\end{tikzcd}$$

This diagram induces the exact sequence of the Reidemeister sets

$$\mathcal{R}(0) \longrightarrow \mathcal{R}(\varphi) \longrightarrow \mathcal{R}(\overline{\varphi}) \longrightarrow 1.$$

Since $\mathcal{R}(0) = 1$ and the same holds for all iterations φ^k, we get the equality of all prime and full Nielsen–Jiang numbers. Thus the results are the same as in the corresponding self-maps of the circle, the base space of the fibration. This was suggested by the referee.

Remark 5.7 ($\omega = -1$ and n odd). Let f be a self-map on K with induced homomorphism φ on Π of type $(F, \mathbf{b}, \omega)$ where $\omega = -1$, n odd. In this case $N(f^n) = |\det(I-F^n)| + |\det(I+F^n)|$. Note also that φ^n is of type $(F^n, \mathbf{b}_n, -1)$. Observe that

$$\varphi^n(s^{\mathbf{x}}\alpha^y) = \begin{cases} s^{F^n\mathbf{x}+\mathbf{b}_n}\alpha^{-y} & \text{when } y \text{ is odd;} \\ s^{F^n\mathbf{x}}\alpha^{-y} & \text{when } y \text{ is even,} \end{cases}$$

and so

$$\varphi^n(s^{\mathbf{x}}\alpha^y)^{-1} = \begin{cases} s^{F^n\mathbf{x}+\mathbf{b}_n}\alpha^{y} & \text{when } y \text{ is odd;} \\ s^{-F^n\mathbf{x}}\alpha^{y} & \text{when } y \text{ is even.} \end{cases}$$

For each $\mathbf{i} \in \mathbb{Z}^2$ and $j \in \mathbb{Z}$, we then have

$$(s^{\mathbf{x}}\alpha^y)(s^{\mathbf{i}}\alpha^j)\varphi^n(s^{\mathbf{x}}\alpha^y)^{-1} = \begin{cases} s^{\mathbf{x}-\mathbf{i}-(-1)^j(F^n\mathbf{x}+\mathbf{b}_n)}\alpha^{2y+j} & \text{when } y \text{ is odd;} \\ s^{\mathbf{x}+\mathbf{i}-(-1)^j(F^n\mathbf{x})}\alpha^{2y+j} & \text{when } y \text{ is even.} \end{cases}$$

Now we analyze this identity more as follows:

Case 1. $j \equiv 0$ or $2 \mod 4$.

We choose y to be even or odd respectively and then we can assume $j = 0$. The exponents of s_1 and s_2 are then, respectively,

$$(I - F^n)\mathbf{x} + \mathbf{i} \quad \text{or} \quad (I - F^n)\mathbf{x} - \mathbf{i} - \mathbf{b}_n.$$

Case 2. $j \equiv 1$ or $3 \mod 4$.

We choose y to be even or odd, respectively, and then we can assume $j = 1$. The exponents of s_1 and s_2 are then, respectively

$$(I + F^n)\mathbf{x} + \mathbf{i} \quad \text{or} \quad (I + F^n)\mathbf{x} - \mathbf{i} + \mathbf{b}_n.$$

Since $\det(I \pm F^n) \neq 0$, we have that $\mathrm{Im}(I - F^n)$ and $\mathrm{Im}(I + F^n)$ have finite indices in $\mathbb{Z}^2$, and $|\mathbb{Z}^2/\mathrm{Im}(I \pm F^n)| = |\det(I \pm F^n)|$.

Consequently, we can choose a complete set of representatives of the Reidemeister classes $[s^{\mathbf{i}}\alpha^j]_n$ in the quotient groups $\mathbb{Z}^2/\mathrm{Im}(I \pm F^n)$. Namely,

$$\mathcal{R}[\varphi^n] = \{[s^{\bar{\mathbf{i}}}]_n \mid \bar{\mathbf{i}} \in \mathbb{Z}^2/\mathrm{Im}(I - F^n)\} \bigcup \{[s^{\bar{\mathbf{i}}}\alpha]_n \mid \bar{\mathbf{i}} \in \mathbb{Z}^2/\mathrm{Im}(I + F^n)\}.$$

When $\det(I \pm F^n) \neq 0$, we have that $\mathrm{Im}(I - F^n)$ and $\mathrm{Im}(I + F^n)$ have finite indices in $\mathbb{Z}^2$, and $|\mathbb{Z}^2/\mathrm{Im}(I \pm F^n)| = |\det(I \pm F^n)|$. Thus in this case f^n is weakly Jiang.

Remark 5.8 ($\omega \neq \pm 1$ odd). Let f be a self-map on K with induced homomorphism φ on Π of type $(F, \mathbf{b}, \omega)$ where ω is odd, $\omega \neq \pm 1$. In this case

$$N(f^n) = \frac{1}{2}|1 - \omega^n|(|\det(I - F^n)| + |\det(I + F^n)|).$$

Note also that φ^n is of type $(F^n, \mathbf{b}_n, \omega^n)$. Observe that

$$\varphi^n(s^{\mathbf{x}}\alpha^y) = \begin{cases} s^{F^n\mathbf{x} + \mathbf{b}_n}\alpha^{y\omega^n} & \text{when } y \text{ is odd;} \\ s^{F^n\mathbf{x}}\alpha^{y\omega^n} & \text{when } y \text{ is even,} \end{cases}$$

and so

$$\varphi^n(s^{\mathbf{x}}\alpha^y)^{-1} = \begin{cases} s^{F^n\mathbf{x} + \mathbf{b}_n}\alpha^{-y\omega^n} & \text{when } y \text{ is odd;} \\ s^{-F^n\mathbf{x}}\alpha^{-y\omega^n} & \text{when } y \text{ is even.} \end{cases}$$

For each $\mathbf{i} \in \mathbb{Z}^2$ and $j \in \mathbb{Z}$, we then have

$$(s^{\mathbf{x}}\alpha^y)(s^{\mathbf{i}}\alpha^j)\varphi^n(s^{\mathbf{x}}\alpha^y)^{-1} = \begin{cases} s^{\mathbf{x} - \mathbf{i} - (-1)^j(F^n\mathbf{x} + \mathbf{b}_n)}\alpha^{y(1-\omega^n)+j} & \text{when } y \text{ is odd;} \\ s^{\mathbf{x} + \mathbf{i} - (-1)^jF^n\mathbf{x}}\alpha^{y(1-\omega^n)+j} & \text{when } y \text{ is even.} \end{cases}$$

Noting that $1 - \omega^n$ is even, we will consider this identity according to j modulo $2(1 - \omega^n)$. Let $j \equiv j'$ or $j \equiv j' + |1 - \omega^n|$ so that $0 \leq j' < |1 - \omega^n|$. We choose y to be even or odd respectively and then we can assume $j = j'$. The exponents of s_1 and s_2 are then, respectively,

$$(I - (-1)^{j'}F^n)\mathbf{x} + \mathbf{i} \quad \text{or} \quad (I - (-1)^{j'}F^n)\mathbf{x} - \mathbf{i} - (-1)^{j'}\mathbf{b}_n.$$

This implies that

$$\begin{aligned}\mathcal{R}[\varphi^n] &= \{[s^{\bar{\mathbf{i}}}\alpha^j]_n \mid \bar{\mathbf{i}} \in \mathbb{Z}^2/\mathrm{Im}(I-F^n);\ j=0,2,\dots,|1-\omega^n|-2\} \\ &\quad \cup \{[s^{\bar{\mathbf{i}}}\alpha^j]_n \mid \bar{\mathbf{i}} \in \mathbb{Z}^2/\mathrm{Im}(I+F^n);\ j=1,3,\dots,|1-\omega^n|-1\}.\end{aligned}$$

6. The Nielsen type numbers: non-weakly Jiang case I

Now we will evaluate the Nielsen type number of periodic points of f in the case when f^n is not weakly Jiang, i.e. $0 \neq N(f^n) \neq R(f^n)$. Explicitly, we should consider the following two cases in this subsection and the next subsection:

- n is odd, $\omega = -1$, and exactly one of $I - F^n$ and $I + F^n$ has zero determinant.
- $\omega \neq \pm 1$ is odd and exactly one of $I-F^n$ and $I+F^n$ has zero determinant.

NOTATION 6.1. Let $n = p_0^{e_0} \dots p_t^{e_t}$ be the prime decomposition of the positive integer n so that $p_0 = 2$ with $e_0 \geq 0$ and the other p_j's are distinct odd primes. Then a proper maximal divisor m of n is of the form n/p_j. Write for each $j = 0, \dots, t$ and for each *nonempty* subset $\{k_0, \dots, k_s\} \subset \{0, \dots, t\}$,

$$n_j = \frac{n}{p_j} \quad \text{and} \quad (n_{k_0}, \dots, n_{k_s}) = \gcd(n_{k_0}, \dots, n_{k_s}) = \frac{n}{p_{k_0} \cdots p_{k_s}}.$$

Before computing the Nielsen type numbers, we observe the following necessary and elementary facts.

LEMMA 6.2. *If F is any integer square matrix and p, q are relatively prime numbers, then*

$$\begin{aligned}\mathrm{Im}(I + F^p + \dots + F^{pq-p}) &\cap \mathrm{Im}(I + F^q + \dots + F^{pq-q}) \\ &= \mathrm{Im}(I + F + \dots + F^{pq-1}).\end{aligned}$$

PROOF. The following observation

$$\begin{aligned}(I + F + \dots + F^{pq-1}) &= (I + F^p + \dots + F^{pq-p})(I + F + \dots + F^{p-1}) \\ &= (I + F^q + \dots + F^{pq-q})(I + F + \dots + F^{q-1}),\end{aligned}$$

implies that the right-hand side is contained in the left-hand side.

For the reverse inclusion, choose any element $\mathbf{x}$ in the left-hand side. Then, for some $\mathbf{y}, \mathbf{z} \in \mathbb{Z}^2$,

$$\mathbf{x} = (I + F^p + \dots + F^{pq-p})\mathbf{y}, \qquad \mathbf{x} = (I + F^q + \dots + F^{pq-q})\mathbf{z}.$$

Thus

$$\begin{aligned}(I + F + \dots + F^{p-1})\mathbf{x} &= (I + F + \dots + F^{p-1})(I + F^p + \dots + F^{pq-p})\mathbf{y} \\ &= (I + F + \dots + F^{pq-1})\mathbf{y} \in \mathrm{Im}(I + F + \dots + F^{pq-1}).\end{aligned}$$

Similarly, we have

$$(I + F + \ldots + F^{q-1})\,\mathbf{x} \in \operatorname{Im}(I + F + \ldots + F^{pq-1}).$$

Since $\operatorname{Im}(I + F + \ldots + F^{pq-1})$ is a subgroup of $\mathbb{Z}^2$ and is invariant under F^k for all non-negative integer k. We may assume that $p < q$. Write

$$q = \ell p + r, \quad 0 < r < p, \quad (\ell, r \in \mathbb{Z}).$$

Then $\gcd(p, r) = 1$ and

$$\begin{aligned}
&(I + F + \ldots + F^{r-1})\,\mathbf{x} \\
&= (I + F + \ldots + F^{q-1})\,\mathbf{x} - (F^r + F^{r+p} + \ldots + F^{q-p})(I + F + \ldots + F^{p-1})\,\mathbf{x} \\
&\qquad\qquad\qquad\qquad\qquad\qquad \in \operatorname{Im}(I + F + \ldots + F^{pq-1}).
\end{aligned}$$

Similarly, writing $p = \ell' r + r'$, $0 < r' < r$, $(\ell', r' \in \mathbb{Z})$, we have

$$\gcd(r, r') = 1, \qquad (I + F + \ldots + F^{r'-1})\,\mathbf{x} \in \operatorname{Im}(I + F + \ldots + F^{pq-1}).$$

Continuing this process, we can show that $\mathbf{x} \in \operatorname{Im}(I + F + \ldots + F^{pq-1})$ as it is required, and hence completes the proof. $\square$

Corollary 6.3. *If F is any integer square matrix and p, q are relatively prime numbers, then*

(a) $\operatorname{Im}(I + F^{pm} + \ldots + F^{pqm-pm}) \cap \operatorname{Im}(I + F^{qm} + \ldots + F^{pqm-qm})$ $= \operatorname{Im}(I + F^{m} + \ldots + F^{pqm-m})$.

(b) $\operatorname{Im}(I - F^{pm} + F^{2pm} - \ldots + (-1)^{q-1}F^{(q-1)pm}) \cap \operatorname{Im}(I - F^{qm} + F^{2qm} - \ldots + (-1)^{q-1}F^{(p-1)qm}) = \operatorname{Im}(I - F^{m} + F^{2m} - \ldots + (-1)^{pq-1}F^{(pq-1)m})$.

(c) $1 + x + \ldots + x^{pq-1}$ *is the least common multiple of* $1 + x^p + \ldots + x^{(q-1)p}$ *and* $1 + x^q + \ldots + x^{(p-1)q}$.

Proof. We obtain (a) and(b) by replacing F by F^m and $-F^m$, respectively, in Lemma 6.2.

When $F = [x]$ is an integer 1×1 matrix, $\operatorname{Im}(I + F^p + \ldots + F^{pq-p})$ is the set of all multiples of $1 + x^p + \ldots + x^{pq-p}$. Thus we can get (c) by Lemma 6.2. $\square$

Lemma 6.4. *If $n = 2pm$ where p is an odd prime number, then*

$$\operatorname{Im}(I - F^{pm}) \cap \operatorname{Im}(I + F^{2m} + \ldots + F^{n-2m}) = \operatorname{Im}((I - F^{m})(I + F^{2m} + \ldots + F^{n-2m})).$$

In particular, $(1 - x^{2pm})/(1 + x^m) = (1 - x^m)(1 + x^{2m} + \ldots + x^{(p-1)2m})$ *is the least common multiple of* $1 - x^{pm}$ *and* $1 + x^{2m} + \ldots + x^{(p-1)2m}$.

Proof. Note that

$$\begin{aligned}
I - F^{pm} &= (I - F^{m})(I + F^{m} + \ldots + F^{pm-m}), \\
I + F^{2m} + \ldots + F^{2pm-2m} &= (I - F^{m} + \ldots + |!F^{pm-m})(I + F^{m} + \ldots + F^{pm-m}).
\end{aligned}$$

Thus the right-hand side is contained in the left-hand side. Write the right-hand side by

$$\begin{aligned} Z &= \mathrm{Im}((I - F^m)(I + F^{2m} + \ldots + F^{n-2m})) \\ &= \mathrm{Im}((I - F^{2m})(I - F^m + F^{2m} - \ldots + F^{pm-m})). \end{aligned}$$

Choose any element $\mathbf{x}$ in the left-hand side. Then for some $\mathbf{y}, \mathbf{z} \in \mathbb{Z}^2$

$$\mathbf{x} = (I - F^{pm})\,\mathbf{y}, \qquad \mathbf{x} = (I + F^{2m} + \ldots + F^{n-2m})\,\mathbf{z}.$$

Hence

$$(I - F^m + \ldots + F^{pm-m})\,\mathbf{x} \in Z, \qquad (I - F^m)\,\mathbf{x} \in Z.$$

Since the group Z is invariant under F^k for all non-negative integer k,

$$\mathbf{x} = (I - F^m + \ldots + F^{pm-m})\,\mathbf{x} + F^m(I + F^{2m} + \ldots + F^{(p-3)m})(I - F^m)\,\mathbf{x} \in Z.$$

Hence the left-hand side is contained in the right-hand side. □

THEOREM 6.5. *Let f be a self-map on K of type $(F, \mathbf{b}, -1)$. If $n = p_1^{e_1} \ldots p_t^{e_t}$ is odd and* $\det(I + F^n) = 0$ *but* $\det(I - F^n) \neq 0$, *then the prime Nielsen–Jiang periodic number of period n is*

$$N\mathrm{P}_n(f) = |\det(I - F^n)| + \sum_{\{k_1,\ldots,k_s\} \subset \{1,\ldots,t\}} (-1)^s |\det(I - F^{(n_{k_1},\ldots,n_{k_s})})|,$$

where the summation runs through the family of nonempty subsets $\{k_1, \ldots, k_s\}$ of $\{1, \ldots, t\}$.

PROOF. Suppose that $\det(I - F^n) \neq 0$ and $\det(I + F^n) = 0$. It is Remark 5.7 that

$$\mathcal{R}[\varphi^n] = \{[s^{\bar{\mathbf{i}}}]_n \mid \bar{\mathbf{i}} \in \mathbb{Z}^2/\mathrm{Im}(I - F^n)\} \cup \{[s^{\bar{\mathbf{i}}}\alpha]_n \mid \bar{\mathbf{i}} \in \mathbb{Z}^2/\mathrm{Im}(I + F^n)\}$$

where the first set is finite but the second set is infinite. Note also that for $m \mid n$, $I - F^m$ is a factor of $I - F^n$ and hence if $\det(I - F^n) \neq 0$ then $\det(I - F^m) \neq 0$. But one cannot expect that if $\det(I + F^n) = 0$ then $\det(I + F^m) = 0$.

We will show first that $\{[s^{\bar{\mathbf{i}}}]_n \,|, \bar{\mathbf{i}} \in \mathbb{Z}^2/(I - F^n)(\mathbb{Z}^2)\}$ is the set of all essential classes. The Reidemeister class $[s^{\bar{\mathbf{i}}}]_n$ corresponds to the fixed point class $p(\mathrm{Fix}(s^{\mathbf{i}}\widetilde{f}^n))$ of f^n. Recalling from Lemma 4.6 that

$$\widetilde{f} = (\delta, D) = \left(\begin{bmatrix} \mathbf{b}/2 \\ * \end{bmatrix}, \begin{bmatrix} F & 0 \\ 0 & \omega \end{bmatrix} \right) = \left(\begin{bmatrix} \mathbf{b}/2 \\ * \end{bmatrix}, \begin{bmatrix} F & 0 \\ 0 & -1 \end{bmatrix} \right),$$

we have

$$s^{\mathbf{i}}\widetilde{f}^n = \left(\begin{bmatrix} \mathbf{i} \\ 0 \end{bmatrix} + \begin{bmatrix} \mathbf{b}_n/2 \\ * \end{bmatrix}, \begin{bmatrix} F^n & 0 \\ 0 & -1 \end{bmatrix} \right).$$

By a simple computation we see that $\mathrm{Fix}(s^{\mathbf{i}}\widetilde{f}^n)$ and hence $p(\mathrm{Fix}(s^{\mathbf{i}}\widetilde{f}^n))$ consist of a single element. This tells that the correspondence Reidemeister class $[s^{\bar{\mathbf{i}}}]_n$

is essential. Since $N(f^n) = |\det(I - F^n)|$, it follows that all other classes are inessential.

Next we will observe which are irreducible among the essential classes $[s^{\bar{\mathbf{i}}}]_n$. Suppose $[s^{\mathbf{i}}]_n$ is reducible to the level m. Then $\iota_{m,n}([s^{\mathbf{x}}\alpha^y]_m) = [s^{\mathbf{i}}]_n$ for some $[s^{\mathbf{x}}\alpha^y]_m \in \mathcal{R}[\varphi^m]$. So $m \,|\, n$ and $y = 0$. Note that as n is odd, both m and n/m are odd.

Now a calculation shows that

$$\iota_{m,n}([s^{\mathbf{x}}]_m) = [s^{\mathbf{i}}\varphi^m(s^{\mathbf{x}})\varphi^{2m}(s^{\mathbf{x}})\ldots\varphi^{n-m}(s^{\mathbf{x}})]_n = [s^{\mathbf{x}_{m,n}}]_n$$

where $\mathbf{x}_{m,n} = (I + F^m + F^{2m} + \ldots + F^{n-m})\mathbf{x}$. If $\mathbf{x} \in \mathrm{Im}(I - F^m)$ then clearly $\mathbf{x}_{m,n} \in \mathrm{Im}(I - F^n)$; if $\mathbf{x}_{m,n} \in \mathrm{Im}(I - F^n)$ then $\mathbf{x} \in \mathrm{Im}(I - F^m)$. For if $\mathbf{x}_{m,n} = (I - F^n)\mathbf{j}$ for some $\mathbf{j}$ then $(I + F^m + F^{2m} + \ldots + F^{n-m})((I - F^m)\mathbf{j} - \mathbf{x}) = \mathbf{0}$ and as $I - F^n$ is invertible it follows that $(I - F^m)\mathbf{j} - \mathbf{x} = \mathbf{0}$. This observation shows that the homomorphism defined by multiplication by $I + F^m + \ldots + F^{n-m}$ is an isomorphism

$$\begin{aligned}(6.1)\quad & \mathbf{x} \in \mathbb{Z}^2/\mathrm{Im}(I - F^m) \\ & \qquad\mapsto \mathbf{x}_{m,n} \in \mathrm{Im}(I + F^m + F^{2m} + \ldots + F^{n-m})/\mathrm{Im}(I - F^n)\end{aligned}$$

and that the essential class $[s^{\bar{\mathbf{i}}}]_n$ is reducible to m if and only if

$$\bar{\mathbf{i}} \in \mathrm{Im}(I + F^m + \ldots + F^{n-m})/\mathrm{Im}(I - F^n) \subset \mathbb{Z}^2/\mathrm{Im}(I - F^n).$$

Obviously, when $n = 1$ all the essential classes $[s^{\bar{\mathbf{i}}}]_1$ are irreducible. Furthermore, for $n > 1$ if $m \,|\, m' \mid n$ and the essential class $[s^{\bar{\mathbf{i}}}]_n$ is reducible to m, then it is reducible to m'. Consequently, the essential class $[s^{\bar{\mathbf{i}}}]_n$ is reducible if and only if for some proper maximal divisor m of n

$$\bar{\mathbf{i}} \in C_i := \mathrm{Im}(I + F^m + \ldots + F^{n-m})/\mathrm{Im}(I - F^n).$$

Thus the set of essential irreducible classes is a one-to-one correspondence with the set

$$\mathbb{Z}^2/\mathrm{Im}(I - F^n) - \bigcup_{j=1}^{t} C_i.$$

By (6.1), $\#C_i = |\det(I - F^{(n_i)})|$. For $1 \le i < i' \le t$, by Corollary 6.3(a),

$$C_i \cap C_{i'} = \mathrm{Im}(I + F^{(n_i,n_{i'})} + \ldots + F^{n-(n_i,n_{i'})})/\mathrm{Im}(I - F^n)$$

and so, by (6.1) again, $\#(C_i \cap C_{i'}) = |\det(I - F^{(n_i,n_{i'})})|$. Using induction, we have

$$\#\bigg(\bigcap_{i=1}^{s} C_{k_i}\bigg) = |\det(I - F^{(n_{k_1},\ldots,n_{k_s})})|.$$

Consequently,

$$\#\Big(\bigcup_{i=1}^{t} C_i\Big) = \sum_{\{k_1,\dots,k_s\}\subset\{1,\dots,t\}} (-1)^{s-1}|\det(I - F^{(n_{k_1},\dots,n_{k_s})})|.$$

Finally, we will find the length of each φ-orbit of the essential irreducible classes $[s^{\bar{\mathbf{i}}}]_n$. By definition,

$$\langle[s^{\bar{\mathbf{i}}}]_n\rangle = \{[s^{\bar{\mathbf{i}}}]_n, [\varphi]([s^{\bar{\mathbf{i}}}]_n), \dots, [\varphi]^{\ell-1}([s^{\bar{\mathbf{i}}}]_n)\} = \Big\{[s^{\bar{\mathbf{i}}}]_n, [s^{F\bar{\mathbf{i}}}]_n, \dots, [s^{F^{\ell-1}\bar{\mathbf{i}}}]_n\Big\},$$

where $\ell = \ell([s^{\bar{\mathbf{i}}}]_n)$ is the length of $[s^{\bar{\mathbf{i}}}]_n$. Then $\ell \mid n$ and $[s^{\bar{\mathbf{i}}}]_n = [s^{F^{\ell}\bar{\mathbf{i}}}]_n$. This implies that $(I-F^{\ell})\mathbf{i} = (I-F^n)\mathbf{j}$ for some $\mathbf{j} \in \mathbb{Z}^2$. So, $(I+F^{\ell}+\dots+F^{n-\ell})\mathbf{j} = \mathbf{i}$ or $\bar{\mathbf{i}} \in \operatorname{Im}(I+F^{\ell}+\dots+F^{n-\ell})/\operatorname{Im}(I-F^n)$. Thus this shows that all the irreducible essential Reidemeister classes have the same length n.

In all, we obtain that

$$\begin{aligned} O_n(\varphi) &= \frac{1}{n}\#\Big(\mathbb{Z}^2/\operatorname{Im}(I-F^n) - \bigcup_{i=1}^{t} C_i\Big) \\ &= \frac{1}{n}\Big(\#(\mathbb{Z}^2/\operatorname{Im}(I-F^n)) - \#\Big(\bigcup_{i=1}^{t} C_i\Big)\Big) \\ &= \frac{1}{n}\Big(|\det(I-F^n)| + \sum_{\{k_1,\dots,k_s\}\subset\{1,\dots,t\}} (-1)^{s}|\det(I - F^{(n_{k_1},\dots,n_{k_s})}|\Big) \end{aligned}$$

The conclusion now follows from the definition: $N\mathrm{P}_n(f) = n \times O_n(\varphi)$. □

Theorem 6.6. *Let f be a self-map on K of type $(F,\mathbf{b},-1)$. If $n = p_1^{e_1}\dots p_t^{e_t}$ is odd and $\det(I-F^n) = 0$ but $\det(I+F^n) \neq 0$, then the prime Nielsen–Jiang periodic number of period n is*

$$N\mathrm{P}_n(f) = |\det(I+F^n)| + \sum_{\{k_1,\dots,k_s\}\subset\{1,\dots,t\}} (-1)^{s}|\det(I + F^{(n_{k_1},\dots,n_{k_s})})|,$$

where the summation runs through the family of nonempty subsets $\{k_1,\dots,k_s\}$ of $\{1,\dots,t\}$.

Proof. Let f be a self-map on K of type $(F,\mathbf{b},-1)$. Suppose that n is odd and $\det(I-F^n) = 0$ but $\det(I+F^n) \neq 0$. As it has been observed before, we may replace F by $-F$. This means that f is homotopic to a self-map on K of type $(-F,\mathbf{b},-1)$, see Theorem 4.4. Since n is odd, the conditions $\det(I-F^n) = 0$ and $\det(I+F^n) \neq 0$ become $\det(I+(-F)^n) = 0$ and $\det(I-(-F)^n) \neq 0$. Using the fact that the prime Nielsen periodic number is a homotopy invariant and applying Theorem 6.5, we have

$$N\mathrm{P}_n(f) = |\det(I-(-F)^n)| + \sum_{\{k_1,\dots,k_s\}\subset\{1,\dots,t\}} (-1)^{s}|\det(I - (-F)^{(n_{k_1},\dots,n_{k_s})})|.$$

This proves our result. □

7. The Nielsen type numbers: non-weakly Jiang case II

In this subsection we will discuss the remaining case where $\omega \neq \pm 1$ is odd and exactly one of $I-F^n$ and $I+F^n$ has zero determinant. Our computation problem of the Nielsen type numbers can be done in a similar, but much complicated way as we have done in the previous section by the following general four steps.

Step 1. Finding the Reidemeister classes $\mathcal{R}[\varphi^n]$.
By Remark 5.8, we have

$$\begin{aligned}\mathcal{R}[\varphi^n] = &\{[s^{\bar{\mathbf{i}}}\alpha^j]_n \mid \bar{\mathbf{i}} \in \mathbb{Z}^2/\mathrm{Im}(I-F^n);\ j = 0, 2, \dots, |1-\omega^n|-2\} \\ &\cup \{[s^{\bar{\mathbf{i}}}\alpha^j]_n \mid \bar{\mathbf{i}} \in \mathbb{Z}^2/\mathrm{Im}(I+F^n);\ j = 1, 3, \dots, |1-\omega^n|-1\}.\end{aligned}$$

Step 2. Finding the essential Reidemeister classes.
Recalling from Lemma 4.6 that

$$\widetilde{f} = (\delta, D) = \left(\begin{bmatrix} \mathbf{b}/2 \\ * \end{bmatrix}, \begin{bmatrix} F & 0 \\ 0 & \omega \end{bmatrix} \right),$$

we have

$$s^{\mathbf{i}}\alpha^j \widetilde{f}^n = \left(\begin{bmatrix} \mathbf{i} \\ j/2 \end{bmatrix} + \begin{bmatrix} (-1)^j \mathbf{b}_n/2 \\ *' \end{bmatrix}, \begin{bmatrix} (-1)^j F^n & 0 \\ 0 & \omega^n \end{bmatrix} \right),$$

where $*' = (1 + \omega + \ldots + \omega^{n-1})*$. By a simple computation we see that $\mathrm{Fix}(s^{\mathbf{i}}\alpha^j \widetilde{f}^n)$ is:
when j is even,

$$\mathrm{Fix}(s^{\mathbf{i}}\alpha^j \widetilde{f}^n) = \left\{ \begin{bmatrix} \mathbf{x} \\ y \end{bmatrix} \;\middle|\; (I-F^n)\mathbf{x} = \mathbf{i} + \frac{1}{2}\,\mathbf{b}_n, (1-\omega^n)y = \frac{j}{2} + *' \right\};$$

when j is odd,

$$\mathrm{Fix}(s^{\mathbf{i}}\alpha^j \widetilde{f}^n) = \left\{ \begin{bmatrix} \mathbf{x} \\ y \end{bmatrix} \;\middle|\; (I+F^n)\mathbf{x} = \mathbf{i} - \frac{1}{2}\,\mathbf{b}_n,\ (1-\omega^n)y = \frac{j}{2} + *' \right\}.$$

We consider first the case that $\det(I-F^n) \neq 0$ and $\det(I+F^n) = 0$. When j is even $\mathrm{Fix}(s^{\mathbf{i}}\alpha^j \widetilde{f}^n))$ and hence $p(\mathrm{Fix}(s^{\mathbf{i}}\alpha^j \widetilde{f}^n))$ consist of a single element. This tells that the correspondence Reidemeister classes $[s^{\bar{\mathbf{i}}}\alpha^j]_n$ are essential. Since $N(f^n) = |1-\omega^n||\det(I-F^n)|/2$, it follows that all other classes are inessential. Thus the essential classes are

$$EC_1 := \{[s^{\bar{\mathbf{i}}}\alpha^j]_n \mid \bar{\mathbf{i}} \in \mathbb{Z}^2/\mathrm{Im}(I-F^n);\ j = 0, 2, \dots, |1-\omega^n|-2\}.$$

If the other case occurs, in a similar way we can see that the essential classes are

$$EC_2 := \{[s^{\bar{\mathbf{i}}}\alpha^j]_n \mid \bar{\mathbf{i}} \in \mathbb{Z}^2/\mathrm{Im}(I+F^n);\ j = 1, 3, \dots, |1-\omega^n|-1\}.$$

Before moving to the next steps, we recall that

$$\varphi^k(s^{\mathbf{x}}\alpha^y) = \begin{cases} s^{F^k\mathbf{x}}\alpha^{\omega^k y} & \text{when } y \text{ is even;} \\ s^{F^k\mathbf{x}+\mathbf{b}_k}\alpha^{\omega^k y} & \text{when } y \text{ is odd.} \end{cases}$$

Let $m \mid n$. When y is even,

$$\begin{aligned} \iota_{m,n}([s^{\mathbf{x}}\alpha^y]_m) &= [(s^{\mathbf{x}}\alpha^y)\varphi^m(s^{\mathbf{x}}\alpha^y)\varphi^{2m}(s^{\mathbf{x}}\alpha^y)\dots\varphi^{n-m}(s^{\mathbf{x}}\alpha^y)]_n \\ &= [(s^{\mathbf{x}}\alpha^y)(s^{F^m\mathbf{x}}\alpha^{\omega^m y})(s^{F^{2m}\mathbf{x}}\alpha^{\omega^{2m}y})\dots(s^{F^{n-m}\mathbf{x}}\alpha^{\omega^{n-m}y})]_n \\ &= [s^{(I+F^m+F^{2m}+\dots+F^{n-m})\mathbf{x}}\alpha^{(1+\omega^m+\omega^{2m}+\dots+\omega^{n-m})y}]_n \\ &:= [s^{\mathbf{x}'_{m,n}}\alpha^{y_{m,n}}]_n; \end{aligned}$$

when y is odd,

$$\begin{aligned} &\iota_{m,n}([s^{\mathbf{x}}\alpha^y]_m) \\ &\quad= [(s^{\mathbf{x}}\alpha^y)\varphi^m(s^{\mathbf{x}}\alpha^y)\varphi^{2m}(s^{\mathbf{x}}\alpha^y)\dots\varphi^{n-m}(s^{\mathbf{x}}\alpha^y)]_n \\ &\quad= [(s^{\mathbf{x}}\alpha^y)(s^{F^m\mathbf{x}+\mathbf{b}_m}\alpha^{\omega^m y})(s^{F^{2m}\mathbf{x}+\mathbf{b}_{2m}}\alpha^{\omega^{2m}y})\dots(s^{F^{n-m}\mathbf{x}+\mathbf{b}_{n-m}}\alpha^{\omega^{n-m}y})]_n \\ &\quad:= [s^{\mathbf{x}''_{m,n}+\mathbf{b}_{m,n}}\alpha^{y_{m,n}}]_n, \end{aligned}$$

where

$$\begin{aligned} \mathbf{x}'_{m,n} &= (I+F^m+F^{2m}+\dots+F^{n-m})\mathbf{x}, \\ y_{m,n} &= (1+\omega^m+\omega^{2m}+\dots+\omega^{n-m})y, \\ \mathbf{x}''_{m,n} &= (I-F^m+\dots+(-1)^{n/m-1}F^{n-m})\mathbf{x}, \\ \mathbf{b}_{m,n} &= -\mathbf{b}_m+\mathbf{b}_{2m}-\dots+(-1)^{n/m-1}\mathbf{b}_{n-m}. \end{aligned}$$

Note here that $1+\omega^m+\dots+\omega^{n-m}$ is an odd integer if n/m is odd; otherwise it is an even integer.

LEMMA 7.1. *If $n = pqm$ where m is even and p, q are relatively prime odd numbers, then*

$$\begin{aligned} (\mathbf{b}_{pm,n} &+ \mathrm{Im}(I - F^{pm} + F^{2pm} - \dots + F^{n-pm})) \\ &\cap (\mathbf{b}_{qm,m} + \mathrm{Im}(I - F^{qm} + F^{2qm} - \dots + F^{n-qm})) \\ &\qquad= \mathbf{b}_{m,n} + \mathrm{Im}(I - F^m + F^{2m} - \dots + F^{n-m}). \end{aligned}$$

PROOF. By a simple computation, we can show that

$$I - F^m + \dots + F^{n-m} = (I - F^{pm} + \dots + F^{n-pm})(I - F^m + \dots + F^{pm-m}),$$

and

$$\begin{aligned}\mathbf{b}_{m,n} &= (-\mathbf{b}_m + \mathbf{b}_{2m}) + \ldots + (-\mathbf{b}_{(p-2)m} + \mathbf{b}_{(p-1)m}) - \mathbf{b}_{pm} \\ &\quad + (\mathbf{b}_{(p+1)m} - \mathbf{b}_{(p+2)m}) + \ldots \\ &\quad + (\mathbf{b}_{(2p-2)m} - \mathbf{b}_{(2p-1)m}) + \mathbf{b}_{2pm} + \ldots \\ &\quad + (\mathbf{b}_{((q-2)p+1)m} - \mathbf{b}_{((q-2)p+2)m}) + \ldots \\ &\quad + (\mathbf{b}_{((q-1)p-2)m} - \mathbf{b}_{((q-1)p-1)m}) + \mathbf{b}_{n-pm} \\ &\quad + (-\mathbf{b}_{((q-1)p+1)m} + \mathbf{b}_{((q-1)p+2)m}) + \ldots + (-\mathbf{b}_{(pq-2)m} + \mathbf{b}_{(pq-1)m}) \\ &= \mathbf{b}_{pm,n} + (I - F^{pm} + \ldots + F^{n-pm})F^m(I + F^{2m} + \ldots + F^{(p-3)m})\mathbf{b}_m.\end{aligned}$$

This implies that

$$\mathbf{b}_{m,n} - \mathbf{b}_{pm,n} \in \mathrm{Im}(I - F^{pm} + \ldots + F^{n-pm}). \tag{7.1}$$

A similar computation yields that

$$\mathbf{b}_{m,n} - \mathbf{b}_{qm,n} \in \mathrm{Im}(I - F^{qm} + \ldots + F^{n-qm}). \tag{7.2}$$

Now it follows that the right-hand side is contained in the left-hand side.

For the reverse inclusion, we assume

$$\begin{aligned}\mathbf{i} - \mathbf{b}_{pm,n} &\in \mathrm{Im}(I - F^{pm} + \ldots + F^{n-pm}), \\ \mathbf{i} - \mathbf{b}_{qm,n} &\in \mathrm{Im}(I - F^{qm} + \ldots + F^{n-qm}).\end{aligned}$$

By (7.1) and (7.2), we have

$$\mathbf{i} - \mathbf{b}_{m,n} \in \mathrm{Im}(I - F^{qm} + \ldots + F^{n-qm}) \cap \mathrm{Im}(I - F^{pm} + \ldots + F^{n-pm}).$$

By Corollary 6.3-(2),

$$\begin{aligned}\mathbf{i} - \mathbf{b}_{m,n} &\in \mathrm{Im}(I - F^{pm} + \ldots + F^{n-pm}) \cap \mathrm{Im}(I - F^{qm} + \ldots + F^{n-qm}) \\ &= \mathrm{Im}(I - F^{m} + \ldots + F^{n-m}).\end{aligned}$$

This shows the reverse inclusion. □

Lemma 7.2. *Let $n = 2^{e_0} p_1^{e_1} \ldots p_t^{e_t}$ be even. Let*

$$\begin{aligned}J_0' &= \{j \mid 0 \le j < |1 - \omega^n|,\ j = (2k+1)(1 + \omega^{(n_0)}) \text{ for some } k\}, \\ J_i &= \{j \mid 0 \le j < |1 - \omega^n|,\ j = 2\ell(1 + \omega^{(n_i)} + \ldots + \omega^{n-(n_i)}) \text{ for some } \ell\}.\end{aligned}$$

For each $i = 1, \ldots, t$, we have

$$\begin{aligned}J_0' \cap J_i = \{c\,|1 + \omega^{(n_0,n_i)} + \ldots + \omega^{n-(n_0,n_i)}| \mid \\ c \text{ is odd with } 0 < c < |1 - \omega^{(n_0,n_i)}|\}.\end{aligned}$$

In particular, $J_0' \cap J_i$ has $|1 - \omega^{(n_0,n_i)}|/2$ elements.

PROOF. Fix $i \in \{1, \dots, t\}$, and let $p = p_i$, $n = 2pm$ and $x = \omega^m = \omega^{(n_0, n_i)}$. By Lemma 6.4,

$$\begin{aligned}\gcd(1 + x^p, 1 + x^2 + \dots + x^{2(p-1)}) &= \frac{(1 + x^p)(1 + x^2 + \dots + x^{2(p-1)})}{1 + x + x^2 + \dots + x^{2p-1}} \\ &= \frac{1 + x^2 + \dots + x^{2(p-1)}}{1 + x + x^2 + \dots + x^{p-1}} = 1 - x + x^2 - \dots + x^{p-1}.\end{aligned}$$

Thus

$$(2k + 1)|1 + x^p| = 2\ell|1 + x^2 + \dots + x^{2(p-1)}|, \qquad \text{with } 0 \le 2k + 1 < |1 - x^p| \text{ and } 0 \le 2\ell < |1 - x^2|$$

turns into

$$(2k + 1)|1 + x| = 2\ell|1 + x + x^2 + \dots + x^{p-1}|.$$

Since $\gcd(1 + x, 1 + x + x^2 + \dots + x^{p-1}) = 1$, we must have

$$2k + 1 = c|1 + x + x^2 + \dots + x^{p-1}|, \qquad 2\ell = c|1 + x|$$

for some odd c with $0 < c < |1 - x|$. This finishes the proof. □

COROLLARY 7.3. *The set* $J'_0 \cap \Big(\bigcap_{i=1}^{s} J_{k_i}\Big)$ *has* $|1-\omega^{(n_0,n_{k_1},\dots,n_{k_s})}|/2$ *elements.*

PROOF. By Corollary 6.3, $j \in \bigcap_{i=1}^{s} J_{k_i}$ if and only if $0 \le j < |1 - \omega^n|$ and j is a multiple of $2(1 + \omega^{(n_{k_1},\dots,n_{k_s})} + \dots + \omega^{n-(n_{k_1},\dots,n_{k_s})})$. Next, we simply apply Lemma 7.2 and finish the proof. □

Now, according as n is odd or even we will consider the next two steps.

7.1. When n is odd. For any m with $m \mid n$, both m and n/m are odd.

CASE 1. $\det(I - F^n) \ne 0$ and $\det(I + F^n) = 0$.

In this case, we will consider the next two steps in a row.

Step 3. Finding the irreducible essential Reidemeister classes.

The essential classes are

$$EC := \big\{[s^{\bar{\mathbf{i}}}\alpha^j]_n \bigm| \bar{\mathbf{i}} \in \mathbb{Z}^2/\mathrm{Im}(I - F^n);\ j = 0, 2, \dots, |1 - \omega^n| - 2\big\}$$

whose cardinality is $|\det(I - F^n)| \times |1 - \omega^n|/2$. Suppose $[s^{\bar{\mathbf{i}}}\alpha^j]_n$ is reducible to the level m. Then $[s^{\mathbf{i}}\alpha^j]_n = \iota_{m,n}([s^{\mathbf{x}}\alpha^y]_m) = [s^{\mathbf{x}'_{m,n}}\alpha^{y_{m,n}}]_n$ for some $[s^{\mathbf{x}}\alpha^y]_m \in \mathcal{R}[\varphi^m]$. Because n is odd and j is even, as we observed above, $y_{m,n}$ is even if and only if y is even. This tells that we only need to consider the essential Reidemeister classes of the form $[s^{\mathbf{x}}\alpha^y]_m$ where y is even and $\bar{\mathbf{i}} \in \mathbb{Z}^2/\mathrm{Im}(I - F^m)$ under the boost function $\iota_{m,n}$.

We observe that $\mathbf{x} \in \mathrm{Im}(I - F^m)$ if and only if $\mathbf{x}'_{m,n} \in \mathrm{Im}(I - F^n)$ as before (see the proof of Theorem 6.5). Moreover, if $0 \le y < |1 - \omega^m|$ then

$0 \le y_{m,n} < |1-\omega^n|$, and the converse holds. This observation gives rise to an one-one correspondence between

$$\mathbb{Z}^2/\mathrm{Im}(I-F^m) \times \{y \mid y \text{ even}, \ 0 \le y < |1-\omega^m|\}$$

and

$$\begin{aligned}&\mathrm{Im}(I+F^m+\ldots+F^{n-m})/\mathrm{Im}(I-F^n)\\ &\qquad \times \{j \mid j \text{ is a multiple of } 2(1+\omega^m+\ldots+\omega^{n-m}), \ 0 \le j < |1-\omega^n|\}.\end{aligned}$$

Furthermore, if $m \mid m' \mid n$ and the essential class $[s^{\bar{\mathbf{i}}}\alpha^j]_n$ is reducible to m, then the essential class $[s^{\bar{\mathbf{i}}}\alpha^j]_n$ is reducible to m'. Consequently, the essential class $[s^{\bar{\mathbf{i}}}\alpha^j]_n$ is reducible if and only if for some $i \in \{1,\ldots,t\}$,

- $\bar{\mathbf{i}} \in \mathrm{Im}(I+F^{n_i}+\ldots+F^{n-n_i})/\mathrm{Im}(I-F^n)$,
- j is a multiple of $2(1+\omega^{n_i}+\ldots+\omega^{n-n_i})$.

For each $i=1,\ldots,t$, let

$$\begin{aligned}C_i = \big\{[s^{\bar{\mathbf{i}}}\alpha^j]_n \in OC \,\big|\, &\bar{\mathbf{i}} \in \mathrm{Im}(I+F^{(n_i)}+\ldots+F^{n-(n_i)})/\mathrm{Im}(I-F^n);\\ &j \text{ is a multiple of } 2(1+\omega^{(n_i)}+\ldots+\omega^{n-(n_i)})\big\}.\end{aligned}$$

Then

$$\#C_i = |\det(I-F^{(n_i)})| \times \frac{1}{2}|1-\omega^{(n_i)}|$$

and for $1 \le i < i' \le t$, we have that $[s^{\bar{\mathbf{i}}}\alpha^j]_n \in C_i \cap C_{i'}$ if and only if

- $\bar{\mathbf{i}} \in \mathrm{Im}(I+F^{(n_i,n_{i'})}+\ldots+F^{n-(n_i,n_{i'})})/\mathrm{Im}(I-F^n)$,
- j is a multiple of $2(1+\omega^{(n_i,n_{i'})}+\ldots+\omega^{n-(n_i,n_{i'})})$.

Here, the conditions above follow from Corollary 6.3(a) and (c). This shows that

$$\#(C_i \cap C_{i'}) = |\det(I-F^{(n_i,n_{i'})})| \times \frac{1}{2}|1-\omega^{(n_i,n_{i'})}|.$$

By induction, we have

$$\#\bigg(\bigcap_{i=1}^{s} C_{k_i}\bigg) = |\det(I-F^{(n_{k_1},\ldots,n_{k_s})})| \times \frac{1}{2}|1-\omega^{(n_{k_1},\ldots,n_{k_s})}|.$$

Thus

$$\#\bigg(\bigcup_{i=1}^{t} C_i\bigg) = \sum_{\{k_1,\ldots,k_s\}\subset\{1,\ldots,t\}} (-1)^{s-1}|\det(I-F^{(n_{k_1},\ldots,n_{k_s})}| \times \frac{1}{2}|1-\omega^{(n_{k_1},\ldots,n_{k_s})}|.$$

Step 4. Finding the length of irreducible essential Reidemeister classes.

Let $[s^{\bar{\mathbf{i}}}\alpha^j]_n$ be an irreducible essential Reidemeister class. By definition,

$$\begin{aligned}\langle[s^{\bar{\mathbf{i}}}\alpha^j]_n\rangle &= \{[s^{\bar{\mathbf{i}}}\alpha^j]_n, [\varphi]([s^{\bar{\mathbf{i}}}\alpha^j]_n), \ldots, [\varphi]^{\ell-1}([s^{\bar{\mathbf{i}}}\alpha^j]_n)\}\\ &= \begin{cases}\{[s^{\bar{\mathbf{i}}}\alpha^j]_n, [s^{F\bar{\mathbf{i}}}\alpha^{\omega j}]_n, \ldots, [s^{F^{\ell-1}\bar{\mathbf{i}}}\alpha^{\omega^{\ell-1}j}]_n\} & \text{when } j \text{ is even;}\\ \{[s^{\bar{\mathbf{i}}}\alpha^j]_n, [s^{F\bar{\mathbf{i}}+\mathbf{b}_1}\alpha^{\omega j}]_n, \ldots, [s^{F^{\ell-1}\bar{\mathbf{i}}+\mathbf{b}_{\ell-1}}\alpha^{\omega^{\ell-1}j}]_n\} & \text{when } j \text{ is odd,}\end{cases}\end{aligned}$$

where $\ell = \ell([s^{\bar{\mathbf{i}}}\alpha^j]_n)$ is the length of $[s^{\bar{\mathbf{i}}}\alpha^j]_n$. Then $\ell \mid n$ and

$$[s^{\bar{\mathbf{i}}}]_n = [\varphi]^\ell([s^{\bar{\mathbf{i}}}\alpha^j]_n) = \begin{cases} [s^{F^\ell \bar{\mathbf{i}}}\alpha^{\omega^\ell j}]_n & \text{when } j \text{ is even;} \\ [s^{F^\ell \bar{\mathbf{i}} + \mathbf{b}_\ell}\alpha^{\omega^\ell j}]_n & \text{when } j \text{ is odd.} \end{cases}$$

Since j is even, $[s^{\bar{\mathbf{i}}}\alpha^j]_n = [s^{F^\ell \bar{\mathbf{i}}}\alpha^{\omega^\ell j}]_n$. This implies that

$$s^{F^\ell \mathbf{i}}\alpha^{\omega^\ell j} = (s^{\mathbf{x}}\alpha^y)(s^{\mathbf{i}}\alpha^j)\varphi^n(s^{\mathbf{x}}\alpha^y)^{-1}$$

for some $s^{\mathbf{x}}\alpha^y \in \Pi$. By Remark 5.8, this identity turns into

$$s^{F^\ell \mathbf{i}}\alpha^{\omega^\ell j} = \begin{cases} s^{-\mathbf{i}+(I-F^n)\mathbf{x}-\mathbf{b}_n}\alpha^{y(1-\omega^n)+j} & \text{when } y \text{ is odd;} \\ s^{\mathbf{i}+(I-F^n)\mathbf{x}}\alpha^{y(1-\omega^n)+j} & \text{when } y \text{ is even.} \end{cases}$$

Since $(1-\omega^\ell)j = (1-\omega^n)(-y)$, we have $j = (1+\omega^\ell+\ldots+\omega^{n-\ell})(-y)$. As n is odd $1+\omega^\ell+\ldots+\omega^{n-\ell}$ is odd; hence as j is even y must be even. As a result, $(I-F^\ell)\,\mathbf{i} = (I-F^n)(-\mathbf{x})$, $(I+F^\ell+\ldots+F^{n-\ell})(-\mathbf{x}) = \mathbf{i}$ or $\bar{\mathbf{i}} \in \mathrm{Im}(I+F^\ell+\ldots+F^{n-\ell})/\mathrm{Im}(I-F^n)$. Thus this shows that the essential class $[s^{\bar{\mathbf{i}}}\alpha^j]_n$ is reducible to ℓ. Hence if it is irreducible its length must be n. That is, all the irreducible essential Reidemeister classes have the same length n.

In all,

$$\begin{aligned} N\mathrm{P}_n(f) &= \#\Big(EC - \bigcup_{i=1}^t C_i\Big) = \#(EC) - \#\Big(\bigcup_{i=1}^t C_i\Big) \\ &= \frac{1}{2}\bigg(|\det(I-F^n)| \times |1-\omega^n| \\ &\qquad + \sum_{\{k_1,\ldots,k_s\}\subset\{1,\ldots,t\}} (-1)^s |\det(I-F^{(n_{k_1},\ldots,n_{k_s})}| \times |1-\omega^{(n_{k_1},\ldots,n_{k_s})}|\bigg). \end{aligned}$$

Case 2. $\det(I-F^n)=0$ and $\det(I+F^n)\neq 0$.

Recall that f is homotopic to a self-map on K of type $(-F,\mathbf{a},\omega)$. Thus in this case it holds that $\det(I-(-F)^n)\neq 0$ and $\det(I+(-F)^n)=0$. Applying the above case for $-F$, we can deduce the following:

- The essential classes are

$$\{[s^{\bar{\mathbf{i}}}\alpha^j]_n \mid \bar{\mathbf{i}} \in \mathbb{Z}^2/\mathrm{Im}(I-(-F)^n);\ j=0,2\ldots,|1-\omega^n|-2\}.$$

- The essential class $[s^{\bar{\mathbf{i}}}\alpha^j]_n$ is reducible if and only if for some $i \in \{1,\ldots,t\}$,
  (a) $\bar{\mathbf{i}} \in \mathrm{Im}(I+(-F)^{n_i}+\ldots+(-F)^{n-n_i})/\mathrm{Im}(I-(-F)^n)$,
  (b) j is a multiple of $2(1+\omega^{n_i}+\ldots+\omega^{n-n_i})$.
- $$\begin{aligned} N\mathrm{P}_n(f) &= \frac{1}{2}\bigg(|\det(I-(-F)^n)| \times |1-\omega^n| \\ &\quad + \sum_{\{k_1,\ldots,k_s\}\subset\{1,\ldots,t\}} (-1)^s |\det(I-(-F)^{(n_{k_1},\ldots,n_{k_s})}| \times |1-\omega^{(n_{k_1},\ldots,n_{k_s})}|\bigg) \end{aligned}$$

$$= \frac{1}{2}\bigg(|\det(I+F^n)| \times |1-\omega^n| + \sum_{\{k_1,\dots,k_s\}\subset\{1,\dots,t\}} (-1)^s|\det(I+F^{(n_{k_1},\dots,n_{k_s})}| \times |1-\omega^{(n_{k_1},\dots,n_{k_s})}|\bigg).$$

7.2. When n is even. In this subsection, we will analyze Step 3 first and then Step 4 in a row.

Step 3. Finding the irreducible essential Reidemeister classes.

Case 1. $\det(I-F^n) \neq 0$ and $\det(I+F^n)=0$.

The essential classes are

$$EC_1 := \big\{[s^{\bar{\mathbf{i}}}\alpha^j]_n \mid \bar{\mathbf{i}} \in \mathbb{Z}^2/\mathrm{Im}(I-F^n); j=0,2,\dots,|1-\omega^n|-2\big\}$$

whose cardinality is $|\det(I-F^n)| \times |1-\omega^n|/2$. Let $n = 2^{e_0}p_1^{e_1}\dots p_t^{e_t}$ be the prime decomposition of the even integer n. We ascertain the following:

Claim. The essential class $[s^{\bar{\mathbf{i}}}\alpha^j]_n$ is reducible if and only if for some $i \in \{0,\dots,t\}$

- $\bar{\mathbf{i}} \in \mathrm{Im}(I+F^{(n_i)}+\dots+F^{n-(n_i)})/\mathrm{Im}(I-F^n)$,
- j is a multiple of $2(1+\omega^{(n_i)}+\dots+\omega^{n-(n_i)})$

or

- $\bar{\mathbf{i}} \in -\overline{\mathbf{b}}_{(n_0)} + \mathrm{Im}(I-F^{(n_0)})/\mathrm{Im}(I-F^n)$,
- $j-(1+\omega^{(n_0)})$ is a multiple of $2(1+\omega^{(n_0)})$.

We recall from Step 2 the following:

$$\iota_{m,n}([s^{\mathbf{x}}\alpha^y]_m) = \begin{cases} [s^{\mathbf{x}'_{m,n}}\alpha^{y_{m,n}}]_n & \text{when } y \text{ is even,} \\ [s^{\mathbf{x}''_{m,n}+\mathbf{b}_{m,n}}\alpha^{y_{m,n}}]_n & \text{when } y \text{ is odd.} \end{cases}$$

Notice that we are concerned with the case where $y_{m,n} = (1+\omega^m+\omega^{2m}+\dots+\omega^{n-m})y$ is even. In this case, we have either y is even or y is odd and n/m is even.

Assume that y is even. We note that $\mathbf{x} \in \mathrm{Im}(I-F^m)$ if and only if $\mathbf{x}'_{m,n} \in \mathrm{Im}(I-F^n)$ (see the proof of Theorem 6.5). Furthermore, if $0 \le y < |1-\omega^m|$ then $0 \le y_{m,n} < |1-\omega^n|$, and the converse holds. This observation gives rise to an one-one correspondence between

$$\mathbb{Z}^2/\mathrm{Im}(I-F^m) \times \{y \mid y \text{ even},\ 0 \le y < |1-\omega^m|\}$$

and

$$\mathrm{Im}(I+F^m+\dots+F^{n-m})/\mathrm{Im}(I-F^n) \times \{j \mid j \text{ a multiple of } 2(1+\omega^m+\dots+\omega^{n-m}),\ 0 \le j < |1-\omega^n|\}.$$

Furthermore, if $m \mid m' \mid n$ and the essential class $[s^{\bar{\mathbf{i}}}\alpha^j]_n$ is reducible to $[s^{\mathbf{x}}\alpha^y]m$ where y is even, then the essential class $[s^{\bar{\mathbf{i}}}\alpha^j]_n$ is reducible to $[s^{\mathbf{x}'}\alpha^{y'}]_{m'}$ where

y' is even. Consequently, the essential class $[s^{\bar{\mathbf{i}}}\alpha^j]_n$ is reduced to an (essential) Reidemeister class of the form $[s^{\mathbf{x}}\alpha^y]_m$ where y is even if and only if for some $i \in \{0, \dots, t\}$

- $\bar{\mathbf{i}} \in \mathrm{Im}(I + F^{n_i} + \dots + F^{n-n_i})/\mathrm{Im}(I - F^n)$,
- j is a multiple of $2(1 + \omega^{n_i} + \dots + \omega^{n-n_i})$.

Assume that n/m even. Then $I + F^m$ is a factor of $I - F^n$ and so $\det(I + F^m) \neq 0$. This implies that $N(f^m) = R(f^m)$ and so f^m is weakly Jiang, even though f^n is not weakly Jiang. Assume in addition that y is odd and so $\bar{\mathbf{x}} \in \mathbb{Z}^2/\mathrm{Im}(I + F^m)$ (see Step 1). If there is $m' < n$ such that $m \mid m' \mid n$ and m'/m is even, then $(1 + \omega^m + \dots + \omega^{m'-m})y$ is even. Hence the boost function $\iota_{m,m'}$ sends a Reidemeister class $[s^{\bar{\mathbf{x}}}\alpha^y]_m$ with y odd to a Reidemeister class $[s^{\bar{\mathbf{x}}'}\alpha^{y'}]_{m'}$ with y' even. Therefore this case turns into the case where y is even, and it was treated in the above. Therefore we have to consider the case where if $m' < n$ with $m \mid m' \mid n$ then $\frac{m'}{m}$ is odd. This happens only when $m = n/2$. The correspondence $\mathbf{x} \mapsto \mathbf{x}''_{m,n} + \mathbf{b}_{m,n} = (I - F^m)\mathbf{x} - \mathbf{b}_m$ gives rise to an one-one correspondence

$$\begin{aligned}\mathbb{Z}^2/\mathrm{Im}(I + F^m) &\xrightarrow{\cong} \mathrm{Im}(I - F^m)/\mathrm{Im}(I - F^n)\\ &\longleftrightarrow -\overline{\mathbf{b}}_m + \mathrm{Im}(I - F^m)/\mathrm{Im}(I - F^n).\end{aligned}$$

Also, there is an one-one correspondence from $\{y \mid y \text{ odd},\ 0 \leq y < |1 - \omega^m|\}$ onto

$$\left\{ j \;\middle|\; j = |1 + \omega^m|(2k+1),\ 0 \leq k < \frac{|1 - \omega^m|}{2} \right\}.$$

Remark that when $m = n/2$, the boost function $\iota_{m,n}$ sends a Reidemeister class $[s^{\bar{\mathbf{x}}}\alpha^y]_m$ with y even to a Reidemeister class $[s^{\bar{\mathbf{x}}'}\alpha^{y'}]_n$, where y' is a multiple of $2(1+\omega^m)$. This tells that any two Reidemeister classes $[s^{\bar{\mathbf{x}}_1}\alpha^{y_1}]_m$ and $[s^{\bar{\mathbf{x}}_2}\alpha^{y_2}]_m$ with y_1 even and y_2 odd are boosted to two distinct Reidemeister classes in level n. Hence the essential class $[s^{\bar{\mathbf{i}}}\alpha^j]_n$ is reducible to $m = (n_0) = n/2$ if and only if either

- $\bar{\mathbf{i}} \in \mathrm{Im}(I + F^{(n_0)})/\mathrm{Im}(I - F^n)$,
- j is a multiple of $2(1 + \omega^{(n_0)})$

or

- $\bar{\mathbf{i}} \in -\overline{\mathbf{b}}_{(n_0)} + \mathrm{Im}(I - F^{(n_0)})/\mathrm{Im}(I - F^n)$,
- $j - (1 + \omega^{(n_0)})$ is a multiple of $2(1 + \omega^{(n_0)})$.

This completes the proof of our claim.

Now let

$$\begin{aligned}C'_0 = \big\{[s^{\bar{\mathbf{i}}}\alpha^j]_n \in EC_1 \mid \bar{\mathbf{i}} \in -\overline{\mathbf{b}}_{(n_0)} &+ \mathrm{Im}(I - F^{(n_0)})/\mathrm{Im}(I - F^n);\\ &j \text{ a multiple of } 2(1 + \omega^{(n_0)}) \text{ plus } |1 + \omega^{(n_0)}|\big\},\end{aligned}$$

and let, for each $i = 0, \dots, t$,

$$C_i = \big\{[s^{\bar{\mathbf{i}}}\alpha^j]_n \in EC_1 \mid \bar{\mathbf{i}} \in \operatorname{Im}(I + F^{(n_i)} + \dots + F^{n-(n_i)})/\operatorname{Im}(I - F^n);\\ j \text{ a multiple of } 2(1 + \omega^{(n_i)} + \dots + \omega^{n-(n_i)})\big\}.$$

Then

$$\begin{aligned} \#C_0' &= |\det(I + F^{(n_0)})| \times \frac{1}{2}|1 - \omega^{(n_0)}|, \\ \#C_i &= |\det(I - F^{(n_i)})| \times \frac{1}{2}|1 - \omega^{(n_i)}|. \end{aligned}$$

As it was observed before,

$$\#(C_0 \cap C_0') = 0, \qquad \#\bigg(\bigcap_{i=1}^{s} C_{k_i}\bigg) = |\det(I - F^{(n_{k_1},\dots,n_{k_s})})| \times \frac{1}{2}|1 - \omega^{(n_{k_1},\dots,n_{k_s})}|.$$

Now to determine $C_0' \cap C_i$, first we claim that

$$(-\mathbf{b}_{(n_0)} + \operatorname{Im}(I - F^{(n_0)})) \cap \operatorname{Im}(I + F^{(n_i)} + \dots + F^{n-(n_i)}) \neq \emptyset.$$

This is equivalent to show that there exist $\mathbf{x}, \mathbf{y}$ such that

$$-(I + F + \dots + F^{(n_0)-1})\mathbf{b} + (I - F^{(n_0)})\mathbf{x} = (I + F^{(n_i)} + \dots + F^{n-(n_i)})\mathbf{y}.$$

Multiplying the inverse of $I + F^{(n_0,n_i)} + \dots + F^{(n_0)-(n_0,n_i)}$ on both sides, we obtain

$$\begin{aligned} -(I + F + \dots + F^{(n_0,n_i)-1})\mathbf{b} + (I - F^{(n_0,n_i)})\mathbf{x}\,[= (I - F^{(n_i)} + \dots + F^{n-(n_i)})\mathbf{y}] \\ = \mathbf{y} - (I - F^{(n_0,n_i)})(F^{(n_0,n_i)} + F^{3(n_0,n_i)} + \dots + F^{(n_0)-(n_i)})\mathbf{y}. \end{aligned}$$

Thus we can choose $\mathbf{x}$, $\mathbf{y}$, in fact we choose

$$\begin{aligned} \mathbf{y} &= -(I + F + \dots + F^{(n_0,n_i)-1})\mathbf{b}, \\ \mathbf{x} &= -(F^{(n_0,n_i)} + F^{3(n_0,n_i)} + \dots + F^{(n_0)-(n_i)})\mathbf{y} \end{aligned}$$

so that the above equality holds. Now we fix $\mathbf{i}_0$ such that

$$\mathbf{i}_0 \in (-\mathbf{b}_{(n_0)} + \operatorname{Im}(I - F^{(n_0)})) \cap (\operatorname{Im}(I + F^{(n_i)} + \dots + F^{n-(n_i)})).$$

Then it can be seen easily that

$$\begin{aligned} (-\mathbf{b}_{n_0} + \operatorname{Im}(I - F^{n_0})) \cap (\operatorname{Im}(I + F^{n_i} + \dots + F^{n-n_i})) \\ = \mathbf{i}_0 + (\operatorname{Im}(I - F^{n_0}) \cap \operatorname{Im}(I + F^{n_i} + \dots + F^{n-n_i})). \end{aligned}$$

By Lemma 6.4, we have

$$\begin{aligned} \operatorname{Im}(I &- F^{n_0})/\operatorname{Im}(I - F^n) \cap \operatorname{Im}(I + F^{n_i} + \dots + F^{n-n_i})/\operatorname{Im}(I - F^n) \\ &= \operatorname{Im}((I - F^{(n_0,n_i)})(I + F^{n_i} + \dots + F^{n-n_i}))/\operatorname{Im}(I - F^n) \\ &\cong \mathbb{Z}^2/\operatorname{Im}(I + F^{(n_0,n_i)}). \end{aligned}$$

By Corollary 6.4 and 7.3, we can conclude that

$$\#(C_0' \cap C_i) = |\det(I + F^{(n_0,n_i)})| \times \frac{1}{2}|1 - \omega^{(n_0,n_i)}|.$$

By induction, we have

$$\#\bigg(\bigcap_{i=1}^{u}(C_0' \cap C_{k_i})\bigg) = |\det(I + F^{(n_0,n_{k_1},\dots,n_{k_u})}| \times \frac{1}{2}|1 - \omega^{(n_0,n_{k_1},\dots,n_{k_u})}|.$$

Consequently, we have

$$\begin{aligned}
\#\bigg(C_0' \cup \bigg(\bigcup_{i=0}^{t} C_i\bigg)\bigg) &= \#C_0' + \#\bigg(\bigcup_{i=0}^{t} C_i\bigg) - \#\bigg(C_0' \cap \bigg(\bigcup_{i=0}^{t} C_i\bigg)\bigg) \\
&= \#C_0' + \#\bigg(\bigcup_{i=0}^{t} C_i\bigg) - \#\bigg(\bigcup_{i=1}^{t}(C_0' \cap C_i)\bigg) \\
&= \frac{1}{2}|\det(I + F^{n_0})| \times |1 - \omega^{n_0}| \\
&\quad - \frac{1}{2}\sum_{\{k_1,\dots,k_s\}\subset\{0,\dots,t\}} (-1)^s|\det(I - F^{(n_{k_1},\dots,n_{k_s})})| \times |1 - \omega^{(n_{k_1},\dots,n_{k_s})}| \\
&\quad + \frac{1}{2}\sum_{\{k_1,\dots,k_u\}\subset\{1,\dots,t\}} (-1)^u|\det(I + F^{(n_0,n_{k_1},\dots,n_{k_u})})| \times |1 - \omega^{(n_0,n_{k_1},\dots,n_{k_u})}|.
\end{aligned}$$

CASE 2. $\det(I - F^n) = 0$ and $\det(I + F^n) \neq 0$.

The essential classes are

$$EC_2 := \big\{[s^{\bar{\mathbf{i}}}\alpha^j]_n \mid \bar{\mathbf{i}} \in \mathbb{Z}^2/\mathrm{Im}(I + F^n);\ j = 1, 3, \dots, |1 - \omega^n| - 1\big\}$$

whose cardinality is $|\det(I + F^n)| \times |1 - \omega^n|/2$. Recall again from Step 2 that

$$\iota_{m,n}([s^{\mathbf{x}}\alpha^y]_m) = \begin{cases} [s^{\mathbf{x}'_{m,n}}\alpha^{y_{m,n}}]_n & \text{when } y \text{ is even;} \\ [s^{\mathbf{x}''_{m,n}+\mathbf{b}_{m,n}}\alpha^{y_{m,n}}]_n & \text{when } y \text{ is odd,} \end{cases}$$

where $y_{m,n} = (1 + \omega^m + \dots + \omega^{n-m})y$ is odd. So, we must have both y and n/m are odd. Thus we need to observe the identity

$$\iota_{m,n}([s^{\mathbf{x}}\alpha^y]_m) = [s^{\mathbf{x}''_{m,n}+\mathbf{b}_{m,n}}\alpha^{y_{m,n}}]_n.$$

We observe that $\mathbf{x} \in \mathrm{Im}(I + F^m)$ if and only if $\mathbf{x}''_{m,n} \in \mathrm{Im}(I + F^n)$ as before. And if $0 \le y < |1 - \omega^m|$ then $0 \le y_{m,n} < |1 - \omega^n|$, and the converse holds. Furthermore, if $m \mid m' \mid n$ and the essential class $[s^{\bar{\mathbf{i}}}\alpha^j]_n$ is reducible to m, then $[s^{\bar{\mathbf{i}}}\alpha^j]_n$ is reducible to m'. This observation implies that the essential class $[s^{\bar{\mathbf{i}}}\alpha^j]_n$ is reducible if and only if for some proper maximal divisor m of n with n/m odd,

- $\bar{\mathbf{i}} - \bar{\mathbf{b}}_{m,n} \in \mathrm{Im}(I - F^m + \dots + F^{n-m})/\mathrm{Im}(I + F^n)$,
- j is odd and is multiple of $(1 + \omega^m + \dots + \omega^{n-m})$.

For each $i=1,\dots,t$, let

$$C_i=\{(\bar{\mathbf{i}},j)\in EC_2\mid \bar{\mathbf{i}}-\overline{\mathbf{b}}_{n_i,n}\in \mathrm{Im}(I-F^{n_i}+\ldots+F^{n-n_i})/\mathrm{Im}(I+F^n),\\ j \text{ is odd and a multiple of } (1+\omega^{n_i}+\ldots+\omega^{n-n_i})\}.$$

Then

$$\#C_i=|\det(I+F^{n_i})|\times\frac{1}{2}|1-\omega^{n_i}|.$$

By Lemma 7.1 and Corollary 6.3, we have that $[s^{\bar{\mathbf{i}}}\alpha^j]_n\in\bigcap_{i=1}^{u}C_{k_i}$ if and only if

- $\bar{\mathbf{i}}-\overline{\mathbf{b}}_{(n_{k_1},\dots,n_{k_u}),n}\in\mathrm{Im}(I-F^{(n_{k_1},\dots,n_{k_u})}+\ldots+F^{n-(n_{k_1},\dots,n_{k_u})})/\mathrm{Im}(I-F^n)$,
- j is odd and a multiple of $1+\omega^{(n_{k_1},\dots,n_{k_u})}+\ldots+\omega^{n-(n_{k_1},\dots,n_{k_u})}$.

Thus we can conclude that

$$\#\bigg(\bigcap_{i=1}^{u}C_{k_i}\bigg)=|\det(I+F^{(n_{k_1},\dots,n_{k_u})}|\times\frac{1}{2}|1-\omega^{(n_{k_1},\dots,n_{k_u})}|.$$

Consequently, we have

$$\#\bigg(\bigcup_{i=1}^{t}C_i\bigg)=\frac{1}{2}\sum_{\{k_1,\dots,k_s\}\subset\{1,\dots,t\}}(-1)^{s-1}|\det(I+F^{(n_{k_1},\dots,n_{k_s})})|\times|1-\omega^{(n_{k_1},\dots,n_{k_s})}|.$$

Step 4. Finding the length of irreducible essential Reidemeister classes.

Let $[s^{\bar{\mathbf{i}}}\alpha^j]_n$ be an irreducible essential Reidemeister class. By definition,

$$\begin{aligned}\langle[s^{\bar{\mathbf{i}}}\alpha^j]_n\rangle&=\{[s^{\bar{\mathbf{i}}}\alpha^j]_n,[\varphi]([s^{\bar{\mathbf{i}}}\alpha^j]_n),\dots,[\varphi]^{\ell-1}([s^{\bar{\mathbf{i}}}\alpha^j]_n)\}\\&=\begin{cases}\{[s^{\bar{\mathbf{i}}}\alpha^j]_n,[s^{F\bar{\mathbf{i}}}\alpha^{\omega j}]_n,\dots,[s^{F^{\ell-1}\bar{\mathbf{i}}}\alpha^{\omega^{\ell-1}j}]_n\} & \text{when } j \text{ is even;}\\ \{[s^{\bar{\mathbf{i}}}\alpha^j]_n,[s^{F\bar{\mathbf{i}}+\mathbf{b}_1}\alpha^{\omega j}]_n,\dots,[s^{F^{\ell-1}\bar{\mathbf{i}}+\mathbf{b}_{\ell-1}}\alpha^{\omega^{\ell-1}j}]_n\} & \text{when } j \text{ is odd,}\end{cases}\end{aligned}$$

where $\ell=\ell([s^{\bar{\mathbf{i}}}\alpha^j]_n)$ is the length of $[s^{\bar{\mathbf{i}}}\alpha^j]_n$. Then $\ell\mid n$ and

$$[s^{\bar{\mathbf{i}}}]_n=[\varphi]^{\ell}([s^{\bar{\mathbf{i}}}\alpha^j]_n)=\begin{cases}[s^{F^{\ell}\bar{\mathbf{i}}}\alpha^{\omega^{\ell}j}]_n & \text{when } j \text{ is even;}\\ [s^{F^{\ell}\bar{\mathbf{i}}+\mathbf{b}_{\ell}}\alpha^{\omega^{\ell}j}]_n & \text{when } j \text{ is odd.}\end{cases}$$

Case 3. $\det(I-F^n)\neq0$ and $\det(I+F^n)=0$.

In this case j must be even. Thus $[s^{\bar{\mathbf{i}}}\alpha^j]_n=[s^{F^{\ell}\bar{\mathbf{i}}}\alpha^{\omega^{\ell}j}]_n$. This implies that

$$s^{F^{\ell}\mathbf{i}}\alpha^{\omega^{\ell}j}=(s^{\mathbf{x}}\alpha^y)(s^{\mathbf{i}}\alpha^j)\varphi^n(s^{\mathbf{x}}\alpha^y)^{-1}$$

for some $s^{\mathbf{x}}\alpha^y\in\Pi$. By Remark 5.8, this identity turns into

$$s^{F^{\ell}\mathbf{i}}\alpha^{\omega^{\ell}j}=\begin{cases}s^{-\mathbf{i}+(I-F^n)\mathbf{x}-\mathbf{b}_n}\alpha^{y(1-\omega^n)+j} & \text{when } y \text{ is odd;}\\ s^{\mathbf{i}+(I-F^n)\mathbf{x}}\alpha^{y(1-\omega^n)+j} & \text{when } y \text{ is even.}\end{cases}$$

Since $(1-\omega^\ell)j=(1-\omega^n)(-y)$, we have $j=(1+\omega^\ell+\ldots+\omega^{n-\ell})(-y)$. Thus either n/ℓ is even or n/ℓ is odd and y is even.

Assume that n/ℓ is even. Then $\ell\mid(n/2)$ and so we may assume that $\ell=n/2$. By the above identity, we can see that

$$\begin{cases}(I+F^\ell)\mathbf{i}=(I-F^n)\mathbf{x}-\mathbf{b}_n=(I+F^\ell)\{(I-F^\ell)\mathbf{x}-\mathbf{b}_\ell\} & \text{when } y \text{ is odd;}\\ (I-F^\ell)\mathbf{i}=(I-F^n)(-\mathbf{x})=(I-F^\ell)(I+F^\ell)(-\mathbf{x}) & \text{when } y \text{ is even.}\end{cases}$$

This yields that

$$\mathbf{i}=\begin{cases}(I-F^\ell)\mathbf{x}-\mathbf{b}_\ell & \text{when } y \text{ is odd;}\\ (I+F^\ell)(-\mathbf{x}) & \text{when } y \text{ is even.}\end{cases}$$

These tell that $[s^{\mathbf{i}}\alpha^j]_n$ is reducible to $\ell=n/2$, a contradiction.

Assume that n/ℓ is odd and y is even. The fact that y is even implies that $(I-F^\ell)\mathbf{i}=(I-F^n)(-\mathbf{x})$, $\mathbf{i}=(I+F^\ell+\ldots+F^{n-\ell})(-\mathbf{x})$ or $\bar{\mathbf{i}}\in\mathrm{Im}(I+F^\ell+\ldots+F^{n-\ell})/\mathrm{Im}(I-F^n)$. Thus this shows that the essential class $[s^{\bar{\mathbf{i}}}\alpha^j]_n$ is reducible to ℓ. Hence as it is irreducible its length must be n.

In all, all the irreducible essential Reidemeister classes have the same length n. Therefore, we obtain that

$$\begin{aligned}
N\mathrm{P}_n(f)&=\#\bigg(EC_1-\bigg(C_0'\cup\bigg(\bigcup_{i=0}^t C_i\bigg)\bigg)\bigg)\\
&=\#(EC_1)-\#\bigg(C_0'\cup\bigg(\bigcup_{i=0}^t C_i\bigg)\bigg)\\
&=\frac{1}{2}\bigg(|\det(I-F^n)|\times|1-\omega^n|-|\det(I+F^{n_0})|\times|1-\omega^{n_0}|\\
&\quad+\sum_{\{k_1,\ldots,k_s\}\subset\{0,1,\ldots,t\}}(-1)^s|\det(I-F^{(n_{k_1},\ldots,n_{k_s})})|\times|1-\omega^{(n_{k_1},\ldots,n_{k_s})}|\\
&\quad-\sum_{\{k_1,\ldots,k_u\}\subset\{1,\ldots,t\}}(-1)^u|\det(I+F^{(n_0,n_{k_1},\ldots,n_{k_u})})|\times|1-\omega^{(n_0,n_{k_1},\ldots,n_{k_u})}|\bigg).
\end{aligned}$$

CASE 4. $\det(I-F^n)=0$ and $\det(I+F^n)\neq 0$.

In this case j must be odd. Thus $[s^{\bar{\mathbf{i}}}\alpha^j]_n=[s^{F^\ell\bar{\mathbf{i}}+\bar{\mathbf{b}}_\ell}\alpha^{\omega^\ell j}]_n$. This implies that

$$s^{F^\ell\mathbf{i}+\mathbf{b}_\ell}\alpha^{\omega^\ell j}=(s^{\mathbf{x}}\alpha^y)(s^{\mathbf{i}}\alpha^j)\varphi^n(s^{\mathbf{x}}\alpha^y)^{-1}$$

for some $s^{\mathbf{x}}\alpha^y\in\Pi_1$. By Remark 5.8, this identity turns into

$$s^{F^\ell\mathbf{i}+\mathbf{b}_\ell}\alpha^{\omega^\ell j}=\begin{cases}s^{-\mathbf{i}+(I+F^n)\mathbf{x}+\mathbf{b}_n}\alpha^{y(1-\omega^n)+j} & \text{when } y \text{ is odd;}\\ s^{\mathbf{i}+(I+F^n)\mathbf{x}}\alpha^{y(1-\omega^n)+j} & \text{when } y \text{ is even.}\end{cases}$$

This yields that $j = y(1+\omega^\ell+\ldots+\omega^{n-\ell})$. As j is odd, so are y and n/ℓ. Furthermore,

$$(I+F^\ell)\,\mathbf{i}+\mathbf{b}_\ell-\mathbf{b}_n=(I+F^n)\,\mathbf{x}=(I+F^\ell)(I-F^\ell+\ldots+F^{n-\ell})\,\mathbf{x}$$

or

$$(I+F^\ell)\,\mathbf{i}-(I+F^\ell)\,\mathbf{b}_{\ell,n}=(I+F^\ell)(I-F^\ell+\ldots+F^{n-\ell})\,\mathbf{x}.$$

Hence $\mathbf{i}-\mathbf{b}_{\ell,n}\in\mathrm{Im}(I-F^\ell+\ldots+F^{n-\ell})$.

In all, the irreducible essential class $[s^{\bar{\mathbf{i}}}\alpha^j]_n$ is reducible to ℓ and so $\ell=n$. Thus all the irreducible essential Reidemeister classes have the same length n. Therefore, we obtain that

$$\begin{aligned}
N\mathrm{P}_n(f) &= \#\bigg(EC_2-\bigcup_{i=1}^t C_i\bigg)=\#(EC_2)-\#\bigg(\bigcup_{i=1}^t C_i\bigg)\\
&=\frac{1}{2}\bigg(|\det(I+F^n)|\times|1-\omega^n|\\
&\quad+\sum_{\{k_1,\ldots,k_s\}\subset\{1,\ldots,t\}}(-1)^s|\det(I+F^{(n_{k_1},\ldots,n_{k_s})})|\times|1-\omega^{(n_{k_1},\ldots,n_{k_s})}|\bigg).
\end{aligned}$$

With these observation so far, we have:

Theorem 7.4. *Let f be a self-map on K of type $(F,\mathbf{b},\omega)$. If $\omega\neq\pm1$ is odd and $\det(I-F^n)\neq0$ but $\det(I+F^n)=0$, then the prime Nielsen–Jiang periodic number of f of period n is given as follows*:

(a) *when $n=p_1^{e_1}\ldots p_t^{e_t}$ is odd,*

$$\begin{aligned}
N\mathrm{P}_n(f)=\frac{1}{2}\bigg(&|\det(I-F^n)||1-\omega^n|\\
&+\sum_{\{k_1,\ldots,k_s\}\subset\{1,\ldots,t\}}(-1)^s|\det(I-F^{(n_{k_1},\ldots,n_{k_s})}||1-\omega^{(n_{k_1},\ldots,n_{k_s})}|\bigg);
\end{aligned}$$

(b) *when $n=2^{e_0}p_1^{e_1}\ldots p_t^{e_t}$ is even,*

$$\begin{aligned}
N\mathrm{P}_n(f)=\frac{1}{2}\Big(&|\det(I-F^n)||1-\omega^n|-|\det(I+F^{(n_0)})||1-\omega^{(n_0)}|\\
&+\sum_{\substack{\{k_1,\ldots,k_s\}\\ \subset\{0,1,\ldots,t\}}}(-1)^s|\det(I-F^{(n_{k_1},\ldots,n_{k_s})})||1-\omega^{(n_{k_1},\ldots,n_{k_s})}|\\
&-\sum_{\substack{\{k_1,\ldots,k_u\}\\ \subset\{1,\ldots,t\}}}(-1)^u|\det(I+F^{(n_0,n_{k_1},\ldots,n_{k_u})})||1-\omega^{(n_0,n_{k_1},\cdots,n_{k_u})}|\Big).
\end{aligned}$$

Here the summation runs through the family of nonempty subsets $\{k_1,\ldots,k_s\}$ of $\{1,\ldots,t\}$.

THEOREM 7.5. *Let f be a self-map on K of type $(F,\mathbf{b},\omega)$. If $\omega\neq\pm1$ is odd and* $\det(I-F^n)=0$ *but* $\det(I+F^n)\neq 0$, *then the prime Nielsen–Jiang periodic number of f of period n is*

$$NP_n(f)=\frac{1}{2}\bigg(|\det(I+F^n)||1-\omega^n| + \sum_{\{k_1,\dots,k_s\}\subset\{1,\dots,t\}}(-1)^s|\det(I+F^{(n_{k_1},\dots,n_{k_s})}||1-\omega^{(n_{k_1},\dots,n_{k_s})}|\bigg),$$

where the summation runs through the family of nonempty subsets $\{k_1,\dots,k_s\}$ of $\{1,\dots,t\}$.

8. The Nielsen type numbers: weakly Jiang case

In this subsection we will finish our evaluation of the Nielsen type number of weakly Jiang maps.

When f^n is weakly Jiang with $N(f^n)=R(f^n)$, due to Theorems 3.2 and 3.3 and Corollary 5.2 we can immediately state what the Nielsen type numbers are. Namely,

COROLLARY 8.1 (Case $N(f^n)=R(f^n)$). *Let f be a self-map on K of type $(F,\mathbf{b},\omega)$ with $N(f^n)=R(f^n)$. Then*

$$N\Phi_n(f)=N(f^n),\qquad NP_n(f)=\sum_{m|n}\mu(m)N(f^{\frac{n}{m}})$$

where $N(f^k)=|1-\omega^k|(|\det(I-F^k)|+|\det(I+F^k)|)/2$.

Now we will work for the case where $N(f^n)=0$. If this is the case, there are no essential classes and thus $NP_n(f)=0$. We are left to find $N\Phi_n(f)$. For this we will use Theorem 3.3: $N\Phi_n(f)=\sum_{k|n}NP_k(f)$.

COROLLARY 8.2 (Case $\omega=1$). *Let f be a self-map on K of type $(F,\mathbf{b},1)$. Then, for all n, $N\Phi_n(f)=NP_n(f)=0$.*

PROOF. Since $\omega=1$, $N(f^k)=0$ and thus $NP_k(f)=0$ for all k; hence $N\Phi_n(f)=0$. □

COROLLARY 8.3 (Case $\det(I\pm F^n)=0$). *Let f be a self-map on K of type $(F,\mathbf{b},\omega)$ such that* $\det(I\pm F^n)=0$. *Then n is odd and $N\Phi_n(f)=NP_n(f)=0$.*

PROOF. If $\det(I\pm F^n)=0$ then Corollary 5.3 states that n is odd and $N(f^k)=0=NP_k(f)$ for all odd k. By Theorem 3.3, we have

$$N\Phi_n(f)=\sum_{k|n}NP_k(f)=0.$$

The last identity follows from the fact that n is odd and so its factors k must be odd and $NP_k(f)=0$. □

Corollary 8.4 (Case $\omega = -1$ and n is even). *Let f be a self-map on K of type $(F, \mathbf{b}, -1)$. Then for all even n,*

$$NP_n(f) = 0, \qquad N\Phi_n(f) = \sum_{\substack{m \text{ odd} \\ m|n}} NP_m(f)$$

where

$$NP_m(f) = \begin{cases} \sum_{k|m} \mu(k) N(f^{m/k}) & \text{when } \det(I - F^n) \neq 0, \\ & \text{or } \det(I - F^n) = 0 \text{ but } \det(I \pm F^m) \neq 0; \\ |\det(I + F^m)| + \displaystyle\sum_{\{k_1,\dots,k_s\} \subset \{1,\dots,t\}} (-1)^s |\det(I + F^{(m_{k_1},\dots,m_{k_s})}| \\ & \text{when } \det(I - F^m) = 0 \neq \det(I + F^m); \\ |\det(I - F^m)| + \displaystyle\sum_{\{k_1,\dots,k_s\} \subset \{1,\dots,t\}} (-1)^s |\det(I - F^{(m_{k_1},\dots,m_{k_s})}| \\ & \text{when } \det(I - F^m) \neq 0 = \det(I + F^m); \\ 0 & \text{when } \det(I \pm F^m) = 0. \end{cases}$$

Here the summation runs through the family of nonempty subsets $\{k_1, \dots, k_s\}$ of $\{1, \dots, t\}$.

Proof. If $\omega = -1$ then for all even m, $N(f^m) = 0$ and thus as before $NP_m(f) = 0$. By Theorem 3.3,

$$N\Phi_n(f) = \sum_{m|n} NP_m(f) = \sum_{\substack{m \text{ odd} \\ m|n}} NP_m(f).$$

Since n is even, by Corollary 5.3 both $\det(I - F^n)$ and $\det(I + F^n)$ cannot be zero. Let m be odd and $m \,|\, n$; since n is even, $m < n$ and n/m is even. Since n/m is even, $I \pm F^m$ is a factor of $I - F^n$.

Assume $\det(I - F^n) \neq 0$. Then $\det(I \pm F^m) \neq 0$. This is the case where f^k is weakly Jiang with $N(f^k) = R(f^k)$ for any $k \mid m$. By Corollary 8.1,

$$N\Phi_n(f) = \sum_{\substack{m \text{ odd} \\ m|n}} NP_m(f) = \sum_{\substack{m \text{ odd} \\ m|n}} \sum_{k|m} \mu(k) N(f^{m/k}).$$

Assume $\det(I - F^n) = 0$.

Case 1. $\det(I \pm F^m) = 0$.

This happens only when the eigenvalues λ_i of F satisfy $\lambda_1 = -\lambda_2 = \pm 1$. Furthermore, $N(f^m) = 0$ and so $NP_m(f) = 0$.

Case 2. $\det(I \pm F^m) \neq 0$.

Then f^m is weakly Jiang with

$$N(f^m) = R(f^m) = 2(|\det(I - F^m)| + |\det(I + F^m)|).$$

By Corollary 8.1,

$$NP_m(f) = \sum_{k|m} \mu(k) N(f^{m/k}).$$

CASE 3. $\det(I - F^m) = 0 \neq \det(I + F^m)$.
By Theorem 6.6,

$$NP_m(f) = |\det(I + F^m)| + \sum_{\{k_1,\ldots,k_s\}\subset\{1,\ldots,t\}} (-1)^s |\det(I + F^{(m_{k_1},\ldots,m_{k_s})}|.$$

CASE 4. $\det(I - F^m) \neq 0 = \det(I + F^m)$.
By Theorem 6.5,

$$NP_m(f) = |\det(I - F^m)| + \sum_{\{k_1,\ldots,k_s\}\subset\{1,\ldots,t\}} (-1)^s |\det(I - F^{(m_{k_1},\ldots,m_{k_s})}|.$$

Consequently, all the observations give rise to our result. □

9. Summary

We can tabulate where to find the formula for the prime Nielsen–Jiang periodic number of f, $NP_n(f)$, of maps f on the flat Riemannian manifold K as follows:

<table>
<tr><th colspan="2">w</th><th>n</th><th>$\det(I \pm F^n)$</th><th>$NP_n(f)$</th></tr>
<tr><td>even</td><td></td><td></td><td></td><td>Corollary 8.1</td></tr>
<tr><td rowspan="10">odd</td><td>1</td><td></td><td></td><td>Corollary 8.2</td></tr>
<tr><td rowspan="5">-1</td><td>even</td><td></td><td>Corollary 8.4</td></tr>
<tr><td rowspan="4">odd</td><td>$\det(I \pm F^n) = 0$</td><td>Corollary 8.3</td></tr>
<tr><td>$\det(I - F^n) \neq 0$
$\det(I + F^n) = 0$</td><td>Theorem 6.5</td></tr>
<tr><td>$\det(I - F^n) = 0$
$\det(I + F^n) \neq 0$</td><td>Theorem 6.6</td></tr>
<tr><td>$\det(I \pm F^n) \neq 0$</td><td>Corollary 8.1</td></tr>
<tr><td rowspan="4">$\neq \pm 1$</td><td>odd
(Corollary 5.3)</td><td>$\det(I \pm F^n) = 0$</td><td>Corollary 8.3</td></tr>
<tr><td rowspan="3"></td><td>$\det(I - F^n) = 0$
$\det(I + F^n) \neq 0$</td><td>Theorem 7.5</td></tr>
<tr><td>$\det(I - F^n) \neq 0$
$\det(I + F^n) = 0$</td><td>Theorem 7.4</td></tr>
<tr><td>$\det(I \pm F^n) \neq 0$</td><td>Corollary 8.1</td></tr>
</table>

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References

[1] H. Brown, R. Bülow, J. Neubüser, H. Wondratschek and H. Zassenhaus, *Crystallographic Groups of Four-Dimensional Spaces*, John Wiley & Sons, Inc., New York, 1978.

[2] V.V. Gorbatsevich, *Compact solvmanifolds of dimension at most* 4, Siberian Math. J. **50** (2009), 239–252.

[3] K.Y. Ha, J.B. Lee and P. Penninckx, *Formulas for the Reidemeister, Lefschetz and Nielsen coincidence number of maps between infra-nilmanifolds*, Fixed Point Theory Appl., 2012, **2012**:39.

[4] P.R. Heath, *A survey of Nielsen periodic point theory*, Nielsen theory and Reidemeister torsion (Warsaw, 1996), 159–188, Banach Center Publ., 49, Polish Acad. Sci., Warsaw, 1999.

[5] ______, *Fibre techniques in Nielsen theory calculations*, Handbook of topological fixed point theory, Springer, Dordrecht, 2005, 489–554.

[6] P.R. Heath and E. Keppelmann, *Fibre techniques in Nielsen periodic point theory on nil and solvmanifolds*, R. Mat. Rep. Acad. Sci. Canada, **16** (1994), 229–234.

[7] ______, *Fibre techniques in Nielsen periodic point theory on nil and solvmanifolds* I, Topology Appl., **76** (1997), 217–247.

[8] ______, *Fibre techniques in Nielsen periodic point theory on nil and solvmanifolds* II, Topology Appl., **106** (2000), 149–167.

[9] B. Jiang, *Lectures on Nielsen fixed point theory*, Contemporary Mathematics Amer. Math. Soc., **14**, 1983.

[10] E.C. Keppelmann and C.K. McCord, *The Anosov theorem for exponential solvmanifolds*, Pacific J. Math., **170** (1995), 143–159.

[11] S.W. Kim, J.B. Lee and K.B. Lee, *Averaging formula for Nielsen numbers*, Nagoya Math. J., **178** (2005), 37–53.

[12] H.J. Kim, J.B. Lee and W.S. Yoo, *Computation of the Nielsen type numbers for maps on the Klein bottle*, J. Korean Math. Soc. **45** (2008), 1483–1503.

[13] J.B. Lee and K.B. Lee, *Lefschetz numbers for continuous maps, and periods for expanding maps on infra-nilmanifolds*, J. Geom. Phys. **56** (2006), 2011–2023.

[14] K.B. Lee, *Maps on infra-nilmanifolds*, Pacific J. Math. **168** (1995), 157–166.

[15] C. McCord, *Nielsen numbers and Lefschetz numbers on solvmanifolds*, Pacific J. Math. **147** (1991), 153–164.

[16] J. Wolf, *Spaces of Constant Curvature*, 5th ed., Publish or Perish, Wilmington, 1984.

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