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ON THE SPACE OF EQUIVARIANT LOCAL MAPS

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Abstract. We introduce the space of equivariant local maps and present the full proof of the splitting theorem for the set of otopy classes of such maps in the case of a representation of a compact Lie group.

Introduction

The notion of *equivariant local maps* appears first [8] and, independently, in [9]. The space of equivariant proper maps is introduced in [8] and [4]. In [7] authors introduce the topology on the set of local maps in the nonequivariant case. By the exponential law, this topology allows one to interpret *otopies*, which generalize homotopies, as paths in the space of local maps and, in consequence, to identify otopy classes of local maps with path-components of this space. In this paper we introduce and study the space of equivariant local maps. It is worth pointing out that the main motivation for studying the space of local maps (both in the equivariant and the nonequivariant case) is that it forms the natural environment for the topological degree theories (see [4]–[7] for details). For most of the paper, we restrict ourselves to the case of a representation of a compact Lie group.

The organization of the paper is as follows. Section 1 presents some preliminaries. In Section 2 we introduce the space of equivariant local maps, prove the exponential law for such maps and show that the inclusion of the space of

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equivariant proper maps into the space of equivariant local maps is a weak homotopy equivalence. In Section 3 we prove that for free actions of compact Lie groups of positive dimension the set of otopy classes is trivial. Section 4 contains some facts concerning equivariant tubular neighbourhoods which will be needed in the next section. In Section 5 we establish a natural decomposition of the set of otopy classes of equivariant local maps with respect to a maximal orbit type. This decomposition result appears implicitly in [4] and explicitly in [10]. As the proofs given there are rather sketchy in some parts, we present here the full proof of this result. Finally, in Section 6 we prove the splitting theorem for the set of path-components of the space of equivariant (proper) local maps. Let us note that results of this type are well-known in the equivariant homotopy theory (see for example [1], [11]).

1. Preliminaries

The notation $A \Subset B$ means that A is a compact subset of B. For a topological space X, let $\tau(X)$ denote the topology on X. Recall that if A, B are topological spaces, then $\mathrm{Map}(A,B)$ denotes the set of all continuous maps of A into B equipped with the usual compact-open topology i.e. having as subbasis all the sets $\Gamma(C,U) = \{f \in \mathrm{Map}(A,B) \mid f(C) \subset U\}$ for $C \Subset A$ and U open in B. For any pointed topological spaces A and B, let $\mathrm{Map}^*(A,B)$ be the subspace of $\mathrm{Map}(A,B)$ consisting of all base-point preserving maps.

For any topological spaces X and Y, let $\mathcal{M}(X,Y)$ be the set of all continuous maps $f\colon D_f \to Y$ such that D_f is an open subset of X. Let $\mathcal{R}$ be a family of subsets of Y. We define

$$\mathrm{Loc}(X,Y,\mathcal{R}) := \{f \in \mathcal{M}(X,Y) \mid f^{-1}(R) \Subset D_f \text{ for all } R \in \mathcal{R}\}.$$

We introduce a topology in $\mathrm{Loc}(X,Y,\mathcal{R})$ generated by the subbasis consisting of all sets of the form

- $H(C,U) := \{f \in \mathrm{Loc}(X,Y,\mathcal{R}) \mid C \subset D_f,\ f(C) \subset U\}$ for $C \Subset X$ and $U \in \tau(Y)$,
- $M(V,R) := \{f \in \mathrm{Loc}(X,Y,\mathcal{R}) \mid f^{-1}(R) \subset V\}$ for $V \in \tau(X)$ and $R \in \mathcal{R}$.

Elements of $\mathrm{Loc}(X,Y,\mathcal{R})$ will be called *local maps*. The natural base point of $\mathrm{Loc}(X,Y,\mathcal{R})$ is the empty map. The set-theoretic union of two local maps f and g with disjoint domains will be denoted by $f \sqcup g$. We define the space of *proper* maps $\mathrm{Prop}(X,Y)$ to be $\mathrm{Loc}(X,Y,\mathcal{K})$, where $\mathcal{K} = \{K \mid K \Subset Y\}$. Moreover, in the case when $\mathcal{R} = \{\{y\}\}$ we will write $\mathrm{Loc}(X,Y,y)$ omitting double curly brackets.

Recall that the formula $\{[\theta(f)](z)\}(x) = f(z,x)$, where f is a function of $Z \times X$ to Y, defines a one-to-one correspondence θ between the set of all (not necessarily continuous) functions of $Z \times X$ to Y and the set of all functions of Z to the set of all functions of X to Y. This correspondence is called the *exponential function.*

The following result called the exponential law for local maps was proved in [7, Theorem 3.1]. Recall that Z^* denotes the Alexandrov one-point compactification of Z.

Theorem 1.1. *If Z and X are locally compact Hausdorff, then the exponential function*

$$\theta\colon \operatorname{Loc}(Z \times X, Y, \mathcal{R}) \to \operatorname{Map}^*(Z^*, \operatorname{Loc}(X, Y, \mathcal{R}))$$

is a homeomorphism.

Assume V is a real finite dimensional orthogonal representation of a compact Lie group G and H is a closed subgroup of G. Recall that $G_x = \{g \in G \mid gx = x\}$, (H) stands for a conjugacy class of H and $WH = NH/H$, where NH is a normalizer of H in G. Let Ω be an open invariant subset of V. We define the following subsets of Ω:

$$\begin{aligned} \Omega^H &= \{x \in \Omega \mid H \subset G_x\}, \\ \Omega_H &= \{x \in \Omega \mid H = G_x\}, \\ \Omega_{(H)} &= \{x \in X \mid (H) = (G_x)\}. \end{aligned}$$

Let

$$\begin{aligned} \Phi(G) &= \{(H) \mid H \text{ is a closed subgroup of } G\}, \\ \operatorname{Iso}(\Omega) &= \{(H) \in \Phi(G) \mid \Omega_{(H)} \neq \emptyset\}. \end{aligned}$$

The set $\operatorname{Iso}(\Omega)$ is partially ordered. Namely, $(H) \leq (K)$ if H is conjugate to a subgroup of K. Throughout the paper we will make use of the following well-known facts (most of them are true in more general setting):

- $\operatorname{Iso}(\Omega)$ is finite,
- WH is a compact Lie group,
- V^H is a linear subspace of V and orthogonal representation of WH,
- the action of WH on Ω_H is free,
- Ω_H is open and dense in Ω^H,
- $\Omega_{(H)}$ is a G-invariant submanifold of Ω,
- $\Omega_{(H)} = G\Omega_H$ and Ω_H is closed in $\Omega_{(H)}$,
- if (H) is maximal in $\operatorname{Iso}(\Omega)$ then $\Omega_{(H)}$ is closed in Ω.

2. Spaces of local and proper G-maps

Assume that G is a topological group and X, Y are two G-spaces. We will denote by $\operatorname{Map}_G(X, Y)$ the subset of $\operatorname{Map}(X, Y)$ consisting of all maps f such that

$$f(gx) = gf(x) \quad \text{for all } x \in X \text{ and } g \in G$$

endowed with the relative topology. Elements of $\mathrm{Map}_G(X,Y)$ are called *equivariant maps* or *G-maps*. Moreover, if G-spaces X and Y are pointed then $\mathrm{Map}^*_G(X,Y)$ denotes the subspace of $\mathrm{Map}_G(X,Y)$ consisting of all base-point preserving maps. Similarly, let $\mathrm{Loc}_G(X,Y,\mathcal{R})$ (resp. $\mathrm{Prop}_G(X,Y)$) be the subspace of $\mathrm{Loc}(X,Y,\mathcal{R})$ (resp. $\mathrm{Prop}(X,Y)$) consisting of equivariant maps with invariant domains and equipped with the induced topology.

Elements of $\mathrm{Loc}_G(X,Y,\mathcal{R})$ will be called *equivariant local maps* or *local G-maps*. Assume that G acts diagonally on Cartesian products of spaces and by conjugation on all mapping spaces. That is, for a map f and $g \in G$ we define $g\cdot f$ by $(g\cdot f)(x) = gf(g^{-1}x)$.

The next result is an immediate consequence of Theorem 1.1.

Theorem 2.1. *Assume that X, Y, Z are G-spaces. If Z and X are locally compact Hausdorff, then the exponential function*

$$\theta\colon \mathrm{Loc}_G(Z\times X,Y,\mathcal{R}) \to \mathrm{Map}^*_G(Z^*,\mathrm{Loc}_G(X,Y,\mathcal{R}))$$

is a G-homeomorphism.

Corollary 2.2. *Assume that X, Y, Z are G-spaces. If Z is compact Hausdorff and X is locally compact Hausdorff, then the exponential function*

$$\theta\colon \mathrm{Loc}_G(Z\times X,Y,\mathcal{R}) \to \mathrm{Map}_G(Z,\mathrm{Loc}_G(X,Y,\mathcal{R}))$$

is a G-homeomorphism.

Remark 2.3. It is worth pointing out that in this paper, in fact, we use the above results only in the case when G acts trivially on Z. Observe that in this case we have

$$\mathrm{Map}^*_G(Z^*,\mathrm{Loc}_G(X,Y,\mathcal{R})) = \mathrm{Map}^*(Z^*,\mathrm{Loc}_G(X,Y,\mathcal{R})),$$
$$\mathrm{Map}_G(Z,\mathrm{Loc}_G(X,Y,\mathcal{R})) = \mathrm{Map}(Z,\mathrm{Loc}_G(X,Y,\mathcal{R})).$$

Assume V is a real finite dimensional orthogonal representation of a compact Lie group G. Throughout the paper $\mathbb{R}^k$ denotes a trivial representation of G. Let Ω be an open invariant subset of $\mathbb{R}^k\oplus V$.

Let us introduce the following notation:

$$\mathcal{F}_G(\Omega) := \mathrm{Loc}_G(\Omega,V,0), \qquad \mathcal{P}_G(\Omega) := \mathrm{Prop}_G(\Omega,V).$$

Let $I=[0,1]$. We assume that the action of G on I is trivial. Any element of $\mathrm{Loc}_G(I\times\Omega,V,0)$ is called an *otopy* and any element of $\mathrm{Prop}_G(I\times\Omega,V)$ is called a *proper otopy*. By Corollary 2.2, each otopy (resp. proper otopy) corresponds to a path in $\mathcal{F}_G(\Omega)$ (resp. $\mathcal{P}_G(\Omega)$) and vice versa.

Given a (proper) otopy $h\colon \Lambda\subset I\times\Omega\to V$ we can define for each $t\in I$ sets $\Lambda_t=\{x\in\Omega \mid (t,x)\in\Lambda\}$ and maps $h_t\colon\Lambda_t\to V$ with $h_t(x)=h(t,x)$. Note that from the above h_t may be the empty map. If h is a (proper) otopy, we say that

h_0 and h_1 are *(proper) otopic*. Of course, (proper) otopy gives an equivalence relation on $\mathcal{F}_G(\Omega)$ ($\mathcal{P}_G(\Omega)$). The set of (proper) otopy classes will be denoted by $\mathcal{F}_G[\Omega]$ ($\mathcal{P}_G[\Omega]$). Observe that if $f \in \mathcal{F}_G(\Omega)$ and V is an open invariant subset of D_f such that $f^{-1}(0) \subset V$, then f and $f|_V$ are otopic. In particular, if $f^{-1}(0) = \emptyset$ then f is otopic to the empty map.

Observe that, by Theorem 2.1, there are natural homeomorphisms

$$\mathcal{F}_G(\mathbb{R}^m \times \Omega) \approx \Omega^m(\mathcal{F}_G(\Omega)) \quad \text{and} \quad \mathcal{P}_G(\mathbb{R}^m \times \Omega) \approx \Omega^m(\mathcal{P}_G(\Omega)), \tag{2.1}$$

and, in consequence, isomorphisms

$$\mathcal{F}_G[\mathbb{R}^m \times \Omega] \approx \pi_m(\mathcal{F}_G(\Omega)) \quad \text{and} \quad \mathcal{P}_G[\mathbb{R}^m \times \Omega] \approx \pi_m(\mathcal{P}_G(\Omega)). \tag{2.2}$$

In particular, $\mathcal{F}_G[\Omega] \approx \pi_0(\mathcal{F}_G(\Omega))$ and $\mathcal{P}_G[\Omega] \approx \pi_0(\mathcal{P}_G(\Omega))$.

We omit the proof of the next proposition, since it is very similar to the proof of Proposition 5.1 in [7]. The crucial observation here is that that all otopies that appear in that proof are equivariant if so are the respective maps.

Proposition 2.4. *Let Ω be an open invariant subset of $\mathbb{R}^k \oplus V$. Then the map $\mathcal{P}_G[\Omega] \to \mathcal{F}_G[\Omega]$ induced by the inclusion is a bijection.*

We can now formulate the main result of this section.

Theorem 2.5. *Let Ω be an open invariant subset of $\mathbb{R}^k \oplus V$. Then the inclusion $\mathcal{P}_G(\Omega) \hookrightarrow \mathcal{F}_G(\Omega)$ is a weak homotopy equivalence.*

Proof. Using (2.1), (2.2) and Proposition 2.4 and repeating the reasoning used in the proof of Theorem 5.1 in [7], we are done. □

Remark 2.6. Let us denote by $\mathrm{Map}^*_G(S^{k+V}, S^V)$ the space of pointed equivariant maps between *representation spheres* i.e. one-point compactifications of representations $\mathbb{R}^k \oplus V$ and V, respectively. The set of G-homotopy classes of such maps will be denoted by $[S^{k+V}; S^V]^*_G$. Observe that the function

$$\kappa\colon \mathcal{P}_G(\mathbb{R}^k \oplus V) \to \mathrm{Map}^*_G(S^{k+V}, S^V)$$

given by $\kappa(f) := f^+$, where

$$f^+(x) = \begin{cases} f(x) & \text{if } x \in D_f, \\ * & \text{otherwise,} \end{cases}$$

establishes a natural homeomorphism

$$\mathcal{P}_G(\mathbb{R}^k \oplus V) \approx \mathrm{Map}^*_G(S^{k+V}, S^V).$$

By the above and Proposition 2.4, we obtain the following sequence of bijections

$$\mathcal{F}_G[\mathbb{R}^k \oplus V] \approx \mathcal{P}_G[\mathbb{R}^k \oplus V] \approx [S^{k+V}; S^V]^*_G. \tag{2.3}$$

3. Free actions of compact Lie groups of positive dimension

Assume V is a real finite dimensional orthogonal representation of a compact Lie group G. The following result is an immediate consequence of Proposition 3.2 and Theorem 3.4, which will be proved below.

Theorem 3.1. *If* $\dim G > 0$, Ω *is an open invariant subset of* V *and* G *acts freely on* Ω *then the set* $\mathcal{F}_G[\Omega]$ *has a single element.*

All manifolds considered are without boundary. Assume that E and M are smooth (i.e., C^1) manifolds of dimensions q and m respectively and $p\colon E \to M$ is a smooth vector bundle of rank n. We will identify M with the zero section of E. A *local cross section* of a bundle $p\colon E \to M$ is a continuous map $s\colon U \to E$, where U is open in M, $s^{-1}(M)$ is compact and $p \circ s = \mathrm{id}_U$. Let $\Gamma_{\mathrm{loc}}(M, E)$ denote the set of all local cross sections of E over M. A *fiber otopy* is a continuous map $h\colon \Lambda \to E$ such that Λ is open in $I \times M$, $h^{-1}(M)$ is compact and $p(h(t, x)) = x$ for all $(t, x) \in \Lambda$. Let $s', s'' \in \Gamma_{\mathrm{loc}}(M, E)$. We say that s' and s'' are *fiber otopic* provided there is a fiber otopy h such that $h_0 = s'$ and $h_1 = s''$, where $h_t(x) = h(t, x)$. Let $\Gamma_{\mathrm{loc}}[M, E]$ denote the set of fiber otopy classes of local cross sections of $p\colon E \to M$.

Proposition 3.2. *If* $n > m$ *then* $\Gamma_{\mathrm{loc}}[M, E]$ *has a single element.*

Roughly speaking, the proof of Proposition 3.2 is based on two simple observations

- under our assumption on dimensions a generic section does not meet the zero section,
- a section with no zeroes is fiber otopic to the empty section.

In this proof we will need the following transversality result for sections of a smooth vector bundle, which can be easily derived from the transversality theorem for maps.

Lemma 3.3. *Arbitrarily close* (*in the* C^1 *sense*) *to any smooth local cross section of a smooth vector bundle there exists a local cross section. which is transverse to to the zero section.*

Proof of Proposition 3.2. Let $[s] \in \Gamma_{\mathrm{loc}}[M, E]$. Without loss of generality we can assume that s is smooth. Notice that it suffices to show that s is fiber otopic to a nowhere vanishing local cross section, because any nowhere vanishing local cross section is fiber otopic to the empty section. It follows from Lemma 3.3 that there is a smooth local cross section $\widetilde{s}$ which is transverse to the zero section and C^1-close to s (so fiber otopic to s). By the definition of transversality and the assumption on dimensions, $\widetilde{s}^{-1}(M)$ is empty, which completes the proof. $\square$

Let Ω be an open invariant subset of V, G acts freely on Ω, $M := \Omega/G$, $E := (\Omega \times V)/G$. The trivial vector bundle $\Omega \times V \to \Omega$ factorizes to the vector bundle $p\colon E \to M$ with the typical fiber $F = \mathbb{R}^n$. We omit the proof of the next result since it is very similar to that of Proposition 7.2 in [12].

Theorem 3.4. *The function $\Phi\colon \mathcal{F}_G(\Omega) \to \Gamma_{\mathrm{loc}}(M, E)$ given by $\Phi(f) := s_f$, where $s_f([x]) := [x, f(x)]$, is bijective. Moreover, Φ induces a bijection between $\mathcal{F}_G[\Omega]$ and $\Gamma_{\mathrm{loc}}[M, E]$.*

Remark 3.5. It is easy to see that the above theorem also holds if Ω is an open invariant subset of $\mathbb{R}^k \oplus V$ with trivial action of G on $\mathbb{R}^k$ and free action of G on Ω.

The following slight generalization of Theorem 3.1 follows immediately from Proposition 3.2 and Remark 3.5.

Theorem 3.6. *If $\dim G > k$ and Ω is an open invariant subset of $\mathbb{R}^k \oplus V$ (G acts trivially on $\mathbb{R}^k$) and G acts freely on Ω then the set $\mathcal{F}_G[\Omega]$ has a single element.*

4. Equivariant tubular neighbourhoods

Let V be real finite dimensional orthogonal representation of a compact Lie group G. Assume that M is a G-invariant submanifold of V. Let $N_x := (T_xM)^\perp \subset V$. Let us denote by $\nu(M)$ the normal bundle i.e.

$$\nu(M) := \{(x, v) \mid x \in M,\ v \in N_x\} \subset M \times V$$

and by $\nu(M, \varepsilon)$ the ε-disc bundle of the normal bundle $\nu(M)$ i.e.

$$\nu(M, \varepsilon) := \{(x, v) \in \nu(M) \mid |v| < \varepsilon\}.$$

For $U \subset M$ let U^ε denote the set $\{x + v \mid x \in U,\ v \in N_x,\ |v| < \varepsilon\}$. The following result is an immediate consequence of the invariant tubular neighbourhood theorem (see for example section 2.4.3 in [2]).

Lemma 4.1. *Let $U \subset M$ be open, bounded and invariant. There is $\varepsilon > 0$ such that the map $n\colon \nu(M, \varepsilon)\restriction_U \to U^\varepsilon$, given by $n(x, v) = x + v$, is a G-diffeomorphism.*

Remark 4.2. The above lemma guarantees that if ε is small enough then each element of U^ε has a unique representation of the form $x + v$, where $x \in U$ and $v \in N_x$.

Lemma 4.3. *Let $f \in \mathcal{F}_G(\Omega)$ and $M = \Omega_{(H)}$. Assume that*

- *(H) is maximal in $\mathrm{Iso}(\Omega)$,*
- *U is open in $D_f \cap M$,*
- *$n\colon \nu(M, \varepsilon)\restriction_U \to U^\varepsilon$ is a diffeomorphism.*

Then, for every $(x,v) \in \nu(M,\varepsilon)\restriction_U$ *and* $\alpha \in \mathbb{R}$*, if* $f(x)+\alpha v = 0$*, then* $\alpha = 0$ *or* $v = 0$*.*

Proof. Assume that $\alpha \neq 0$. Let $K := G_x$. Hence $(K) = (H)$. As $x \in V^K$ we have $f(x) = -\alpha v \in V^K$. Since V^K is a linear subspace of V, $v \in V^K$ and, in consequence, $x + v \in V^K \cap U^\varepsilon \subset \Omega_{(H)} \cap U^\varepsilon$ (the last inclusion follows from the maximality of (H)). Therefore $v = 0$ by Remark 4.2. □

5. Separation of zeros of maximal orbit type

Assume V is a real finite dimensional orthogonal representation of a compact Lie group G and Ω is an open invariant subset of V. In the remainder of this section we also assume that (H) is maximal in $\operatorname{Iso}(\Omega)$. The main goal if this section is to show that, under the above assumptions, there is a natural bijection between the sets $\mathcal{F}_G[\Omega]$ and $\mathcal{F}_{WH}[\Omega_H] \times \mathcal{F}_G[\Omega \setminus \Omega_{(H)}]$. The naive approach suggests to define this bijection simply by taking the otopy classes of the respective restrictions i.e. by the formula

$$[f] \mapsto ([f\restriction_{D_f \cap \Omega_H}], [f\restriction_{D_f \setminus \Omega_{(H)}}])$$

Unfortunately, $f\restriction_{D_f \setminus \Omega_{(H)}}$ does need not to be a local G-map. For this reason, we first have to perturbate the map f within its otopy class so that the restriction of the perturbation to the set $D_f \setminus \Omega_{(H)}$ would be a local G-map. Roughly speaking, our perturbation does not change f on $\Omega_{(H)}$ and separates zeros of maximal orbit type, which lie on $\Omega_{(H)}$, from all other zeros of f. The precise definition of this bijection requires to introduce some notation and definitions. Recall that $N_x := (T_x\Omega_{(H)})^\perp$ may be identified with a linear subspace of V. Throughout the rest of this section we will use the following notation. For a map $k\colon D_k \subset V \to V$ and $X \subset D_k$, let

$$\begin{aligned} Z(k,X) &= \{x \in X \mid k(x) = 0\},\\ Z_\delta(k,X) &= \{x \in X \mid \operatorname{dist}(x, Z(k,X)) < \delta\}. \end{aligned}$$

Let us define the sets:

$$\begin{aligned} P_\delta &= Z_\delta(f, D_f \cap \Omega_{(H)}),\\ P_\delta^\varepsilon &= \{x + v \mid x \in P_\delta,\ v \in N_x,\ |v| < \varepsilon\} \end{aligned}$$

and the homotopy $\overline{f}^s_\delta\colon P^{3\varepsilon}_{2\delta} \to V$ $(s \in [0, 2\varepsilon])$

$$\overline{f}^s_\delta(x+v) := \begin{cases} f(x) + v & \text{if } |v| \le \dfrac{1}{2}s,\\ f(x) + (s - |v|)\dfrac{v}{|v|} & \text{if } \dfrac{1}{2}s \le |v| \le s,\\ f\left(x + \dfrac{3\varepsilon}{s - 3\varepsilon}(s - |v|)\dfrac{v}{|v|}\right) & \text{if } s \le |v| \le 3\varepsilon, \end{cases}$$

where $\delta > 0$ is chosen so that $\mathrm{cl}_{\Omega_{(H)}}(Z_{2\delta}) \Subset \Omega_{(H)}$ and $\varepsilon > 0$ is chosen so that $Z_{2\delta}^{3\varepsilon}$ is a tubular neighbourhood of $Z_{2\delta}$ (see Lemma 4.1) and $\mathrm{cl}_{\Omega}(Z_{2\delta}^{3\varepsilon}) \Subset D_f$ (see Figure 1). Note that such δ and ε always exist and if δ and ε satisfy the above conditions and $\delta \geq \delta' \geq 0$ and $\varepsilon \geq \varepsilon' \geq 0$, so does δ' and ε'. In the next definition we will make use of the following auxiliary function $m\colon \Omega_{(H)} \to \mathbb{R}$ given by

$$m(x) = \frac{\min\{\mathrm{dist}(x, P_\delta), \delta\}}{\delta}.$$

Finally, we define the homotopy $f_\delta^s\colon D_f \to V$ $(s \in [0, 2\varepsilon])$

$$f_\delta^s(z) := \begin{cases} f(z) & \text{if } z \in D_f \setminus P_{2\delta}^{3\varepsilon}, \\ m(x)f(z) + (1 - m(x))\overline{f}_\delta^s(z) & \text{if } z = x + v \in P_{2\delta}^{3\varepsilon}. \end{cases}$$

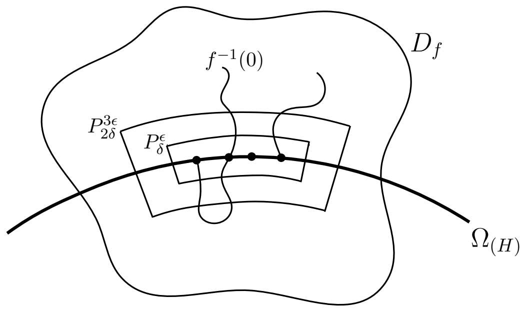

Figure 1. Separation of zeros of maximal orbit type

Remark 5.1. The above construction guarantees that

- $f = f_\delta^0$ is otopic to $f_\delta^{2\varepsilon}$ in $\mathcal{F}_G(\Omega)$,
- $f_\delta^s\restriction_{D_f \cap \Omega_{(H)}} = f\restriction_{D_f \cap \Omega_{(H)}}$ for all $s \in [0, 2\varepsilon]$,
- $(f_\delta^{2\varepsilon}\restriction_{P_\delta^{2\varepsilon}})^{-1}(0) = Z(f, D_f \cap \Omega_{(H)}) \subset \Omega_{(H)}$ by Lemma 4.3.

Remark 5.2. Observe that both $f_\delta^{2\varepsilon}\restriction_{P_\delta^\varepsilon}$ and $f_\delta^{2\varepsilon}\restriction_{D_f \setminus \mathrm{cl}\, P_\delta^\varepsilon}$ are elements of $\mathcal{F}_G(\Omega)$ and

$$[f_\delta^{2\varepsilon}] = [f_\delta^{2\varepsilon}\restriction_{P_\delta^\varepsilon} \sqcup f_\delta^{2\varepsilon}\restriction_{D_f \setminus \mathrm{cl}\, P_\delta^\varepsilon}] \quad \text{in } \mathcal{F}_G[\Omega].$$

Let us define the map $\theta\colon \mathcal{F}_G(\Omega) \to \mathcal{F}_{WH}[\Omega_H] \times \mathcal{F}_G[\Omega \setminus \Omega_{(H)}]$ by the formula $\theta(f) = ([f_H], [\widetilde{f}])$, where $f_H = f\restriction_{D_f \cap \Omega_H}$ and $\widetilde{f} = f_\delta^{2\varepsilon}\restriction_{D_f \setminus \Omega_{(H)}}$.

Lemma 5.3. *The map θ is well-defined.*

Proof. Since $f_H^{-1}(0) = V^H \cap f^{-1}(0) \Subset D_f \cap \Omega_H$, $f_H \in \mathcal{F}_{WH}(\Omega_H)$. By Remark 5.1, $\widetilde{f} \in \mathcal{F}_G(\Omega \setminus \Omega_{(H)})$. So it suffices to show that the otopy class of $f_\delta^{2\varepsilon}\restriction_{D_f \setminus \Omega_{(H)}}$ in $\mathcal{F}_G[\Omega \setminus \Omega_{(H)}]$ does not depend on the choice of ε and δ. We can certainly assume that $\varepsilon' \leq \varepsilon$ and $\delta' \leq \delta$, since otherwise we replace ε' and δ' by $\min\{\varepsilon, \varepsilon'\}$ and $\min\{\delta, \delta'\}$ and repeat our reasoning twice. Under that assumption the desired otopy (homotopy) is given by $f^{2\varepsilon_t}_{\delta_t}\restriction_{D_f\setminus\Omega_{(H)}}$, where $\varepsilon_t = (1-t)\varepsilon + t\varepsilon'$ and $\delta_t = (1-t)\delta + t\delta'$. □

Consider the map $\Theta\colon \mathcal{F}_G[\Omega] \to \mathcal{F}_{WH}[\Omega_H] \times \mathcal{F}_G[\Omega \setminus \Omega_{(H)}]$ induced by θ on otopy classes. We can now formulate the main result of this section.

Theorem 5.4. *The map Θ is a well-defined bijection.*

We have divided the proof of Theorem 5.4 into a sequence of lemmas.

Lemma 5.5. *The map Θ is well-defined.*

Proof. We have to show that if $[f] = [g]$ in $\mathcal{F}_G[\Omega]$ then

(1) $[f_H] = [g_H]$ in $\mathcal{F}_{WH}[\Omega_H]$,
(2) $[\widetilde{f}] = [\widetilde{g}]$ in $\mathcal{F}_G[\Omega \setminus \Omega_{(H)}]$.

By assumption, there is an otopy $h\colon \Lambda \subset I \times \Omega \to V$ such that $f = h_0$ and $g = h_1$. The proof of (1) is straightforward. Namely, let $A = \Lambda \cap (I \times \Omega_H)$ and $k = h\restriction_A$. Then $k\colon A \subset I \times \Omega_H \to V^H$ is an otopy such that $k_0 = f_H$ and $k_1 = g_H$, and (1) is proved.

To show (2) we will proceed analogously to the construction of the map $f^{2\varepsilon}_\delta$, but this time we will perturbate the otopy h instead of the map f. Let us define the sets:

$$Q_\delta = Z_\delta(h, \Lambda_{(H)}),$$
$$Q^\varepsilon_\delta = \{(t, x+v) \mid (t,x) \in Q_\delta,\ v \in N_x,\ |v| < \varepsilon\}$$

and the map $\overline{h}\colon Q^{3\varepsilon}_{2\delta} \to V$

$$\overline{h}(t, x+v) := \begin{cases} h(t,x) + v & \text{if } |v| \le \varepsilon, \\ h(t,x) + (2\varepsilon - |v|)\dfrac{v}{|v|} & \text{if } \varepsilon \le |v| \le 2\varepsilon, \\ h\left(t, x + 3(|v| - 2\varepsilon)\dfrac{v}{|v|}\right) & \text{if } 2\varepsilon \le |v| \le 3\varepsilon. \end{cases}$$

where $\delta > 0$ is chosen so that $\mathrm{cl}_{\Lambda_{(H)}}(Q_{2\delta}) \Subset \Lambda_{(H)}$ and $\varepsilon > 0$ is chosen so that $Q^{3\varepsilon}_{2\delta}$ is a tubular neighbourhood of $Q_{2\delta}$ (see Lemma 4.1) and $\mathrm{cl}_\Lambda(Q^{3\varepsilon}_{2\delta}) \Subset \Lambda$. The auxiliary function $M\colon \Lambda_{(H)} \to \mathbb{R}$ is defined by the formula

$$M(p) = \frac{\min\{\mathrm{dist}(p, Q_\delta), \delta\}}{\delta}.$$

Finally, we define the map $H\colon \Lambda \to V$

$$H(t,z) := \begin{cases} h(t,z) & \text{if } z \in \Lambda \setminus Q^{3\varepsilon}_{2\delta}, \\ M(t,x)h(t,z) + (1 - M(t,x))\overline{h}(t,z) & \text{if } (t,z) = (t, x+v) \in Q^{3\varepsilon}_{2\delta}. \end{cases}$$

It is easy to see that H is an otopy and, more importantly, so is $H\restriction_{\Lambda\setminus\Lambda_{(H)}}$. Note that, in general, the restriction $h\restriction_{\Lambda\setminus\Lambda_{(H)}}$ does not need to be an otopy. The following facts:

- $H_0(x) = f_\delta^{2\varepsilon}(x) = f(x)$ for all $x \in D_f \cap \Omega_{(H)}$,
- $H_0(x) = f_\delta^{2\varepsilon}(x) \neq 0$ for all $x \in P_\delta^\varepsilon \setminus \Omega_{(H)}$,

imply that the straight-line homotopy between $\widetilde{f}$ and $H_0\restriction_{D_f\setminus\Omega_{(H)}}$ is in fact an otopy. The same is true for $\widetilde{g}$ and $H_1\restriction_{D_g\setminus\Omega_{(H)}}$. Consequently, we obtain the following sequence of otopy relations in $\mathcal{F}_G\left(\Omega\setminus\Omega_{(H)}\right)$

$$\widetilde{f} \sim H_0\restriction_{D_f\setminus\Omega_{(H)}} \sim H_1\restriction_{D_g\setminus\Omega_{(H)}} \sim \widetilde{g},$$

which is the desired conclusion. □

Lemma 5.6. *The map* Θ *is surjective.*

Proof. Let $[a] \in \mathcal{F}_{WH}[\Omega_H]$ and $[b] \in \mathcal{F}_G[\Omega\setminus\Omega_{(H)}]$. We will construct a map $f \in \mathcal{F}_G(\Omega)$ such that $\Theta([f]) = ([a],[b])$. Set $A = GD_a$. First we extend the map a to a map $a_{(H)}\colon A \to V$ by the formula $a_{(H)}(gx) = ga(x)$ for $x \in D_a$ and $g \in G$. Define the sets:

$$R_\delta = Z_\delta(a_{(H)}, A), \qquad R_\delta^\varepsilon = \{x + v \mid x \in R_\delta,\ v \in N_x,\ |v| < \varepsilon\},$$

where $\delta > 0$ is chosen so that $\operatorname{cl}_{\Omega_{(H)}}(Z_\delta) \Subset A$ and $\varepsilon > 0$ is chosen so that

$$\operatorname{dist}(b^{-1}(0), \operatorname{cl} R_\delta^\varepsilon) > 0$$

and $R_{2\delta}^{3\varepsilon}$ is a tubular neighbourhood of $R_{2\delta}$. Now we extend the map $a_{(H)}\restriction_{R_\delta}$ to a map $a_{(H)}^\varepsilon\colon R_\delta^\varepsilon \to V$ by the formula $a_{(H)}^\varepsilon(x+v) = a_{(H)}(x)+v$. Let $\widehat{b} = b\restriction_{D_b\setminus\operatorname{cl} R_\delta^\varepsilon}$. By the above, both $a_{(H)}^\varepsilon$ and $\widehat{b}$ are equivariant local maps, and so is $f := a_{(H)}^\varepsilon \sqcup \widehat{b}$. It remains to prove that $([f_H],[\widetilde{f}]) = ([a],[b])$. Since $f_H = a\restriction_{D_a\cap R_\delta}$, we have $[f_H] = [a]$ in $\mathcal{F}_{WH}[\Omega_H]$. Throughout the rest of the proof, we will write $\sim$ for the otopy relation in $\mathcal{F}_G(\Omega\setminus\Omega_{(H)})$. From the definition of f, we have $\widetilde{f} \sim f\restriction_{D_f\setminus\Omega_{(H)}}$ by means of the straight-line homotopy. Moreover, $f\restriction_{D_f\setminus\Omega_{(H)}} \sim \widehat{b} \sim b$ (both otopies come from obvious restrictions). We thus get $\widetilde{f} \sim b$, which completes the proof. □

The proof of injectivity is similar in spirit to that of surjectivity.

Lemma 5.7. *The map* Θ *is injective.*

Proof. Assume that $\Theta([f]) = \Theta([g])$, that is, there are two otopies:

(1) $h_H\colon B \subset I \times \Omega_H \to V^H$ joining f_H and g_H in $\mathcal{F}_{WH}(\Omega_H)$,
(2) $\widetilde{h}\colon \Lambda \subset I \times \left(\Omega\setminus\Omega_{(H)}\right) \to V$ joining $\widetilde{f}$ and $\widetilde{g}$ in $\mathcal{F}_G\left(\Omega\setminus\Omega_{(H)}\right)$.

Define the map $h_{(H)}\colon GB \subset I\times\Omega_{(H)} \to V$ by the formula $h_{(H)}(t,gx) = gh_H(t,x)$ for $(t,x) \in B$ and $g \in G$. Set

$$S_\delta = Z_\delta(h_{(H)}, GB), \quad S_\delta^\varepsilon = \{(t, x+v) \mid (t,x) \in S_\delta,\ v \in N_x,\ |v| < \varepsilon\},$$

where $\delta > 0$ is chosen so that $\operatorname{cl} S_\delta \Subset GA$ and ε is chosen so that

$$\operatorname{dist}(\widetilde{h}^{-1}(0), \operatorname{cl} S_\delta^\varepsilon) > 0.$$

and S_δ^ε is a tubular neighbourhood of S_δ contained in $I \times \Omega$. Let us define the map $h^\perp\colon S_\delta^\varepsilon \subset I \times \Omega \to V$ by $h^\perp(t, x+v) := h_{(H)}(t,x) + v$. Note that both $h^\perp$ and $\widehat{h} := \widetilde{h}\restriction_{\Lambda \setminus \operatorname{cl} S_\delta^\varepsilon}$ are otopies. Moreover, since the domains of $h^\perp$ and $\widehat{h}$ are disjoint, the disjoint union $h^\perp \sqcup \widehat{h}$ is also an otopy. Here and subsequently, the symbol $\sim$ denotes the otopy relation in $\mathcal{F}_G(\Omega)$. By Remark 5.2, we get

$$f \sim f^{2\varepsilon} \sim f^{2\varepsilon}\restriction_{P_\delta^\varepsilon} \sqcup \widetilde{f}\restriction_{D_f \setminus \operatorname{cl} P_\delta^\varepsilon}.$$

On the other hand, by the definitions of $h^\perp$ and $\widehat{h}$, we obtain

$$h_0^\perp \sim f^{2\varepsilon}\restriction_{P_\delta^\varepsilon} \quad \text{and} \quad \widehat{h}_0 \sim \widetilde{f}\restriction_{D_f \setminus \operatorname{cl} P_\delta^\varepsilon},$$

and consequently $f \sim h_0^\perp \sqcup \widehat{h}_0$. Similarly, $g \sim h_1^\perp \sqcup \widehat{h}_1$, and finally

$$f \sim h_0^\perp \sqcup \widehat{h}_0 \sim h_1^\perp \sqcup \widehat{h}_1 \sim g,$$

which proves the lemma. □

Remark 5.8. It is easily seen that in the case when Ω is an open invariant subset of $\mathbb{R}^k \oplus V$ both the definition of

$$\Theta\colon \mathcal{F}_G[\Omega] \to \mathcal{F}_{WH}[\Omega_H] \times \mathcal{F}_G[\Omega \setminus \Omega_{(H)}]$$

and the proof that Θ is a bijection are essentially the same.

6. Splitting of the set of otopy classes of equivariant local maps

Assume V is a real finite dimensional orthogonal representation of a compact Lie group G and Ω is an open invariant subset of $\mathbb{R}^k \oplus V$. Let

$$\Phi_k(G) := \{(H) \in \Phi(G) \mid \dim WH \le k\}.$$

It is well-known that the set $\operatorname{Iso}(\Omega)$ is finite and so is $\operatorname{Iso}(\Omega) \cap \Phi_k(G)$. Recall that if X, Y are G-spaces and A (resp. B) is a G-subspace of X (resp. Y) then the set of relative G-homotopy classes of G-maps from (X,A) to (Y,B) is denoted by $[X,A;Y,B]_G$. We finish this paper with a series of splitting results. Recall that the sets of (proper) otopy classes of equivariant (proper) local maps can be identified with the sets of path-components of the spaces of equivariant (proper) local maps.

Theorem 6.1. *There are bijections*

$$\mathcal{F}_G[\Omega] \approx \prod_{(H)} \mathcal{F}_{WH}[\Omega_H], \tag{6.1}$$

$$\mathcal{P}_G[\Omega] \approx \prod_{(H)} \mathcal{P}_{WH}[\Omega_H], \tag{6.2}$$

$$[S^{k+V}; S^V]_G^* \approx \prod_{(H)} \left[S^{k+V^H}, S^{k+V^H} \setminus (\mathbb{R}^k \times V_H); S^{V^H}, *\right]_{WH}, \tag{6.3}$$

where the products are taken over the set $\operatorname{Iso}(\Omega) \cap \Phi_k(G)$.

Proof. Theorem 3.6, Remark 5.8 and obvious induction on orbit types give immediately (6.1). In turn, Proposition 2.4 and (6.1) imply (6.2). Finally, combining (2.3) and (6.2) with the observation that the map defined in the same way as κ in Remark 2.6 induces a bijection

$$\mathcal{P}_{WH}[\mathbb{R}^k \times V_H] \approx \left[S^{k+V^H}, S^{k+V^H} \setminus (\mathbb{R}^k \times V_H); S^{V^H}, *\right]_{WH},$$

we obtain (6.3). □

Remark 6.2. As you can see from the above proof, the main difficulty in proving Theorem 6.1 lies in Theorem 5.4.

Remark 6.3. Recall that the extreme case of the trivial action is covered by [7]. Namely, if G acts trivially on V and Ω is an open subset of V, then

$$\mathcal{F}_G[\Omega] = \mathcal{F}_{\{e\}}[\Omega] \approx \sum_{\alpha} \mathbb{Z},$$

where the direct sum is taken over all connected components α of the set Ω. Similarly, if G acts trivially on $\mathbb{R}^{n+k}$, then

$$\mathcal{F}_G[\mathbb{R}^{n+k}] = \mathcal{F}_{\{e\}}[\mathbb{R}^{n+k}] \approx \pi_{n+k}(S^n).$$

Remark 6.4. A more thorough description of $\mathcal{F}_G[\Omega]$ based on the formula (6.1) and the detailed analysis of the factors $\mathcal{F}_{WH}[\Omega_H]$ is given in [3], which continues and develops the approach presented here.

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