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COHOMOLOGICAL DECOMPOSITION OF COMPLEX NILMANIFOLDS

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Dedicated to Yuli Rudyak on the occasion of his 65th birthday

Abstract. We study *pureness* and *fullness* of invariant complex structures on nilmanifolds. We prove that in dimension six, apart from the complex torus, there exist only two non-isomorphic complex structures satisfying both properties, which live on the real nilmanifold underlying the Iwasawa manifold. We also show that the product of two almost complex manifolds which are pure and full is not necessarily full.

1. Introduction

Nilmanifolds are compact homogeneous spaces $M = \Gamma \setminus G$, where G is a simply connected nilpotent Lie group and Γ is a lattice of maximal rank in G. Any structure on the Lie algebra $\mathfrak{g}$ of G gives rise to a left-invariant structure on G and thus, it defines a structure on M. The invariant geometry of nilmanifolds is an important source of examples in differential geometry. For instance, up to dimension 6, Goze and Khakimdjanov [15] classified the nilmanifolds admitting a symplectic form, whereas the question of which nilmanifolds admit an invariant complex structure was settled by Salamon in [21]. Moreover, there are five classes of 6-dimensional nilmanifolds which do not admit either symplectic forms or invariant complex structures, although Cavalcanti and Gualtieri [7] proved that all of them admit a generalized complex structure.

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Nilmanifolds with invariant structures have provided explicit examples of compact manifolds satisfying interesting and unusual geometrical and/or topological properties. For instance, the Kodaira-Thurston (nil)manifold [22] was the first example of a symplectic manifold with no Kähler metric, and more generally, Benson and Gordon proved [5] that a symplectic nilmanifold satisfies the Hard Lefschetz Condition (HLC) if and only if it is a torus. Regarding the HLC, given a symplectic manifold (M,ω), Mathieu [19] proved that any de Rham cohomology class of M has a symplectically harmonic representative (in the sense of Brylinski [6]) if and only if (M,ω) satisfies the HLC, and in [16] symplectic nilmanifolds were used to find the first examples of 6-dimensional compact manifolds that are symplectically flexible, giving in this way an affirmative answer to a question raised by Khesin and McDuff (see [24]). There are other interesting constructions in symplectic geometry where nilmanifolds are involved (see [23]). Nilmanifolds with invariant complex structure also play an important role in complex geometry (see the recent book [1]).

A key ingredient in many of the geometrical/topological applications of nilmanifolds is a result due to Nomizu [20], asserting that the Chevalley–Eilenberg complex $(\bigwedge^* \mathfrak{g}^*, d)$ of the Lie algebra $\mathfrak{g}$ underlying the nilmanifold $M = \Gamma \setminus G$ is quasi-isomorphic to the de Rham complex of M. In this paper we study the cohomological decomposition of complex nilmanifolds, with special attention to dimension 6, motivated by the study of the pureness and fullness properties that we explain below.

Let M be a $2n$-dimensional manifold and J an almost complex structure on M. Following [2], [18], let $H^k_{\mathrm{dR}}(M;\mathbb{C})$ be the (complex) kth de Rham cohomology group of M and, for any (p,q), denote by $H^{(p,q)}_J(M)$ the subgroup of $H^{p+q}_{\mathrm{dR}}(M;\mathbb{C})$ consisting of the de Rham cohomology classes of total degree $p+q$ that have a representative of bidegree (p,q) with respect to J. The almost complex structure J is said to be *complex-$\mathcal{C}^\infty$-pure-and-full at the kth stage* if there is a direct sum decomposition

$$H^k_{\mathrm{dR}}(M;\mathbb{C}) = \bigoplus_{p+q=k} H^{(p,q)}_J(M). \tag{1.1}$$

If the sum on the right hand side of (1.1) is direct, although not necessarily equal to $H^k_{\mathrm{dR}}(M;\mathbb{C})$, then J is called *complex-$\mathcal{C}^\infty$-pure at the kth stage*, and if

$$H^k_{\mathrm{dR}}(M;\mathbb{C}) = \sum_{p+q=k} H^{(p,q)}_J(M),$$

the sum not necessarily being direct, then J is said to be *complex-$\mathcal{C}^\infty$-full at the kth stage*.

Taking $H^{(p,q),(q,p)}_J(M)$ as the subgroup of $H^{p+q}_{\mathrm{dR}}(M;\mathbb{C})$ consisting of the classes of total degree $p+q$ that have a representative which is the sum of forms of bidegrees (p, q) and (q, p) with respect to J, one can consider the subgroups

$$H_J^{(p,q),(q,p)}(M)_{\mathbb{R}} = H_J^{(p,q),(q,p)}(M) \cap H^k_{\mathrm{dR}}(M;\mathbb{R})$$

of the (real) de Rham cohomology group $H^k_{\mathrm{dR}}(M;\mathbb{R})$ and define similar concepts of $\mathcal{C}^\infty$*-pure*, $\mathcal{C}^\infty$*-full* and $\mathcal{C}^\infty$*-pure-and-full* at the kth stage. Also similar subgroups $H^J_{(p,q)}(M)$ and definitions of *pure* or *full* almost complex structures can be given by using the space of currents instead of the space of differential forms, and the de Rham homology instead of the de Rham cohomology (for more details and related results see [2], [3], [4], [11], [12], [13], [14], [18]).

Some relations among several pureness and fullness conditions have been proved in the presence of additional geometric structures. For instance, by a result of Fino and Tomassini [14, Theorem 4.1], given an almost-Kähler structure (J, ω) on a compact manifold, if the almost complex structure J is C^∞-pure-and-full and the symplectic form ω satisfies the HLC, then J is also pure-and-full.

Of special interest is the case $k = 2$. Motivated by a question of Donaldson [10, Question 2], Li and Zhang studied in [18] almost complex manifolds (M, J) which are $\mathcal{C}^\infty$-pure-and-full (at the second stage), i.e.

$$H^2_{\mathrm{dR}}(M;\mathbb{R}) = H^+_J(M) \oplus H^-_J(M),$$

where $H^\pm_J(M)$ are the J-invariant and the J-anti-invariant cohomologies (see Section 2 for details). They showed that $H^-_J(M)$ measures the difference between the tamed cone and the compatible cone. Moreover, Drăghici, Li and Zhang proved in [11] that every compact 4-dimensional almost complex manifold is $\mathcal{C}^\infty$-pure-and-full.

It is well-known that if M is a compact Kähler manifold then (1.1) holds for any k [9], so the relation (1.1) can be seen as a generalization in (almost) complex geometry of the Hodge decomposition theorem for compact Kähler manifolds. In general, a compact complex manifold (M, J) of dimension greater than or equal to six may be neither complex-$\mathcal{C}^\infty$ pure nor full. Our main goal in this paper is to investigate which invariant complex structures J on nilmanifolds are complex-$\mathcal{C}^\infty$-pure-and-full at every stage.

In Section 2, we first show that a Nomizu type theorem holds for the spaces $H_J^{(p,q)}(M)$ of any nilmanifold M endowed with an invariant (almost) complex structure J. Proposition 2.2 gives conditions on the complex structures J that ensure complex-$\mathcal{C}^\infty$-pureness at some stages. Of special importance are the abelian and the complex parallelizable structures, because they are always complex-$\mathcal{C}^\infty$-pure at the second stage. On the other hand, Drăghici, Li and Zhang proved in [13, Proposition 2.7] that the product $(M_1\times M_2, J_1+J_2)$ of two compact almost complex manifolds (M_1, J_1) and (M_2, J_2) which are $\mathcal{C}^\infty$-pure-and-full is also $\mathcal{C}^\infty$-pure-and-full provided $b_1(M_1) = 0$ or $b_1(M_2) = 0$, and they asked if the statement holds without any assumption on the first Betti number. Using

Proposition 2.2, in Examples 2.6 and 2.7 we consider the Kodaira–Thurston manifold to construct some product manifolds which give a negative answer to the previous question.

In Section 3 we study pureness and fullness at every stage of any invariant complex structure on a nilmanifold of dimension 6, using the classification of complex structures given in [8]. Our results can be found in the tables given at the end of the section, in terms of the Lie algebra underlying the nilmanifold together with the parameters defining the complex structure. Several consequences are deduced from this general study. Firstly, apart from a complex torus, there are only two complex structures (up to isomorphism) which are complex-$\mathcal{C}^\infty$-pure-and-full at every stage (see Theorem 3.1). One of them is the Iwasawa manifold, whose complex structure is complex parallelizable, and its pureness and fullness were already known [3]. However, as far as we know, the other structure, which is abelian and lives on the real nilmanifold underlying the Iwasawa manifold, provides a new example of a complex structure being complex-$\mathcal{C}^\infty$-pure-and-full at every stage. Finally, as a consequence of our general study we also arrive at a duality result between pureness and fullness at different stages (see Proposition 3.2).

2. Pureness and fullness of complex nilmanifolds

Let $M = \Gamma \setminus G$ be a nilmanifold of dimension $2n$. An *invariant* complex structure J on M is a complex structure that comes from a left-invariant complex structure on the nilpotent Lie group G. Equivalently, J is an endomorphism $J\colon \mathfrak{g} \to \mathfrak{g}$ of the Lie algebra $\mathfrak{g}$ of G such that $J^2 = -\,\mathrm{Id}$ and satisfying the integrability condition

$$[JX, JY] = J[JX, Y] + J[X, JY] + [X, Y], \quad \text{for any } X, Y \in \mathfrak{g}.$$

Important classes of invariant complex structures on nilmanifolds are the *complex-parallelizable* structures, for which $[JX, Y] = J[X, Y]$, and the *abelian* structures, which satisfy the condition $[JX, JY] = [X, Y]$. A Lie algebra $\mathfrak{g}$ has a complex-parallelizable structure if and only if $\mathfrak{g}$ can be endowed with a complex Lie algebra structure. On a compact complex parallelizable nilmanifold there exists a global basis of holomorphic vector fields, the Iwasawa manifold being the first non-trivial example of this kind (see Section 3).

Let us denote by $\mathfrak{g}_{\mathbb{C}}$ the complexification of $\mathfrak{g}$ and by $\mathfrak{g}_{\mathbb{C}}^*$ its dual. Given an endomorphism $J\colon \mathfrak{g} \to \mathfrak{g}$ such that $J^2 = -\mathrm{Id}$, we denote by $\mathfrak{g}^{1,0}$ and $\mathfrak{g}^{0,1}$ the eigenspaces corresponding to the eigenvalues $\pm i$ of J as an endomorphism of $\mathfrak{g}_{\mathbb{C}}^*$, respectively. The decomposition $\mathfrak{g}_{\mathbb{C}}^* = \mathfrak{g}^{1,0} \oplus \mathfrak{g}^{0,1}$ induces a natural bigraduation on $\bigwedge^*(\mathfrak{g}_{\mathbb{C}}^*) = \oplus_{p,q} \bigwedge^{p,q}(\mathfrak{g}_{\mathbb{C}}^*) = \oplus_{p,q} \bigwedge^p(\mathfrak{g}^{1,0}) \otimes \bigwedge^q(\mathfrak{g}^{0,1})$. If d denotes the Chevalley-Eilenberg differential of the Lie algebra, we shall also denote by d its extension to the complexified exterior algebra, i.e. $d\colon \bigwedge^*(\mathfrak{g}_{\mathbb{C}}^*) \to \bigwedge^{*+1}(\mathfrak{g}_{\mathbb{C}}^*)$.

It is well-known that J is a complex structure if and only if $d(\mathfrak{g}^{1,0}) \subset \bigwedge^{2,0}(\mathfrak{g}^*_{\mathbb{C}}) \oplus \bigwedge^{1,1}(\mathfrak{g}^*_{\mathbb{C}})$. Notice that abelian structures satisfy $d(\mathfrak{g}^{1,0}) \subset \bigwedge^{1,1}(\mathfrak{g}^*_{\mathbb{C}})$, and they are characterized by the fact that the complex Lie algebra $\mathfrak{g}^{1,0}$ is abelian, whereas complex-parallelizable structures satisfy $d(\mathfrak{g}^{1,0}) \subset \bigwedge^{2,0}(\mathfrak{g}^*_{\mathbb{C}})$.

Salamon proved in [21] the following equivalent condition for the endomorphism J to be integrable: J is a complex structure if and only if $\mathfrak{g}^{1,0}$ has a basis $\{\omega^j\}_{j=1}^n$ such that $d\omega^1 = 0$ and

$$d\omega^j \in \mathcal{I}(\omega^1, \ldots, \omega^{j-1}), \quad \text{for } j = 2, \ldots, n, \tag{2.1}$$

where $\mathcal{I}(\omega^1, \ldots, \omega^{j-1})$ is the ideal in $\bigwedge^*(\mathfrak{g}^*_{\mathbb{C}})$ generated by $\{\omega^1, \ldots, \omega^{j-1}\}$. From now on, we shall denote $\omega^j \wedge \omega^k$ and $\omega^j \wedge \overline{\omega^k}$ simply by ω^{jk} and $\omega^{j\overline{k}}$, respectively.

Let J be an invariant (almost) complex structure on a $2n$-dimensional nilmanifold M, and let ν be a volume element on M induced by a bi-invariant one on the Lie group G such that, after rescaling, M has volume equal to 1. Given any k-form $\alpha \in \Omega^k(M)$, one can define an element $\widetilde{\alpha} \in \bigwedge^k(\mathfrak{g}^*)$ by

$$\widetilde{\alpha}(X_1, \ldots, X_k) = \int_{m \in M} \alpha_m(X_1|_m, \ldots, X_k|_m)\, \nu, \quad \text{for } X_1, \ldots, X_k \in \mathfrak{g},$$

where $X_j|_m$ is the value at the point $m \in M$ of the projection on M of the left-invariant vector field X_j on the Lie group G. This defines a linear map $\sim\colon \Omega^k(M) \to \bigwedge^k(\mathfrak{g}^*)$, which is known as the symmetrization process. It is clear that $\widetilde{\alpha} = \alpha$ for any invariant k-form α on M. Moreover, $\widetilde{d\alpha} = d\widetilde{\alpha}$ for any form α on M, that is, the map $\sim$ commutes with the differential d. Extending the symmetrization process to the space of complex forms $\Omega^*_{\mathbb{C}}(M)$, one has that if α is a form of pure type (p, q) then $\widetilde{\alpha}$ is again of pure type (p, q) (see [8] for more details and other references about the symmetrization process and its applications).

By [20] we know that the natural inclusion $(\bigwedge^*(\mathfrak{g}^*_{\mathbb{C}}), d) \hookrightarrow (\Omega^*_{\mathbb{C}}(M), d)$ induces an isomorphism $\iota\colon H^*(\mathfrak{g}_{\mathbb{C}}) \to H^*_{\mathrm{dR}}(M; \mathbb{C})$ between the Lie algebra cohomology and the (complex) de Rham cohomology of the nilmanifold. This implies that any closed complex k-form α on M is cohomologous to the invariant k-form $\widetilde{\alpha}$, that is, the inverse map of ι is induced by the symmetrization process. Now, if we let

$$\begin{aligned} H_J^{(p,q)}(\mathfrak{g}) = \{&\mathbf{a} \in H^{p+q}(\mathfrak{g}_{\mathbb{C}}) \mid \text{there exists a closed } (p,q)\text{-form } \alpha \\ &\text{such that } [\alpha] = \mathbf{a}\}, \\ H_J^{(p,q)}(M) = \{&\mathbf{a} \in H^{p+q}_{\mathrm{dR}}(M; \mathbb{C}) \mid \text{there exists a closed form } \alpha \text{ of bidegree } (p,q) \\ &\text{such that } [\alpha] = \mathbf{a}\}, \end{aligned}$$

then the following Nomizu type result comes straightforward (see also [4]).

Proposition 2.1. *Let J be an invariant (almost) complex structure on a nilmanifold M. Then, the restriction to $H_J^{(p,q)}(\mathfrak{g})$ of the isomorphism*

$$\iota\colon H^{p+q}(\mathfrak{g}_{\mathbb{C}}) \to H^{p+q}_{\mathrm{dR}}(M;\mathbb{C})$$

is an isomorphism onto $H_J^{(p,q)}(M)$, with inverse mapping

$$\sim\colon H_J^{(p,q)}(M) \to H_J^{(p,q)}(\mathfrak{g})$$

given by the symmetrization process.

By [17, Lemma 2], any invariant complex structure J on a $2n$-dimensional nilmanifold M is complex-$\mathcal{C}^\infty$-pure at the first stage, and if $b_1(M) = 2n-1$ then J is not complex-$\mathcal{C}^\infty$-full at the first stage. In the following result we study pureness at higher stages.

Proposition 2.2. *Let J be an invariant complex structure on a $2n$-dimensional nilmanifold M. Then*:

(a) $H_J^{(n,0)}(M) \cap H_J^{(0,n)}(M) = \{[0]\}$, *and* $H_J^{(n,0)}(M) \cap H_J^{(n-l,l)}(M) = \{[0]\} = H_J^{(n-l,l)}(M) \cap H_J^{(0,n)}(M)$, *for any* $1 \le l \le n-1$.

(b) *If J is abelian or complex parallelizable, then for any $2 \le k \le n$ the complex structure is complex-$\mathcal{C}^\infty$-pure at the kth stage if and only if the sum $H_J^{(k-1,1)}(M) + \ldots + H_J^{(1,k-1)}(M)$ is direct; in particular, J is always complex-$\mathcal{C}^\infty$-pure at the second stage.*

Proof. By Proposition 2.1, we reduce the proof to the level of the Lie algebra $\mathfrak{g}$. Let us first see (a). Fix some $l = 1, \ldots, n$ and consider $\mathbf{a} \in H_J^{(n,0)}(\mathfrak{g}) \cap H_J^{(n-l,l)}(\mathfrak{g})$. Then, there exist closed elements $\beta \in \bigwedge^{n,0}(\mathfrak{g}_{\mathbb{C}}^*)$ and $\gamma \in \bigwedge^{n-l,l}(\mathfrak{g}_{\mathbb{C}}^*)$ satisfying $\beta - \gamma = d\alpha$ for some $\alpha \in \bigwedge^{n-1}(\mathfrak{g}_{\mathbb{C}}^*)$. Due to the action of $d = \partial + \overline{\partial}$ on the elements of total degree $n-1$, necessarily $\beta = \partial\alpha^{n-1,0}$, where $\alpha^{n-1,0}$ is the component of α of bidegree $(n-1,0)$. Notice that the integrability condition (2.1) implies that there is a $(1,0)$-basis $\{\omega^j\}_{j=1}^n$ such that the $(n,0)$-form $\omega^{1\ldots n}$ is closed, so there exists $\lambda \in \mathbb{C}$ such that $\beta = \lambda\,\omega^{1\ldots n}$. Therefore,

$$|\lambda|^2\,\omega^{1\ldots n\overline{1}\ldots\overline{n}} = \beta \wedge \overline{\beta} = \partial\alpha^{n-1,0} \wedge \overline{\beta} = d(\alpha^{n-1,0} \wedge \overline{\beta}).$$

But the Lie algebra $\mathfrak{g}$ is unimodular and therefore $b_{2n}(\mathfrak{g}) = 1$, so there cannot exist a non-zero element of top degree which is exact, that is, necessarily $\lambda = 0$ and thus $\beta = 0$. This implies that $\mathbf{a} = [\gamma] = -[d\alpha]$, i.e. $\mathbf{a} = 0$, and so $H_J^{(n,0)}(\mathfrak{g}) \cap H_J^{(n-l,l)}(\mathfrak{g}) = \{[0]\}$ for any $l = 1, \ldots, n$. The proof of $H_J^{(n-l,l)}(M) \cap H_J^{(0,n)}(M) = \{[0]\}$ for any $l = 1, \ldots, n-1$ is similar.

For the proof of (b), note that one of the implications is trivial due to the definition of complex-$\mathcal{C}^\infty$-pure. For the other one, it suffices to check that for each $2 \le k \le n$, we have that $H_J^{(k,0)}(\mathfrak{g}) \cap H_J^{(0,k)}(\mathfrak{g}) = \{[0]\}$ and,

$$H_J^{(k,0)}(\mathfrak{g}) \cap H_J^{(k-l,l)}(\mathfrak{g}) = \{[0]\}, \qquad H_J^{(0,k)}(\mathfrak{g}) \cap H_J^{(k-l,l)}(\mathfrak{g}) = \{[0]\},$$

for any $l = 1, \dots, k-1$. The case $k = n$ is a consequence of (a).

For $k = 2, \dots, n-1$, we prove first that any class $\mathbf{a} \in H_J^{(k,0)}(\mathfrak{g}) \cap H_J^{(k-l,l)}(\mathfrak{g})$ is zero. Let $\beta \in \bigwedge^{k,0}(\mathfrak{g}_{\mathbb{C}}^*)$ and $\gamma \in \bigwedge^{k-l,l}(\mathfrak{g}_{\mathbb{C}}^*)$ be closed forms such that $[\beta] = \mathbf{a} = [\gamma]$. Then, there exists $\alpha = \alpha^{k-1,0} + \ldots + \alpha^{0,k-1} \in \bigwedge^{k-1}(\mathfrak{g}_{\mathbb{C}}^*)$ satisfying $d\alpha = \beta - \gamma$, which implies $\beta = \partial\alpha^{k-1,0}$. However, when J is abelian the Lie algebra differential d satisfies $d(\mathfrak{g}_{\mathbb{C}}^*) \subset \bigwedge^{1,1}(\mathfrak{g}_{\mathbb{C}}^*)$, thus $d\alpha^{k-1,0} \in \bigwedge^{k-1,1}(\mathfrak{g}_{\mathbb{C}}^*)$ and $\beta = \partial\alpha^{k-1,0} = 0$. In the same way, when J is complex parallelizable one has $d(\mathfrak{g}^{1,0}) \subset \bigwedge^{2,0}(\mathfrak{g}_{\mathbb{C}}^*)$, which in particular implies $\overline{\partial}(\bigwedge^{k-1,0}(\mathfrak{g}_{\mathbb{C}}^*)) = 0$, and therefore $\beta = \partial\alpha^{k-1,0} = d\alpha^{k-1,0}$. We conclude that, in any case, the form β is cohomologous to zero, i.e. $\mathbf{a} = [\beta] = 0$. A similar argument allows us to prove that $H_J^{(0,k)}(\mathfrak{g}) \cap H_J^{(k-l,l)}(\mathfrak{g}) = \{[0]\}$ and $H_J^{(k,0)}(\mathfrak{g}) \cap H_J^{(0,k)}(\mathfrak{g}) = \{[0]\}$. $\square$

In [18], Li and Zhang introduced for any almost complex structure J on a manifold M, the concept of $\mathcal{C}^\infty$-pure-and-full associated to the second (real) de Rham cohomology group $H^2_{\mathrm{dR}}(M;\mathbb{R})$. Let

$$\begin{aligned} H_J^+(M) = \{\mathbf{a} \in H^2_{\mathrm{dR}}(M;\mathbb{R}) \mid &\text{ there exists a closed } J\text{-invariant } \alpha \\ &\text{such that } [\alpha] = \mathbf{a}\}, \\ H_J^-(M) = \{\mathbf{a} \in H^2_{\mathrm{dR}}(M;\mathbb{R}) \mid &\text{ there exists a closed } J\text{-anti-invariant } \alpha \\ &\text{such that } [\alpha] = \mathbf{a}\}. \end{aligned}$$

If $H_J^+(M) \cap H_J^-(M) = \{[0]\}$ then J is called $\mathcal{C}^\infty$-*pure*, and if $H^2_{\mathrm{dR}}(M;\mathbb{R}) = H_J^+(M) + H_J^-(M)$ then J is said to be $\mathcal{C}^\infty$-*full*. The almost complex structure is called $\mathcal{C}^\infty$-*pure-and-full* when $H^2_{\mathrm{dR}}(M;\mathbb{R}) = H_J^+(M) \oplus H_J^-(M)$.

Remark 2.3. Notice that $H_J^+(M) = H_J^{(1,1)}(M) \cap H^2_{\mathrm{dR}}(M;\mathbb{R})$ and $H_J^-(M) = H_J^{(2,0),(0,2)}(M) \cap H^2_{\mathrm{dR}}(M;\mathbb{R})$. As it is observed in [3, Remark 2], being complex-$\mathcal{C}^\infty$-full at the second stage is a stronger condition that being $\mathcal{C}^\infty$-full. Moreover, if J is integrable then complex-$\mathcal{C}^\infty$-pure at the second stage implies $\mathcal{C}^\infty$-pure.

Remark 2.4. Complex nilmanifolds are used in [2] to prove that being $\mathcal{C}^\infty$-pure and being $\mathcal{C}^\infty$-full are non-related properties. Moreover, Angella, Tomassini and Zhang construct in [4, Proposition 4.1] an almost-Kähler structure (J, ω, g) on the nilmanifold underlying the Iwasawa manifold which is $\mathcal{C}^\infty$-pure but not $\mathcal{C}^\infty$-full.

Every compact 4-dimensional almost complex manifold is $\mathcal{C}^\infty$-pure-and-full (see [11]). In [13] Drăghici, Li and Zhang proved the following result:

Proposition 2.5 [13, Proposition 2.7]. *Suppose (M_1, J_1) and (M_2, J_2) are compact almost complex manifolds, both $\mathcal{C}^\infty$-pure-and-full. Assume $b_1(M_1) = 0$ or $b_1(M_2) = 0$. Then $(M_1 \times M_2, J_1 + J_2)$ is $\mathcal{C}^\infty$-pure-and-full.*

They asked if the statement holds without any assumption on b_1. As a consequence of Proposition 2.2, we next show that even in the complex case, the previous result does not hold if both $b_1(M_1)$ and $b_1(M_2)$ are not zero. For the construction we will consider the Kodaira–Thurston manifold [22].

Recall that the Kodaira–Thurston manifold KT is endowed with the invariant complex structure defined by the equations

$$d\omega^1 = 0, \quad d\omega^2 = \omega^{1\bar{1}}. \tag{2.2}$$

This complex structure is abelian and any other invariant complex structure on KT is isomorphic to (2.2). Since KT has (real) dimension 4, we have by [11] that it is $\mathcal{C}^\infty$-pure-and-full (this can also be seen directly using Proposition 2.1).

A complex m-dimensional torus $\mathbb{T}^m$ is trivially $\mathcal{C}^\infty$-pure-and-full because it is Kähler. In the following example we show that the product $KT \times \mathbb{T}^m$ is not $\mathcal{C}^\infty$-full.

Example 2.6. For any $m \geq 1$, the compact complex manifold $N = KT \times \mathbb{T}^m$ is $\mathcal{C}^\infty$-pure but not $\mathcal{C}^\infty$-full.

Writing the complex structure equations on $\mathbb{T}^m$ as $d\omega^l = 0$, for $3 \leq l \leq m+2$, we have that the structure equations for the complex nilmanifold N are

$$d\omega^1 = 0, \quad d\omega^2 = \omega^{1\bar{1}}, \quad d\omega^3 = \ldots = d\omega^{m+2} = 0. \tag{2.3}$$

Let us first see the case $m = 1$. We get

$$\begin{aligned} H_J^+(N) &= H_J^{(1,1)}(N) \cap H^2_{\mathrm{dR}}(N;\mathbb{R}) \\ &= \langle [\omega^{1\bar{2}} - \omega^{2\bar{1}}], [i\,\omega^{1\bar{2}} + i\,\omega^{2\bar{1}}], [\omega^{1\bar{3}} - \omega^{3\bar{1}}], [i\,\omega^{1\bar{3}} + i\,\omega^{3\bar{1}}], [i\,\omega^{3\bar{3}}]\rangle, \\ H_J^-(N) &= H_J^{(2,0),(0,2)}(N) \cap H^2_{\mathrm{dR}}(N;\mathbb{R}) \\ &= \langle [\omega^{12} + \omega^{\bar{1}\bar{2}}], [i\,\omega^{12} - i\,\omega^{\bar{1}\bar{2}}], [\omega^{13} + \omega^{\bar{1}\bar{3}}], [i\,\omega^{13} - i\,\omega^{\bar{1}\bar{3}}]]\rangle, \end{aligned}$$

since Proposition 2.1 also holds for $H_J^{(2,0),(0,2)}(N)$. However, $H^2_{\mathrm{dR}}(N;\mathbb{R}) = H_J^+(N) \oplus H_J^-(N) \oplus \langle [\omega^{23} + \omega^{2\bar{3}} - \omega^{3\bar{2}} + \omega^{\bar{2}\bar{3}}], [i\,\omega^{23} - i\,\omega^{2\bar{3}} - i\,\omega^{3\bar{2}} - i\,\omega^{\bar{2}\bar{3}}]\rangle$, that is, the complex product manifold N is $\mathcal{C}^\infty$-pure but not $\mathcal{C}^\infty$-full.

In general, i.e. for any $m \geq 1$, since (2.3) implies that the complex structure on N is abelian, by Proposition 2.2(b) and Remark 2.3 we have that N is $\mathcal{C}^\infty$-pure because it is complex-$\mathcal{C}^\infty$-pure at the second stage. However, N is not $\mathcal{C}^\infty$-full because the de Rham cohomology classes

$$[\omega^{2l} + \omega^{2\bar{l}} - \omega^{l\bar{2}} + \omega^{\bar{2}\bar{l}}], \quad [i\,\omega^{2l} - i\,\omega^{2\bar{l}} - i\,\omega^{l\bar{2}} - i\,\omega^{\bar{2}\bar{l}}], \quad 3 \leq l \leq m+2,$$

do not belong to the sum $H_J^+(N) \oplus H_J^-(N)$: in fact, this is a direct consequence of the fact that the invariant real exact 2-forms on N belong to the space generated by $i\,\omega^{1\bar{1}}$.

Another example in (real) dimension 8 can be obtained by using the product of two Kodaira–Thurston manifolds.

Example 2.7. The compact complex manifold $N = KT \times KT$ is $\mathcal{C}^\infty$-pure but not $\mathcal{C}^\infty$-full.

We write the complex structure equations for N as

$$d\omega^1 = d\omega^2 = 0, \quad d\omega^3 = \omega^{1\bar{1}}, \quad d\omega^4 = \omega^{2\bar{2}}.$$

Notice that the complex structure of N is again abelian, so by Proposition 2.2(b) and Remark 2.3 we have that N is $\mathcal{C}^\infty$-pure. However, N is not $\mathcal{C}^\infty$-full because the de Rham cohomology classes

$$\begin{array}{ll} [\omega^{14} + \omega^{1\bar{4}} - \omega^{4\bar{1}} + \omega^{\bar{1}\bar{4}}], & [\omega^{23} + \omega^{2\bar{3}} - \omega^{3\bar{2}} + \omega^{\bar{2}\bar{3}}], \\ [\omega^{34} + \omega^{3\bar{4}} - \omega^{4\bar{3}} + \omega^{\bar{3}\bar{4}}], & [i\,\omega^{14} - i\,\omega^{1\bar{4}} - i\,\omega^{4\bar{1}} - i\,\omega^{\bar{1}\bar{4}}], \\ [i\,\omega^{23} - i\,\omega^{2\bar{3}} - i\,\omega^{3\bar{2}} - i\,\omega^{\bar{2}\bar{3}}], & [i\,\omega^{34} - i\,\omega^{3\bar{4}} - i\,\omega^{4\bar{3}} - i\,\omega^{\bar{3}\bar{4}}], \end{array}$$

do not belong to the direct sum $H_J^+(N) \oplus H_J^-(N)$. This is due to the fact that the invariant real exact 2-forms on N belong to the space generated by $i\,\omega^{1\bar{1}}$ and $i\,\omega^{2\bar{2}}$.

Notice that in Examples 2.6 and 2.7 the first Betti numbers are far from being zero; in fact, $b_1(KT) = 3$ and $b_1(\mathbb{T}^m) = 2m$. We do not know if the statement of Proposition 2.5 holds if $b_1(M_1) = 1$ or $b_1(M_2) = 1$.

3. Cohomological decomposition of 6-dimensional complex nilmanifolds

In this section we study which invariant complex structures on 6-dimensional nilmanifolds are complex-$\mathcal{C}^\infty$-pure or full at every stage. For this purpose we use the classification of invariant complex structures given in [8].

Recall that a 6-dimensional nilmanifold $M = \Gamma \setminus G$ admits an invariant complex structure J if and only if its underlying Lie algebra $\mathfrak{g}$ is isomorphic to one in the following list [21]:

$$\begin{array}{ll} \mathfrak{h}_1 = (0,0,0,0,0,0), & \mathfrak{h}_{10} = (0,0,0,12,13,14), \\ \mathfrak{h}_2 = (0,0,0,0,12,34), & \mathfrak{h}_{11} = (0,0,0,12,13,14+23), \\ \mathfrak{h}_3 = (0,0,0,0,0,12+34), & \mathfrak{h}_{12} = (0,0,0,12,13,24), \\ \mathfrak{h}_4 = (0,0,0,0,12,14+23), & \mathfrak{h}_{13} = (0,0,0,12,13+14,24), \\ \mathfrak{h}_5 = (0,0,0,0,13+42,14+23), & \mathfrak{h}_{14} = (0,0,0,12,14,13+42), \\ \mathfrak{h}_6 = (0,0,0,0,12,13), & \mathfrak{h}_{15} = (0,0,0,12,13+42,14+23), \\ \mathfrak{h}_7 = (0,0,0,12,13,23), & \mathfrak{h}_{16} = (0,0,0,12,14,24), \\ \mathfrak{h}_8 = (0,0,0,0,0,12), & \mathfrak{h}_{19}^- = (0,0,0,12,23,14-35), \\ \mathfrak{h}_9 = (0,0,0,0,12,14+25), & \mathfrak{h}_{26}^+ = (0,0,12,13,23,14+25), \end{array}$$

where, for instance, the notation $\mathfrak{h}_2 = (0,0,0,0,12,34)$ means that there exists a basis $\{e^i\}_{i=1}^6$ of real 1-forms such that $de^1 = de^2 = de^3 = de^4 = 0$, $de^5 = e^1 \wedge e^2$, $de^6 = e^3 \wedge e^4$.

It is well-known that in dimension 6 there are, up to isomorphism, two complex-parallelizable structures defined by the complex equations

$$d\omega^1 = d\omega^2 = 0, \quad d\omega^3 = \rho\,\omega^{12}, \tag{3.1}$$

with $\rho = 0$ or 1, whose Lie algebras are $\mathfrak{h}_1$ (for $\rho = 0$) and $\mathfrak{h}_5$ (for $\rho = 1$), where the latter case corresponds to the Iwasawa manifold.

The remaining complex structures in dimension 6 are parametrized, up to equivalence, by the following three families [8]:

$$\text{Family I}: \begin{cases} d\omega^1 = d\omega^2 = 0, \\ d\omega^3 = \rho\,\omega^{12} + \omega^{1\bar{1}} + \lambda\,\omega^{1\bar{2}} + D\,\omega^{2\bar{2}}, \end{cases} \tag{3.2}$$

where $\rho \in \{0,1\}$, $\lambda \in \mathbb{R}^{\geq 0}$ and $D \in \mathbb{C}$ with $\mathfrak{Im}\,D \geq 0$;

$$\text{Family II}: \begin{cases} d\omega^1 = 0, \\ d\omega^2 = \omega^{1\bar{1}}, \\ d\omega^3 = \rho\,\omega^{12} + B\,\omega^{1\bar{2}} + c\,\omega^{2\bar{1}}, \end{cases} \tag{3.3}$$

where $\rho \in \{0,1\}$, $B \in \mathbb{C}$, $c \in \mathbb{R}^{\geq 0}$ and $(\rho, B, c) \neq (0,0,0)$; and

$$\text{Family III}: \begin{cases} d\omega^1 = 0, \\ d\omega^2 = \omega^{13} + \omega^{1\bar{3}}, \\ d\omega^3 = \varepsilon\, i\,\omega^{1\bar{1}} \pm i(\omega^{1\bar{2}} - \omega^{2\bar{1}}), \end{cases} \tag{3.4}$$

where $\varepsilon \in \{0,1\}$.

A complex structure J is abelian if and only if it belongs to Families I or II with $\rho = 0$. Moreover, Family I corresponds to complex structures on $\mathfrak{h}_2, \dots, \mathfrak{h}_6$ or $\mathfrak{h}_8$, Family II corresponds to complex structures on $\mathfrak{h}_7$ or $\mathfrak{h}_9, \dots, \mathfrak{h}_{16}$, and Family III to $\mathfrak{h}_{19}^-$ or $\mathfrak{h}_{26}^+$. The possible values of the complex parameters for each Lie algebra appear in the tables below (they provide actually a classification of complex structures as proved in [8]).

By Nomizu theorem one can compute de Rham cohomology groups in terms of the complex structure equations (3.2), (3.3) and (3.4). From now on, the notation $\delta_{\text{expression}}$ means that $\delta_{\text{expression}} = 1$ if expression $= 0$ is satisfied, and $\delta_{\text{expression}} = 0$ otherwise.

Cohomology groups of complex nilmanifolds M in Family I.

$H^1_{\text{dR}}(M;\mathbb{C}) = \langle [\omega^1],\, [\omega^2],\, [\omega^{\bar{1}}],\, [\omega^{\bar{2}}],\, \delta_\rho\delta_\lambda\delta_{\mathfrak{Im}\,D}[\omega^3 + \omega^{\bar{3}}] \rangle$,

$$
H^2_{dR}(M;\mathbb{C}) = \begin{cases}
\text{if} \quad D = 0 : \\
\quad \Big\langle [\omega^{13}],\ [\omega^{1\bar{2}}],\ [\omega^{2\bar{2}}], [\omega^{\bar{1}\bar{2}}],\ [\omega^{\bar{1}\bar{3}}], \\
\quad [\rho\,\omega^{23} - \omega^{3\bar{1}} - \lambda\,\omega^{3\bar{2}}], [\omega^{1\bar{3}} + \lambda\,\omega^{2\bar{3}} + \rho\,\omega^{\bar{2}\bar{3}}], \\
\quad \begin{cases} [\omega^{12}],\ \delta_\lambda[\omega^{2\bar{1}}],\ \delta_\lambda[\omega^{23} + \omega^{2\bar{3}}],\ \delta_\lambda[\omega^{3\bar{2}} - \omega^{\bar{2}\bar{3}}],\ \rho = 0 \\ [\omega^{2\bar{1}}],\ \delta_{\lambda-1}[\omega^{2\bar{3}} + \omega^{3\bar{1}} + \omega^{\bar{2}\bar{3}}],\ \rho = 1 \end{cases} \Big\rangle, \\
\\
\text{if} \quad D \neq 0 : \\
\quad \Big\langle [\omega^{1\bar{2}}],\ [\omega^{\bar{1}\bar{2}}],\ [\lambda\bar{D}\,\omega^{13} + |D|^2\omega^{23} + |D|^2\omega^{2\bar{3}} - \rho D\,\omega^{\bar{1}\bar{3}}], \\
\quad [\bar{D}\,\omega^{13} + D\,\omega^{1\bar{3}} + \lambda\,D\,\omega^{2\bar{3}} + \rho\,D\,\omega^{\bar{2}\bar{3}}], \\
\quad [\rho\bar{D}\,\omega^{13} + |D|^2\omega^{3\bar{2}} - \lambda D\,\omega^{\bar{1}\bar{3}} - |D|^2\omega^{\bar{2}\bar{3}}], \\
\quad [\rho\,\overline{D}\,\omega^{23} - \overline{D}\,\omega^{3\bar{1}} - \lambda\,\overline{D}\,\omega^{3\bar{2}} + D\,\omega^{\bar{1}\bar{3}}], \\
\quad \begin{cases} [\omega^{12}],\ \delta_\lambda[\omega^{2\bar{1}}],\ (\delta_{\lambda-1} + \delta_\lambda\delta_{\mathfrak{Im}\,D})[\omega^{2\bar{2}}], \quad \rho = 0 \\ [\omega^{2\bar{1}}],\ [\omega^{2\bar{2}}], \quad \rho = 1 \end{cases} \Big\rangle,
\end{cases}
$$

$$
\begin{aligned}
H^3_{dR}(M;\mathbb{C}) = \Big\langle & [\omega^{123}],\ [\omega^{13\bar{2}}],\ [\omega^{2\bar{1}\bar{3}}],\ [\omega^{\bar{1}\bar{2}\bar{3}}],\ [\omega^{12\bar{3}} - \rho\,\omega^{2\bar{2}\bar{3}}],\ [\omega^{23\bar{1}} + \lambda\,\omega^{2\bar{2}\bar{3}}], \\
& [\omega^{23\bar{2}} - \omega^{2\bar{2}\bar{3}}],\ [\omega^{1\bar{1}\bar{3}} - \overline{D}\,\omega^{2\bar{2}\bar{3}}], [\omega^{1\bar{2}\bar{3}} + \lambda\,\omega^{2\bar{2}\bar{3}}],\ [\omega^{3\bar{1}\bar{2}} - \rho\,\omega^{2\bar{2}\bar{3}}], \\
& \begin{cases} [\omega^{12\bar{2}}],\ [\omega^{13\bar{3}}],\ [\omega^{2\bar{1}\bar{2}}],\ [\omega^{3\bar{1}\bar{3}}], \quad \rho = \lambda = D = 0, \\ [\omega^{12\bar{2}}],\ [\omega^{13\bar{3}} + \omega^{23\bar{3}} - \omega^{3\bar{1}\bar{3}} - \omega^{3\bar{2}\bar{3}}], \quad \rho = 1,\ D = 0,\ \lambda = 1 \end{cases} \Big\rangle,
\end{aligned}
$$

$$
\begin{aligned}
H^4_{dR}(M;\mathbb{C}) = \Big\langle & [\omega^{123\bar{2}}],\ [\omega^{2\bar{1}\bar{2}\bar{3}}],\ [\omega^{123\bar{3}} + \rho\,(\omega^{23\bar{2}\bar{3}} - \omega^{3\bar{1}\bar{2}\bar{3}})], \\
& [\rho\,\lambda\,\omega^{123\bar{3}} - \omega^{13\bar{2}\bar{3}} + (\rho - 1)(\omega^{23\bar{1}\bar{3}} + \lambda\,\omega^{23\bar{2}\bar{3}})], \\
& \begin{cases} [\omega^{12\bar{1}\bar{3}}],\ [\omega^{13\bar{1}\bar{2}}],\ [\omega^{3\bar{1}\bar{2}\bar{3}}],\ \delta_\lambda[\omega^{13\bar{2}\bar{3}}], \delta_\lambda\delta_D[\omega^{12\bar{2}\bar{3}}], \\ \delta_\lambda\delta_D[\omega^{23\bar{1}\bar{2}}],\ \delta_{\mathfrak{Im}\,D}[\omega^{13\bar{1}\bar{3}} - D\,\omega^{23\bar{2}\bar{3}}], \quad \rho = 0 \\ [\omega^{23\bar{1}\bar{2}}],\ [\overline{D}\,\omega^{123\bar{3}} + \omega^{13\bar{1}\bar{3}} - D\,\omega^{3\bar{1}\bar{2}\bar{3}}], \\ [\omega^{23\bar{1}\bar{3}} + \lambda\,\omega^{3\bar{1}\bar{2}\bar{3}}],\ \delta_{\lambda-1}[\omega^{13\bar{1}\bar{2}}], \\ (\delta_D\delta_{\lambda-1} + (1 - \delta_{\lambda-1}))[\omega^{12\bar{2}\bar{3}}], \quad \rho = 1 \end{cases} \Big\rangle,
\end{aligned}
$$

$$
H^5_{dR}(M;\mathbb{C}) = \langle [\omega^{123\bar{1}\bar{3}}],\ [\omega^{123\bar{2}\bar{3}}],\ [\omega^{13\bar{1}\bar{2}\bar{3}}],\ [\omega^{23\bar{1}\bar{2}\bar{3}}],\ \delta_\rho\delta_\lambda\delta_{\mathfrak{Im}\,D}[\omega^{12\bar{1}\bar{2}\bar{3}}] \rangle.
$$

Cohomology groups of complex nilmanifolds M in Family II.

$$H^1_{\mathrm{dR}}(M;\mathbb{C}) = \langle [\omega^1],\, [\omega^{\bar1}],\, [\omega^2+\omega^{\bar2}],\, \delta_\rho\delta_{c-|B|}[c\,\omega^3 + B\,\omega^{\bar3}]\rangle,$$

$$H^2_{\mathrm{dR}}(M;\mathbb{C}) = \begin{cases} \text{if}\quad \rho=0: \\ \quad \langle [\omega^{12}],\, [\omega^{\bar1\bar2}],\, [\omega^{13} - c\,\omega^{2\bar2} - \omega^{\bar1\bar3}], \\ \quad [\omega^{1\bar3} + B\,\omega^{2\bar2} + B\,\omega^{3\bar1}],\, [c\,\omega^{3\bar1} - B\,\omega^{\bar1\bar3}], \\ \quad\quad \delta_{c-|B|}[\omega^{2\bar1}],\, \delta_{c-|B|}[\omega^{23} + \omega^{2\bar3} - \omega^{3\bar2} + \omega^{\bar2\bar3}]\rangle, \\ \text{if}\quad \rho=1: \\ \quad \Bigg\langle [\omega^{1\bar2}],\, [\omega^{2\bar1}],\, [\omega^{13} - c\,\omega^{2\bar2} - \omega^{\bar1\bar3}], \\ \quad \begin{cases} [\omega^{1\bar3} + \omega^{2\bar2}],\, [\omega^{2\bar2} + \omega^{3\bar1}], \\ \quad \delta_c[\omega^{13}],\, \delta_c[\omega^{23} - \omega^{3\bar2}],\, \delta_c[\omega^{2\bar3} + \omega^{\bar2\bar3}], \quad B=1 \\ [(B-1)\,\omega^{1\bar3} + (|B|^2-1)\,\omega^{2\bar2} + (\overline{B}-1)\,\omega^{3\bar1}], \\ \quad [c\,\omega^{2\bar2} + c\,\omega^{3\bar1} - (B-1)\,\omega^{\bar1\bar3}], \\ \quad \delta_{c-|B-1|}[c\,\omega^{23} + (B-1)\,\omega^{2\bar3} \\ \quad\quad -c\,\omega^{3\bar2} + (B-1)\,\omega^{\bar2\bar3}], \quad B\neq 1 \end{cases} \Bigg\rangle, \end{cases}$$

$$H^3_{\mathrm{dR}}(M;\mathbb{C}) = \Bigg\langle [\omega^{123}],\, [\omega^{\bar1\bar2\bar3}],\, \delta_{c-|B-\rho|}[\omega^{2\bar1\bar2}],$$
$$\delta_{c-|B-\rho|}[(B-\rho)\,\omega^{13\bar3} + \overline{B}\,(B-\rho)\,\omega^{23\bar2} - c\,B\,\omega^{2\bar2\bar3} - c\,\omega^{3\bar1\bar3}],$$
$$\begin{cases} [\omega^{12\bar3}],\, [\omega^{3\bar1\bar2}],\, [\overline{B}\,\omega^{13\bar2} - c\,\omega^{1\bar2\bar3}], \\ \quad \delta_B[\omega^{23\bar1}],\, (1-\delta_B)[c\,\omega^{1\bar2\bar3} - \overline{B}\omega^{2\bar1\bar3}], \quad \rho=0, \\ [\omega^{13\bar2} + c\,\omega^{3\bar1\bar2}],\, [\omega^{23\bar1} + B\,\omega^{3\bar1\bar2}],\, [\omega^{1\bar2\bar3} + \overline{B}\,\omega^{3\bar1\bar2}], \\ \quad [\omega^{2\bar1\bar3} + c\,\omega^{3\bar1\bar2}],\, \delta_c\delta_{B-1}[\omega^{12\bar2}], \\ \quad \delta_c\delta_{B-1}[\omega^{13\bar3} + \omega^{23\bar2}],\, \delta_c\delta_{B-1}[\omega^{23\bar3} - \omega^{3\bar2\bar3}], \\ \quad \delta_c\delta_{B-1}[\omega^{2\bar2\bar3} + \omega^{3\bar1\bar3}], \quad \rho=1 \end{cases} \Bigg\rangle,$$

$$H^4_{\mathrm{dR}}(M;\mathbb{C}) = \Bigg\langle [\omega^{123\bar2}],\, [\omega^{12\bar2\bar3}],\, \delta_{B-\rho}[\omega^{23\bar1\bar2}],\, \delta_{c-|B-\rho|}[\omega^{13\bar1\bar2}],$$
$$\begin{cases} [\omega^{123\bar3}],\, [\omega^{3\bar1\bar2\bar3}],\, \delta_{B-1}[\omega^{2\bar1\bar2\bar3}], \\ \quad \delta_{c-|B|}[c\,\omega^{13\bar2\bar3} - \overline{B}\,\omega^{23\bar1\bar3}], \quad \rho=0 \\ [\overline{B}\,\omega^{123\bar3} - \omega^{13\bar2\bar3} - c\,\omega^{3\bar1\bar2\bar3}], \\ [c\,\omega^{123\bar3} - \omega^{23\bar1\bar3} - B\,\omega^{3\bar1\bar2\bar3}],\, \delta_c\delta_{B-1}[\omega^{12\bar1\bar3}], \\ \quad (\delta_{c-|B-1|} + (1-\delta_{B-1})(1-\delta_{c-|B-1|}))[\omega^{2\bar1\bar2\bar3}], \quad \rho=1 \end{cases} \Bigg\rangle,$$

$H^5_{\mathrm{dR}}(M;\mathbb{C}) = \langle[\omega^{123\bar1\bar3}],\, [\omega^{123\bar2\bar3}],\, [\omega^{23\bar1\bar2\bar3}],\, \delta_\rho\delta_{c-|B|}[\omega^{12\bar1\bar2\bar3}]\rangle.$

Cohomology groups of complex nilmanifolds M in Family III.

$H^1_{\mathrm{dR}}(M;\mathbb{C}) = \langle[\omega^1],\, [\omega^{\bar1}],\, \delta_\varepsilon[\omega^3+\omega^{\bar3}]\rangle,$

$H^2_{\mathrm{dR}}(M;\mathbb{C}) = \langle[\omega^{12}],\, [\omega^{\bar1\bar2}],\, [\omega^{23}\pm 2\,\varepsilon\,\omega^{1\bar3}+\omega^{2\bar3}],\, [\pm 2\,\varepsilon\,\omega^{3\bar1}+\omega^{3\bar2}-\omega^{\bar2\bar3}],\, \delta_\varepsilon[\omega^{1\bar1}]\rangle,$

$H^3_{\mathrm{dR}}(M;\mathbb{C}) = \langle[\omega^{123}],\, [\omega^{12\bar3}],\, [\omega^{1\bar1\bar3}],\, [\omega^{3\bar1\bar2}],\, [\omega^{\bar1\bar2\bar3}],\, [\omega^{23\bar2}\mp\varepsilon(\omega^{1\bar2\bar3}-\omega^{2\bar1\bar3})-\omega^{2\bar2\bar3}]\rangle,$

$H^4_{\mathrm{dR}}(M;\mathbb{C}) = \langle[\omega^{123\bar1}],\, [\omega^{123\bar3}],\, [\omega^{1\bar1\bar2\bar3}],\, [\omega^{3\bar1\bar2\bar3}],\, \delta_\varepsilon[\omega^{23\bar2\bar3}]\rangle,$

$H^5_{\mathrm{dR}}(M;\mathbb{C}) = \langle[\omega^{123\bar2\bar3}],\, [\omega^{23\bar1\bar2\bar3}],\, \delta_\varepsilon[\omega^{12\bar1\bar2\bar3}]\rangle.$

Studying complex-$\mathcal{C}^\infty$-pureness and fullness of each 6-dimensional complex nilmanifold M is reduced by Proposition 2.1 to the Lie algebra level $(\mathfrak{g}, J)$. Despite this reduction, it is not an easy task since it requires a thorough analysis in terms of the parameters which define the different complex structures on each one of the Lie algebras.

In order to illustrate the procedure followed to attain the results given in the subsequent tables, here we briefly describe how to study complex-$\mathcal{C}^\infty$-pureness and fullness at the 2nd stage for the complex nilmanifolds corresponding to $(\mathfrak{h}_4, J)$. First note that the complex structures J on $\mathfrak{h}_4$ belong to Family I and they correspond to $\lambda = 1$ and either $(\rho, D) = (0, 1/4)$ or $(\rho, D) = (1, x)$ with $x \in \mathbb{R}\setminus\{0\}$. A direct calculation shows that

$$\begin{aligned} H^2((\mathfrak{h}_4)_{\mathbb{C}}) = \langle\, & [\omega^{1\bar2}],\, [\omega^{2\bar2}],\, [\omega^{\bar1\bar2}],\, [\omega^{13}+D\,\omega^{23}+D\,\omega^{2\bar3}-\rho\,\omega^{\bar1\bar3}],\\ & [\rho\,\omega^{13}+D\,\omega^{3\bar2}-\omega^{\bar1\bar3}-D\,\omega^{\bar2\bar3}],\, [\rho\,\omega^{23}-\omega^{3\bar1}-\omega^{3\bar2}+\omega^{\bar1\bar3}],\\ & [\omega^{13}+\omega^{1\bar3}+\omega^{2\bar3}+\rho\,\omega^{\bar2\bar3}],\, \delta_\rho\,[\omega^{12}],\, \delta_{\rho-1}[\omega^{2\bar1}]\,\rangle. \end{aligned}$$

Notice that the cohomology classes satisfy some relations which depend on the value of ρ:

- if $\rho = 0$, i.e. J is abelian, then $[\omega^{1\bar1}] = -[\omega^{1\bar2}] - D\,[\omega^{2\bar2}]$ and $[\omega^{2\bar1}] = [\omega^{1\bar2}]$;
- if $\rho = 1$, then $[\omega^{1\bar1}] = -[\omega^{2\bar1}] - D\,[\omega^{2\bar2}] + [\omega^{\bar1\bar2}]$ and $[\omega^{12}] = -[\omega^{1\bar2}] + [\omega^{2\bar1}] - [\omega^{\bar1\bar2}]$.

In any case, the second Betti number is equal to 8.

On the other hand, one can see that

$$\begin{aligned} H^{(2,0)}_J(\mathfrak{h}_4) &= \langle\, [\omega^{12}]\,\rangle,\\ H^{(0,2)}_J(\mathfrak{h}_4) &= \langle\, [\omega^{\bar1\bar2}]\,\rangle,\\ H^{(1,1)}_J(\mathfrak{h}_4) &= \langle\, [\omega^{1\bar2}],\, [\omega^{2\bar2}],\, \delta_{\rho-1}[\omega^{1\bar1}],\, \delta_{\rho-1}[\omega^{2\bar1}],\\ &\qquad \delta_{\rho-1}\delta_{D+2}[\omega^{1\bar3}+2\omega^{2\bar3}+\omega^{3\bar1}+2\omega^{3\bar2}]\,\rangle. \end{aligned}$$

Therefore, counting dimensions, we conclude that none of the complex structures is complex-$\mathcal{C}^\infty$-full at the second stage as the sum of $H^{(2,0)}_J(\mathfrak{h}_4)$, $H^{(1,1)}_J(\mathfrak{h}_4)$

and $H_J^{(0,2)}(\mathfrak{h}_4)$ never generates the whole second complex de Rham cohomology group.

Furthermore, when $\rho = 0$, we have

$$H_J^{(1,1)}(\mathfrak{h}_4) = \langle\, [\omega^{1\bar{2}}],\, [\omega^{2\bar{2}}]\,\rangle$$

and the complex structure is complex-$\mathcal{C}^\infty$-pure at the 2nd stage (this also follows from Proposition 2.2(b)). Finally, it is easy to see that the complex structures with $\rho = 1$ are not complex-$\mathcal{C}^\infty$-pure at the 2nd stage, since for example the element $[\omega^{\bar{1}\bar{2}}] \in H_J^{(0,2)}(\mathfrak{h}_4)$ also belongs to $H_J^{(1,1)}(\mathfrak{h}_4)$ because

$$[\omega^{\bar{1}\bar{2}}] = [\omega^{1\bar{1}} + \omega^{2\bar{1}} + D\,\omega^{2\bar{2}}].$$

In the tables below we sum up the behaviour of any invariant complex structure J in terms of the Lie algebra underlying the nilmanifold M and depending on the parameters which define J. Notice that the complex-parallelizable structures (3.1) do not appear in the tables because it is well-known that they are complex-$\mathcal{C}^\infty$-pure-and-full at every stage [3].

Family I				Stages									
				1st		2nd		3rd		4th		5th	
$\mathfrak{g}$	ρ	λ	$D = x + iy$	pure	full	pure	full	pure	full	pure	full	pure	full
$\mathfrak{h}_2$	0	0	$y = 1$	✓	✓	✓	–	–	–	–	✓	✓	✓
	1	1	$y > 0$	✓	✓	–	–	–	–	–	–	✓	✓
$\mathfrak{h}_3$	0	0	± 1	✓	–	✓	–	–	–	–	✓	–	✓
$\mathfrak{h}_4$	0	1	$\frac{1}{4}$	✓	✓	✓	–	–	–	–	✓	✓	✓
	1	1	$D \in \mathbb{R} \setminus \{0\}$	✓	✓	–	–	–	–	–	–	✓	✓
$\mathfrak{h}_5$	0	1	0	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
			$D \in \left(0, \frac{1}{4}\right)$	✓	✓	✓	–	–	–	–	✓	✓	✓
	1	0	0	✓	✓	–	–	✓	✓	–	–	✓	✓
		$\lambda \neq 0$		✓	✓	–	–	–	–	–	–	✓	✓
		any allowed structure satisfying $D \neq 0$		✓	✓	–	–	–	–	–	–	✓	✓
$\mathfrak{h}_6$	1	1	0	✓	✓	–	–	–	–	–	–	✓	✓
$\mathfrak{h}_8$	0	0	0	✓	–	✓	–	–	–	–	✓	–	✓

As a consequence of our study we have that the direct sum decomposition (1.1) is satisfied for every k, only for the two complex structures (3.1) and for one abelian complex structure on $\mathfrak{h}_5$, concretely:

Theorem 3.1. *Let J be an invariant complex structure on a 6-dimensional nilmanifold M. Then, J is complex-$\mathcal{C}^\infty$-pure-and-full at every stage if and only if J is isomorphic to the complex-parallelizable structures* (3.1) *or to the abelian complex structure defined by the equations $d\omega^1 = d\omega^2 = 0$ and $d\omega^3 = \omega^{1\bar{1}} + \omega^{1\bar{2}}$.*

In [3] (see also [1]) it is proved that complex-$\mathcal{C}^\infty$-full at the kth stage implies complex-$\mathcal{C}^\infty$-pure at the $(2n-k)$th stage. In fact, suppose that there is a nonzero class $\mathbf{b} \in H_J^{(p_1,q_1)}(M) \cap H_J^{(p_2,q_2)}(M)$ with $(p_1, q_1) \neq (p_2, q_2)$ such that $p_1 + q_1 = 2n - k = p_2 + q_2$. Since the pairing $p\colon H^k_{\mathrm{dR}}(M;\mathbb{C}) \times H^{2n-k}_{\mathrm{dR}}(M;\mathbb{C}) \to \mathbb{C}$, defined by

$$p(\mathbf{a}, \mathbf{b}) = \int_M \alpha \wedge \beta \quad \text{for } \mathbf{a} = [\alpha] \text{ and } \mathbf{b} = [\beta],$$

is non-degenerate, there is a nonzero class $\mathbf{a} \in H^k_{\mathrm{dR}}(M;\mathbb{C})$ such that $p(\mathbf{a}, \mathbf{b}) \neq 0$. It is easy to check that the class $\mathbf{a} \notin H_J^{(k,0)}(M) + \ldots + H_J^{(0,k)}(M)$. In conclusion, if J is not complex-$\mathcal{C}^\infty$-pure at the $(2n-k)$th stage then J is not complex-$\mathcal{C}^\infty$-full at the kth stage.

However, in general it is not clear when the converse holds, that is, when pure at the kth stage implies full at the $(2n-k)$th stage. As a consequence of our study we get the following duality result:

Proposition 3.2. *Let J be an invariant complex structure on a 6-dimensional nilmanifold M. Then, for any $1 \le k \le 5$, J is complex-$\mathcal{C}^\infty$-full at the kth stage if and only if it is complex-$\mathcal{C}^\infty$-pure at the $(6-k)$th stage.*

Family II				Stages									
				1st		2nd		3rd		4th		5th	
$\mathfrak{g}$	ρ	B	c	pure	full	pure	full	pure	full	pure	full	pure	full
$\mathfrak{h}_7$	1	1	0	✓	–	–	–	–	–	–	–	–	✓
$\mathfrak{h}_9$	0	1	1	✓	–	✓	–	–	–	–	✓	–	✓
$\mathfrak{h}_{10}$	1	0	1	✓	–	–	–	–	–	–	–	–	✓
$\mathfrak{h}_{11}$	1	$B \in \mathbb{R} \setminus \{0,1\}$	$\lvert B-1\rvert$	✓	–	–	–	–	–	–	–	–	✓
$\mathfrak{h}_{12}$	1	$\mathfrak{Im}\, B \neq 0$	$\lvert B-1\rvert$	✓	–	–	–	–	–	–	–	–	✓
$\mathfrak{h}_{13}$	1	$c \neq \lvert B-1\rvert,\ (c,\lvert B\rvert) \neq (0,1)$ $\mathcal{S}(B,c) < 0$		✓	–	–	–	–	–	–	–	–	✓
$\mathfrak{h}_{14}$	1	$c \neq \lvert B-1\rvert,\ (c,\lvert B\rvert) \neq (0,1)$ $\mathcal{S}(B,c) = 0$		✓	–	–	–	–	–	–	–	–	✓
$\mathfrak{h}_{15}$	0	0	1	✓	–	✓	–	✓	✓	–	✓	–	✓
		1	0	✓	–	✓	✓	✓	✓	✓	✓	–	✓
			$c \neq 0,1$	✓	–	✓	–	–	–	–	✓	–	✓
	1	0	0	✓	–	✓	✓	✓	✓	✓	✓	–	✓
			$c \neq 0,1$	✓	–	–	–	–	–	–	–	–	✓
		$\lvert B\rvert \neq 0,1$	0	✓	–	–	✓	–	–	✓	–	–	✓
		$c \neq \lvert B-1\rvert,\ (c,\lvert B\rvert) \neq (0,1)$, $cB \neq 0,\ \ \mathcal{S}(B,c) > 0$		✓	–	–	–	–	–	–	–	–	✓
$\mathfrak{h}_{16}$	1	$\lvert B\rvert = 1,\ B \neq 1$	0	✓	–	–	✓	–	–	✓	–	–	✓

where $\mathcal{S}(B,c) = c^4 - 2\,(|B|^2+1)\,c^2 + (|B|^2-1)^2$.

Family III		Stages									
		1st		2nd		3rd		4th		5th	
$\mathfrak{g}$	ε	pure	full	pure	full	pure	full	pure	full	pure	full
$\mathfrak{h}_{19}^{-}$	0	✓	–	✓	–	–	–	–	✓	–	✓
$\mathfrak{h}_{26}^{+}$	1	✓	✓	✓	–	–	–	–	✓	✓	✓

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