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DENSE PERIODICITY ON GRAPHS

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ABSTRACT. We establish a Barge-Martin type theorem for graph self-maps for which the set of periodic points is dense.

1. Introduction. The purpose of this paper is to describe graph self-maps for which the set of periodic points is dense. Barge and Martin [**2**] established a structure theorem for maps on the interval with dense periodic points; that is, the twice iterate of such a map is topologically mixing on some countable subintervals and is identical on the other. A similar theorem was proved for tree maps in [**7**].

In this paper, we extend the above to graph self-maps, see Section 3. A motivation for studying graph maps is that higher-dimensional dynamics can often be reduced to a one-dimensional dynamics: this is the case in the study of the structure of attractors of a diffeomorphism, the quotient maps generated by maps on manifolds with an invariant foliation of codimension one and the dynamics of pseudo-Anosov homeomorphisms on a surface.

Throughout this paper, by a *graph*, we mean a *connected* compact one-dimensional polyhedron, and a *tree* is a graph which contains no loops. For a graph G, we denote the sets of endpoints and of branch points of G by $\mathrm{E}(G)$ and $\mathrm{B}(G)$, respectively. A *map* f is a continuous function; f^0 is the identity map, and for every $n \geq 0$, $f^{n+1} = f^n \circ f$. We denote by $\mathrm{Fix}(f)$ and $\mathrm{Per}(f)$ the sets of fixed points and of periodic points of f, respectively. A subset K of X is invariant under $f : X \to X$ if $f(K) \subseteq K$, $\mathrm{Int}\, K$ and $\mathrm{Cl}\, K$ denote the interior and closure of K in X, and the orbit of $x \in X$ under f is $\mathrm{Orb}_f(x) = \{f^n(x) \mid n \geq 0\}$.

For a natural number S, N_S denotes the least common multiple of the positive integers less than or equal to S.

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2. Preliminaries. An onto map $f : X \to X$ is called (*topologically*) *transitive* if any of the following equivalent conditions holds.

(i) There exists a point with dense orbit.

(ii) Whenever U, V are nonempty open sets, there exists an $n \geq 1$ such that $f^{-n}(U) \cap V \neq \varnothing$.

(iii) The only closed invariant set K with $\mathrm{Int}\, K \neq \varnothing$ is $K = X$.

We note that if f^n is transitive for some n, then so is f.

A map f is *totally transitive* if f^n is transitive for all $n \geq 1$. A transitive map is not always totally transitive. On the other hand, it is well known that for a transitive graph map with periodic points, the set of periodic points is dense. Therefore, for such a map f, the nth power f^n has dense periodic points for $n \geq 1$.

A map $f : X \to X$ is called *topologically mixing* if for every pair of nonempty open sets U and V, there exists an $N \geq 1$ such that $f^{-n}(U) \cap V \neq \varnothing$ for $n \geq N$. A topological mixing map on a compactum is in general totally transitive. It is also known that a totally transitive graph map with periodic points is topologically mixing.

We first recall some basic results for maps with dense periodic points.

Proposition 2.1 [**3**, Lemma 2.3]. *Let $f : X \to X$ be a map from a compactum to itself for which the set of periodic points is dense. Then, for every connected set E of X with* $\mathrm{Int}\, E \neq \varnothing$, $\mathrm{Cl} \cup_{n \geq 0} f^n(E)$ *has finitely many components. These components have nonempty interior and are permuted by f.*

Proposition 2.2 (cf. [**4**, Lemma 1]). *Let $f : X \to X$ be a map from a compactum to itself for which the set of periodic points is dense. If Y is closed in X and invariant under f, then* $\mathrm{Cl}\,(X \setminus Y)$ *is also invariant under f.*

Roe [**7**] showed a decomposition theorem for tree maps for which the set of periodic points is dense. It is slightly reworded here. The case of interval maps was proved earlier by Barge-Martin [**2**].

Theorem 2.3 ([**7**, Theorem 5]). *Let $f : T \to T$ be a tree map for which the set of periodic points is dense. Let $N_{\mathrm{E}\,(T)} = \mathrm{LCM}\,\{2, 3, \ldots, \#\,\mathrm{E}\,(T)\}$. Then there exists a collection (perhaps finite or empty) $\{J_1, J_2, \cdots\}$ of subtrees of T with disjoint interiors such that*

(i) $f^{N_{\mathrm{E}\,(T)}}(J_i) = J_i$ *for* $i \geq 1$,

(ii) $f^{N_{\mathrm{E}\,(T)}}|_{J_i} : J_i \to J_i$ *is totally transitive for* $i \geq 1$, *and*

(iii) $f^{N_{\mathrm{E}\,(T)}}(x) = x$ *for* $x \in T \setminus \cup_i J_i$.

Remark. His proof of Theorem 5 and Lemma 1 in [**7**] show totally transitivity of $f^{N_{\mathrm{E}\,(T)}}|_{J_i} : J_i \to J_i$ above.

Let G be a graph, $x \in G$, and U an open connected neighborhood of x in G whose closure is a tree. The number of components of $U \setminus \{x\}$ is called the *valence* of x and is denoted by $\mathrm{v}\,(x)$, and we set $\mathrm{v}\,(G) = \max\{\mathrm{v}\,(x) \mid x \in G\}$. A point of valence ≥ 3 is a branch point and of valence 1 is an endpoint.

The following theorem is a direct generalization of [**1**, Lemma 2] or [**6**, Corollary 3.2]; we omit the straightforward proof.

Theorem 2.4 (cf. [**6**, Corollary 3.2]). *Let $f : G \to G$ be a graph map satisfying* $\mathrm{Fix}\,(f) \neq \varnothing$ *and* $\mathrm{Cl\ Orb}_{\,f}(x) = G$ *for some $x \in X$. Let $N_{\mathrm{v}\,(G)} = \mathrm{LCM}\,\{2, 3, \ldots, \mathrm{v}\,(G)\}$. Then one of the following occurs*:

(i) $\mathrm{Cl\ Orb}_{\,f^{N_{\mathrm{v}\,(G)}}}(x) = G$, *in which case* $\mathrm{Cl\ Orb}_{\,f^s}(f^k(x)) = G$ *for $s \geq 1$ and $k \geq 0$, i.e., f is totally transitive.*

(ii) $\mathrm{Cl\ Orb}_{\,f^{N_{\mathrm{v}\,(G)}}}(x) \neq G$, *in which case there exists a number p, $2 \leq p \leq \mathrm{v}\,(G)$ such that*

(a) $G = \cup_{i=0}^{p-1} \mathrm{Cl\ Orb}_{\,f^p}(f^i(x))$,

(b) $\mathrm{Cl\ Orb}_{\,f^p}(f^i(x))$ *is a subgraph of G for* $0 \leq i \leq p-1$,

(c) $\mathrm{Int\ Cl\ Orb}_{\,f^p}(f^i(x)) \cap \mathrm{Int\ Cl\ Orb}_{\,f^p}(f^j(x)) = \varnothing$ *for* $0 \leq i < j \leq p-1$,

(d) $f(\mathrm{Cl\ Orb}_{\,f^p}(f^i(x))) = \mathrm{Cl\ Orb}_{\,f^p}(f^{i+1}(x)) \pmod p$, *and*

(e) $\mathrm{Cl\ Orb}_{\,f^{pk}}(f^i(x)) = \mathrm{Cl\ Orb}_{\,f^p}(f^i(x))$ *for $k \geq 1$ and $0 \leq i \leq p-1$, i.e., $f|_{\mathrm{Cl\ Orb}_{\,f^p}(f^i(x))}$ is totally transitive for $0 \leq i \leq p-1$.*

3. Results. Here is our main theorem.

Theorem 3.1. *Let $f : G \to G$ be a graph map for which the set of periodic points is dense. Then there exist a natural number N and a collection (perhaps finite or empty) $\{G_1, G_2, \dots\}$ of subgraphs of G with disjoint interiors such that*

(i) $f^N(G_i) = G_i$ *for* $i \geq 1$,

(ii) $f^N|_{G_i} : G_i \to G_i$ *is totally transitive (i.e., topologically mixing) for* $i \geq 1$, *and*

(iii) $f^N(x) = x$ *for* $x \in G \setminus \cup_i G_i$.

Before showing our main theorem, we need lemmas used later.

Lemma 3.2. *Let $g : G \to G$ be a nontransitive graph map for which the set of periodic points is dense, and let $g(x) = x$ for some $x \in G$. Then there exists a proper subgraph K of G such that $x \in K$ and $g(K) \subseteq K$.*

Proof. Since g is not transitive, there exists a proper closed set F of G such that $\mathrm{Int}\, F \neq \varnothing$ and $g(F) \subseteq F$.

Let E be a component of F with $\mathrm{Int}\, E \neq \varnothing$. Then it follows from Proposition 2.1 that $E^* \equiv \mathrm{Cl} \cup_{n \geq 0} g^n(E)$ is the direct sum of nondegenerate subgraphs $E_0, \dots, E_{\ell-1}$ satisfying $g(E_i) = E_{i+1}$ (mod ℓ) for $0 \leq i \leq \ell - 1$.

If $x \in E^*$, ℓ must be 1, and then put $K = E^*$. On the other side, let $x \notin E^*$. By Proposition 2.2, we note that $g(\mathrm{Cl}\,(G \setminus E^*)) \subseteq \mathrm{Cl}\,(G \setminus E^*) \subsetneqq G$. It is clear that $\mathrm{Cl}\,(G \setminus E^*)$ is the direct sum of nondegenerate subgraphs $\widetilde{E_0}, \dots, \widetilde{E_m}$. Here we put $K = \widetilde{E_{i_0}}$, where $x \in \mathrm{Int}\, \widetilde{E_{i_0}}$. This is the set that we required. □

Lemma 3.3. *Let G be a graph which includes a loop. Let $g : G \to G$ be a nontransitive map for which the set of periodic points is dense, and let $x_1, \dots, x_s \in G \setminus (\mathrm{E}\,(G) \cup \mathrm{B}\,(G))$ such that $g(x_i) = x_i$ for $1 \leq i \leq s$ and $G \setminus \{x_1, \dots, x_s\}$ is homotopic to a point. Then there* *exist a natural number L and collections $\{G_{\infty 1}, \dots, G_{\infty s}\}$ and (perhaps finite or empty) $\{G_1, G_2, \dots\}$ of subcontinua in G such that*

(i) *any two sets of $\{G_{\infty 1}, \dots, G_{\infty s}\}$ coincide or have disjoint interiors,*

(ii) *any two sets of $\{G_1, G_2, \dots\}$ have disjoint interiors,*

(iii) $\mathrm{Int}\, G_{\infty i} \cap \mathrm{Int}\, G_j = \varnothing$ *for* $1 \le i \le s$, $j \ge 1$,

(iv) *each of $\{G_{\infty i}\}$ is invariant under g,*

(v) *each of $\{G_j\}$ is invariant under g^L,*

(vi) $g^L|_{G_j} : G_j \to G_j$ *is totally transitive for j,*

(vii) $g^L(x) = x$ *for* $x \in G \setminus (G_{\infty 1} \cup \dots \cup G_{\infty s} \cup \cup_j G_j)$, *and*

(viii) *for $1 \le i \le s$, either $G_{\infty i} = \varnothing$ or*

(a) *$G_{\infty i}$ is proper,*

(b) *$G_{\infty i}$ includes a loop having x_i, and*

(c) *$g|_{G_{\infty i}} : G_{\infty i} \to G_{\infty i}$ is not transitive.*

Remark. In the Lemma above, the number s must be the rank of the first homology group of G.

Proof. We have proper subgraphs K_i, $1 \le i \le s$, of G such that

(1) $x_i \in K_i$,

(2) $g(K_i) \subseteq K_i$, and

(3) $K_i = K_j$ or $\mathrm{Int}\, K_i \cap \mathrm{Int}\, K_j = \varnothing$ for $i \neq j$.

Indeed, it follows from Lemma 3.2 that there exists a proper subgraph K_1 of G such that $x_1 \in K_1$ and $g(K_1) \subseteq K_1$. By Proposition 2.2, we have $g(\mathrm{Cl}\,(G \setminus K_1)) \subseteq \mathrm{Cl}\,(G \setminus K_1) \subsetneqq G$. Next, for $2 \le i \le s$, if $x_i \in K_1$, then put $K_i = K_1$. If not, choose the connected component K_i of $\mathrm{Cl}\,(G \setminus K_1)$ satisfying $x_i \in \mathrm{Int}\, K_i$. Then these K_is are proper subgraphs with $x_i \in K_i$ and $g(K_i) \subseteq K_i$.

For $1 \le i \le s$, we here put

$$G_{\infty i} = \begin{cases} \varnothing & \text{if } K_i \text{ contains no loops having } x_i \text{ or} \\ & g|_{K_i} : K_i \to K_i \text{ is transitive}, \\ K_i & \text{otherwise.} \end{cases}$$

We now consider the *distinct* graphs in the collection $\{K_1, \ldots, K_s\}$, recall (3).

In the case when K_i is a tree; since (2) and $\mathrm{Cl}\,\mathrm{Per}\,(g|_{K_i}) = K_i$, we have by Theorem 2.3, a collection (perhaps finite or empty) $\{J_1^{K_i}, J_2^{K_i}, \ldots\}$ of subtrees of K_i with disjoint interiors such that

- $g^{N_{\mathrm{E}\,(K_i)}}(J_j^{K_i}) = J_j^{K_i}$ for $j \geq 1$,
- $g^{N_{\mathrm{E}\,(K_i)}}|_{J_j^{K_i}} : J_j^{K_i} \to J_j^{K_i}$ is totally transitive for $j \geq 1$, and
- $g^{N_{\mathrm{E}\,(K_i)}}(x) = x$ for $x \in K_i \setminus \bigcup_j J_j^{K_i}$.

In the case when K_i includes a loop and $g|_{K_i} : K_i \to K_i$ is transitive, it follows from (1) and Theorem 2.4 that there exist a natural number $p_i \leq \mathrm{v}\,(K_i)$ and subgraphs $K_0^i, \ldots, K_{p_i-1}^i$ with disjoint interiors such that

- $K_i = K_0^i \cup \cdots \cup K_{p_i-1}^i$,
- $\mathrm{Int}\ K_j^i \neq \varnothing$ for $0 \leq j \leq p_i - 1$, and
- $g^{p_i}|_{K_j^i} : K_j^i \to K_j^i$ is totally transitive for $0 \leq j \leq p_i - 1$.

Since p_i divides $N_{\mathrm{v}\,(K_i)}$, we note that $g^{N_{\mathrm{v}\,(K_i)}}|_{K_j^i} : K_j^i \to K_j^i$ is totally transitive for $0 \leq j \leq p_i - 1$.

Let $H = \mathrm{Cl}\,(G \setminus \cup_{i=1}^s K_i)$, and assume that the set H is nonempty. Represent $H = \oplus_{k=1}^{\ell(H)} H_k$ by connected (*tree*) components H_k. Since $g(H) \subseteq H$ by Proposition 2.2 and $\mathrm{Cl}\,\mathrm{Per}\,(g) = G$, there exists a natural number t such that

$$g^t(H_k) \subseteq H_k \quad \text{for} \quad 1 \leq k \leq \ell(H). \tag{4}$$

Indeed, let

$$E^G = \max\{\#\,\mathrm{E}\,(T) \mid T \text{ is a tree in } G\}.$$

We note that $E^G = 2\mathrm{rank}\,[H_1(G)] + \#\,\mathrm{E}\,(G)$, where $H_1(G)$ is the first homology group of G. As $\ell(H) \leq E^G - 1$, the number

$$N_{E^G-1} = \mathrm{LCM}\,\{2, 3, \ldots, E^G - 1\} \tag{5}$$

is suitable for t in (4) and note that this depends only on G.

Since (4) and $\mathrm{Cl\,Per}\,(g^t|_{H_k}) = H_k$ for $1 \le k \le \ell(H)$, it follows from Theorem 2.3 that for $1 \le k \le \ell(H)$, there exists a collection, perhaps finite or empty, $\{J_1^k, J_2^k, \dots\}$ of subtrees of H_k with disjoint interiors such that

- $(g^t)^{N_{\mathrm{E}\,(H_k)}}(J_i^k) = J_i^k$ for $i \ge 1$,
- $(g^t)^{N_{\mathrm{E}\,(H_k)}}|_{J_i^k} : J_i^k \to J_i^k$ is totally transitive for $i \ge 1$, and
- $(g^t)^{N_{\mathrm{E}\,(H_k)}}(x) = x$ for $x \in H_k \setminus \cup_i J_i^k$.

Finally, we represent the collection $\{J_j^{K_i}\} \cup \{K_j^i\} \cup \{J_i^k\}$ by $\{G_1, G_2, \dots\}$ and put

$$L = N_{E^{G}-1} \times N_{E^{G}}. \tag{6}$$

Since the numbers $N_{\mathrm{E}\,(K_i)}$, $N_{\mathrm{V}\,(K_i)}$ and $t \times N_{\mathrm{E}\,(H_k)}$ divide L, we conclude our proof. □

Remark. Let H be a subgraph of G, and let L' be the number with respect to H defined by (6) in the proof of Lemma 3.3. Then we note that L' divides L.

Proof of Theorem 3.1. We may assume that G includes a loop by the benefits of Theorem 2.3. It is clear from $\mathrm{Cl\,Per}\,(f) = G$ that we have a natural number k and $x_1, \dots, x_s \in G \setminus (\mathrm{E}\,(G) \cup \mathrm{B}\,(G))$ such that $f^k(x_i) = x_i$ for $1 \le i \le s$ and $G \setminus \{x_1, \dots, x_s\}$ is homotopic to a point. If $f^k : G \to G$ is transitive, then we carry out our purpose by Theorem 2.4.

We assume that $f^k : G \to G$ is not transitive and let $g = f^k$ below for simplicity. We note that $\mathrm{Cl\,Per}\,(g) = G$.

Let

$$L = N_{E^{G}-1} \times N_{E^{G}},$$

as (6) in the proof of Lemma 3.3.

Using Lemma 3.3 recursively, for any countable ordinal number α, we shall construct $\{G_{\infty 1}^{\alpha}, \dots, G_{\infty s}^{\alpha}\}$ and (perhaps finite or empty) $\{G_1^{\alpha}, G_2^{\alpha}, \dots\}$ of subcontinua in G such that

(1) any two sets of $\{G_{\infty 1}^{\alpha}, \dots, G_{\infty s}^{\alpha}\}$ coincide or have disjoint interiors,

(2) any two sets of $\cup_{\beta \le \alpha}\{G_1^{\beta}, G_2^{\beta}, \dots\}$ have disjoint interiors,

(3) Int $G_{\infty i}^{\alpha} \cap \text{Int } G_j^{\beta} = \varnothing$ for $\beta \leq \alpha$, $1 \leq i \leq s$, and $j \geq 1$,

(4) $g(G_{\infty i}^{\alpha}) \subseteq G_{\infty i}^{\alpha}$ for $1 \leq i \leq s$ and $g^L(G_j^{\alpha}) \subseteq G_j^{\alpha}$ for j,

(5) $g^L|_{G_j^{\alpha}} : G_j^{\alpha} \to G_j^{\alpha}$ is totally transitive for j,

(6) $g^L(x) = x$ for $x \in G \setminus (G_{\infty 1}^{\alpha} \cup \cdots \cup G_{\infty s}^{\alpha} \cup \cup_{\beta \leq \alpha}(G_1^{\beta} \cup G_2^{\beta} \cup \cdots))$, and

(7) for $1 \leq i \leq s$, either $G_{\infty i}^{\alpha} = \varnothing$ or

(a) $G_{\infty i}^{\alpha} \subsetneqq G_{\infty i}^{\beta}$ for $\beta < \alpha$,

(b) $G_{\infty i}^{\alpha}$ includes a loop having x_i, and

(c) $g|_{G_{\infty i}^{\alpha}} : G_{\infty i}^{\alpha} \to G_{\infty i}^{\alpha}$ is not transitive.

Our construction is by transfinite induction on $\alpha < \omega_1$, where ω_1 is the first uncountable ordinal number.

Let $\alpha = 0$. Then we put $G_{\infty i}^{0} = G$ for $1 \leq i \leq s$, and $G_j^0 = \varnothing$ for $j \geq 1$.

For $\beta < \alpha$, suppose that we have constructed collections $\{G_{\infty 1}^{\beta}, \dots, G_{\infty s}^{\beta}\}$ and $\{G_1^{\beta}, G_2^{\beta}, \dots\}$ of subcontinua in G satisfying properties (1)–(7).

Now let α be a limit ordinal number. Put $\widetilde{G}_{\infty i}^{\alpha} = \cap_{\beta<\alpha} G_{\infty i}^{\beta}$ for $1 \leq i \leq s$. We shall construct $G_{\infty i}^{\alpha}$ and G_j^{α} for $1 \leq i \leq s$ and j. Consider now the nonempty *distinct* graphs in $\{\widetilde{G}_{\infty i}^{\alpha} \mid 1 \leq i \leq s\}$, and in every other graph, let $G_{\infty i}^{\alpha}$ be empty.

In the case when $g|_{\widetilde{G}_{\infty i}^{\alpha}} : \widetilde{G}_{\infty i}^{\alpha} \to \widetilde{G}_{\infty i}^{\alpha}$ is not transitive, put $G_{\infty i}^{\alpha} = \widetilde{G}_{\infty i}^{\alpha}$ and $\widetilde{G}_{i,j}^{\alpha} = \varnothing$ for $j \geq 1$.

In the case when $g|_{\widetilde{G}_{\infty i}^{\alpha}} : \widetilde{G}_{\infty i}^{\alpha} \to \widetilde{G}_{\infty i}^{\alpha}$ is transitive, put $G_{\infty i}^{\alpha} = \varnothing$. It follows from $g(x_i) = x_i \in \widetilde{G}_{\infty i}^{\alpha}$ and Theorem 2.4 that there exist a natural number $p_i \leq \text{v}\,(\widetilde{G}_{\infty i}^{\alpha})$ and subgraphs $K_{i,0}^{\alpha}, \dots, K_{i,p_i-1}^{\alpha}$ with disjoint interiors such that

(8) $\widetilde{G}_{\infty i}^{\alpha} = K_{i,0}^{\alpha} \cup \cdots \cup K_{i,p_i-1}^{\alpha}$,

(9) Int $K_{i,j}^{\alpha} \neq \varnothing$ for $0 \leq j \leq p_i - 1$, and

(10) $g^{p_i}|_{K_{i,j}^{\alpha}} : K_{i,j}^{\alpha} \to K_{i,j}^{\alpha}$ is totally transitive for $0 \leq j \leq p_i - 1$.

Since p_i divides L, we note that $g^L|_{K^\alpha_{i,j}} : K^\alpha_{i,j} \to K^\alpha_{i,j}$ is totally transitive for $0 \le j \le p_i - 1$. Then represent $\{\widetilde{G}^\alpha_{i,j}\} \cup \{K^\alpha_{i,j}\}$ by $\{G^\alpha_i\}$.

Let $\alpha = \alpha' + 1$ be a nonlimit ordinal number. Put $G^\alpha_{\infty i} = \varnothing$ and $G^\alpha_{i,j} = \varnothing$ for $j \ge 1$ if $G^{\alpha'}_{\infty i} = \varnothing$. Next, represent the nonempty *distinct* graphs in $\{G^{\alpha'}_{\infty i} \mid 1 \le i \le s\}$ by $\{G^{\alpha'}_{\infty i_1}, \dots, G^{\alpha'}_{\infty i_q}\}$. We note that $G^{\alpha'}_{\infty i_k}$ includes a loop having x_{i_k} for $1 \le k \le q$.

Let $1 \le k \le q$. We have a set $I_k \subseteq J_k \equiv \{j \in \{1, \dots, s\} \mid x_j \in G^{\alpha'}_{\infty i_k}\}$ such that $\{x_i \mid i \in I_k\} \cap \mathrm{E}(G^{\alpha'}_{\infty i_k}) = \varnothing$ and $G^{\alpha'}_{\infty i_k} \setminus \{x_i \mid i \in I_k\}$ is homotopic to a point. We recall that $I_{k_1} \cap I_{k_2} = \varnothing$ for $k_1 \ne k_2$. Then apply Lemma 3.3 to $g|_{G^{\alpha'}_{\infty i_k}} : G^{\alpha'}_{\infty i_k} \to G^{\alpha'}_{\infty i_k}$ and $\{x_i \mid i \in I_k\}$. We note again the Remark following the proof of Lemma 3.3. Then we have collections $\{G^\alpha_{\infty i} \mid i \in I_k\}$ and $\{G^\alpha_{i_k,1}, G^\alpha_{i_k,2}, \dots\}$ of subcontinua in $G^{\alpha'}_{\infty i_k}$ such that

(11) any two sets of $\{G^\alpha_{\infty i} \mid i \in I_k\}$ coincide or have disjoint interiors,

(12) any two sets of $\{G^\alpha_{i_k,1}, G^\alpha_{i_k,2}, \dots\}$ have disjoint interiors,

(13) $\mathrm{Int}\, G^\alpha_{\infty i} \cap \mathrm{Int}\, G^\alpha_{i_k,j} = \varnothing$ for $i \in I_k$ and $j \ge 1$,

(14) $g(G^\alpha_{\infty i}) \subseteq G^\alpha_{\infty i}$ for $i \in I_k$ and $g^L(G^\alpha_{i_k,j}) \subseteq G^\alpha_{i_k,j}$ for $j \ge 1$,

(15) $g^L|_{G^\alpha_{i_k,j}} : G^\alpha_{i_k,j} \to G^\alpha_{i_k,j}$ is totally transitive for $j \ge 1$,

(16) $g^L(x) = x$ for $x \in G^{\alpha'}_{\infty i_k} \setminus (\cup_{i \in I_k} G^\alpha_{\infty i} \cup \cup_{j \ge 1} G^\alpha_{i_k,j})$, and

(17) for $i \in I_k$, either $G^\alpha_{\infty i} = \varnothing$ or

(a) $G^\alpha_{\infty i}$ is proper,

(b) $G^\alpha_{\infty i}$ includes a loop having x_i, and

(c) $g|_{G^\alpha_{\infty i}} : G^\alpha_{\infty i} \to G^\alpha_{\infty i}$ is not transitive.

We put $G^\alpha_{\infty i} = \varnothing$ for $i \in J_k \setminus I_k$ and enumerate $\{G^\alpha_{i_k,j} \mid 1 \le k \le q, j \ge 1\} \cup \{G^\alpha_{i,j} \mid i \notin \cup J_k, j \ge 1\}$ as $\{G^\alpha_j \mid j \ge 1\}$.

Then, in any case, we are able to go to the next stage. Thus, we have finished our construction.

Finally, there exists an ordinal number $\alpha_0 < \omega_1$ such that $G^{\alpha_0}_i = \varnothing$ for $1 \le i \le s$. Indeed, by our construction,

$$G^0_{\infty i} \supseteq G^1_{\infty i} \supseteq \cdots \supseteq G^\alpha_{\infty i} \supseteq \cdots, \ \alpha < \omega_1.$$

It follows from [**5**, Theorem 2, page 258] that there exists an ordinal number $\alpha_0 < \omega_1$ such that $G_{\infty i}^{\alpha_0} = G_{\infty i}^{\alpha_0+1} = \cdots$ for $1 \le i \le s$. Then $G_{\infty i}^{\alpha_0} = \varnothing$, $1 \le i \le s$, follows from (7). Therefore, we obtain a countable collection $\cup_{\beta \le \alpha_0}\{G_1^\beta, G_2^\beta, \dots\}$ and f^N, where $N = L \times k$, as required, and the proof of our main theorem is finally finished. $\square$

Remark. For simplicity, we used the Roe decomposition theorem for tree maps (Theorem 2.3) in our proof. We are able to prove Theorem 3.1 (essentially, Lemma 3.3) by use of the Barge-Martin decomposition theorem for interval maps [**2, 4**].

4. Examples.

Example 1. Let $f : [0,1] \to [0,1]$ be the map whose graph appears in Figure 1, where copies of the small square converge to $\{(0,1)\}$ or $\{(1,0)\}$, $f(0) = 1$, and $f(1) = 0$. Then the closed intervals J_i which are the projective images of those squares to the first coordinate have that $f^2(J_i) = J_i$, $f^2|_{J_i}$ is totally transitive, and $f^2(x) = x$ for $x \in [0,1]\backslash \cup_i J_i$.

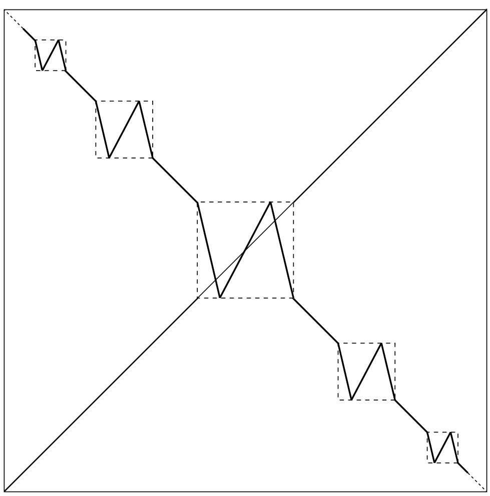

FIGURE 1.

Example 2. Let S^1 be the unit circle on the complex plane. Using the map $f : [0,1] \to [0,1]$ in Example 1, we define the continuous map $g : S^1 \to S^1$ by $g(e^{2\pi i\theta}) = e^{2\pi i f(\theta)}$, where $0 \leq \theta \leq 1$. Put $H_i = \{e^{2\pi i\theta} \mid \theta \in J_i\}$. Then we have that $g^2(H_i) = H_i$, $g^2|_{H_i}$ is totally transitive, and $g^2(x) = x$ for $x \in S^1 \setminus \cup_i H_i$.

Example 3. Let B_3 be the bouquet defined by the one-point union on the origins of the three copies S_0, S_1 and S_2 of the unit circle S^1. Here we may write any element of B_3 by the productive coordinate $(e^{2\pi i\theta}, j)$, $0 \leq \theta \leq 1$, $j = 0,1,2$. Using $g : S^1 \to S^1$ in Example 2, define $h : B_3 \to B_3$ by $h((e^{2\pi i\theta}, j)) = (g(e^{2\pi i\theta}), j+1 \ _{(\mathrm{mod}\ 3)})$, where $j = 0,1,2$. Put $K_i^j = H_i \times \{j\}$, where H_i as in Example 2, $i \geq 1$, and $j = 0,1,2$. Then we see that $h^6(K_i^j) = K_i^j$, $h^6|_{K_i^j}$ is totally transitive, and $h^6(x) = x$ for $x \in B_3 \setminus \cup_{j=0,1,2} \cup_i K_i^j$.

The decomposition theorem does not always hold for general spaces.

Example 4. Let B_n be the bouquet with n-petals generated by the unit circle for $n \geq 1$. Define $h_n : B_n \to B_n$ as h in Example 3. Attach, for each $n \geq 1$, the origin of B_n to the point $\{n\}$ of the half real line $\mathbf{R}^{\geq 0}$, and the one-dimensional locally finite (noncompact) polyhedron is denoted by B. The map $\hat{h} : B \to B$ is defined by $\hat{h}|_{B_n} = h_n$ for $n \geq 1$ and $\hat{h}(x) = x$ for $x \in B \setminus \cup_{n\geq 1} B_n$. Then the map has no decomposition in the conclusion of our theorem.

Example 5. Let $h_n : B_n \to B_n$, $n \geq 1$, be as in Example 3. Attach, for each $n \geq 1$, the origin of B_n to the point $\{1/n\}$ of the unit interval $[0,1]$ on the condition that the diameter of B_n is less than or equal to $1/n$, and the one-dimensional Peano continuum is denoted by C. The map $\check{h} : C \to C$ is defined by $\check{h}|_{B_n} = h_n$ for $n \geq 1$ and $\check{h}(x) = x$ for $x \in C \setminus \cup_{n\geq 1} B_n$. Then the map also has no decomposition in the conclusion of our theorem.

REFERENCES

1. M. Barge and J. Martin, *Chaos, periodicity, and snakelike continua*, Trans. Amer. Math. Soc. **289** (1985), 355–365.

2. M. Barge and J. Martin, *Dense periodicity on the interval*, Proc. Amer. Math. Soc. **94** (1985), 731–735.

3. E.M. Coven and I. Mulvey, *Transitivity and the centre for maps of the circle*, Ergodic Theory Dynam. Systems **6** (1986), 1–8.

4. ———, *The Barge-Martin decomposition theorem for pointwise nonwandering maps of the interval*, Lecture Notes in Math. **1342**, (1988), Springer, Berlin.

5. K. Kuratowski, *Topology* Vol. I, Academic Press, PWN, Warsaw, 1966.

6. R.P. Roe, *Dynamics and indecomposable inverse limit spaces of maps on finite graphs*, Topology Appl. **50** (1993), 117–128.

7. ———, *Dense periodicity on finite trees*, Topology Proc. **19** (1994), 237–248.

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