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ON A CONJECTURE OF KUFNER AND PERSSON

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ABSTRACT. The L^p-L^q boundedness of the (conjugate) Hardy operator $(L^*f)(x) = \int_x^\infty l(t,x)f(t)\,dt$ for the case $0 < q < 1$ has been studied and thereby a conjecture of Kufner and Persson is proved. An important role here is played by level type functions.

1. Introduction. The boundedness of the Hardy operator $(Hf)(x) = \int_0^x f(t)\,dt$ between weighted Lebesgue spaces $L^p((0,\infty),v)$ and $L^q((0,\infty),u)$, $p \in (1,\infty)$, $q \in (0,\infty)$, has been studied quite extensively during the last decades. Simultaneously, as a natural case, the corresponding conjugate Hardy operator $(H^*f)(x) = \int_x^\infty f(t)\,dt$ was also considered and studied. A good account of such work can be found in [**3, 4, 7**]. In order to obtain the boundedness of H^*, generally, two methods are employed. One is using the duality arguments and the other is by making suitable variable transformations in the Hardy inequality

$$\left(\int_0^\infty [(Hf)(x)]^q u(x)\,dx\right)^{1/q} \le C\left(\int_0^\infty f^p(x)v(x)\,dx\right)^{1/p}.$$

In the case $1 < p$, $q < \infty$, the two methods yield the same necessary and sufficient conditions. However, when $0 < q < 1$, we cannot use duality arguments. Moreover, in this case, the proof requires quite a different approach than the other cases. This case is due to Sinnamon [**8**] who made use of "level functions" introduced by Halperin [**2**].

Further, Bloom and Kerman [**1**] and Oinarov [**5, 6**], see also [**3**], studied the boundedness of the generalized Hardy operator $(Lf)(x) = \int_0^x l(x,t)f(t)\,dt$ and its corresponding conjugate operator $(L^*f)(x) = \int_x^\infty l(t,x)f(t)\,dt$ involving the so-called "Oinarov kernel" $l(x,t)$. It

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can be seen that, for any p, q, the L^p-L^q boundedness of L^* cannot be obtained by variable transformations in the corresponding Hardy inequality. Here also the case $0 < q < 1$ needs special attention. In this case, as before, the boundedness of L has been obtained using level functions and the boundedness of L^* was still open since duality arguments are already ruled out for this case. However, Kufner and Persson [**3**, page 113] gave a conjecture in this regard. The aim of this paper is to give a proof of their conjecture, see Conjecture 3.1 below. The technique used in the proof is similar to the one used by Stepanov [**10, 11**], cf. also the technique of proof in [**3**, pages 110–114].

2. C-level intervals and C-level functions. As mentioned in Section 1, the L^p-L^q boundedness of the operators H and L, for the case $0 < q < 1$, requires the use of level functions. As regards the boundedness of L^*, Halperin's theory of level functions is not adequate. In fact, the construction of level functions is based on the "left part" of the interval (a, b) whereas what we require is a level function which is based on the "right part" of it. So, in this section, we develop this theory and give certain results similar to the ones available in the framework of level functions.

Remark 2.1. In this section, we shall define some level type intervals (would be called complementary level intervals or C-level intervals) and level type functions (would be called complementary level functions or C-level functions) and state some of the properties possessed by these intervals (functions) similar to level intervals (functions). The proofs of the results presented in this section can be obtained exactly on the same lines as those of Halperin's results [**2**]. Also, in [**9**], Sinnamon has developed the idea of level functions in a more general setting and, in a personal communication, he informed us that his idea of level functions is independent of the "left" or "right" part of the interval considered. Moreover, the primary aim of this paper is not to develop this section but to apply these results in the subsequent sections. Therefore, we prefer not to give the proofs.

Let us fix some notation and terminology. Let $-\infty \leq a, b \leq \infty$. By a weight function on (a, b), we shall mean a function which is measurable and positive almost everywhere on (a, b). Let f be a nonnegative measurable function on (a, b). The symbol $f(a, b)$ will denote the integral $\int_a^b f(t)\,dt$. Similarly, for a weight function w, the symbol $w(a, b)$ will have the corresponding meaning. For a weight w on (a, b), we shall write

$$R(a,b) = \begin{cases} f(a,b)/w(a,b) & \text{if } \int_a^b w(t)\,dt < \infty \\ \lim_{s\to b}\sup f(a,s)/w(a,s) & \text{if } \int_a^b w(t)\,dt = \infty. \end{cases} \tag{2.1}$$

We begin with the following:

Definition 2.2. Let f be a nonnegative measurable function on (a, b) and w a weight function on (a, b). An interval $(\alpha, \beta) \subset (a, b)$ is called a C-level interval of f with respect to w, written $(\alpha, \beta) \in C = C(a, b, f, w)$, if

$$R(x, \beta) \le R(\alpha, \beta), \quad \text{for every} \quad x \in (\alpha, \beta).$$

If the C-level interval (α, β) is not contained in a larger C-level interval, then it is called a *maximal C-level interval*, written $(\alpha, \beta) \in C_M = C_M(a, b, f, w)$.

We have the following results:

Theorem 2.3. *Every C-level interval is contained in a maximal C-level interval.*

Theorem 2.4. *Let $I_k \in C$ with $I_k \cap I_{k+1} \neq \varnothing$, $k = 1, 2, \dots$. Then*

$$\bigcup_k I_k \in C.$$

Remark 2.5. In view of Theorems 2.3 and 2.4, the system $C_M = C_M(a,b,f,w)$ of all maximal C-level intervals is either empty or it is a denumerable system of nonoverlapping intervals. Thus, if we assume that $C_M \neq \varnothing$, then

$$C_M = \{I_n = (a_n, b_n); n = 1, 2, \dots\},$$

with $I_i \cap I_j = \varnothing$, $i \neq j$.

Definition 2.6. Let f be a nonnegative measurable function on (a,b) and

$$I = \begin{cases} \cup_n I_n & \text{if } C_M \neq \varnothing \\ \varnothing & \text{if } C_M = \varnothing, \end{cases}$$

where I_ns are the intervals from Remark 2.5. We define the *C-level function* f_c of f with respect to a weight function w on (a,b) by

$$f_c(x) = \begin{cases} R(a_n, b_n)w(x) & \text{if } x \in I_n \\ f(x) & \text{if } x \in (a,b)\backslash I. \end{cases}$$

Similar to the notation (2.1), we shall write

$$R_c(a,b) = \begin{cases} f_c(a,b)/w(a,b) & \text{if } \int_a^b w(t)\,dt < \infty \\ \lim_{s\to b} \sup f_c(a,s)/w(a,s) & \text{if } \int_a^b w(t)\,dt = \infty. \end{cases}$$

Lemma 2.7. *Let $C_M = C_M(a,b,f,w) \neq \varnothing$, f_c the C-level function of f and $I_n = (a_n, b_n)$ a maximal C-level interval. Then*

(i) $f(x, b_n) \leq f_c(x, b_n)$, $x \in I_n$.

(ii) $f(a_n, b_n) = f_c(a_n, b_n)$.

(iii) $f(x, \beta) \leq f_c(x, \beta)$, $\beta \in (a,b)\backslash I$, $x \in (a, \beta)$.

(iv) $f(x,\beta) = f_c(x,\beta)$, $x, \beta \in (a,b)\backslash I$, $x < \beta$.

We also have the following :

Theorem 2.8. *Let f_c be the C-level function of f. Then*

(i) *Every C-level interval of f is a C-level interval of f_c, i.e., $C(a,b,f,w) \subset C(a,b,f_c,w)$.*

(ii) *Every maximal C-level interval of* f_c *is a C-level interval of* f, *i.e.*, $C_M(a,b,f_c,w) \subset C(a,b,f,w)$.

(iii) *The functions* f *and* f_c *have the same maximal C-level intervals.*

(iv) *For each C-level interval* J *of* f_c, *there exists a constant* K (*depending upon* J) *such that* $f_c(x) = Kw(x)$, $x \in J$.

(v) $(f_c)_c = f_c$.

Theorem 2.9. *Let* $(\alpha,\beta) \subset (a,b)$ *and* $x \in (\alpha,\beta)$. *Then*

(i) $R_c(\alpha,\beta) \le R_c(x,\beta)$.

(ii) $R_c(\alpha,x) \le R_c(\alpha,\beta)$.

Lemma 2.10. *Let* (α_1,β_1), $(\alpha_2,\beta_2) \subset (a,b)$ *such that* $\alpha_1 \le \alpha_2$, $\beta_1 \le \beta_2$. *Then*

$$R_c(\alpha_1,\beta_1) \le R_c(\alpha_2,\beta_2).$$

Theorem 2.11. *Let* f_c *be the C-level function of* f *with respect to* w. *Then for almost every* $x \in (a,b)$, $f_c(x)/w(x)$ *is a nondecreasing function.*

3. The conjecture. For a weight function u on $(0,\infty)$, we shall denote by $L^p((0,\infty),u)$, $1 \le p < \infty$, the weighted Lebesgue space which is the set of all measurable functions f defined on Ω such that

$$\|f\|_{p,(0,\infty),u} := \left(\int_0^\infty |f(x)|^p u(x)\, dx \right)^{1/p} < \infty.$$

It is known that for $1 \le p < \infty$, $L^p((0,\infty),u)$ is a Banach space and for $1 < p < \infty$, it is reflexive too.

Consider the generalized Hardy operator

$$L : L^p((0,\infty),v) \longrightarrow L^q((0,\infty),u)$$

defined by

$$(Lf)(x) := \int_0^x l(x,t) f(t)\, dt, \quad x > 0,$$

where the kernel $l(x,t)$ is defined for $0 < t < x < \infty$ and $l(x,t) \ge 0$.

The conjugate operator L^* to L is given by

$$(L^*g)(x) := \int_x^\infty l(t,x)g(t)\,dt, \quad x > 0.$$

For the sake of convenience we shall use the following notations. We denote for $s \geq 0$

$$(L_s f)(x) := \int_0^x l^s(x,t)f(t)\,dt,$$

and

$$(L_s^* g)(x) := \int_x^\infty l^s(t,x)g(t)\,dt.$$

For example, L_0 is the standard Hardy operator $\int_0^x f(t)\,dt$.

The aim of this section is to prove the following conjecture of Kufner and Persson [**3**, page 113] regarding the boundedness of the operator L^* for the case $0 < q < 1 < p < \infty$.

Conjecture 3.1. *Suppose $0 < q < 1 < p < \infty$ and denote $1/r = 1/q - 1/p$. Let L^* be defined by*

$$(L^*g)(x) := \int_x^\infty l(t,x)g(t)\,dt, \quad x > 0;$$

where the kernel $l(t,x)$ is defined for $0 < x < t < \infty$, $l(t,x) \geq 0$, and is increasing in the first variable, i.e.,

$$l(t_1,x) \leq l(t_2,x), \quad \text{for} \quad 0 < t_1 < t_2. \tag{3.1}$$

If

$$B_1^* := \left(\int_0^\infty (L_0^* v^{1-p'})^{r/q'}(t)(L_q u)^{r/q}(t) v^{1-p'}(t)\,dt \right)^{1/r} < \infty,$$

then

$$\|L^*g\|_{q,(0,\infty),u} \le C\|g\|_{p,(0,\infty),v} \tag{3.2}$$

for all $g \ge 0$ *with* $C \le B_1^*$.

Conversely, if (3.2) *holds for all* $g \ge 0$, *then* $B_2^* \le C < \infty$ *where*

$$B_2^* := \left(\int_0^\infty (L_q u)^{p'/q}(t) v^{1-p'}(t)\, dt\right)^{1/p'}.$$

The boundedness of the operator L for the case $0 < q < 1 < p < \infty$ has been studied by Sinnamon [**8**], see also [**3**]. As mentioned in Section 1, the boundedness of the operator L^* cannot be obtained using duality arguments applied on the boundedness of L since $q < 1$. So, a direct approach is needed. In this direction, the basic tools have been developed in Section 2 in terms of C-level functions. Using those tools, we first prove the following important lemma which is needed in order to prove Conjecture 3.1.

Lemma 3.2. *Let* (a, b) *be an interval and* w *a weight function on* (a, b). *Suppose that* $\int_a^b w(x)\, dx < \infty$. *Then for each measurable function* $f \ge 0$ *there exists a nonnegative function* g *such that*

(i) $\int_x^b f(t)\, dt \le \int_x^b g(t)\, dt$, $x \in (a, b)$.

(ii) $g(x)/w(x)$ *is increasing on* (a, b).

(iii) $\int_a^b (g(x)/w(x))^p w(x)\, dx \le \int_a^b (f\ (x)/w(x))^p w(x)\, dx$, $p \ge 1$.

Proof. Let f_c be the C-level function of f with respect to w. We show that $g \equiv f_c$ satisfies (i)–(iii). We find that (i) and (ii) follow immediately by using Lemma 2.7 and Theorem 2.11, respectively.

Further, if $x \in (a, b) \setminus I$, then $f_c(x) = f(x)$. Therefore it suffices to show that

$$\int_{a_n}^{b_n} \left(\frac{f_c(x)}{w(x)}\right)^p w(x)\, dx \le \int_{a_n}^{b_n} \left(\frac{f(x)}{w(x)}\right)^p w(x)\, dx,$$

where (a_n, b_n) is a maximal C-level interval of f with respect to w. The case $p = 1$ is disposed of by Lemma 2.7 (ii). For $p > 1$, put

$$J_n = \int_{a_n}^{b_n} \left(\frac{f_c(x)}{w(x)} \right)^p w(x)\, dx.$$

If $J_n = 0$, we are done trivially. Otherwise, since

$$f_c(x) = c_n w(x) \quad \text{with} \quad c_n = \left(\int_{a_n}^{b_n} f(t)\, dt \right) \Big/ \left(\int_{a_n}^{b_n} w(t)\, dt \right),$$

using Lemma 2.7 and Hölder's inequality, we have

$$\begin{aligned} J_n &= \int_{a_n}^{b_n} c_n^{p-1} f_c(x)\, dx \\ &= \int_{a_n}^{b_n} c_n^{p-1} f(x)\, dx \\ &= \int_{a_n}^{b_n} \left(\frac{f_c(x)}{w(x)} \right)^{p-1} w^{1/p'}(x)\, \frac{f(x)}{w(x)} w^{1/p}(x)\, dx \\ &\le J_n^{1/p'} \left(\int_{a_n}^{b_n} \left(\frac{f(x)}{w(x)} \right)^p w(x)\, dx \right)^{1/p} \end{aligned}$$

and (iii) follows. □

Proof of Conjecture 3.1. Suppose that (3.2) is satisfied for all functions $g \ge 0$. Then, by the reverse Minkowski integral inequality, we have

$$\begin{aligned} C\|g\|_{p,(0,\infty),v} &\ge \left(\int_0^\infty u(x) \left(\int_x^\infty l(t,x) g(t)\, dt \right)^q dx \right)^{1/q} \\ &\ge \int_0^\infty g(t) \left(\int_0^t l^q(t,x) u(x)\, dx \right)^{1/q} dt, \end{aligned}$$

and the duality of $L^p((0,\infty),v)$ gives

$$\begin{aligned} C &\geq \sup_{\|g\|_{p,(0,\infty),v}=1} \int_0^\infty g(t)\bigg(\int_0^t l^q(t,x)u(x)\,dx\bigg)^{1/q} dt \\ &= \sup_{\|g\|_{p,(0,\infty),v}=1} \int_0^\infty g(t)(L_q u)^{1/q}(t)\,dt \\ &= \|(L_q u)^{1/q}\|_{p',(0,\infty),v^{1-p'}} \\ &= \bigg(\int_0^\infty (L_q u)^{p'/q}(t) v^{1-p'}(t)\,dt\bigg)^{1/p'} \\ &= B_2^*. \end{aligned}$$

To prove the other implication, we may assume, without any loss of generality, that g is compactly supported in $(0,\infty)$ and that $\int_0^\infty v^{1-p'} < \infty$. Using Lemma 3.2 (i), we get

$$\begin{aligned} (L^*g)(x) &= \int_x^\infty l(t,x)g(t)\,dt = \int_x^\infty l(t,x)d\bigg(-\int_t^\infty g(s)\,ds\bigg) \\ &= l(x,x)\int_x^\infty g(s)\,ds + \int_x^\infty \bigg(\int_t^\infty g(s)\,ds\bigg)\,d_t(l(t,x)) \\ &\leq l(x,x)\int_x^\infty g_c(s)\,ds + \int_x^\infty \bigg(\int_t^\infty g_c(s)\,ds\bigg)\,d_t(l(t,x)) \\ &= (L^*g_c)(x), \end{aligned}$$

where g_c is the C-level function of g. Therefore

$$\begin{aligned} &\int_0^\infty (L^*g)^q(x)u(x)\,dx \\ &\quad\leq \int_0^\infty (L^*g_c)^q(x)u(x)\,dx \\ &\quad= \int_0^\infty u(x)(L^*g_c)^{q-1}(x)\int_x^\infty l(t,x)g_c(t)\,dt\,dx \\ &\quad= \int_0^\infty g_c(t)\int_0^t l(t,x)u(x)\bigg[\bigg(\int_x^t + \int_t^\infty\bigg)l(s,x)g_c(s)\,ds\bigg]^{q-1} dx\,dt \\ &\quad\leq \int_0^\infty g_c(t)\int_0^t l(t,x)u(x)\bigg[\int_t^\infty l(s,x)g_c(s)\,ds\bigg]^{q-1} dx\,dt \\ &\quad\leq \int_0^\infty g_c(t)\int_0^t l^q(t,x)u(x)\bigg[\int_t^\infty g_c(s)\,ds\bigg]^{q-1} dx\,dt \end{aligned}$$

because $l(s,x) \geq l(t,x)$ for $t \leq s < \infty$ and $0 < q < 1$. By Lemma 3.2 (ii) with $w = v^{1-p'}$,

$$\int_t^\infty g_c(s)\,ds = \int_t^\infty \frac{g_c(s)}{v^{1-p'}(s)} v^{1-p'}(s)\,ds \geq g_c(t)v^{p'-1}(t)\int_t^\infty v^{1-p'}(s)\,ds$$

so that

$$\int_0^\infty u(x)(L^*g)^q(x)\,dx \leq \int_0^\infty (g_c(t))^q v^{(p'-1)(q-1)-q/p}(t)v^{q/p}(t) \times \left(\int_t^\infty v^{1-p'}(s)\,ds\right)^{q-1}(L_q u)(t)\,dt.$$

Since $((p'-1)(q-1)-q/p)r/q = ((p'-1/q')-(1/p))r = 1-p'$, Hölder's inequality with exponents p/q and $(p/q)' = r/q$ yields

$$\begin{aligned}\int_0^\infty & u(x)(L^*g)^q(x)\,dx \\ &\leq \left(\int_0^\infty (g_c)^p(x)v(x)\,dx\right)^{q/p}\left(\int_0^\infty v^{1-p'}(x) \times \left(\int_x^\infty v^{1-p'}(t)\,dt\right)^{(q-1)(r/q)}(L_q u)^{r/q}(x)\,dx\right)^{q/r} \\ &= \left(\int_0^\infty (g_c)^p(x)v(x)\,dx\right)^{q/p} \times \left(\int_0^\infty v^{1-p'}(x)(L_0^*v^{1-p'})^{r/q'}(x)(L_q u)^{r/q}(x)\,dx\right)^{q/r} \\ &= \|g_c\|^q_{p,(0,\infty),v}.(B_1^*)^q.\end{aligned}$$

The result now follows from the fact that due to Lemma 3.2 (iii) we have

$$\begin{aligned}\|g_c\|^p_{p,(0,\infty),v} &= \int_0^\infty (g_c)^p(x)v(x)\,dx = \int_0^\infty \left(\frac{g_c(x)}{v^{1-p'}(x)}\right)^p v^{(1-p')p+1}(x)\,dx \\ &= \int_0^\infty \left(\frac{g_c(x)}{v^{1-p'}(x)}\right)^p v^{1-p'}(x)\,dx \\ &\leq \int_0^\infty \left(\frac{g(x)}{v^{1-p'}(x)}\right)^p v^{1-p'}(x)\,dx \\ &= \int_0^\infty g^p(x)v(x)\,dx = \|g\|^p_{p,(0,\infty),v}.\end{aligned}$$

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