THE ASYMPTOTIC
BROWDER HARTMAN STAMPACCHIA CONDITION
AND INTERIOR BANDS OF ε-SOLUTIONS
FOR NONLINEAR COMPLEMENTARITY PROBLEMS

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ABSTRACT. In this paper we study two remarkable properties of the interior band of ε-solutions of a general nonlinear complementarity problem in the n-dimensional Euclidean space. The study is based on the notion of scalar derivative, on the notion of infinitesimal interior point ε-exceptional family of elements for a function and on the asymptotic Browder Hartman Stampacchia condition.

1. Introduction. Let $(\mathbf{R}^n, \langle \cdot, \cdot \rangle)$ be the n-dimensional Euclidean space ordered by the closed pointed convex cone $\mathbf{R}^n_+$ and $f : \mathbf{R}^n \to \mathbf{R}^n$ a continuous function. We consider the following nonlinear complementarity problem:

$$NCP(f, \mathbf{R}^n_+) : \begin{cases} \text{find } x^0 \in \mathbf{R}^n_+ \text{ such that} \\ f(x^0) \in \mathbf{R}^n_+ \text{ and } \langle x^0, f(x^0) \rangle = 0. \end{cases}$$

It is well known that the $NCP(f, \mathbf{R}^n_+)$ has many applications in optimization, economics, engineering, game theory and mechanics [**6, 10, 14, 16, 17**].

We note that there exist several equivalent formulations of the $NCP(f, \mathbf{R}^n_+)$. In particular, several formulations are in the form of a nonlinear equation of the form $F(x) = 0$, where $F : \mathbf{R}^n \to \mathbf{R}^n$ is a continuous function. By using such formulations, several techniques proposed by some authors are based on the idea to perturb F to a certain $F(x, \varepsilon)$, where ε is a positive parameter and to consider the equation $F(x, \varepsilon) = 0$. If $F(x, \varepsilon) = 0$ has a unique solution,

denoted by $x(\varepsilon)$ and $x(\varepsilon)$ is continuous in ε, then the solutions de-
scribe (depending on the properties of F) a short path denoted by
$\{x(\varepsilon) : \varepsilon \in \,]0, \varepsilon_0]\}$ or a long path $\{x(\varepsilon) : \varepsilon \in \,]0, \infty[\}$. We note that, if
a short path $\{x(\varepsilon) : \varepsilon \in \,]0, \varepsilon_0]\}$ is bounded, then for any sequence $\{\varepsilon_k\}$
with $\{\varepsilon_k\} \to 0$, the sequence $\{x(\varepsilon_k)\}$ has at least one accumulation
point, which by continuity is a solution to the $NCP(f, \mathbf{R}^n_+)$. Based
on this fact, several numerical methods for solving the $NCP(f, \mathbf{R}^n_+)$
have been developed, as for example the interior point path-following
methods, regularization methods and noninterior path-following meth-
ods among others. About such methods the reader can see the papers
[1, 2, 3–5, 8–15, 21–25, 28]. The most common interior point path-
following method is based on the notion of *central path*. We recall [29]
that the curve $\{x(\varepsilon) : \varepsilon \in \,]0, \infty[\}$ is said to be the central path if for
each $\varepsilon > 0$ the vector $x(\varepsilon)$ is the unique solution to the system

$$
(1) \qquad \begin{cases} x(\varepsilon) > 0, \ f(x(\varepsilon)) > 0 \\ \text{and } X(\varepsilon)f(x(\varepsilon)) = \varepsilon e, \end{cases}
$$

where the inequality "$>$" means that the components of the vector
are strictly positive, $e = (1, \ldots, 1)^T$, $X(\varepsilon) = $ the matrix $\operatorname{diag}(x(\varepsilon))$
and $x(\varepsilon)$ is continuous on $]0, \infty[$. It is well known that, for a general
$NCP(f, \mathbf{R}^n_+)$, the system (1) may have multiple solutions for a given
$\varepsilon > 0$, and even if the solution is unique it is not necessarily continuous
in ε.   Therefore, the existence of the central path is not always
guaranteed.

We consider in this paper the multi-valued mapping $\mathcal{U} : \,]0, \infty[\, \to$
$\mathcal{S}(\mathbf{R}^n_{++})$ defined by

$$
\mathcal{U}(\varepsilon) = \{x \in \mathbf{R}^n_{++} : f(x) > 0, Xf(x) = \varepsilon e\},
$$

where $X = $ the matrix $\operatorname{diag}(x)$, $\mathcal{S}(\mathbf{R}^n_{++})$ is the collection of all subsets
of $\mathbf{R}^n_{++}$ and $\mathbf{R}^n_{++} = \{x = (x_1, \ldots, x_n) : x_1 > 0, \ldots, x_n > 0\}$. We say
that $\mathcal{U}$ is the *interior band mapping*. The multi-valued mapping $\mathcal{U}$ was
studied from several points of view in [29].

Now, the main goal of our paper is to study under what conditions
the multi-valued mapping $\mathcal{U}$ has the following properties:

(a) $\mathcal{U}(\varepsilon) \neq \varnothing$ for each $\varepsilon \in \,]0, \infty[$.

(b) For any fixed $\varepsilon_0 > 0$, the set $\cup_{\varepsilon \in \,]0, \varepsilon_0]} \mathcal{U}(\varepsilon)$ is bounded.

Conditions (a) and (b) were defined in [**29**].

Our study is based on the *asymptotic Browder Hartman Stampacchia condition*, on the notion of *scalar derivative* and the notion of *infinitesimal interior point ε-exceptional family of elements* for the *inversion of f*. By the results presented in this paper we show another utility of the Browder Hartman Stampacchia condition, never put in evidence by other authors.

2. Preliminaries

For more details about the notions and the results presented in this section, the reader is referred to [**29**].

Definition 2.1. Let $f : \mathbf{R}^n \to \mathbf{R}^n$ be a continuous function. Given a scalar $\varepsilon > 0$ we say that a family $\{x^r\}_{r>0} \subset \mathbf{R}^n_{++}$ is an *interior point ε-exceptional family* for f if $\|x^r\| \to \infty$ as $r \to \infty$ and for each x^r there exists a positive number $0 < \mu_r < 1$ such that

$$(2) \qquad f_i(x^r) = \frac{1}{2}\left(\mu_r - \frac{1}{\mu_r}\right)x_i^r + \frac{\varepsilon\mu_r}{x_i^r} \quad \text{for all} \quad i = 1, 2, \ldots, n.$$

Theorem 2.1. *Let $f : \mathbf{R}^n \to \mathbf{R}^n$ be a continuous function. Then for each $\varepsilon > 0$ there exists either a point $x(\varepsilon)$ such that*

$$(3) \quad x(\varepsilon) > 0, \quad f(x(\varepsilon)) > 0, \quad x_i(\varepsilon)f_i(x(\varepsilon)) = \varepsilon, \quad i = 1, 2, \ldots, n.$$

or an interior point ε-exceptional family for f.

Proof. This result is proved in [**29**]. $\square$

Let $K \subset \mathbf{R}^n$ be a closed pointed convex cone and $f : \mathbf{R}^n \to \mathbf{R}^n$ a continuous function. The following notion was used in [**17, 18**].

Definition 2.2. We say that a family of elements $\{x^r\}_{r>0} \subset K$ is an *exceptional family of elements*, denoted by EFE, for f with respect to K if $\|x^r\| \to \infty$ as $r \to \infty$ and for every real number $r > 0$, there

exists a real number $\mu_r > 0$ such that the vector $u^r = \mu_r x^r + f(x^r)$ satisfies the following conditions:

(1) $u^r \in K^*$ (the dual of K),

(2) $\langle u^r, x^r \rangle = 0$.

If in the definition of the problem $NCP(f, \mathbf{R}_+^n)$ we replace the cone $\mathbf{R}_+^n$ by the cone K, we obtain the problem $NCP(f, K)$, i.e.,

$$NCP(f, K) : \begin{cases} \text{find } x^0 \in K \text{ such that} \\ f(x^0) \in K^* \text{ and } \langle x^0, f(x^0) \rangle = 0. \end{cases}$$

Theorem 2.2. *If $f : \mathbf{R}^n \to \mathbf{R}^n$ is a continuous function and $K \subset \mathbf{R}^n$ is a closed pointed convex cone, then there exists either a solution to the $NCP(f, K)$ or an exceptional family of elements for f with respect to K.*

Proof. A proof of this result is given in [**17**].   □

The following definition and theorem can be found in [**17**].

Definition 2.3. Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space, $K \subset H$ a closed convex cone and $f : H \to H$ a mapping. We say that the mapping f satisfies *condition* Θ with respect to K if

$$(4) \quad \begin{cases} \text{there exists } \rho > 0 \text{ such that for each } x \in K \text{ with } \|x\| > \rho, \\ \text{there exists } p \in K \text{ with } \|p\| < \|x\| \text{ such that} \\ \langle x - p, f(x) \rangle \geq 0. \end{cases}$$

Theorem 2.3. *Let H be a Hilbert space, $K \subset H$ a closed convex cone and $f : H \to H$ a mapping. If f satisfies condition Θ with respect to K, then it is without exceptional family of elements with respect to K.*

3. The Browder Hartman Stampacchia condition. First we recall a general classical result. Let $(E, \|\cdot\|)$ be a reflexive Banach space

and $f : E \to E^*$. We say that f is *hemi-continuous* if it is continuous from the line segment to the weak topology of E^*.

We say that f is *monotone* if for any $x, y \in E$ we have that

$$\langle x - y, f(x) - f(y) \rangle \geq 0,$$

and we say that f is *strongly monotone* if there exists a constant $\alpha > 0$ such that for any $x, y \in E$ we have

$$\langle x - y, f(x) - f(y) \rangle \geq \alpha \|x - y\|^2.$$

Definition 3.1. We say that f satisfies the *Browder Hartman Stampacchia condition*, denoted by BHS, on a closed convex cone $K \subset E$ if there exists $\rho > 0$ such that $\langle x, f(x) \rangle > 0$, for any $x \in K$ with $\|x\| = \rho$.

The following result is to support the importance of condition (BHS).

Theorem 3.1 (Browder Hartman Stampacchia). *Let $(E, \|\cdot\|)$ be a reflexive Banach space, $f : E \to E^*$ a monotone hemi-continuous mapping and $K \subset E$ a closed convex cone. If f satisfies condition BHS on K, then the problem $NCP(f, K)$ has a solution.*

Proof. A proof of this result is given in [**17**, Theorem 4.6]. $\square$

Proposition 3.1. *If $f : E \to E^*$ is strongly monotone on K, then f satisfies condition BHS on K.*

Proof. Indeed, if we take $y = 0$, we have

$$\langle x, f(x) \rangle \geq \langle x, f(0) \rangle + \alpha \|x\|^2.$$

If $f(0) = 0$ we deduce that $\langle x, f(x) \rangle \geq \alpha \|x\|^2$, for any $x \in K$. If we take an arbitrary $\rho > 0$, we have that $\langle x, f(x) \rangle > 0$, for any $x \in K$ with $\|x\| = \rho$.

If $f(0) \neq 0$, then in this case we consider the set

$$D = \left\{ x \in K : \|x\| \leq \frac{\|f(0)\|}{\alpha} \right\},$$

which is nonempty and bounded. For any $x \in K \setminus D$ we have

$$\alpha \|x\|^2 > \|x\| \cdot \|f(0)\| \geq -\langle x, f(0) \rangle,$$

which implies $\langle x, f(x) \rangle > 0$ for any $x \in K \setminus D$. Because D is bounded, there exists $\rho > 0$ such that $D \subset B(0, \rho)$ and for any $x \in K$ with $\|x\| = \rho$ we have $\langle x, f(x) \rangle > 0$. $\square$

By the next result we can obtain many functions which satisfy condition (BHS). Let $K \subset \mathbf{R}^n$ be a closed convex cone. We say that a function $T : K \to \mathbf{R}^n$ satisfies condition (β) if there exists a real number $\beta(T) > 0$ such that for all $x \in K$ with $\|x\| \geq 1$, we have $\|T(x)\| \leq \beta(T)\|x\|$.

Examples. (1) Any linear continuous operator $T : \mathbf{R}^n \to \mathbf{R}$ satisfies condition (β).

(2) If $T : K \to \mathbf{R}^n$ is a k-Lipschitz mapping, then T satisfies condition (β) with $\beta(T) = k + \beta_0$, where $\beta_0 = k\|x^0\| + \|T(x^0)\|$ and x^0 is an arbitrary element in K.

Theorem 3.2.   *Let $f : K \to \mathbf{R}^n$ be a continuous function and $T : K \to \mathbf{R}^n$ a function satisfying condition (β). If the following conditions are satisfied:*

(1) $\liminf_{\|x\| \to \infty} \langle f(x) - T(x), x \rangle / \|x\|^2 \geq k_0 > 0$,

(2) $\beta(T) < k_0$,

then there exists $\rho > 0$ such that $\langle x, f(x) \rangle > 0$, for all $x \in K$ with $\|x\| = \rho$.

Proof. Let $\varepsilon > 0$ be such that $\beta(T) + \varepsilon < k_0$. From assumption (1) we have that there exists $\rho_0 > 0$ such that for all $x \in K$ with $\|x\| > \rho_0$ we have

$$\frac{\langle f(x) - T(x), x \rangle}{\|x\|^2} > k_0 - \varepsilon,$$

which implies
$$\langle f(x) - T(x), x \rangle > (k_0 - \varepsilon)\|x\|^2$$
and finally,
$$\langle f(x), x \rangle > \langle T(x), x \rangle + (k_0 - \varepsilon)\|x\|^2.$$
From the last inequality we obtain

$$\langle f(x), x \rangle \geq -\beta(T)\|x\|^2 + (k_0 - \varepsilon)\|x\|^2 = \|x\|^2(-\beta(T) - \varepsilon + k_0) > 0,$$

for all $x \in K$ with $\|x\| > \rho_0$. If we take $\rho > \rho_0$, the proof is complete.
$\square$

Remark 3.1. Theorem 3.2 is applicable in the following cases

(i) $f(x) = T(x) + ax + b$, where $a > 0$, $b \in \mathbf{R}^n$ is an arbitrary vector and T satisfies condition (β) with $\beta(T) < a$.

(ii) $f(x) = T(x) + L(x) + b$, where $b \in \mathbf{R}^n$ is an arbitrary vector, L is a linear operator from $\mathbf{R}^n$ into $\mathbf{R}^n$ such that $\langle L(x), x \rangle \geq k_0\|x\|^2$, for any $x \in K$ and T satisfies condition β with $\beta(T) < k_0$.

In the n-dimensional Euclidean space Theorem 3.1 has the following form:

Theorem 3.3. *Let $K \subset \mathbf{R}^n$ be a closed convex cone and $f : \mathbf{R}^n \to \mathbf{R}^n$ a continuous function. If there exists $\rho > 0$ such that $\langle x, f(x) \rangle \geq 0$ for any $x \in K$ with $\|x\| = \rho$, then the problem $NCP(f, K)$ has a solution x^* such that $\|x^*\| \leq \rho$.*

Proof. Let T_ρ be the radial retraction onto the ball $B(0, \rho) = \{x \in \mathbf{R}^n : \|x\| \leq \rho\}$, i.e.,

$$T_\rho(x) = \begin{cases} x & \text{if } \|x\| \leq \rho \\ \rho x/\|x\| & \text{if } \|x\| > \rho. \end{cases}$$

It is known that T_ρ is continuous. If we denote by $K_\rho = B(0, \rho) \cap K$, we have that T_ρ is also a continuous retraction of the cone K onto K_ρ.

We denote by $F : \mathbf{R}^n \to \mathbf{R}^n$ the continuous mapping defined by

$$F(x) = f(T_\rho(x)) + \|x - T_\rho(x)\|x.$$

For any $x \in K$ with $\|x\| > \rho$ we have $\langle x, F(x) \rangle > 0$. Indeed, we have

$$
\begin{aligned}
\langle x, F(x) \rangle &= \langle x, f(T_\rho(x)) \rangle + \|x - T_\rho(x)\| \cdot \|x\|^2 \\
&= \left\langle \frac{\|x\|}{\rho} T_\rho(x), f(T_\rho(x)) \right\rangle + \|x - T_\rho(x)\| \cdot \|x\|^2 > 0.
\end{aligned}
$$

Because of the fact that $\langle x, F(x) \rangle > 0$ for any $x \in K$ with $\|x\| > \rho$, it is easy to show that F satisfies condition Θ with respect to K. Hence, by Theorem 2.3, F is without EFE with respect to K. Applying Theorem 2.2, we have that $NCP(F, K)$ has a solution x^* which must satisfy the inequality $\|x^*\| \leq \rho$. Therefore, $F(x^*) = f(x^*)$ and x^* is a solution to $NCP(f, K)$. $\square$

Remark 3.2. Theorem 3.3 is known in complementarity theory, but our proof presented here is different than other proofs.

Remark 3.3. If $f : \mathbf{R}^n \to \mathbf{R}^n$ is a continuous function such that there exists $\rho > 0$ with the property that $\langle x, f(x) \rangle \geq 0$ for any $x \in K$ with $\|x\| = \rho$, then the function $F(x) = f(T_\rho(x)) + \|x - T_\rho(x)\|x$ is such that

$$
\liminf_{\substack{\|x\| \to +\infty \\ x \in K}} \langle x, F(x) \rangle = +\infty.
$$

Indeed, we have, for any $x \in K$ with $\|x\| > \rho$,

$$
\langle x, F(x) \rangle = \langle x, f(T_\rho(x)) + \|x - T_\rho(x)\| \cdot \|x\|^2
$$

$$
\begin{aligned}
&= \frac{\|x\|}{\rho} \langle T_\rho(x), f(T_\rho(x)) \rangle + \|x - T_\rho(x)\| \cdot \|x\|^2 \\
&\geq \|x - T_\rho(x)\| \cdot \|x\|^2 \geq \rho^2 \|x - T_\rho(x)\| \geq \rho^2 [\|x\| - \rho].
\end{aligned}
$$

Computing $\liminf_{\substack{\|x\| \to +\infty \\ x \in K}}$, we obtain that

$$
\liminf_{\substack{\|x\| \to +\infty \\ x \in K}} \langle x, F(x) \rangle = +\infty.   \quad \square
$$

4. The asymptotic Browder Hartman Stampacchia condition. Let $(\mathbf{R}^n, \langle \cdot, \cdot \rangle)$ be the n-dimensional Euclidean space, $K \subset \mathbf{R}^n$ a closed pointed convex cone and $f : \mathbf{R}^n \to \mathbf{R}^n$ a continuous function. We introduce the following condition.

Definition 4.1. We say that f satisfies the *asymptotic Browder Hartman Stampacchia condition*, denoted by ABHS, with respect to K if

$$\liminf_{\substack{\|x\| \to +\infty \\ x \in K}} \langle x, f(x) \rangle = +\infty.$$

The relations between conditions (BHS) and (ABHS) is given by the following result.

Proposition 4.1. *Let $K \subset \mathbf{R}^n$ be a closed pointed convex cone and $f : \mathbf{R}^n \to \mathbf{R}^n$ a continuous function. If f satisfies condition (ABHS), then f satisfies condition (BHS). If f satisfies condition (BHS), then the function $F(x) = f(T_\rho(x) + \|x - T_\rho(x)\|x$ satisfies condition (ABHS).*

Proof. We suppose that f satisfies condition (ABHS), i.e., we have that

$$\liminf_{\substack{\|x\| \to +\infty \\ x \in K}} \langle x, f(x) \rangle = +\infty.$$

In this case, given $r > 0$, there exists $\rho > 0$ such that for any $x \in K$ with $\|x\| = \rho$ we have $\langle x, f(x) \rangle > r$. Indeed, if this is not true, then for any $n \in \mathbf{N}$, there exists $x_n \in K$ with $\|x_n\| = n$ such that $\langle x_n, f(x_n) \rangle \leq r$. Therefore, condition (ABHS) is not satisfied. This contradiction implies that f satisfies condition (BHS). Conversely, if f satisfies condition (BHS), then by Remark 3.3 F satisfies condition (ABHS). $\square$

Corollary 4.1. *If $K \subset \mathbf{R}^n$ is a closed pointed convex cone and $f : \mathbf{R}^n \to \mathbf{R}^n$ is a continuous function satisfying condition (ABHS), then $NCP(f, K)$ has a solution.*

Proof. This corollary is a consequence of Proposition 4.1 and Theorem 3.3.   □

In the following we will show that condition (ABHS) is a good mathematical tool for the study of properties (a) and (b) of the interior band mapping $\mathcal{U}$ with respect to the cone $\mathbf{R}_+^n$.

In regards to the existence of a solution to the $NCP(f, K)$ in r^n we cite also the following result.

Proposition 4.2. *Let $K \subset \mathbf{R}^n$ be a closed pointed convex cone and $f : \mathbf{R}^n \to \mathbf{R}^n$ a continuous function. If f satisfies the condition*

$$(5) \qquad \liminf_{\substack{\|x\| \to \infty \\ x \in K}} \langle x, f(x) \rangle > 0,$$

then $NCP(f, K)$ has a solution.

Proof. It is sufficient to prove that f is without EFE with respect to K. Indeed, if we suppose that f has an EFE, namely, $\{x^r\}_{r>0} \subset K$, then we have

$$\langle x^r, f(x^r) \rangle = \langle x^r, u^r - \mu_r x^r \rangle = \langle x^r, u^r \rangle - \mu_r \langle x^r, x^r \rangle = -\mu_r \|x^r\|^2 < 0,$$

which implies

$$\liminf_{\|x^r\| \to \infty} \langle x^r, f(x^r) \rangle \leq 0.$$

This relation is impossible because we supposed condition (5).   □

Theorem 4.1. *If $f : \mathbf{R}_+^n \to \mathbf{R}^n$ is a continuous function, then*

$$\liminf_{\substack{\|x\| \to \infty \\ x \in \mathbf{R}_{++}^n}} \langle x, f(x) \rangle = +\infty$$

if and only if

$$\liminf_{\substack{\|x\| \to \infty \\ x \in \mathbf{R}_+^n}} \langle x, f(x) \rangle = +\infty.$$

Proof. Obviously if

$$\liminf_{\substack{\|x\| \to \infty \\ x \in \mathbf{R}_+^n}} \langle x, f(x) \rangle = +\infty,$$

then

$$\liminf_{\substack{\|x\| \to \infty \\ x \in \mathbf{R}_{++}^n}} \langle x, f(x) \rangle = +\infty.$$

The converse follows if we show that if

$$\liminf_{\substack{\|x\| \to \infty \\ x \in \mathbf{R}_{++}^n}} \langle x, f(x) \rangle = +\infty,$$

then

$$\liminf_{\substack{\|x\| \to \infty \\ x \in \partial\mathbf{R}_+^n}} \langle x, f(x) \rangle = +\infty.$$

Let $\{x^n\}$ be a sequence such that $\|x^n\| \to \infty$ as $n \to \infty$ and for any $n \in \mathbf{N}$, $x^n \in \partial\mathbf{R}_+^n$. Let n be fixed. Since $x^n \in \partial\mathbf{R}_+^n$, there exists $\{y^m\} \subset \mathbf{R}_{++}^n$ such that $\{y^m\} \to x^n$ as $m \to \infty$. For any $n \in \mathbf{N}$ we select such a sequence $\{y^m\}$. Let $\varepsilon_0 > 0$ be an arbitrary real number. For any $n \in \mathbf{N}$ we can select the sequence $\{y^m\}$ such that $\|y^m - x^n\| < \varepsilon_0$, for any $m \in \mathbf{N}$. We can suppose that for any $n \in \mathbf{N}$, $\|f(x^n)\| > 0$. Because $\lim_{m \to \infty} \|x^n - y^m\| = 0$, there exists $m_1 \in \mathbf{N}$ such that

$$\|x^n - y^m\| < \frac{1}{\|f(x^n)\|},$$

for any $m > m_1$.

Because f is continuous, there exists an $m_2 \in \mathbf{N}$ such that

$$\|f(x^n) - f(y^m)\| < \frac{1}{\varepsilon_0 + \|x^n\|},$$

for any $m > m_2$. For any $m > \max\{m_1, m_2\}$ we have

$$\|y^m\| \le \|x^n - y^m\| + \|x^n\| < \varepsilon_0 + \|x^n\|$$

and

$$\|f(x^n) - f(y^m)\| < \frac{1}{\varepsilon_0 + \|x^n\|} < \frac{1}{\|y^m\|}.$$

We take an arbitrary $m > \max\{m_1, m_2\}$ and we have

$$\|x^n - y^m\| < \frac{1}{\|f(x^n)\|}$$

and

$$\|f(x^n) - f(y^m)\| < \frac{1}{\|y^m\|}.$$

If we denote by $y^{n,*} = y^m$, then we have

$$\|x^n - y^{n,*}\| < \frac{1}{\|f(x^n)\|}$$

and

$$\|f(x^n) - f(y^{n,*})\| < \frac{1}{\|y^{n,*}\|}.$$

We have

$$\langle y^{n,*}, f(y^{n,*})\rangle = [\langle y^{n,*}, f(y^{n,*})\rangle - \langle x^n, f(x^n)\rangle] + \langle x^n, f(x^n)\rangle$$

and

$$\begin{aligned}
\|\langle y^{n,*}, &f(y^{n,*})\rangle - \langle x^n, f(x^n)\rangle\| \\
&\leq \|\langle x^n - y^{n,*}, f(x^n)\rangle\| + \|\langle y^{n,*}, f(x^n) - f(y^{n,*})\rangle\| \\
&< \frac{1}{\|f(x^n)\|} \cdot \|f(x^n)\| + \|y^{n,*}\| \cdot \frac{1}{\|y^{n,*}\|} = 2.
\end{aligned}$$

Therefore,

(6) $$\langle y^{n,*}, f(y^{n,*})\rangle < 2 + \langle x^n, f(x^n)\rangle.$$

Since $\|x^n\| - \|y^{n,*}\| \leq \|x^n - y^{n,*}\| < \varepsilon_0$, for any $n \in \mathbf{N}$, we have $\|x^n\| < \varepsilon_0 + \|y^{n,*}\|$, for any $n \in \mathbf{N}$, which implies $\|y^{n,*}\| \to +\infty$, because $\|x^n\| \to +\infty$.

Computing lim inf in (6), we have

$$\liminf_{n \to \infty} \langle x^n, f(x^n) \rangle = +\infty.$$

Therefore,

$$\liminf_{\substack{\|x\| \to +\infty \\ x \in \partial \mathbf{R}_+^n}} \langle x, f(x) \rangle = +\infty$$

and the proof is complete.   □

5. Scalar derivatives. Let $C \subseteq \mathbf{R}^n$ be a set which contains at least one nonisolated point: $F : \mathbf{R}^n \to \mathbf{R}^n$ and x_0 a nonisolated point of C. The following definition is an extension of Definition 2.2 [**26**]:

Definition 5.1. The limit

$$\underline{F}^{\#,C}(x_0) = \liminf_{\substack{x \to x_0 \\ x \in C}} \frac{\langle F(x) - F(x_0), x - x_0 \rangle}{\|x - x_0\|^2}$$

is called the *lower scalar derivative* of f at x_0 along C. Taking lim sup in place of lim inf, we can define the upper scalar derivative $\overline{F}^{\#,C}(x_0)$ of F at x_0 along C similarly.

Scalar derivatives were studied in [**26, 27**] and successfully applied to fixed point theorems in [**19, 20**].

6. Inversions. The following definition can be found in [**7**, Example 5.1, page 169]:

Definition 6.1. The operator

$$i : \mathbf{R}^n \setminus \{0\} \longrightarrow \mathbf{R}^n \setminus \{0\}; \quad i(x) = \frac{x}{\|x\|^2}$$

is called *inversion* (of pole 0).

It is easy to see that i is one to one and $i^{-1} = i$.

Let $f : \mathbf{R}^n \to \mathbf{R}^n$ be a mapping. Since $\mathbf{R}^n \setminus \{0\}$ is an invariant set of i, the following definition makes sense.

Definition 6.2. The *inversion* (of pole 0) of the mapping f is the mapping $\mathcal{I}(f) : \mathbf{R}^n \to \mathbf{R}^n$ defined by:

$$\mathcal{I}(f)(x) = \begin{cases} \|x\|^2 (f \circ i)(x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

It is easy to see that the inversion of mappings $\mathcal{I}$ is a one to one operator on the set of mappings $\{f \mid f : \mathbf{R}^n \to \mathbf{R}^n; f(0) = 0\}$ and $\mathcal{I}^{-1} = \mathcal{I}$, i.e., $\mathcal{I}(\mathcal{I}(f)) = f$.

The properties of inversions were studied in detail in [**19**].

7. Infinitesimal interior point ε exceptional families.

Definition 7.1. Let $g : \mathbf{R}^n \to \mathbf{R}^n$ be a function. Given a scalar $\varepsilon > 0$, we say that a sequence $\{y^r\}_{r>0} \subset \mathbf{R}^n_{++}$ is an *infinitesimal interior point ε exceptional family* for g if $\|y^r\| \to 0$ as $r \to +\infty$ and for each y^r there exists a positive number $0 < \mu_r < 1$ such that

$$(7) \qquad g_l(y^r) = \frac{1}{2}\left(\mu_r - \frac{1}{\mu_r}\right)y^r_l + \frac{\varepsilon \mu_r}{y^r_l}\|y^r\|^4,$$

for all $l = 1, 2, \ldots, n$.

Proposition 7.1. *If $f : \mathbf{R}^n \to \mathbf{R}^n$ is a continuous function and $g = \mathcal{I}(f)$ is the inversion of f, then $\{x^r\} \subset \mathbf{R}^n_{++}$ is an interior point ε-exceptional family for f if and only if $\{y^r\} \subset \mathbf{R}^n_{++}$ is an infinitesimal interior point ε-exceptional family for g, where $y^r = i(x^r)$ is the inversion of x^r, for all $r > 0$.*

Proof. Suppose that $\{x^r\}_{r>0} \subset \mathbf{R}^n_{++}$ is an interior point ε-exceptional family for f, and let

$$(8) \qquad\qquad y^r = i(x^r),$$

for all $r > 0$. Since $i^{-1} = i$, equations (2) and (8) imply that

$$(9) \qquad f_l(i(y^r)) = \frac{1}{2}\left(\mu_r - \frac{1}{\mu_r}\right)i(y^r)_l + \frac{\varepsilon\mu_r}{i(y^r)_l},$$

for all $l = 1, 2, \ldots, n$. Multiplying both sides of equation (9) by $\|y^r\|^2$ we obtain equation (7). Hence, $\{y^r\}_{r>0} \subset \mathbf{R}_{++}^n$ is an infinitesimal interior point ε-exceptional family for g. Similarly it can be proved that if $\{y^r\}_{r>0} \subset \mathbf{R}_{++}^n$ is an infinitesimal interior point ε-exceptional family for g, then $\{x^r\}_{r>0} \subset \mathbf{R}_{++}^n$ is an interior point ε-exceptional family for f. $\quad\square$

Theorem 7.1. *Let $f : \mathbf{R}^n \to \mathbf{R}^n$ be a continuous function and $\varepsilon > 0$. If there is no infinitesimal interior point ε-exceptional family for $g = \mathcal{I}(f)$, then there exists a point $x(\varepsilon)$ such that*

$$(10) \qquad x(\varepsilon) > 0, \ f(x(\varepsilon)) > 0, \ x_l(\varepsilon)f_l(x(\varepsilon)) = \varepsilon,$$

for all $l = 1, 2, \ldots, n$.

Proof. Suppose, to the contrary, that there is no point $x(\varepsilon)$ which satisfies relation (10). Then, by Theorem 2.1, the function f has an interior point ε-exceptional family $\{x^r\}_{r>0} \subset \mathbf{R}_{++}^n$. Hence, Proposition 7.1 implies that $\{y^r\}_{r>0} \subset \mathbf{R}_{++}^n$ is an infinitesimal interior point ε-exceptional family for g, where $y^r = i(x^r)$, for all $r > 0$. But this is in contradiction with our assumption. $\quad\square$

By Theorem 7.1 it is interesting to find conditions under which the inversion of a continuous function does not possess an infinitesimal interior point ε-exceptional family for all $\varepsilon > 0$. For such functions $\mathcal{U}(\varepsilon) \neq \varnothing$, for each $\varepsilon > 0$.

8. Results related to properties (a) and (b) of the interior band mapping $\mathcal{U}$.

Theorem 8.1. *Let $f : \mathbf{R}^n \to \mathbf{R}^n$ be a continuous function. If*

$$\liminf_{\substack{\|x\| \to +\infty \\ x \in \mathbf{R}_{++}^n}} \langle x, f(x) \rangle = +\infty,$$

then

   (1) *the problem $NCP(f, \mathbf{R}^n_+)$ has a solution,*

   (2) $\mathcal{U}(\varepsilon) \neq \varnothing$*, for any* $\varepsilon > 0$,

   (3) *for any fixed* $\varepsilon_0 > 0$ *the set* $\cup_{\varepsilon \in]0,\varepsilon_0]}\mathcal{U}(\varepsilon)$ *is bounded.*

Proof. (1) By Theorem 4.1 we have that

$$\liminf_{\substack{\|x\| \to +\infty \\ x \in \mathbf{R}^n_+}} \langle x, f(x) \rangle = +\infty$$

and by Proposition 4.1 we have that f satisfies condition (BHS). Applying Theorem 3.3 we obtain that $NCP(f, \mathbf{R}^n_+)$ has a solution.

(2) By using Theorem 2.1 it is sufficient to show that f does not have an interior point ε-exceptional family $\{x^r\}_{r>0} \subset \mathbf{R}^n_{++}$. Indeed, we suppose that f has an interior point ε-exceptional family $\{x^r\}_{r>0} \subset \mathbf{R}^n_{++}$. Multiplying formula (2) given in Definition 2.1 by x^r_l and summing with l from 1 to n, we obtain

$$\langle x^r, f(x^r) \rangle = \frac{1}{2}\left(\mu_r - \frac{1}{\mu_r}\right)\|x^r\|^2 + n\varepsilon\mu_r,$$

where $0 < \mu_r < 1$, for any $r > 0$. From the last equality we deduce

$$\langle x^r, f(x^r) \rangle + \frac{1}{2}\left(\frac{1}{\mu_r} - \mu_r\right)\|x^r\|^2 < n\varepsilon.$$

Let $r_0 > 0$ such that $\|x^{r_0}\| > 0$. Because $\|x^r\| \to +\infty$ as $r \to +\infty$, we can consider a subsequence $\{x^{r_i}\}$ such that $\|x^{r_0}\| < \|x^{r_i}\|$ and $\|x^{r_i}\| \to +\infty$ as $i \to \infty$. For this subsequence we have

$$\frac{1}{2}\left(\frac{1}{\mu_{r_i}} - \mu_{r_i}\right)\|x^{r_0}\|^2 + \langle x^{r_i}, f(x^{r_i}) \rangle < n\varepsilon.$$

Computing $\liminf$ and using the assumption of our Theorem, we obtain a contradiction. Therefore, by Theorem 2.1, we have that $\mathcal{U}(\varepsilon) \neq \varnothing$, for any $\varepsilon > 0$.

(3) We observe that for any $x(\varepsilon) \in \mathcal{U}(\varepsilon)$ we have $\langle x(\varepsilon), f(x(\varepsilon)) \rangle = n\varepsilon$. Now, we suppose that there is an $\varepsilon > 0$ such that $\cup_{\varepsilon \in]0,\varepsilon_0]}\mathcal{U}(\varepsilon)$ is not

bounded. Then, there is a sequence $x(\varepsilon_k) \in \mathcal{U}(\varepsilon_k)$ with $\varepsilon_k \in {]}0, \varepsilon_0]$ which is not bounded. Hence, by the assumption of our theorem we have

$$\liminf_{k \to \infty} \langle x(\varepsilon_k), f(x(\varepsilon_k)) \rangle = +\infty.$$

On the other hand,

$$\langle x(\varepsilon_k), f(x(\varepsilon_k)) \rangle = n\varepsilon_k \leq n\varepsilon_0,$$

which implies

$$\liminf_{k \to \infty} \langle x(\varepsilon_k), f(x(\varepsilon_k)) \rangle \leq n\varepsilon_0$$

and we have a contradiction. Therefore, $\cup_{\varepsilon \in]0,\varepsilon_0]}\mathcal{U}(\varepsilon)$ is bounded for all $\varepsilon_0 > 0$. $\square$

We remark that in Theorem 8.1 conclusion (1) is (a) of $\mathcal{U}$ and conclusion (2) is property (b).

Theorem 8.2. *Let $f : \mathbf{R}^n \to \mathbf{R}^n$ be a continuous function If*

$$\liminf_{\substack{\|x\| \to +\infty \\ x \in \mathbf{R}^n_{++}}} \frac{\langle x, f(x) \rangle}{\|x\|^2} > 0,$$

then the interior band mapping $\mathcal{U}$ has properties (a) *and* (b).

Proof. We denote by

$$r = \liminf_{\substack{\|x\| \to +\infty \\ x \in \mathbf{R}^n_{++}}} \frac{\langle x, f(x) \rangle}{\|x\|^2}$$

and we take r_0 such that $0 < r_0 < r$. There exists $\rho > 0$ such that for any $x \in \mathbf{R}^n_{++}$ with $\|x\| > \rho$,

$$\frac{\langle x, f(x) \rangle}{\|x\|^2} > r_0.$$

Indeed if this is not true, then for any $n \in \mathbf{N}$, there exists $x^n \in \mathbf{R}_{++}^n$ with $\|x^n\| > n$ and such that

$$\frac{\langle x^n, f(x^n) \rangle}{\|x^n\|^2} \le r_0.$$

Because $\|x^n\| \to +\infty$ as $n \to \infty$ we have that

$$\liminf_{n \to \infty} \frac{\langle x^n, f(x^n) \rangle}{\|x^n\|^2} \le r_0 < r,$$

which is impossible. Therefore, for any $x \in \mathbf{R}_{++}^n$ with $\|x\| > \rho$ we have $\langle x, f(x) \rangle > r_0 \|x\|^2$ which implies that

$$\liminf_{\substack{\|x\| \to +\infty \\ x \in \mathbf{R}_{++}^n}} \langle x, f(x) \rangle = +\infty.$$

Applying Theorem 8.1 we obtain that the interior band mapping $\mathcal{U}$ has properties (a) and (b). $\qquad \square$

Theorem 8.3. *Let $f : \mathbf{R}^n \to \mathbf{R}^n$ be a continuous function and $g = \mathcal{I}(f)$. If the lower scalar derivative of g in 0 along $\mathbf{R}_{++}^n$ is positive, then the interior band mapping $\mathcal{U}$ has properties (a) and (b).*

Proof. We have

$$(11) \qquad \underline{g}^{\#}(0) = \liminf_{\substack{y \to 0 \\ y \in \mathbf{R}_{++}^n}} \frac{\langle g(y), y \rangle}{\|y\|^2}.$$

Let $y = i(x)$. Then, we have

$$(12) \qquad \liminf_{\substack{y \to 0 \\ y \in \mathbf{R}_{++}^n}} \frac{\langle g(y), y \rangle}{\|y\|^2} = \liminf_{\substack{\|x\| \to \infty \\ x \in \mathbf{R}_{++}^n}} \frac{\langle f(x), x \rangle}{\|x\|^2}.$$

Equations (11) and (12) imply

$$\underline{g}^{\#}(0) = \liminf_{\substack{\|x\| \to \infty \\ x \in \mathbf{R}_{++}^n}} \frac{\langle f(x), x \rangle}{\|x\|^2}.$$

Hence, the result follows by using Theorem 8.1. $\qquad \square$

The following definition extends the notion of copositive functions, see [**17**]. We remark that the notion can be introduced along an arbitrary convex cone in a Hilbert space too, but for our investigation it is sufficient to consider the case of $\mathbf{R}^n$ with the cone $\mathbf{R}^n_{++}$.

Definition 8.1. The function $f : \mathbf{R}^n \to \mathbf{R}^n$ is called *asymptotically copositive* along $\mathbf{R}^n_{++}$ if there is a $\rho > 0$ such that

$$\langle f(x), x \rangle \geq 0,$$

for all $x \in \mathbf{R}^n_{++}$ with $\|x\| > \rho$.

The following definition can be formulated along an arbitrary convex cone in a Hilbert space too, but for the same reason as above we would consider the case of $\mathbf{R}^n$ with the cone $\mathbf{R}^n_{++}$ only.

Definition 8.2. The function $f : \mathbf{R}^n \to \mathbf{R}^n$ is called *strongly asymptotically copositive* along $\mathbf{R}^n_{++}$ if there are $\beta, \rho > 0$ such that

$$\langle f(x), x \rangle \geq \beta \|x\|^2,$$

for all $x \in \mathbf{R}^n_{++}$ with $\|x\| > \rho$.

We remark that f is strongly asymptotically copositive along $\mathbf{R}^n_{++}$ if and only if there is a $\beta > 0$ such that the function $f - \beta I$ is asymptotically copositive along $\mathbf{R}^n_{++}$, where I is the identity function of $\mathbf{R}^n$.

The following theorem follows directly from Theorem 8.1 and Definition 8.2.

Theorem 8.4. *If $f : \mathbf{R}^n \to \mathbf{R}^n$ is a continuous strongly asymptotically copositive function along $\mathbf{R}^n_{++}$, then the interior band mapping $\mathcal{U}$ has properties* (a) *and* (b).

Corollary 8.1. *Let $f : \mathbf{R}^n \to \mathbf{R}^n$ be a continuous function. If there is a $\rho > 0$ and $\beta > 0$ such that $f(x) - \beta x \in \mathbf{R}^n_+$, for all $x \in \mathbf{R}^n_{++}$ with $\|x\| > \rho$, then the interior band mapping $\mathcal{U}$ has properties* (a) *and* (b).

Proof. We have

$$\langle f(x) - \beta x, x \rangle \geq 0$$

for all $x \in \mathbf{R}^n_{++}$ with $\|x\| > \rho$. Hence,

$$\langle f(x), x \rangle \geq \beta \|x\|^2,$$

for all $x \in \mathbf{R}^n_{++}$ with $\|x\| > \rho$. Therefore, f is strongly asymptotically copositive along $\mathbf{R}^n_{++}$ and the result follows from Theorem 8.4. $\quad\square$

At the end of our paper we present two results: Theorem 8.5 and Corollary 8.3, which show that the coercivity condition of Theorem 8.2 can be satisfied by a large class of functions. For this we need the following corollary which is a particularization of Corollary 3.1 [**19**]:

Corollary 8.2. *Let* $D = \{x \in \mathbf{R}^n : \|x\| \leq 1\}$ *and* $f : \mathbf{R}^n \to \mathbf{R}^n$; $f(0) = 0$. *There exists a unique extension* $\tilde{f} : \mathbf{R}^n \to \mathbf{R}^n$ *of* $f|_D$ *such that* $\tilde{f}$ *is a fixed point of* $\mathcal{I}$, *i.e.,* $\tilde{f} = \mathcal{I}(\tilde{f})$.

We proved in our paper [**19**] that this extension has the form

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } \|x\| \leq 1 \\ \mathcal{I}(f)(x) & \text{if } \|x\| > 1. \end{cases}$$

Since $\tilde{f} = \mathcal{I}(\tilde{f})$ and

$$\liminf_{\|x\| \to \infty} \frac{\langle \tilde{f}(x), x \rangle}{\|x\|^2} = \underline{\mathcal{I}(\tilde{f})}^{\#}(0) \quad (\text{by } [\mathbf{19}, \text{Lemma 4.1}]),$$

we have

$$\liminf_{\|x\| \to \infty} \frac{\langle \tilde{f}(x), x \rangle}{\|x\|^2} = \underline{\tilde{f}}^{\#}(0) = \underline{f}^{\#}(0).$$

Hence,

$$(13) \qquad\qquad \liminf_{\|x\| \to \infty} \frac{\langle \tilde{f}(x), x \rangle}{\|x\|^2} > 0$$

if and only if

$$\underline{f}^{\#}(0) > 0.$$

On the other hand, by using the Cauchy inequality it is easy to see that if $\tilde{f}$ satisfies condition (13), then for any $b \in \mathbf{R}^n$, $\tilde{f} + b$ also satisfies condition (13). Hence, we have the following result:

Theorem 8.5. *Let $b \in \mathbf{R}^n$, $f : \mathbf{R}^n \to \mathbf{R}^n$ with $f(0) = 0$ and $\underline{f}^{\#}(0) > 0$, $\tilde{f} : \mathbf{R}^n \to \mathbf{R}^n$ with*

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } \|x\| \leq 1 \\ \mathcal{I}(f)(x) & \text{if } \|x\| > 1 \end{cases}$$

and $F = \tilde{f} + b$. Then, F satisfies the condition

$$\liminf_{\|x\| \to \infty} \frac{\langle F(x), x \rangle}{\|x\|^2} > 0.$$

Corollary 8.3. *Let $a, b \in \mathbf{R}^n$, $f : \mathbf{R}^n \to \mathbf{R}^n$ with $f(0) = a$ and $\underline{f}^{\#}(0) > 0$, and $F : \mathbf{R}^n \to \mathbf{R}^n$ with*

$$F(x) = \begin{cases} f(x) - a + b & \text{if } \|x\| \leq 1 \\ \mathcal{I}(f)(x) - \|x\|^2 a + b & \text{if } \|x\| > 1. \end{cases}$$

Then, F satisfies the condition

$$\liminf_{\|x\| \to \infty} \frac{\langle F(x), x \rangle}{\|x\|^2} > 0.$$

Proof. Let $f_0(x) = f(x) - a$. It is easy to see that $f_0(0) = 0$ and $\underline{f_0}^{\#}(0) = \underline{f}^{\#}(0) > 0$. Hence, we can apply Theorem 8.5 to the function f_0 to obtain the desired result. $\square$

For $A : \mathbf{R}^n \to \mathbf{R}^n$ a linear operator, we denote by A_s the operator $(A + A^*)/2$, where A^* is the adjoint of A. Let $\sigma(A_s)$ be the spectrum of A_s. With these notations we have as follows:

Remark 8.1. If f satisfies supplementary conditions in 0, the paper [**27**] provides useful computational formulae for checking the condition $\underline{f}^{\#}(0) > 0$:

(1) By Theorem 1.1 [**27**] if f is locally Lipschitz in 0 and the directional derivative $f'(0; h)$ exists for each h, then

$$\underline{f}^{\#}(0) = \inf_{\|h\|=1} \langle f'(0; h),\, h \rangle$$

(2) By Theorem 1.2 [**27**] if f is Frechét differentiable in 0, with the differential $df(0)$, then

$$\underline{f}^{\#}(0) = \min_{\|h\|=1} \langle df(0)(h),\, h \rangle,$$

(3) By Theorem 1.5 [**27**] if f is Frechét differentiable in 0 with the differential $df(0)$, then

$$\underline{f}^{\#}(0) = \min \sigma((df(0)))_s.$$

9. Comments. In this paper we studied the interior band of ε-solutions of the nonlinear complementarity problem defined by a continuous function from $\mathbf{R}^n$ to $\mathbf{R}^n$ and by the cone $\mathbf{R}^n_+$.

By the results presented in this paper we put in evidence the importance of the asymptotic Browder Hartman Stampacchia condition. By using this condition and the scalar derivative, we obtained some new results related to the interior band of ε-solutions.

By our method we do not need to suppose that the mapping f is uniformly P-mapping or monotone mapping as in several papers cited in our references.

Our ideas presented in this paper may be a starting point for new developments.

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