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SEQUENCES OF CONSECUTIVE HAPPY NUMBERS

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1. Introduction. Let $S_2 : \mathbf{Z}^+ \to \mathbf{Z}^+$ denote the function that takes a positive integer to the sum of the squares of its decimal digits. A *happy number* is a positive integer a such that $S_2^m(a) = 1$ for some $m \geq 0$. In [**2**], happy numbers were generalized as follows: For $e \geq 2$, $b \geq 2$, and $0 \leq a_i < b$, define $S_{e,b} : \mathbf{Z}^+ \to \mathbf{Z}^+$ by

$$S_{e,b}\left(\sum_{i=0}^{n} a_i b^i\right) = \sum_{i=0}^{n} a_i^e.$$

If $S_{e,b}^m(a) = 1$ for some $m \geq 0$, then a is an *e-power b-happy number.*

Using a computer search, it is easy to find examples of short sequences of consecutive happy numbers. The least examples of sequences of lengths 1–5 are given in Table 1. A natural question to ask is whether or not there exist arbitrarily long finite sequences of consecutive happy numbers. In 2000, El-Sedy and Siksek [**1**] showed that the answer is yes.

One can also ask more generally for what values of e and b do there exist arbitrarily long sequences of consecutive e-power b-happy numbers. Some results are already known. For example, for all $e \geq 2$,

TABLE 1. CONSECUTIVE HAPPY NUMBERS.

Length	Least happy number sequence
1	1
2	31, 32
3	1880, 1881, 1882
4	7839, 7840, 7841, 7842
5	44488, 44489, 44490, 44491, 44492

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every positive integer is an e-power 2-happy number [**2**, Theorem 4], while for each odd b, there are no consecutive 2-power b-happy numbers. The latter follows from [**2**, Theorem 10], which states that if p is prime and $b \equiv 1 \pmod{p}$, then for any $a \in \mathbf{Z}^+$ and $k \in \mathbf{Z}^+$,

$$S_{p,b}^k(a) \equiv a \pmod{p}. \tag{1}$$

In particular, all 2-power b-happy numbers are congruent to 1 modulo d, where $d = \gcd(2, b-1)$. With this in mind, we define a *d-consecutive sequence* to be an arithmetic sequence with constant difference d.

In Section 2, we show that, for each base b, letting $d = \gcd(2, b-1)$, there exist arbitrarily long finite d-consecutive sequences of 2-power b-happy numbers. The result is based on a theorem due to Lenstra [**3**]. We then, in Section 3, consider such sequences of 3-power b-happy numbers, proving their existence for $2 \le b \le 13$ with $d = \gcd(6, b-1)$.

2. Lenstra's theorem. In this section we present Theorem 1, which is due to H. Lenstra. Since his proof has not been published elsewhere, we include a slightly modified version of it here.

Let $U_{e,b}$ denote the set of all fixed points and cycles of $S_{e,b}$, that is, let

$$U_{e,b} = \{a \in \mathbf{Z}^+ \mid \text{for some } m \in \mathbf{Z}^+,\ S_{e,b}^m(a) = a\}.$$

Following Lenstra [**3**], a finite set T is (e,b)-*good* if, for each $u \in U_{e,b}$, there exist $n, k \in \mathbf{Z}^+$ such that for all $t \in T$, $S_{e,b}^k(t+n) = u$.

Theorem 1 [**3**]. *Fix $b \ge 2$, and let $d = \gcd(2, b-1)$. A finite set T of positive integers is $(2,b)$-good if and only if all of the elements of T are congruent modulo d.*

As immediate corollaries we get the following.

Corollary 2. *There exist arbitrarily long finite d-consecutive sequences of b-happy numbers.*

Corollary 3. *If there exists at least one positive integer that is not a b-happy number, then there exist arbitrarily long finite d-consecutive sequences of numbers that are not b-happy numbers.*

The proof of Theorem 1 is constructive in that it provides an algorithm for finding example sequences corresponding to any $(2, b)$-good set. We illustrate this with an example at the end of this section.

Let $I : \mathbf{Z}^+ \to \mathbf{Z}^+$ be defined by $I(t) = t + 1$. We begin with two lemmas, which we prove in general for any exponent, e.

Lemma 4. *Fix e, $b \geq 2$. Let $F : \mathbf{Z}^+ \to \mathbf{Z}^+$ be the composition of a finite sequence of the functions $S_{e,b}$ and I. If $F(T)$ is (e, b)-good, then T is (e, b)-good.*

Proof. It follows immediately from the definition of (e, b)-good that for e, $b \geq 2$, if $I(T)$ is (e, b)-good, then T is (e, b)-good. Using induction on the length of the sequence of functions, it suffices to show that if $S_{e,b}(T)$ is (e, b)-good, then T is (e, b)-good.

Suppose that $S_{e,b}(T)$ is (e, b)-good, and let $u \in U_{e,b}$. Then there exist n' and k' such that for all $s \in S_{e,b}(T)$, $S_{e,b}^{k'}(s + n') = u$. Let

$$n = \underbrace{11 \ldots 11}_{n'} 00 \ldots 00,$$

where the number of zeros is the number of base b digits of the largest element of T. Then $S_{e,b}(n) = n'$ and, for each $t \in T$, $S_{e,b}(t + n) = S_{e,b}(t) + n'$. Let $k = k' + 1$. Then, for all $t \in T$,

$$S_{e,b}^{k}(t + n) = S_{e,b}^{k'}(S_{e,b}(t + n)) = S_{e,b}^{k'}(S_{e,b}(t) + n') = u.$$

So T is (e, b)-good. $\square$

Lemma 5. *Fix e, $b \geq 2$. If $T = \{t\} \subseteq \mathbf{Z}^+$, then T is (e, b)-good.*

Given $u \in U_{e,b}$, there exist x and k such that $S_{e,b}^{k}(x) = u$. Let $r \in \mathbf{Z}^+$ such that $t \leq b^r x$. Then, since $S_{e,b}^{k}(t + (b^r x - t)) = S_{e,b}^{k}(x) = u$, T is (e, b)-good. $\square$

We now prove Theorem 1 by induction on the number of elements in T.

Proof of Theorem 1. Let $d = \gcd(2, b-1)$. Since every positive integer is 2-power 2-happy, we fix $b > 2$. It follows from Congruence (1) that if, for each t in a set T, $S_{2,b}^k(t+n) = u$, then the elements of T are all congruent modulo d.

For the converse, let T be a finite set of positive integers all of which are congruent modulo d. Note that if T is empty, then vacuously it is $(2, b)$-good. If T has exactly one element, then, by Lemma 5, T is $(2, b)$-good.

Fix $N > 1$, and assume that any set of fewer than N elements all of which are congruent modulo d is $(2, b)$-good. Suppose T has exactly N elements, and let $t_1 > t_2 \in T$. We will show that there exists a function F of the type described in Lemma 4 such that $F(t_1) = F(t_2)$.

There are three cases.

Case 1. If t_1 and t_2 have the same nonzero digits, $S_{2,b}(t_1) = S_{2,b}(t_2)$. Thus, it suffices to let $F = S_{2,b}$.

Case 2. If $t_1 \equiv t_2 \pmod{b-1}$, then $t_1 - t_2 = (b-1)v$ for some $v \in \mathbf{Z}^+$. Choose $r \in \mathbf{Z}^+$ so that $b^r > bv + t_2$, and let $m = b^r + v - t_2 > 0$. Then

$$I^m(t_1) = t_1 + b^r + v - t_2 = b^r + v + (b-1)v = b^r + bv$$

and

$$I^m(t_2) = t_2 + b^r + v - t_2 = b^r + v.$$

Since $b^r > bv$, it follows that $I^m(t_1)$ and $I^m(t_2)$ have the same nonzero digits. Thus, referring back to Case 1, it suffices to let $F = S_{2,b}I^m$.

Case 3. If neither of the above holds, let $u = t_1 - t_2$.

If b is odd, then $d = \gcd(2, b-1) = 2$ and therefore, from the conditions on T, u must be even. It follows from Congruence (1) that $S_{2,b}(u-1) \equiv u-1 \pmod 2$ and thus is odd. Therefore, $-S_{2,b}(u-1)-1$ is even. Since $b-1$ is even, there exists a $c \in \mathbf{Z}$, $0 \le c < (b-1)/2$, such that

$$2c \equiv -S_{2,b}(u-1) - 1 \pmod{b-1}. \tag{2}$$

If b is even, then 2 is invertible modulo $b-1$. Hence (defining $S_{2,b}(0) = 0$, when $u = 1$) there exists a $c \in \mathbf{Z}$, $0 \le c < b-1$, such that $2c \equiv -S_{2,b}(u-1) - 1 \pmod{b-1}$. So, in either case, there exists a $c \in \mathbf{Z}$, $0 \le c < b-1$ satisfying Congruence (2).

Choose $r' \in \mathbf{Z}^+$ such that $(c+1)b^{r'} > t_1$. Let $m' = (c+1)b^{r'} - t_2 - 1 \geq 0$. Then $S_{2,b}(t_1+m') = S_{2,b}((c+1)b^{r'}+u-1) = (c+1)^2 + S_{2,b}(u-1) = c^2+2c+1+S_{2,b}(u-1) \equiv c^2 \pmod{b-1}$, by Congruence (2). Further, $S_{2,b}(t_2+m') = S_{2,b}((c+1)b^{r'}-1) = c^2 + r'(b-1)^2 \equiv c^2 \pmod{b-1}$. Therefore, $S_{2,b}I^{m'}(t_1) \equiv S_{2,b}I^{m'}(t_2) \pmod{b-1}$. Using the argument in Case 2, it suffices to let $F = S_{2,b}I^m S_{2,b}I^{m'}$, for some appropriately chosen $m \in \mathbf{Z}^+$.

Hence, in any case, there exists a map F as in Lemma 4 such that $F(t_1) = F(t_2)$. This implies that $F(T)$ has fewer elements than does T. Further, since the elements of T are congruent modulo d, the same holds for $I(T)$ and, by Congruence (1), for $S_{2,b}(T)$, implying that the same holds for $F(T)$. Therefore, by the induction hypothesis, $F(T)$ is $(2,b)$-good. Hence by Lemma 4, T is $(2,b)$-good.

This completes the proof of Theorem 1. □

To illustrate the construction provided by this proof, we use it to find a pair of consecutive happy numbers ($e = 2$, $b = 10$). Starting with $T = \{1,2\}$, the method involves finding the function F in the proof of the theorem, then for each occurrence of $S_{2,10}$ in F, using the method from the proof of Lemma 4 to find a suitable value of n such that $t+n$ is a happy number, for each appropriate t.

Using the notation of the proof, let $t_1 = 2$ and $t_2 = 1$. Case 3 of the proof applies to T with $u = 1$. Note that $c = 4$ satisfies Congruence (2). Since $(4+1) > 2$, we take $r' = 1$ and so $m' = 48$. This yields $T_1 = S_{2,10}I^{48}(T) = \{97, 25\}$.

Now to find m, we look at Case 2 of the proof with $t_1 = 97$, $t_2 = 25$, $v = 8$. Since $10^3 > 10 \cdot 8 + 25$, let $r = 3$. Then $m = 983$ and we get $T_2 = S_{2,10}I^{983}(T_1) = \{65\}$.

We now have a set with only one element and so use the proof of Lemma 5. Since $S_{2,10}(1) = 1$ and $t = 65$, we take $x = 1$ and $r = 2$. So, $n = 100 - 65 = 35$ and we see that $S_{2,10}I^{35}(T_2) = \{1\}$.

Finally, we use the proof of Lemma 4 to find the corresponding pair of consecutive happy numbers: Since $S_{2,10}I^{35}(T_2) = \{1\}$, T_2 is $(2,10)$-good with $n_2 = 35$ and $k_2 = 1$. Following the proof, $T_2 = S_{2,10}(I^{983}(T_1))$ and each element of $I^{983}(T_1)$ has four digits, so, letting

$$n_1 = \underbrace{11\dots11}_{35}0000,$$

each element of $I^{983}(T_1) + n_1 = T_1 + (n_1 + 983)$ is a happy number. Now, $T_1 = S_{2,10}(I^{48}(T))$ and each element of $I^{48}(T)$ has 2 digits, so we let

$$n_0 = \underbrace{11\dots11}_{n_1+983}00,$$

and see that each element of $T + (n_0 + 48)$ is happy. Hence

$$\underbrace{11111\dots11}_{\underbrace{11\dots11}_{35}0983}49 \quad \text{and} \quad \underbrace{11111\dots11}_{\underbrace{11\dots11}_{35}0983}50$$

are consecutive happy numbers.

Recall from Table 1 that the smallest pair of consecutive happy numbers is 31 and 32. So Lenstra's algorithm clearly does not construct the smallest examples of consecutive b-happy numbers.

3. The cubic case. In this section, we generalize to 3-power (or *cubic*) b-happy numbers. Recall that all positive integers are cubic 2-happy.

For cubic happy numbers (the base 10 case), notice that it follows from Congruence (1) that, for all $x, k \in \mathbf{Z}^+$, $S_3^k(x) \equiv x \pmod 3$. Thus, if x is a cubic happy number, $x \equiv 1 \pmod 3$. We begin this section by proving that there exist arbitrarily long finite 3-consecutive sequences of cubic happy numbers.

Theorem 6. *Let T be a finite set of positive integers. The set T is $(3, 10)$-good if and only if all of the elements of T are congruent modulo 3.*

Proof. One direction follows from Congruence (1). For the converse, assume that all of the elements of T are congruent modulo 3. Choose $n \in \{0, 1, 2\}$ such that for all $t \in T$, $t + n \equiv 0 \pmod 3$. It follows from [**2**, Theorem 5] that for each $a \equiv 0 \pmod 3$, there exists a $k_a \in \mathbf{Z}^+$

such that $S_{3,10}^{k_a}(a) = 153$. Hence, since T is finite, there exists a $k \in \mathbf{Z}^+$ such that for each $t \in T$, $S_{3,10}^k(t+n) = 153$. Thus $S_{3,10}^k I^n(T) = \{153\}$. But by Lemma 5, $\{153\}$ is $(3,b)$-good and so by Lemma 4, T is $(3,b)$-good. $\square$

As an immediate corollary, we have the following.

Corollary 7. *There exist arbitrarily long finite* 3*-consecutive sequences of cubic happy numbers.*

For other bases, we recall from [**2**, Theorem 12] that, for any $b \geq 2$ and $a, k \in \mathbf{Z}^+$,

$$S_{3,b}^k(a) \equiv a \pmod{\gcd(6, b-1)}. \tag{3}$$

This leads easily to the following lemma.

Lemma 8. *Fix $b \geq 2$, and let T be a finite set of positive integers. If T is $(3,b)$-good, then all of the elements of T are congruent modulo* $\gcd(6, b-1)$.

We conjecture that the converse is true, which would imply that for each $b \geq 2$, there exist arbitrarily long finite sequences of d-consecutive cubic b-happy numbers for $d = \gcd(6, b-1)$.

Conjecture 1. *Fix $b \geq 2$, and let T be a finite set of positive integers. The set T is $(3,b)$-good if and only if all of the elements of T are congruent modulo* $\gcd(6, b-1)$.

In a series of lemmas, we prove Conjecture 1 for $2 \leq b \leq 13$. Finally, we will show that the conjecture holds for infinitely many bases, b.

Lemma 9 allows for a generalization of the method used in the proof of Theorem 1 to the cubic case. Its proof follows the same format as the proof of that theorem. The generalization is limited in its application in that, as indicated in Lemma 10, it does not apply to every base.

Lemma 9. *Fix $b \geq 2$, let $d = \gcd(6, b-1)$, and let T be a finite set of nonnegative integers all of which are congruent modulo d. If for every integer $w \equiv 0 \pmod d$ there exists a positive integer c such that*

$$3c(c+1) \equiv w \pmod{b-1}, \tag{4}$$

then T is $(3,b)$-good.

Proof. Since every set is $(3,2)$-good, fix $b > 2$. Using Lemma 5, assume that T has $N > 1$ elements and that any set of fewer than N elements all of which are congruent modulo d is $(3,b)$-good. Let $t_1 > t_2 \in T$. Following the proof of Theorem 1, we consider three cases, showing in each that there exists a function F of the type described in Lemma 4 such that $F(t_1) = F(t_2)$.

If t_1 and t_2 have the same nonzero digits, let $F = S_{3,b}$.

If $t_1 \equiv t_2 \pmod{b-1}$, let $v \in \mathbf{Z}^+$ such that $t_1 - t_2 = (b-1)v$. Choose $r \in \mathbf{Z}^+$ so that $b^r > bv + t_2$, and let $m = b^r + v - t_2 > 0$. Then $I^m(t_1)$ and $I^m(t_2)$ have the same nonzero digits, so let $F = S_{3,b}I^m$.

If neither of the above holds, let $u = t_1 - t_2 \equiv 0 \pmod d$. It follows from Congruence (3) that $S_{3,b}(u-1) \equiv u - 1 \pmod d$ and so $-1 - S_{3,b}(u-1) \equiv 0 \pmod d$. By assumption, there exists a c such that

$$3c(c+1) \equiv -1 - S_{3,b}(u-1) \pmod{b-1}. \tag{5}$$

Choose $r' \in \mathbf{Z}^+$ such that $b^{r'} > \max\{u-1, t_2+1\}$. Let $m' = (c+1)b^{r'} - t_2 - 1 \geq 0$.

Then

$$S_{3,b}(t_1 + m') = (c+1)^3 + S_{3,b}(u-1) \equiv c^3 \pmod{b-1}$$

and

$$S_{3,b}(t_2 + m') = c^3 + r'(b-1)^3 \equiv c^3 \pmod{b-1}.$$

Thus, $S_{3,b}(m + t_2) \equiv S_{3,b}(m + t_1) \pmod{b-1}$. Using the argument above, we get $F = S_{3,b}I^m S_{3,b}I^{m'}$, for some appropriately chosen $m \in \mathbf{Z}^+$.

Hence, in any case, there exists a map F as in Lemma 4 such that $F(t_1) = F(t_2)$. As in the proof of Theorem 1, this implies that T is $(3, b)$-good. □

We now use Lemma 9 to prove Conjecture 1 for specific values of b.

Lemma 10. *Let T be a finite set of positive integers, and let*

$$b \in \{3, 4, 5, 7, 9, 13\}.$$

If all of the elements of T are congruent modulo $d = \gcd(6, b-1)$, then T is $(3, b)$-good.

Proof. Fix $b \in \{3, 4, 5, 7, 9, 13\}$, and let T be a finite set of positive integers all of which are congruent modulo d. Let $w \equiv 0 \pmod{d}$. Using Lemma 9, it suffices to show that there exists a c with $3c(c+1) \equiv w \pmod{b-1}$.

If $b \in \{3, 4, 7\}$, $d = b - 1$ and so taking $c = 0$, $3c(c+1) \equiv 0 \equiv w \pmod{b-1}$. If $b = 5$, $d = 2$ and taking $c = 0$ and 1 yields $3c(c+1) \equiv 0$ and $2 \pmod{4}$, respectively. If $b = 9$, $d = 2$ and taking $c = 0$, 2, 3, and 1 yields $3c(c+1) \equiv 0$, 2, 4, and $6 \pmod{8}$, respectively. Finally, if $b = 13$, then $d = 6$ and taking $c = 0$ and 1 yields $3c(c+1) \equiv 0$ and $6 \pmod{12}$, respectively. □

Lemma 9 does not apply to $b \in \{6, 8, 10, 11, 12\}$. We already proved Conjecture 1 for $b = 10$ in Theorem 6. The following lemma addresses $b \in \{6, 8, 12\}$ and, finally, Lemma 12 proves the remaining case of $b = 11$. Recall that $U_{3,b}$ is the set of all cycles and fixed points of $S_{3,b}$. Note that for $b \in \{6, 8, 12\}$, $d = \gcd(6, b-1) = 1$.

Lemma 11. *If T is a finite set of positive integers and $b \in \{6, 8, 12\}$, then T is $(3, b)$-good.*

Proof. Let $b \in \{6, 8, 12\}$, and let T be a finite set of positive integers. It is easy to see that there exists some $k \in \mathbf{Z}^+$ such that $S_{3,b}^k(T) \subseteq U_{3,b}$. By Lemma 4, it therefore suffices to prove that $U_{3,b}$ is $(3, b)$-good.

Let $b = 6$. From [**2**, Theorem 5],

$$U_{3,6} = \{1, 9, 28, 62, 73, 99, 128, 190, 251\}.$$

So,

$$\begin{aligned} S_{3,6}^8 I^2 S_{3,6}^8 I^2 S_{3,6}^8 I^2(U_{3,6}) &= S_{3,6}^8 I^2 S_{3,6}^8 I^2(\{1,44,514\}) \\ &= S_{3,6}^8 I^2(\{1,514\}) = \{1\}. \end{aligned}$$

By Lemmas 4 and 5, $U_{3,6}$ is $(3,6)$-good.

Next let $b = 8$. Again from [**2**, Theorem 5],

$$U_{3,8} = \{1, 92, 133, 307, 432, 433, 434, 440, 469, 476, 559\}.$$

Now,

$$\begin{aligned} S_{3,8}^6 I^{45} S_{3,8}^6 I^{45} S_{3,8}^6 I^{45}(U_{3,8}) &= S_{3,8}^6 I^{45} S_{3,8}^6 I^{45}(\{1,92,432,559,469\}) \\ &= S_{3,8}^6 I^{45}(\{1,92,469\}) = \{1\}. \end{aligned}$$

Thus, $U_{3,8}$ is $(3,8)$-good.

Finally let $b = 12$. Using [**2**, Corollary 9] and a straightforward computer search, we see that $U_{3,12} = \{1, 8, 288, 342, 343, 415, 512, 755, 793, 811, 944, 1001, 1008, 1136, 1344, 1351, 1464, 1539, 1672, 1738, 1855, 2002\}$. So

$$\begin{aligned} &S_{3,12}^{14} I^9 S_{3,12}^{22} I^2 S_{3,12}^{22} I^2 S_{3,12}^{22} I^2(U_{3,8}) \\ &\quad = S_{3,12}^{14} I^9 S_{3,12}^{22} I^2 S_{3,12}^{22} I^2(\{1, 342, 415, 755, 1008, 1351, 1539, 1855\} \\ &\quad = S_{3,12}^{14} I^9 S_{3,12}^{22} I^2(\{1, 415, 1008, 1855\}) \\ &\quad = S_{3,12}^{14} I^9(\{1, 415, 1855\}) = \{1\}. \end{aligned}$$

Thus, $U_{3,12}$ is $(3,12)$-good. $\square$

Now, we consider the base 11 case.

Lemma 12. *Let T be a finite set of positive integers. If all of the elements of T are congruent modulo 2, then T is $(3,11)$-good.*

Proof. Let T be a finite set of positive integers all of which are congruent modulo 2. There exists some $k \in \mathbf{Z}^+$ such that $S_{3,11}^k(T) \subseteq U_{3,11}$. By Lemma 4, it therefore suffices to prove that any subset $V \subseteq U_{3,11}$ of integers all of which are congruent modulo 2 is $(3,11)$-good. We will use induction on the size of V. Again, if V is empty or has only one element, we have already shown that it is $(3,11)$-good.

Using [**2**, Corollary 9] and a computer search,

$$U_{3,11} = \{1, 3, 9, 10, 27, 35, 36, 54, 126, 133, 226, 243, 307, 370, 433, 435, 459, 476, 593, 684, 687, 688, 729, 757, 793, 855, 946, 953, 1000, 1051, 1054, 1064, 1071, 1072, 1133, 1161, 1216, 1280, 1305, 1366, 1415, 1520, 1536, 1584, 1793, 1855\}.$$

Without loss of generality, assume $370 \in V$. (If not, then add an appropriate amount, then apply $S_{3,11}$ and rename the resulting set V, utilizing Lemma 4.) Let $t \in V$, $t \neq 370$. By assumption, t is even.

If $t = 1280$, then $S^6_{3,11}(t+1) = 855 = S^6_{3,11}(370+1)$, and so $S^6_{3,11}I(V)$ is a smaller subset of $U_{3,11}$. If $t \in \{10, 1000, 1520, 226, 946, 1072, 1366, 36, 54, 1064, 1536, 688, 684, 476, 1054, 1584\}$, i.e., if t is in the loop containing 10, then choose k such that $S^k_{3,11}(t) = 1000$. Since $S^4_{3,11}(1001) = 855 = S^4_{3,11}(371)$, we have $S^4_{3,11}IS^k_{3,11}(V)$ is a smaller subset of $U_{3,11}$.

If neither of the above holds, then $t = 126$ or 1216. In this case, we have that $S^6_{3,11}I^{70}(370) = 370$ and $S^6_{3,11}I^{70}(t)$ is in the loop containing 10. Hence, combining with the above, $S^4_{3,11}IS^k_{3,11}S^6_{3,11}I^{70}(V)$ is a smaller subset of $U_{3,11}$.

Thus, by induction, $U_{3,11}$ is $(3, 11)$-good, as needed. □

Combining the results from Theorem 6 and Lemmas 8, 10, 11, and 12, this completes the proof of Conjecture 1 for $2 \leq b \leq 13$:

Theorem 13. *Let T be a finite set of positive integers, and let $2 \leq b \leq 13$. The set T is $(3, b)$-good if and only if all of the elements of T are congruent modulo* $\gcd(6, b-1)$.

Hence, for $2 \leq b \leq 13$ and $d = \gcd(6, b-1)$, there exist arbitrarily long finite d-consecutive sequences of cubic b-happy numbers.

Finally, we show that there are infinitely many bases b such that there exist arbitrarily long finite d-consecutive sequences of cubic b-happy numbers for $d = \gcd(6, b-1)$ by exhibiting two such families of bases.

Theorem 14. *For any $r \in \mathbf{Z}^+$, let $b = 2^r + 1$ or $3 \cdot 2^r + 1$. Then for $d =$ $\gcd(6, b-1)$, there exist arbitrarily long finite d-consecutive sequences of b-happy numbers.*

Proof. Fix $r \in \mathbf{Z}^+$.

Note that for $b = 3 \cdot 2^r + 1$, $d = 6$. Then by Lemma 9, it suffices to show that for every $w = 6v$, $v \in \mathbf{Z}$, there exists a $c \in \mathbf{Z}$ such that $3c(c+1) \equiv w \pmod{3 \cdot 2^r}$.

For $b = 2^r + 1$, $d = 2$, and so it suffices to show that for every even w there exists a c such that $3c(c+1) \equiv w \pmod{2^r}$. Since w is even and 3 is invertible modulo 2^r, we can again let $w = 6v$, for some $v \in \mathbf{Z}$.

Since $1^2 \equiv 8v+1 \pmod{2^3}$, we can apply Hensel's lemma to conclude that for each $n \in \mathbf{Z}^+$, there exists an $a_n \in \mathbf{Z}$ such that $a_n^2 \equiv 8v + 1 \pmod{2^{n+1}}$. Since each a_n is odd, we can let $a_{r+1} = 2c + 1$, with $c \in \mathbf{Z}$. It then follows that $(2c+1)^2 \equiv 8v+1 \pmod{2^{r+2}}$ and therefore $3c(c + 1) \equiv 6v \equiv w \pmod{3 \cdot 2^r}$, satisfying the required condition in either case. □

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