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CONSTRUCTING MAPPINGS ONTO RADIAL SLIT DOMAINS

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ABSTRACT. Formulas are derived for computing a univalent mapping of the unit disc onto a radial slit domain with threefold circular symmetry. Goodman's upper bound for the univalent Bloch constant can be computed to a high degree of accuracy. A new upper bound is given for the universal constant which determines the lowest tone a membrane can produce if it contains no circular membrane with more than a specified radius.

1. Introduction. A particular construction of a conformal mapping of the unit disc onto a planar domain with infinitely many radial slits is used in [**5**] to obtain an upper bound (Goodman's constant) for the *univalent Bloch constant*, and in [**1, 2**] to obtain an upper bound for the universal constant, here named the Makai-Hayman (M-H) constant, which determines the lowest tone a membrane can produce if it contains no circular membrane with more than a specified radius. The goal of this article is to review this construction in order to obtain general formulas by which Goodman's constant and an upper bound for the M-H constant can be computed more easily and accurately than was done in these two articles.

In Section 2 the univalent Bloch constant is denoted $\mathbf{B}_u$, and Goodman's constant is denoted $\mathbf{B}_\infty$. The exact value of $\mathbf{B}_u$ has not been determined, see [**4**]. The best upper bound for $\mathbf{B}_u$ is the bound determined by Beller and Hummel in [**3**]. The lower bound for $\mathbf{B}_u$ is discussed in [**7**]. A lower bound 1/900 for the M-H constant was found by Hayman, see [**6**], in 1976. Unknown to him at the time, a lower bound 1/4 had been found by Makai [**8**] in 1965.

2. Radial slit domains. In this article i denotes the *complex number* with the property $i^2 = -1$, and j, k, m, n are *integer*-valued variables. For $n \geq 1$, let $r_n, l_0, l_1, \dots, l_n$ be any positive real numbers,

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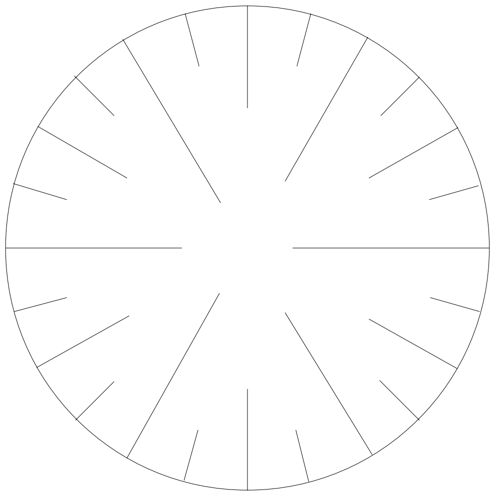

FIGURE 1. Radial slit domain with $n = 2$ and $r_2 < l_0 < l_1 < l_2 < R_2$.

and let R_n be any positive real number or infinity, with $0 < r_n < R_n$ and $0 < l_j < R_n$, for $0 \leq j \leq n$. A *radial slit domain* with three-fold circular symmetry will be denoted

$$\mathbf{G}^n[r_n, l_0, l_1, l_2, \ldots, l_n, R_n]. \tag{1}$$

If $R_n < \infty$ this is a disc $B(0, R_n)$ minus linear segments

$$\bigcup_{k=0}^{2} [r_n\, e^{i2k\pi/3}, R_n\, e^{i2k\pi/3}] \bigcup_{j=0}^{n} \bigcup_{k=0}^{3\cdot 2^j - 1} [l_j\, e^{i(2k+1)\pi/(3\cdot 2^j)}, R_n\, e^{i(2k+1)\pi/(3\cdot 2^j)}],$$

i.e., a simply connected domain, which is a disc of radius R_n centered at the origin, with radial slits corresponding to the $3 \cdot 2^{n+1}$th roots of unity. (The case $n = 0$ may also be considered but will not be done explicitly here.) Such a domain, with $n = 2$, is shown in Figure 1. If $R_n = \infty$, then $\mathbf{G}^n[r_n, l_0, l_1, l_2, \ldots, l_n, \infty]$ is the plane minus infinite rays

$$\bigcup_{k=0}^{2} [r_n\, e^{i2k\pi/3}, \infty) \bigcup_{j=0}^{n} \bigcup_{k=0}^{3\cdot 2^j - 1} [l_j\, e^{i(2k+1)\pi/(3\cdot 2^j)}, \infty).$$

Goodman constructed univalent mappings of the unit disc $\mathbf{D}$ onto domains

$$\mathbf{G}^3\left[1, 2, c, 1+\csc\left(\frac{\pi}{12}\right), 1+\csc\left(\frac{\pi}{24}\right)\right], \quad \mathbf{G}^3[1, 2, c, 1+\csc(\pi/12), \infty], \tag{2}$$

where the constant $2 < c < 1+\csc(\pi/12)$ is chosen so that the first domain in (2) contains no disc with a radius larger than 1. This domain is a subset of the simply connected, unbounded domain

$$\bigcup_{n=2}^{\infty} \mathbf{G}^n\left[1, 2, c, 1+\csc\left(\frac{\pi}{3\cdot 2^2}\right), \dots, 1+\csc\left(\frac{\pi}{3\cdot 2^n}\right), 1+\csc\left(\frac{\pi}{3\cdot 2^{n+1}}\right)\right], \tag{3}$$

which is the plane with an infinite number of radial slits. If F_∞ is a univalent mapping from $\mathbf{D}$ onto the domain (3) such that $F_\infty(0) = 0$, and if $\mathbf{B}_\infty := 1/|F'_\infty(0)|$, then $\mathbf{B}_u \le \mathbf{B}_\infty$. Goodman's analysis of the mappings onto the domains (2) determined that $0.65646 < \mathbf{B}_\infty < 0.65647$.

3. Mappings onto radial slit domains. Univalent (conformal) functions defined on the unit disc $\mathbf{D}$ are from here on considered to be mappings onto domains which are simply connected subsets of the plane determined by introducing radial slits. Furthermore, all functions fix the origin, with positive derivative at the origin. In the special case that a subset of $\mathbf{D}$ is determined by making $3\cdot 2^j$ slits, all having the same length, and corresponding with the $3\cdot 2^j$th roots of unity, for some nonnegative integer j, then this subset of $\mathbf{D}$ will be denoted $\mathbf{D}_j(l)$, where $0 < l < 1$ determines the slits to be of length $1-l$.

The *Koebe function* $h(z) = z/(1+z)^2$ maps $\mathbf{D}$ onto the plane minus the infinite ray $[1/4, \infty)$. Thus, for $j \ge 0$,

$$H_j(z) := \left[h\left(z^{3\cdot 2^j}\right)\right]^{1/(3\cdot 2^j)} = \frac{z}{\left(1+z^{3\cdot 2^j}\right)^{1/(3\cdot 2^{j-1})}} \tag{4}$$

is a univalent mapping of $\mathbf{D}$ onto the plane minus infinite radial slits corresponding to the $3\cdot 2^j$th roots of unity. Furthermore, for a fixed $j \ge 0$, and for any real number $0 < \alpha < 1$, the function $H_j^{-1}(\alpha H_j(z))$

is a mapping of $\mathbf{D}$ onto a domain $\mathbf{D}_j(l)$, where $0 < l < 1$. The inverse of this function is $H_j^{-1}\left((1/\alpha)\, H_j(z)\right)$, and the value of α can be expressed as $\alpha(l) = H_j(l)/H_j(1)$, which is a strictly increasing function of l on $[0,1]$ for each j.

A normalization of the radial slit domain defined in (1) above is given as follows: suppose that $n \geq 1$ and numbers $0 < r_n < 1$ and $0 < l_j < 1$, for $0 \leq j \leq n$ are fixed. Then let

$$\mathbf{G}_1^n := \mathbf{G}^n[r_n, l_0, l_1, l_2, \ldots, l_n, 1], \tag{5}$$

$$\mathbf{G}_\infty^n := \mathbf{G}^n[r_n, l_0, l_1, l_2, \ldots, l_n, \infty]. \tag{6}$$

Now the function $H_n^{-1}\left(\alpha_n\, H_n(z)\right)$ determined by setting

$$\alpha_n = H_n(l_n)/H_n(1) \tag{7}$$

is a mapping of $\mathbf{D}$ onto $\mathbf{D}_n(l_n)$. For the inverse of this function, extended to the boundary point $z = l_n$ of $\mathbf{D}_n(l_n)$, we have $H_n^{-1}\left(1/\alpha_n\, H_n(l_n)\right) = 1$. To continue, let $\gamma_j := \exp\left(i\pi/(3\cdot 2^j)\right)$ for $0 \leq j \leq n$. A rotation by γ_n of the domain $\mathbf{D}_n(l_n)$ determines the (rotated) inverse function:

$$g_n(z) := \gamma_n\, H_n^{-1}\left(\frac{1}{\alpha_n} H_n(\overline{\gamma}_n\, z)\right).$$

The domain of g_n is symmetric by reflection across diameters of $\mathbf{D}$ corresponding to the $3\cdot 2^{n+1}$th roots of unity. Thus, if $g_n(z)$ is restricted to $\mathbf{G}_1^n$ (which has slits at the $3\cdot 2^{n+1}$th roots of unity), then the image of this restricted mapping is a domain with slits corresponding to the $3\cdot 2^n$th roots of unity, contained in $\mathbf{D}$. Specifically, the image of this restricted mapping is the domain

$$\mathbf{G}^{n-1}[g_n(r_n),\, \overline{\gamma}_0\, g_n(\gamma_0\, l_0),\, \overline{\gamma}_1\, g_n(\gamma_1\, l_1),\, \overline{\gamma}_2\, g_n(\gamma_2\, l_2), \ldots, \\ \overline{\gamma}_{n-1}\, g_n(\gamma_{n-1}\, l_{n-1}), 1].$$

This method of reducing the number of slits from $3\cdot 2^{n+1}$ to $3\cdot 2^n$ may now be applied again to reduce the number of slits to $3\cdot 2^{n-1}$. By defining $l_{n-1}^\dagger = \overline{\gamma}_{n-1}\, g_n(\gamma_{n-1}\, l_{n-1})$, and setting

$$\alpha_{n-1} = \frac{H_{n-1}(l_{n-1}^\dagger)}{H_{n-1}(1)} = \frac{H_{n-1}(\overline{\gamma}_{n-1}\, g_n(\gamma_{n-1}\, l_{n-1}))}{H_{n-1}(1)}, \tag{8}$$

the function $H_{n-1}^{-1}(\alpha_{n-1} H_{n-1}(z))$ is a mapping of $\mathbf{D}$ onto the domain $\mathbf{D}_{n-1}(l_{n-1}^{\dagger})$. The inverse of this function, extended to the boundary point $z = l_{n-1}^{\dagger}$ of this domain, has the property

$$H_{n-1}^{-1}\Big((1/\alpha_{n-1})H_{n-1}(l_{n-1}^{\dagger})\Big) = 1.$$

So if

$$g_{n-1}(z) := \gamma_{n-1} H_{n-1}^{-1}\left(\frac{1}{\alpha_{n-1}} H_{n-1}(\overline{\gamma}_{n-1} z)\right),$$

then the composition $g_{n-1} \circ g_n$ can be restricted to a mapping of the domain $\mathbf{G}_1^n$ onto a domain with slits corresponding to the $3 \cdot 2^{n-1}$th roots of unity, contained in $\mathbf{D}$.

In general, for $0 \le j \le n$, let

$$g_j(z) := \gamma_j H_j^{-1}\left(\frac{1}{\alpha_j} H_j(\overline{\gamma}_j z)\right), \tag{9}$$

where α_j is chosen so that the function $H_j^{-1}(\alpha_j H_j(z))$ maps $\mathbf{D}$ onto the domain $\mathbf{D}_j(l_j^{\dagger})$, where $l_j^{\dagger} := \overline{\gamma}_j g_{j+1} \circ \cdots \circ g_{n-1} \circ g_n(\gamma_j l_j)$. The corresponding inverse function has (with extension of its domain to the point $z = l_j^{\dagger}$) the property $H_j^{-1}\Big((1/\alpha_j)H_j(l_j^{\dagger})\Big) = 1$. It follows that

$$\alpha_j = \frac{H_j(l_j^{\dagger})}{H_j(1)} = \frac{H_j(\overline{\gamma}_j g_{j+1} \circ \cdots \circ g_n(\gamma_j l_j))}{H_j(1)}. \tag{10}$$

The function $g_0 \circ g_1 \circ \cdots \circ g_{n-1} \circ g_n(z)$ is a mapping of the domain $\mathbf{G}_1^n$ onto the domain $\mathbf{D}_0(r_n^{\dagger})$ where $r_n^{\dagger} := g_0 \circ g_1 \circ \cdots \circ g_{n-1} \circ g_n(r_n)$. Therefore, let

$$g_{-1}(z) := H_0^{-1}\left(\frac{1}{\alpha_{-1}} H_0(z)\right), \tag{11}$$

where

$$\alpha_{-1} = \frac{H_0(r_n^{\dagger})}{H_0(1)} = \frac{H_0(g_0 \circ g_1 \cdots g_{n-1} \circ g_n(r_n))}{H_0(1)}. \tag{12}$$

Now the function $g_{-1} \circ g_0 \circ g_1 \cdots g_{n-1} \circ g_n(z)$ is a mapping of the domain $\mathbf{G}_1^n$ onto $\mathbf{D}$. The inverse of this function, which may be denoted F_n, is a mapping of $\mathbf{D}$ onto the domain $\mathbf{G}_1^n$. Since $g_j'(0) = 1/\alpha_j$ for $-1 \le j \le n$, it follows that $F_n'(0) = \prod_{j=-1}^{n} \alpha_j$. In Theorem 3.2 it will be shown that this product can be evaluated in terms of a formula involving nested square root terms.

Lemma 3.1. *If $H_j(z)$ is defined as in (4) and $\gamma_j := \exp\left(i\pi/(3 \cdot 2^j)\right)$, for $0 \le j \le n$, and the functions $T_j(z)$ for $j \ge 1$ and $T_0(z)$ are defined as follows,*

$$T_j(z) := \gamma_{j+1}\, z \left(\frac{1}{1 - 4\, z^{3\cdot 2^j}} \right)^{1/(3\cdot 2^{j+1})},$$

$$T_0(z) := \overline{\gamma}_0\, z \left(\frac{1}{1 - 4\, z^3} \right)^{1/3},$$

then

(I) $H_{j+1}\left(\overline{\gamma}_{j+1}\gamma_j\, H_j^{-1}(z)\right) = T_j(z)$, *for $j \ge 1$ and $z \in H_j(\mathbf{D})$*

(II) $H_0\left(\overline{\gamma}_0\, H_0^{-1}(z)\right) = T_0(z)$, *for $z \in H_0(\mathbf{D})$.*

Proof. For part (I),

$$[H_{j+1}(z)]^2 = \frac{z}{\left(1 - iz^{3\cdot 2^j}\right)^{1/(3\cdot 2^{j-1})}} \cdot \frac{z}{\left(1 + iz^{3\cdot 2^j}\right)^{1/(3\cdot 2^{j-1})}}.$$

Therefore,

$$\left[H_{j+1}(\overline{\gamma}_{j+1}\gamma_j\, z)\right]^2 = \gamma_{j+1}^2\, [H_j(z)]^2\, M_j(z),$$

$$\text{where} \quad M_j(z) := \left(\frac{1 + z^{3\cdot 2^j}}{1 - z^{3\cdot 2^j}} \right)^{1/(3\cdot 2^{j-1})}.$$

Now the left side of (I) can be evaluated by means of

$$\left[H_{j+1}\left(\overline{\gamma}_{j+1}\gamma_j H_j^{-1}(z)\right)\right]^2 = \gamma_{j+1}^2\, z^2\, M_j(H_j^{-1}(z)). \tag{13}$$

To continue, let $w = M_j(H_j^{-1}(z))$, then solve for z, getting

$$z = H_j(M_j^{-1}(w)) = \left(\frac{w^{3\cdot 2^j} - 1}{4\, w^{3\cdot 2^j}} \right)^{1/(3\cdot 2^j)}.$$

Solving for w, substituting in (13), then taking the square root of both sides of (13), completes part (I) of the lemma. Proving part (II) is similar. □

Theorem 3.2. *If $n \geq 1$ is specified, and $\mathbf{G}_1^n$ is a radial slit domain as defined in* (5), *in terms of any real numbers $0 < r_n < 1$ and $0 < l_j < 1$, for $0 \leq j \leq n$, then define the following positive numbers, for $0 \leq m \leq n-1$,*

(14)

$$L_{-1} := \frac{\left(1-(r_n)^{3\cdot 2^n}\right)^2}{(r_n)^{3\cdot 2^n}},\ L_m := \frac{\left(1-(l_m)^{3\cdot 2^n}\right)^2}{(l_m)^{3\cdot 2^n}},\ L_n := \frac{\left(1+(l_n)^{3\cdot 2^n}\right)^2}{(l_n)^{3\cdot 2^n}}.$$

If F_n is the univalent mapping of the unit disc $\mathbf{D}$ onto the domain $\mathbf{G}_1^n$, which fixes the origin, i.e., $F_n(0) = 0$, with $F_n'(0) > 0$, then $F_n'(0)$ may be computed by means of the following system of equations:

$$\beta_n := \frac{1}{4} L_n,$$

$$\beta_{n-1} := \frac{1}{4}\sqrt{4\beta_n + L_{n-1}},$$

$$\beta_{n-2} := \frac{1}{2}\sqrt{\beta_{n-1} + \frac{1}{4}\sqrt{4\beta_n + L_{n-2}}},$$

$$\vdots$$

(15)

$$\beta_{n-k} := \frac{1}{2}\sqrt{\beta_{n-k+1} + \frac{1}{2}\sqrt{\beta_{n-k+2} + \cdots + \frac{1}{2}\sqrt{\beta_{n-1} + \frac{1}{4}\sqrt{4\beta_n + L_{n-k}}}}},$$

$$\vdots$$

$$\beta_0 := \frac{1}{2}\sqrt{\beta_1 + \frac{1}{2}\sqrt{\beta_2 + \cdots + \frac{1}{2}\sqrt{\beta_{n-1} + \frac{1}{4}\sqrt{4\beta_n + L_0}}}},$$

i.e., for each $1 \leq k \leq n$ the number β_{n-k} is expressed in terms of k

nested root terms. Finally, we have

(16)
$$\left(\frac{1}{F_n'(0)}\right)^3 = \beta_0 + \frac{1}{2}\sqrt{\beta_1 + \frac{1}{2}\sqrt{\beta_2 + \cdots + \frac{1}{2}\sqrt{\beta_{n-1} + \frac{1}{4}\sqrt{4\beta_n + L_{-1}}}}}.$$

Proof. The proof follows the construction outlined at the beginning of this section and makes use of equations (7), (10) for $1 \le j \le n$, and (12).

Step 1. Determining α_{-1}.

According to (12),

$$\begin{aligned} H_0(1)\,\alpha_{-1} &= H_0(g_0 \circ g_1 \cdots g_{n-1} \circ g_n(r_n)) \\ &= H_0\Bigg(g_0 \circ g_1 \cdots g_{n-2}\Bigg(\gamma_{n-1} H_{n-1}^{-1} \\ &\qquad \left(\frac{1}{\alpha_{n-1}} H_{n-1}(\overline{\gamma}_{n-1} g_n(r_n))\right)\Bigg)\Bigg). \end{aligned} \tag{17}$$

Now let

$$\begin{aligned} x_{-1,n-1} &:= H_{n-1}(\overline{\gamma}_{n-1} g_n(r_n)) \\ &= H_{n-1}\left(\overline{\gamma}_{n-1}\gamma_n\, H_n^{-1}\left(\frac{1}{\alpha_n}\, H_n(\overline{\gamma}_n r_n)\right)\right), \end{aligned} \tag{18}$$

then this is equivalent to

$$H_n(\overline{\gamma}_n \gamma_{n-1}\, H_{n-1}^{-1}(x_{-1,n-1})) = H_n(\overline{\gamma}_n r_n)/\alpha_n.$$

Here, according to part (I) of Lemma 3.1 it follows that

$$\left(\frac{1}{(x_{-1,n-1})^{3\cdot 2^{n-1}}}\right)^2 - 4\,\frac{1}{(x_{-1,n-1})^{3\cdot 2^{n-1}}} = (\alpha_n)^{3\cdot 2^n}\, L_{-1}. \tag{19}$$

From (18), $\arg(x_{-1,n-1}) = \arg(\overline{\gamma}_{n-1}) = -\pi/(3\cdot 2^{n-1})$. We define the (positive) number $u_{-1,n-1} = -1/(x_{-1,n-1})^{3\cdot 2^{n-1}}$, then (19) becomes

$$(u_{-1,n-1})^2 + 4\,u_{-1,n-1} = (\alpha_n)^{3\cdot 2^n}\, L_{-1}. \tag{20}$$

From (17) and (18) we now have

$$H_0(1)\,\alpha_{-1} = H_0\left(g_0 \circ g_1 \cdots g_{n-3}\left(\gamma_{n-2}\, H_{n-2}^{-1}\left(x_{-1,n-2}/\alpha_{n-2}\right)\right)\right), \tag{21}$$

where $x_{-1,n-2} := H_{n-2}\left(\overline{\gamma}_{n-2}\gamma_{n-1}\, H_{n-1}^{-1}\left(x_{-1,n-1}/\alpha_{n-1}\right)\right)$. By repeating the steps above we get another equation

$$(u_{-1,n-2})^2 + 4\, u_{-1,n-2} = (\alpha_{n-1})^{3\cdot 2^{n-1}}\, u_{-1,n-1},$$

where $u_{-1,n-2} := -1/(x_{-1,n-2})^{3\cdot 2^{n-2}}$ is a positive number.

Suppose now that numbers $x_{-1,n-1}$, $x_{-1,n-2}, \dots, x_{-1,j+1}$, and positive numbers $u_{-1,n-1}$, $u_{-1,n-2}, \dots, u_{-1,j+1}$ are defined in this way, then from (21),

$$\begin{aligned} H_0(1)\,\alpha_{-1} &= H_0\left(g_0 \circ g_1 \cdots g_j\left(\gamma_{j+1}\, H_{j+1}^{-1}\left(x_{-1,j+1}/\alpha_{j+1}\right)\right)\right) \\ &= H_0\left(g_0 \circ g_1 \cdots g_{j-1}\left(\gamma_j\, H_j^{-1}\left(x_{-1,j}/\alpha_j\right)\right)\right), \end{aligned} \tag{22}$$

where $x_{-1,j} := H_j\left(\overline{\gamma}_j \gamma_{j+1}\, H_{j+1}^{-1}\left(x_{-1,j+1}/\alpha_{j+1}\right)\right)$.

Now let $u_{-1,j} := -1/(x_{-1,j})^{3\cdot 2^j}$ so that the positive number $u_{-1,j}$ satisfies

$$(u_{-1,j})^2 + 4\, u_{-1,j} = (\alpha_{j+1})^{3\cdot 2^{j+1}} u_{-1,j+1}. \tag{23}$$

In this way, numbers $x_{-1,j}$ and positive numbers $u_{-1,j}$ can be defined, for $0 \le j \le n-2$, so that equation (23) is satisfied for these values of j. Continuing from (22), we have $H_0(1)\,\alpha_{-1} = H_0\left(\gamma_0\, H_0^{-1}\left(x_{-1,0}/\alpha_0\right)\right)$. This implies, by part (II) of Lemma 3.1 the equation

$$u_{-1,0} = \frac{4}{\alpha_0^3}\left(\frac{1}{(\alpha_{-1})^3} - 1\right). \tag{24}$$

Equations (20), (23) for $0 \le j \le n$, together with (24), determine the constant α_{-1} in terms of the constants $\alpha_0, \dots, \alpha_n$, and the number L_{-1}. The next step will be to show how, with similar reasoning as in Step 1, the constant α_0 can be determined. The equations are slightly different.

Step 2. *Determining* α_0. According to (10) with $j = 1$, we have

$$\begin{aligned} H_0(1)\,\alpha_0 &= H_0(\overline{\gamma}_0\, g_1 \circ g_2 \cdots g_{n-1} \circ g_n(\gamma_0\, l_0)) \\ &= H_0\left(\overline{\gamma}_0\, g_1 \circ g_2 \cdots g_{n-2}\left(\gamma_{n-1}\, H_{n-1}^{-1}\right.\right. \\ &\qquad \left.\left.\left((1/\alpha_{n-1}) H_{n-1}(\overline{\gamma}_{n-1}\, g_n(\gamma_0\, l_0))\right)\right)\right), \end{aligned} \tag{25}$$

and we let

$$\begin{aligned} x_{0,n-1} &:= H_{n-1}(\overline{\gamma}_{n-1}\, g_n(\gamma_0\, l_0)) \\ &= H_{n-1}\left(\overline{\gamma}_{n-1}\gamma_n\, H_n^{-1}\left((1/\alpha_n)H_n(\overline{\gamma}_n\gamma_0\, l_0)\right)\right). \end{aligned}$$

This is equivalent to

$$(u_{0,n-1})^2 + 4\, u_{0,n-1} = (\alpha_n)^{3\cdot 2^n}\, L_0, \tag{26}$$

where we define the positive number $u_{0,n-1} = -1/(x_{0,n-1})^{3\cdot 2^{n-1}}$. As in Step 1, numbers $x_{0,j}$ and positive numbers $u_{0,j}$ can be defined, for $1 \le j \le n-2$, so that

$$(u_{0,j})^2 + 4\, u_{0,j} = (\alpha_{j+1})^{3\cdot 2^{j+1}}\, u_{0,j+1}, \quad \text{for} \quad 1 \le j \le n-2. \tag{27}$$

Furthermore, (25) becomes, for $2 \le j \le n-1$,

$$\begin{aligned} H_0(1)\, \alpha_0 &= H_0\left(\overline{\gamma}_0\, g_1 \circ g_2 \cdots g_{j-1}\left(\gamma_j\, H_j^{-1}\left(x_{0,j}/\alpha_j\right)\right)\right) \\ &\vdots \\ &= H_0\left(\overline{\gamma}_0\gamma_1\, H_1^{-1}\left(x_{0,1}/\alpha_1\right)\right). \end{aligned}$$

Since $H_0(1) = 1/(2^{2/3})$, this implies

$$H_1\left(\overline{\gamma}_1\gamma_0\, H_0^{-1}\left(\frac{\alpha_0}{2^{2/3}}\right)\right) = x_{0,1}/\alpha_1.$$

By Lemma 3.1 (I) this is equivalent to

$$u_{0,1} = 16\, \frac{1}{(\alpha_0\alpha_1)^6}\left(1 - (\alpha_0)^3\right). \tag{28}$$

Equations (26) and (27), together with (28), determine the constant α_0 in terms of the constants $\alpha_1 \cdots \alpha_n$ and the number L_0.

Step 3 below is skipped in the case that $n = 1$ or $n = 2$, and Step 4 below is skipped in the case that $n = 1$.

Step 3. *Determining* α_m *for* $0 \le m \le n-2$. As in Step 1 and Step 2, any constant α_m, for $0 \le m \le n-2$, can be determined in terms of constants $\alpha_{m+1} \cdots \alpha_n$. In other words, the equations (26), (27) and (28) can be generalized to the equations:

$$(u_{m,n-1})^2 + 4\, u_{m,n-1} = (\alpha_n)^{3\cdot 2^n} L_m, \tag{29}$$

$$(u_{m,j})^2 + 4\, u_{m,j} = (\alpha_{j+1})^{3\cdot 2^{j+1}} u_{m,j+1}, \quad \text{for} \quad m+1 \le j \le n-2, \tag{30}$$

$$u_{m,m+1} = 16\, \frac{1}{(\alpha_m \alpha_{m+1})^{3\cdot 2^{m+1}}} \,(1 - (\alpha_m)^{3\cdot 2^m}), \tag{31}$$

where $0 \le m \le n-2$. (In the case $m = n-2$, equation (30) is skipped.)

Step 4. *Determining* α_{n-1}. Here (8) determines that

$$\begin{aligned} H_{n-1}(1)\,\alpha_{n-1} &= H_{n-1}(\overline{\gamma}_{n-1}\, g_n(\gamma_{n-1}\, l_{n-1})) \\ &= H_{n-1}\left(\overline{\gamma}_{n-1}\gamma_n\, H_n^{-1}\left((1/\alpha_n) H_n(\overline{\gamma}_n \gamma_{n-1}\, l_{n-1})\right)\right). \end{aligned}$$

Since $H_{n-1}(1) = 1/(2^{2/(3\cdot 2^{n-1})})$, this means that

$$H_n\left(\overline{\gamma}_n \gamma_{n-1}\, H_{n-1}^{-1}\left(\frac{\alpha_{n-1}}{2^{2/(3\cdot 2^{n-1})}}\right)\right) = H_n(\gamma_n\, l_{n-1})\Big/\alpha_n.$$

Now it follows from Lemma 3.1 (I) that

$$\left(\frac{1}{(\alpha_{n-1})^{3\cdot 2^{n-1}}}\right)^2 - \frac{1}{(\alpha_{n-1})^{3\cdot 2^{n-1}}} = \frac{1}{16}\,(\alpha_n)^{3\cdot 2^n}\, L_{n-1}, \tag{32}$$

which determines the value of α_{n-1} in terms of α_n and L_{n-1}.

Step 5. *Determining* α_n. This follows from (7):

$$H_n(1)\,\alpha_n = H_n(l_n) = \frac{l_n}{(1 + (l_n)^{3\cdot 2^n})^{1/(3\cdot 2^{n-1})}}.$$

Since $H_n(1) = 1/(2^{1/(3\cdot 2^{n-1})})$, we get

$$1/(\alpha_n)^{3\cdot 2^n} = L_n/4, \tag{33}$$

which determines α_n in terms of L_n.

Special case: $n = 2$. The theorem will now be proved for the case $n = 2$. The proof for the general case can be inferred from this. Let $u = u_{-1,1}$, $v = u_{-1,0}$ and $w = u_{0,1}$ be the positive numbers occurring in equations (20) and (23) with $j = 0$, (24), and (29) and (31) with $m = 0$. These equations, together with (32) and (33), can now be expressed as the following system of equations:

$$(34)\quad \begin{gathered} (u+2)^2 = (\alpha_2)^{12}\, L_{-1} + 4, \quad (v+2)^2 = (\alpha_1)^6\, u + 4, \\ v = \frac{4}{(\alpha_0)^3}\left(\frac{1}{(\alpha_{-1})^3} - 1\right), \\ (w+2)^2 = (\alpha_2)^{12}\, L_0 + 4, \qquad w = 16\,\frac{1-(\alpha_0)^3}{(\alpha_0\alpha_1)^6}, \\ \frac{1}{(\alpha_1)^{12}} - \frac{1}{(\alpha_1)^6} = \frac{1}{16}\,(\alpha_2)^{12}\, L_1, \\ \frac{1}{(\alpha_2)^{12}} = \frac{L_2}{4}. \end{gathered}$$

Also define the following numbers:
(35)

$$\beta_2 = \frac{1}{(\alpha_2)^{12}},\ \beta_1 = \frac{1}{(\alpha_1\alpha_2)^6} - \frac{1}{2(\alpha_2)^6},\ \beta_0 = \frac{1}{(\alpha_0\alpha_1\alpha_2)^3} - \frac{1}{2(\alpha_1\alpha_2)^3}.$$

It follows from the fifth line of (34) that

$$(36)\qquad \beta_2 = L_2/4.$$

Solving the equation in the fourth line of (34) gives

$$\frac{1}{(\alpha_1)^6} = \frac{1}{2}\left(1 + \sqrt{1 + \frac{(\alpha_2)^{12}\, L_1}{4}}\right).$$

By factoring a term $(\alpha_2)^6$ from the square root, this becomes

$$(37)\qquad \beta_1 = \frac{1}{2}\sqrt{\frac{1}{(\alpha_2)^{12}} + \frac{L_1}{4}} = \frac{1}{4}\sqrt{4\,\beta_2 + L_1}.$$

Solving the second equation in the third line of (34) gives

$$\frac{1}{(\alpha_0)^3} = \frac{1}{2}\left(1 + \sqrt{1 + \frac{(\alpha_1)^6\, w}{4}}\right).$$

Factoring $(\alpha_1)^3$ from the root term, and solving the first equation of the third line of (34) gives

$$\frac{1}{(\alpha_0\alpha_1)^3} - \frac{1}{2(\alpha_1)^3} = \frac{1}{2}\sqrt{\frac{1}{(\alpha_1)^6} + \frac{\sqrt{(\alpha_2)^{12}\,L_0 + 4} - 2}{4}}.$$

By factoring $(\alpha_2)^3$ from the root terms, this simplifies to

$$\beta_0 = \frac{1}{2}\sqrt{\beta_1 + \frac{1}{4}\sqrt{4\,\beta_2 + L_0}}. \tag{38}$$

This procedure applied to the first and second lines of (34) gives

$$\frac{1}{(\alpha_{-1}\alpha_0\alpha_1\alpha_2)^3} = \frac{1}{2}\sqrt{\beta_1 + \frac{1}{4}\sqrt{4\,\beta_2 + L_{-1}}} + \beta_0. \tag{39}$$

Thus, the product $\alpha_{-1}\,\alpha_0\,\alpha_1\,\alpha_2$, which is the derivative of the mapping F_2 of the unit disc $\mathbf{D}$ onto the domain $\mathbf{G}_1^2 = \mathbf{G}^2[r_2, l_0, l_1, l_2, 1]$, is determined by (36), (37), (38) and (39) together with the definitions corresponding to (14):

$$L_{-1} = \frac{(1-(r_2)^{12})^2}{(r_2)^{12}}, \quad L_0 = \frac{(1-(l_0)^{12})^2}{(l_0)^{12}},$$
$$L_1 = \frac{(1-(l_1)^{12})^2}{(l_1)^{12}}, \quad L_2 = \frac{(1+(l_2)^{12})^2}{(l_2)^{12}}. \qquad \square$$

Suppose now that G_n is the mapping of $\mathbf{D}$ onto the domain $\mathbf{G}_\infty^n$ as defined in (6), then we can write

$$G_n(z) = \overline{\gamma}_n\,H_n(\gamma_n\,\mathcal{F}_n(z)),$$

where $\mathcal{F}_n(z)$ is the mapping of $\mathbf{D}$ onto $\mathbf{G}_1^n := \mathbf{G}^n[\tilde{r}_n, \tilde{l}_0, \tilde{l}_1, \tilde{l}_2, \dots, \tilde{l}_n, 1]$, where the numbers $0 < \tilde{r}_n < 1$ and $0 < \tilde{l}_j < 1$ for $0 \le j \le n$ are defined by

$$\overline{\gamma}_n\,H_n(\gamma_n\,\mathcal{F}_n(\tilde{r}_n)) = r_n$$
$$\overline{\gamma}_n\,H_n(\gamma_n\,\mathcal{F}_n(\tilde{l}_j)) = l_j,$$
$$\text{for} \quad 0 \le j \le n.$$

Theorem 3.2 now applies to the mapping $\mathcal{F}_n(z)$. We get the next corollary:

Corollary 3.3. *If $n \geq 1$ is specified, and $\mathbf{G}_\infty^n$ is the radial slit domain as defined in* (6), *in terms of any real numbers $0 < r_n < 1$ and $0 < l_j < 1$, for $0 \leq j \leq n$, then define the following numbers:*

$$\widetilde{L}_{-1} := \frac{1}{(r_n)^{3\cdot 2^n}}, \qquad \widetilde{L}_m := \frac{1}{(l_m)^{3\cdot 2^n}}, \qquad \textit{for} \quad 0 \leq m \leq n.$$

If G_n is the univalent mapping of the unit disc $\mathbf{D}$ onto the domain $\mathbf{G}_\infty^n$, which fixes the origin, i.e., $G_n(0) = 0$, with $G_n'(0) > 0$, then $G_n'(0)$ may be computed by means of the system of equations (15) *and* (16) *with constants $\widetilde{L}_j$ replacing constants L_j for $-1 \leq j \leq n$, and G_n replacing F_n.*

Suppose that $n \geq 2$; then, as in [**5**], we let

$$r_n := \frac{1}{1 + \csc\left(\pi/(3\cdot 2^{n+1})\right)}, \quad l_0 := 2\, r_n,$$
$$l_1 := r_n\, c, \quad l_j := r_n \left(1 + \csc\left(\frac{\pi}{3\cdot 2^j}\right)\right),$$

for $2 \leq j \leq n$. The exact value of the constant c is, see [**1**, page 588]:

$$c := 1/2\left(1 + \sqrt{3} + b\sqrt{3} + \sqrt{3 + b(2\sqrt{3} - 2)}\right),$$
$$\text{where} \quad b := \sqrt{2\sqrt{3} - 3}.$$

With these assigned values, the domain $\mathbf{G}_1^n$ defined in (5) contains no disc with a radius larger than r_n (compare with (3)). By choosing $n = 3$, and by means of Theorem 3.2 and Corollary 3.3 we can compute $\mathbf{B}_\infty$ accurately to 24 decimal digits:

$$0.656468083606318712155841 < \mathbf{B}_\infty < 0.656468083606318712155841 2.$$

4. Computing F_n. The power series expression of $F_n(z)$ may be expressed as $F_n(z) = z\, f_n(z^3)$, where $f_n(z)$ is a nonvanishing mapping of the unit disc having a power series expansion $f_n(z) = \sum_{i=0}^{\infty} c_i\, z^i$. If we write $F_n(z) = \sum_{i=0}^{\infty} b_{3i+1}\, z^{3i+1}$, then $b_{3i+1} = c_i$ for $i \geq 0$.

Let functions $h(z)$ and $H_j(z)$ for $j \geq 0$ be defined as in (4), and let $T_j(z)$ for $j \geq 0$ and $T_0(z)$ be the expressions defined in Lemma 3.1. Furthermore, define

$$t_0(z) = \frac{-z}{1-4\,z}, \quad t(z) = \frac{-z^2}{1-4\,z}, \quad s(z) = \frac{1}{1-4\,z}.$$

Now the functions T_0, H_j, T_j for $j \geq 0$ can be expressed as follows:

$$H_j(z)^{3\cdot 2^j} = h(z^{3\cdot 2^j}), \quad T_0(z)^3 = t_0(z^3), \quad T_j(z)^{3\cdot 2^j} = t(z^{3\cdot 2^j}). \tag{40}$$

Suppose now that $n \geq 1$ is fixed. We obtain from Section 3 that

$$F_n(z) = g_n^{-1}(g_{n-1}^{-1}(\cdots(g_1^{-1}(g_0^{-1}(g_{-1}^{-1}(z))))\cdots)), \tag{41}$$

where $g_{-1}(z)$ is defined in (11), $g_j(z)$ is defined in (9), with α_n defined in (7), α_{-1} defined in (12), and α_j defined in (10).

We now examine the case $n = 3$. From (9), (10), (40), (41) and Lemma 3.1, we get

$$H_3\big(\overline{\gamma}_3\, F_3(z)\big)^{24} = (\alpha_3)^{24}\, t\big(\,(\alpha_2)^{12}\, t\big(\,(\alpha_1)^6\, t\big(\,(\alpha_0)^3\, t_0\big(\,(\alpha_{-1})^3\, h(z^3)\big)\big)\big)\big).$$

Letting $w := z^3$ gives

$$h\big(-w^8 f_3(w)^{24}\big) = (\alpha_3)^{24}\, t\big((\alpha_2)^{12}\, t\big(\,(\alpha_1)^6\, t\big(\,(\alpha_0)^3\, t_0\big(\,(\alpha_{-1})^3\, h(w)\big)\big)\big)\big). \tag{42}$$

To simplify this, define terms $d_0(w)$ and $d_{j+1}(w)$ for $0 \leq j \leq 2$ as follows:

$$d_0(w) = -(\alpha_0\alpha_{-1})^3\, h(w)\; s\big(\,(\alpha_{-1})^3\, h(w)\big),$$
$$d_{j+1}(w) = -(\alpha_{j+1})^{3\cdot 2^{j+1}}\; d_j(w)^2\; s\big(d_j(w)\big).$$

Now (42) becomes $h\big(-w^8 f_3(w)^{24}\big) = d_3(w)$.

We introduce the numbers σ_i for $-1 \leq i \leq 3$, defined as follows:

$$\sigma_{-1} = (\alpha_{-1})^3, \quad \sigma_0 = (\alpha_0\alpha_{-1})^3, \quad \sigma_1 = (\alpha_1\alpha_0\alpha_{-1})^6,$$
$$\sigma_2 = (\alpha_1\alpha_0\alpha_{-1})^{12}, \quad \sigma_3 = (\alpha_2\alpha_1\alpha_0\alpha_{-1})^{24},$$

and let terms $q_0(w)$ and $q_{j+1}(w)$ for $0 \le j \le 2$ be defined as follows:

$$q_0(w) = s(\sigma_{-1}\ h(w)), \quad q_{j+1}(w) = q_j(w)^2\ s\big(-\sigma_j\ h(w)\ q_j(w)\big);$$

then $(1+w)^{16}\, h\big(-w^8 f_3(w)^{24}\big) = -\sigma_3\ w^8\ q_3(w)$. Next define terms $p_0(w)$ and $p_{j+1}(w)$ for $0 \le j \le 2$ as follows:

$$p_0(w) = (1+w)^2/q_0(w) = (1+w)^2 - 4\,\sigma_{-1}\ w,$$
$$p_{j+1}(w) = (1+w)^{2^{j+2}}/q_{j+1}(w) = p_j(w)^2 + 4\,\sigma_j\ w^{2^j}\ p_j(w).$$

This gives us the following expression for $f_3(w)$ in terms of $p_3(w)$:

$$\big(1 - w^8 f_3(w)^{24}\big)^2 = \frac{1}{\sigma_3}\ p_3(w)\ f_3(w)^{24}.$$

This proves case $n = 3$ of Theorem 4.1 stated below. This theorem states that essentially F_n can be computed on its domain by solving a quadratic equation which is expressed in terms of a polynomial $\mathbf{P}_n$. The coefficients of $\mathbf{P}_n$ may be expressed in terms of constants σ_i which can in turn be expressed in terms of the constants β_i introduced in the statement of Theorem 3.2.

Theorem 4.1. *If $n \ge 2$ and F_n is the mapping of the unit disc onto a radial slit domain as defined in* (5)*, such that $F_n(0) = 0$ and $F_n'(0) > 0$, and f_n is the related mapping defined by $F_n(z) = z\ f_n(z^3)$, then*

$$\big(1 - w^{2^n} f_n(w)^{3\cdot 2^n}\big)^2 = \frac{1}{\sigma_n}\ \mathbf{P}_n(w)\ f_n(w)^{3\cdot 2^n}, \tag{43}$$

where $\mathbf{P}_n(w)$ is a polynomial of degree 2^{n+1} determined by means of the following system of equations:

$$\begin{aligned} \mathbf{P}_n(w) = p_n(w) &= p_{n-1}(w)^2 + 4\,\sigma_{n-1}\, w^{2^{n-1}} p_{n-1}(w), \\ &\ \vdots \\ p_{n-j}(w) &= p_{n-j-1}(w)^2 + 4\,\sigma_{n-j-1}\, w^{2^{n-j-1}} p_{n-j-1}(w) \\ &\quad \text{for } 1 \le j \le n-1, \\ &\ \vdots \\ p_0(w) &= 1 + (2 - 4\,\sigma_{-1})w + w^2, \end{aligned} \tag{44}$$

and the constants σ_j for $-1 \le j \le n$ are defined in terms of constants α_j for $-1 \le j \le n$ as follows:

$$\sigma_{-1} = \alpha_{-1}^3, \quad \sigma_j = \left(\prod_{k=-1}^{j} \alpha_k \right)^{3\cdot 2^j}, \quad \text{for } 0 \le j \le n. \tag{45}$$

Note that $\sigma_n = (F_n'(0))^{3\cdot 2^n}$.

The next objective is to obtain explicit expressions for some of the coefficients of the polynomials $\mathbf{P}_n(w)$ for $n \ge 2$. The next proposition will show that these expressions may be derived in terms of the following constants:

$$\delta_0 := 1-2\,(\sigma_{-1})+\sigma_0, \quad \delta_j := 1-2\,(\sigma_{j-1})^2+\sigma_j, \quad \text{for} \quad 1 \le j \le n. \tag{46}$$

We can obtain from the proof of Theorem 3.2, corresponding to (35) in the case $n=2$, the following general expressions for the constants β_j, for $0 \le j \le n$, defined in the statement of Theorem 3.2:

$$\beta_n = \left(\frac{1}{\alpha_n}\right)^{3\cdot 2^n}, \quad \beta_j = \left(\frac{1}{\prod_{k=j}^{n} \alpha_k}\right)^{3\cdot 2^j} - \left(\frac{1}{2\prod_{k=j+1}^{n} \alpha_k}\right)^{3\cdot 2^j}, \quad \text{for} \quad 0 \le j \le n-1. \tag{47}$$

Thus, from (45) we get

$$\delta_j = 1 - 2\, F_n'(0)^{3\cdot 2^j} \beta_j, \quad \text{for} \quad 0 \le j \le n. \tag{48}$$

Proposition 4.2. *The polynomial $\mathbf{P}_n(w)$ in Theorem 4.1 may be expressed, for a fixed value of $n \ge 1$, as*

$$\mathbf{P}_n(w) = 1 + \sum_{k=1}^{2^{n+1}} a_k w^k,$$

for some set of constants $\{a_k\}_{k=1}^{2^{n+1}}$. *Expressions for some of these constants are*:

(49)

$$a_1 = 2^{n+1}\,\delta_0, \quad \textit{if} \quad n \geq 1,$$

$$a_2 = 2^n\,\delta_1 + \frac{(a_1)^2}{2}\,(1 - 1/2^n)\,, \quad \textit{if} \quad n \geq 2,$$

$$a_3 = a_1\,\left(1 + (1 - 1/2^{n-1})\left(a_2 - \frac{(a_1)^2}{3}\,(1 - 1/2^n)\right)\right), \quad \textit{if} \quad n \geq 2,$$

$$a_4 = 2^{n-1}\,\delta_2 + (1 - 1/2^{n-1})\left(a_1 a_3 + \frac{(a_2)^2}{2} - (1 - 1/2^n)\,A_4\right),$$

$$\textit{where} \quad A_4 := \left((a_1)^2 a_2 - \frac{(a_1)^4}{12}\,(3 - 1/2^{n-1})\right), \quad \textit{if} \quad n \geq 3.$$

Proof. We define $r_j(w) = p_j(w)/p_{j-1}(w) - p_{j-1}(w)$ for $1 \leq j \leq n$, then (44) becomes

(50)

$$2\,r_j(w) - r_{j-1}(w)^2 = 8\,w^{2^{j-1}}(\delta_j - 1), \quad \text{for} \quad 2 \leq j \leq n, \quad \text{if} \quad n \geq 2,$$

$$r_1(w) + 2\,p_0(w) = 2 + 4\,\delta_0\,w + 2\,w^2, \quad \text{if} \quad n \geq 1.$$

Furthermore $\mathbf{P}_n(w)$ is determined by means of the following equations:

$$\mathbf{P}_n(w) = p_n(w) = p_{n-1}(w)(r_n(w) + p_{n-1}(w)), \quad \text{if} \quad n \geq 1.$$

$$p_{n-j}(w) = p_{n-j-1}(w)(r_{n-j}(w) + p_{n-j-1}(w)), \quad \text{for } 1 \leq j \leq n-1, \text{ if } n \geq 2. \tag{51}$$

The proof continues by comparing coefficients of the polynomials and rational functions in equations (50) and (51). This is a lengthy, but reasonably straightforward procedure. □

By means of (43) the coefficients $\{c_i\}_{i=0}^4$ of f_n, and thus the coefficients $\{b_{3i+1}\}_{i=0}^4$ of F_n can be obtained explicitly in terms of the coefficients $\{a_i\}_{i=1}^4$ of $\mathbf{P}_n$.

5. The fundamental frequency of a membrane. In the article [**1**], authors Bañuelos and Carroll applied a lemma, see Lemma 5.1 below, of Pólya and Szegö in order to compute an approximate upper bound for the first Dirichlet eigenvalue for the Laplacian in the radial slit domain corresponding to the case $n = 3$ of Goodman's construction. This determines an upper bound for the Makai-Hayman constant. The authors of [**1**] encountered computational difficulties which can now be circumvented by means of the formulas derived in Section 4.

The following lemma, stated in [**1**], is extracted from [**9**]. The first Dirichlet eigenvalue for the Laplacian in a domain D is denoted λ_D.

Lemma 5.1. *Suppose that D is a simply connected domain. Then*

$$\lambda_D \leq (j_0)^2 \inf_F \left\{ \frac{1}{\sum_{k=1}^{\infty} |b_k|^2 \, \mu_k} \right\},$$

where

$$\mu_k = k^2 \, \frac{\int_0^1 J_0^2(j_0 r) \, r^{2k-1} \, dr}{\int_0^1 J_0^2(j_0 r) \, r \, dr}$$

is expressed in terms of the first Bessel function J_0 *and its smallest positive root j_0, and the infimum is taken over all conformal maps $F(w) = \sum_{k=0}^{\infty} b_k w^k$ from the unit disc onto D.*

To apply the lemma, we choose D to be

$$D_n := \mathbf{G}^n \left[1, \frac{l_0}{r_n}, \frac{l_1}{r_n}, \frac{l_2}{r_n}, \dots, \frac{l_n}{r_n}, \frac{R_n}{r_n} \right],$$

see (1), for some value of $n \geq 2$, and constants r_n and l_j, for $0 \leq j \leq n$, defined at the end of Section 3, i.e., D_n is the nth stage of Goodman's construction, and $F = F_n / r_n$, which is the mapping of the unit disc onto D_n.

If $n = 3$, we obtain from

$$\lambda_{D_n} < (j_0)^2 \left\{ \frac{1}{\sum_{k=0}^{4} (b_{3k+1})^2 \, \mu_{3k+1}} \right\},$$

with values

$$b_1 = 1.52330330\ldots, \quad b_4 = 0.78064564\ldots, \quad b_7 = 0.29710937\ldots,$$
$$b_{10} = 0.06702743\ldots, \quad b_{13} = -0.26993984\ldots,$$

computed by means of (14), (15), (43), (49) and (48), that $\lambda_{D_3} \leq 2.11683941$. (This is the upper bound given in [**1**].) A smaller (better) upper bound can be obtained by using the inequality

$$\lambda_{D_n} < (j_0)^2 \left\{ \frac{1}{\sum_{k=0}^{K} (b_{3k+1})^2 \, \mu_{3k+1}} \right\}, \tag{52}$$

for any value $n \geq 1$ which is computationally manageable, and $K > 1$ as large as computationally possible. The coefficients b_{3k+1} for $0 \leq k \leq K$ can be computed by means of equation (43), with the coefficients a_k for $1 \leq k \leq K$ of the polynomial $\mathbf{P}_n$ determined by means of equations (44). The constants σ_j can be calculated in terms of the constants β_j, see (47) and (45). The terms μ_{3k+1} for $0 \leq k \leq K$ in (52) can be calculated by means of the following identity derived by repeated application of *integration by parts* and taking into account that the Bessel function $J_0(j_0 x)$ is the solution of the differential equation $x\,y'' + y' + j_0^2\, x\, y = 0$:

$$\mu_{m+1} = \frac{2\,(m+1)^2\,[m!]^4\,(j_0)^2}{(2m+1)!} \times \left(\frac{(-4)^m}{2\,(j_0)^{2m+2}} + \sum_{p=1}^{m} \frac{(-4)^{p-1}\,(2m-2p+1)!}{(j_0)^{2p}\,(m-p)!\,[(m-p+1)!]^3} \right).$$

Table 1 shows the upper bounds computed, using MAPLE, for values $n = 3, 4, 5$, with $K = 2^{n+1}$ being the number of coefficients computed for each value of n. The digital accuracy needed is about 1000 digits.

In 1976 Hayman [**6**] proved the next theorem relating to the study of vibrating membranes, as stated in [**1**]. The constant a is the Makai-Hayman constant mentioned in Section 1 above.

TABLE 1. Upper bounds for λ_D.

value of n	value of K	upper bound
3	16	2.09595
4	32	2.09518
5	64	2.09479

Theorem 5.2 [**6**]. *Let D be a simply connected domain in the complex plane. Let R_D be the inradius of D, that is, the radius of the largest disc contained in D, and let λ_D be the first Dirichlet eigenvalue for the Laplacian in D. There is a universal constant a such that*

$$\lambda_D \geq \frac{a}{(R_D)^2}.$$

In [**1**] the authors prove that $0.6197 < a < 2.13$. This is now improved to

$$0.6197 < a < 2.095,$$

(since $R_{D_3} = 1$).

6. Conclusion. This article has been an effort to make calculations manageable for mappings onto a special family of radial slit domains with threefold symmetry, which includes Goodman's domain as a special case. The method of working backwards from a target domain, i.e., successively reducing a radial slit domain to another radial slit domain with fewer slits (the idea behind the proof of Theorem 3.2) requires some degree of symmetry of the target domain. The target domain used by Beller and Hummel in their paper [**3**] for a computation of an upper bound for the univalent Bloch constant is a domain with threefold symmetry, having three rings of slits, but with a different positioning of the third ring of slits. Bañuelos and Carroll make the observation at the end of their paper [**1**] that Goodman's domain can be modified, from the third ring onwards, by replacing radial slits with slits along geodesics, and they conjecture that this may be the optimal domain for computing an upper bound for the univalent Bloch constant. Further investigation will show if, and how, the methods used in this article can be applied to these and other target domains.

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