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TWO PARAMETERS FOR RAMANUJAN'S THETA-FUNCTIONS AND THEIR EXPLICIT VALUES

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ABSTRACT. We define two parameters $g_{k,n}$ and $g'_{k,n}$ involving Ramanujan's theta-functions $\psi(q)$ for any positive real numbers k and n. We study several properties of these parameters and find some explicit values of $\psi(q)$ and quotients of $\psi(q)$ and of $\phi(q)$. This work is a sequel to some recent works by J. Yi.

1. Introduction. For $q := e^{2\pi i z}$, $\mathrm{Im}\,(z) > 0$, define $\psi(q)$ as

$$\psi(q) := \sum_{n=0}^{\infty} q^{n(n+1)/2} = 2^{-1} q^{-1/8} \vartheta_2(0, z),$$

where ϑ_2 is one of the classical theta-functions [**15**, page 464]. For $q := e^{2\pi i z}$ and $\mathrm{Im}\,(z) > 0$, we also define

$$\phi(q) := \sum_{n=-\infty}^{\infty} q^{n^2} = \vartheta_3(0, 2z)$$

and

$$f(-q) := (q; q)_\infty = q^{-1/24} \eta(z),$$

where ϑ_3 is another classical theta-function [**15**, page 464] and η denotes the Dedekind eta-function and $(a; q)_\infty$ is defined by

$$(a; q)_\infty := \prod_{k=0}^{\infty} (1 - a q^k).$$

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In his first notebook [**12**, page 248] Ramanujan recorded many elementary values of $\psi(q)$ and $\phi(q)$. Particularly, he recorded $\psi(e^{-n\pi})$ for $n = 1$, 2, 4, 8, 1/2, and 1/4 and $\phi(e^{-n\pi})$ and $\phi(-e^{-n\pi})$ for $n = 1$, 2, 4, 8, 1/2, and 1/4. All these values were proved by Berndt [**7**, page 325]. Ramanujan also recorded nonelementary values of $\phi(e^{-n\pi})$ for $n = 3$, 5, 9, 7, and 45. Berndt and Chan [**8**] found proofs for these. They also found new explicit values of $\phi(e^{-n\pi})$ for $n = 13$, 27, and 63. Recently, Yi [**17, 18**] evaluated many new values of $\phi(q)$ and $f(q)$ using modular identities, transformation formulae for theta-functions and the parameters $h_{k,n}$, $h'_{k,n}$, $r_{k,n}$, and $r'_{k,n}$ defined, respectively, by

$$r_{k,n} := \frac{f(-q)}{k^{1/4}q^{(k-1)/24}f(-q^k)}, \qquad q = e^{-2\pi\sqrt{n/k}}, \tag{1.1}$$

$$r'_{k,n} := \frac{f(q)}{k^{1/4}q^{(k-1)/24}f(q^k)}, \qquad q = e^{-\pi\sqrt{n/k}}, \tag{1.2}$$

$$h_{k,n} := \frac{\phi(q)}{k^{1/4}\phi(q^k)}, \qquad q = e^{-\pi\sqrt{n/k}}, \tag{1.3}$$

and

$$h'_{k,n} := \frac{\phi(-q)}{k^{1/4}\phi(-q^k)}, \qquad q = e^{-2\pi\sqrt{n/k}}. \tag{1.4}$$

In particular, she evaluated $\phi(e^{-n\pi})$ for $n = 1$, 2, 3, 4, 5 and 6 and $\phi(-e^{-n\pi})$ for $n = 1$, 2, 4, 6, 8, 10, and 12. Motivated by Yi's work, we define, for any positive real numbers k and n, the two parameters $g_{k,n}$ and $g'_{k,n}$ of the theta-function $\psi(q)$, by

$$g_{k,n} := \frac{\psi(-q)}{k^{1/4}q^{(k-1)/8}\psi(-q^k)}, \qquad q = e^{-\pi\sqrt{n/k}}, \tag{1.5}$$

and

$$g'_{k,n} := \frac{\psi(q)}{k^{1/4}q^{(k-1)/8}\psi(q^k)}, \qquad q = e^{-\pi\sqrt{n/k}}. \tag{1.6}$$

In this paper, we establish many general properties of these parameters, which are analogous to those of $h_{k,n}$ and $h'_{k,n}$. We also find several general theorems for the explicit evaluations of these parameters by using theta-function identities. In particular, we obtain several new explicit values of the theta-function $\psi(q)$ and quotients of $\psi(q)$ and of $\phi(q)$.

In Section 2, we present the theta-function identities involving $\psi(q)$ and $\phi(q)$ and transformation formulae which are used in the subsequent sections.

In Section 3, we list the explicit values of $r_{k,n}$, and $r'_{k,n}$ from [**17**] for ready reference in the later sections.

In Section 4, we give some general properties of $g_{k,n}$ and $g'_{k,n}$. We also establish relations between $g_{k,n}$, $g'_{k,n}$, $r_{k,n}$, and $r'_{k,n}$.

In Section 5, we give some general theorems for the explicit evaluations of $g_{k,n}$ and $g'_{k,n}$ and find many explicit values of $g_{k,n}$ and $g'_{k,n}$ by using the results in Sections 2–4.

In Section 6, we find several explicit values of the theta-functions $\psi(\pm q)$.

In Section 7, we find several explicit values of quotients of the theta-function $\phi(q)$.

In the last two sections, we briefly discuss about applications of the parameters $g_{k,n}$, $g'_{k,n}$, $h_{k,n}$ and $h'_{k,n}$ to the explicit evaluations of the Rogers-Ramanujan continued fraction and Ramanujan's cubic continued fraction.

2. Theta-function identities and transformation formulae.

In this section, we give some theta-function identities and a transformation formula which will be used in the subsequent sections. We also present proofs of the new theta-function identities.

Theorem 2.1 (Ramanujan [**12**, page 327], Berndt [**6**, page 233]). *If*

$$P = \frac{\psi(q)}{q^{1/2}\psi(q^5)} \quad \textit{and} \quad Q = \frac{\psi(q^3)}{q^{3/2}\psi(q^{15})},$$

then

$$PQ + \frac{5}{PQ} = \left(\frac{Q}{P}\right)^2 - \left(\frac{P}{Q}\right)^2 + 3\left(\frac{Q}{P} + \frac{P}{Q}\right). \tag{2.1}$$

Theorem 2.2 (Baruah [**2**, page 245]). *If*

$$P = \frac{\psi(q)}{q^{1/4}\psi(q^3)} \qquad \textit{and} \qquad Q = \frac{\psi(q^5)}{q^{5/4}\psi(q^{15})},$$

then

$$(2.2) \quad (PQ)^2 + \left(\frac{3}{PQ}\right)^2 = \left(\frac{Q}{P}\right)^3 - \left(\frac{P}{Q}\right)^3 - 5\left(\frac{Q}{P} - \frac{P}{Q}\right) + 5\left(\frac{P}{Q}\right)^2 + 5\left(\frac{Q}{P}\right)^2.$$

Theorem 2.3 (Baruah [**2**, page 250]). *If*

$$P = \frac{\psi(q)}{q^{1/4}\psi(q^3)} \qquad \textit{and} \qquad Q = \frac{\psi(q^7)}{q^{7/4}\psi(q^{21})},$$

then

$$(2.3) \qquad k_1(PQ)^3 + k_2PQ = k_3(PQ)^2 + k_4\left(\frac{P}{Q}\right)^2 - k_5,$$

where $k_1 = (P/Q)^8 - 1$, $k_2 = 14P^4((P/Q)^4 - 1)$, $k_3 = P^4(7 - P^4)$, $k_4 = 7P^4(P^4 - 3)$, *and* $k_5 = 27(P/Q)^4 - 7P^4(3 + 3(P/Q)^4 - P^4)$.

Theorem 2.4. *If*

$$P = \frac{\psi(-q)}{q\psi(-q^9)} \qquad \textit{and} \qquad Q = \frac{\psi(-q^3)}{q^3\psi(-q^{27})},$$

then

$$(2.4) \qquad \left(\frac{3}{Q} + Q + 3\right)\left(\frac{3}{P} + P + 3\right) = \left(\frac{Q}{P}\right)^2.$$

Proof. The proof of the theorem follows easily from [**5**, page 345]. □

Theorem 2.5 (Berndt [**5**, page 306]). *If*

$$\mu = \frac{f^4(-q)}{qf^4(-q^7)} \quad \textit{and} \quad \nu = \frac{f(-q^{1/7})}{q^{2/7}f(-q^7)},$$

then

$$2\mu = 7(\nu^3 + 5\nu^2 + 7\nu) + (\nu^2 + 7\nu + 7)(4\nu^3 + 21\nu^2 + 28\nu)^{1/2}. \tag{2.5}$$

Theorem 2.6 (Adiga et al. [**1**, page 10]; Baruah and Bhattacharyya [**4**, page 2157]). *If*

$$P = \frac{\psi(-q)}{q^{1/4}\psi(-q^3)} \quad \textit{and} \quad Q = \frac{\phi(q)}{\phi(q^3)},$$

then

$$Q^4 + P^4Q^4 = 9 + P^4. \tag{2.6}$$

Theorem 2.7 (Adiga et al. [**1**, page 10]; Baruah and Bhattacharyya [**4**, page 2156]). *If*

$$P = \frac{\psi(-q)}{q\psi(-q^9)} \quad \textit{and} \quad Q = \frac{\phi(q)}{\phi(q^9)},$$

then

$$Q + PQ = 3 + P. \tag{2.7}$$

Theorem 2.8 (Adiga et al. [**1**, page 10]; Baruah and Bhattacharyya [**4**, page 2156]). *If*

$$P = \frac{\psi(-q)}{q^{1/2}\psi(-q^5)} \quad \textit{and} \quad Q = \frac{\phi(q)}{\phi(q^5)},$$

then

$$Q^2 + P^2Q^2 = 5 + P^2. \tag{2.8}$$

Theorem 2.9. *If*

$$P = \frac{\psi(q)}{q^{1/4}\psi(q^3)} \quad \textit{and} \quad Q = \frac{\psi(q^2)}{q^{1/2}\psi(q^6)},$$

then

$$\left(\frac{P}{Q}\right)^2 + \frac{3}{P^2} - P^2 + \left(\frac{Q}{P}\right)^2 = 0. \tag{2.9}$$

Proof. Replacing q by $-q$ in [**5**, page 39], we note that

$$\psi(q) = \frac{f^2(-q^2)}{f(-q)}. \tag{2.10}$$

Thus,

$$P = \frac{f^2(-q^2)f(-q^3)}{q^{1/4}f(-q)f^2(-q^6)} \quad \text{and} \quad Q = \frac{f^2(-q^4)f(-q^6)}{q^{1/2}f(-q^2)f^2(-q^{12})}. \tag{2.11}$$

Set

$$L_1 := \frac{f(-q)}{q^{1/12}f(-q^3)} \quad \text{and} \quad L_2 := \frac{f^2(-q^2)}{q^{1/3}f^2(-q^6)}, \tag{2.12}$$

and

$$M_1 := \frac{f(-q^2)}{q^{1/6}f(-q^6)} \quad \text{and} \quad M_2 := \frac{f^2(-q^4)}{q^{2/3}f^2(-q^{12})}. \tag{2.13}$$

Then, from (2.11), (2.12) and (2.13), we have

$$P = \frac{L_2}{L_1}, \quad Q = \frac{M_2}{M_1}, \quad \text{and} \quad L_2 = M_1^2. \tag{2.14}$$

Now from (2.12), (2.13) and [**6**, page 204], we deduce that

$$(L_1M_1)^2 + \left(\frac{3}{L_1M_1}\right)^2 = \left(\frac{M_1}{L_1}\right)^6 + \left(\frac{L_1}{M_1}\right)^6, \tag{2.15}$$

$$L_2M_2 + \left(\frac{9}{L_2M_2}\right) = \left(\frac{M_2}{L_2}\right)^3 + \left(\frac{L_2}{M_2}\right)^3. \tag{2.16}$$

Employing (2.14) we find that

$$M_1^{12} = \frac{P^{12} - 9P^8}{P^4 - 1} \tag{2.17}$$

and

$$M_1^6 = \frac{Q^6 - 9Q^2}{Q^4 - 1}. \tag{2.18}$$

From (2.17) and (2.18), we conclude that

$$\frac{P^{12} - 9P^8}{P^4 - 1} = \left(\frac{Q^6 - 9Q^2}{Q^4 - 1}\right)^2. \tag{2.19}$$

Simplifying the above equation (2.19), we obtain

(2.20)
$$(P^4 - 3Q^2 + P^4Q^2 + Q^4)(-P^4 - 3Q^2 + P^4Q^2 - Q^4)(9 - P^4 - Q^4 + P^4Q^4) = 0.$$

By examining the behavior of the first and the last factors of the lefthand side of (2.20) near $q = 0$, it can be seen that there is a neighborhood about the origin where these factors are not zero. Then the second factor is zero in this neighborhood. By the identity theorem this factor is identically zero. Thus, we have

$$P^4 + 3Q^2 - P^4Q^2 + Q^4 = 0. \tag{2.21}$$

Dividing the above equation by P^2Q^2, we complete the proof. □

Theorem 2.10. *If*

$$P = \frac{\psi(-q)}{q^{1/4}\psi(-q^3)} \quad \text{and} \quad Q = \frac{\psi(q^2)}{q^{1/2}\psi(q^6)},$$

then

$$\left(\frac{P}{Q}\right)^2 + \frac{3}{P^2} + P^2 - \left(\frac{Q}{P}\right)^2 = 0. \tag{2.22}$$

Proof. Replacing q by $-q$ in Theorem 2.9, we complete the proof. □

Theorem 2.11. *If*

$$P = \frac{\psi(-q)}{q^{1/4}\psi(-q^3)} \quad \textit{and} \quad Q = \frac{\psi(q)}{q^{1/4}\psi(q^3)}$$

then

(2.23)
$$\left(\frac{P}{Q}\right)^4 + \left(\frac{Q}{P}\right)^4 + \left(\left(\frac{P}{Q}\right)^2 - \left(\frac{Q}{P}\right)^2\right)\left(\left(\frac{3}{PQ}\right)^2 - (PQ)^2\right) - 10 = 0.$$

Proof. Invoking (2.10), we obtain

$$P = \frac{f^2(-q^2)f(q^3)}{q^{1/4}f(q)f^2(-q^6)} \quad \text{and} \quad Q = \frac{f^2(-q^2)f(-q^3)}{q^{1/4}f(-q)f^2(-q^6)} \tag{2.24}$$

We set

$$L_1 := \frac{f(q)}{q^{1/12}f(q^3)} \qquad \text{and} \quad L_2 := \frac{f^2(-q^2)}{q^{1/3}f^2(-q^6)}, \tag{2.25}$$

$$M_1 := \frac{f(-q)}{q^{1/12}f(-q^3)} \quad \text{and} \quad M_2 := \frac{f^2(-q^2)}{q^{1/3}f^2(-q^6)}. \tag{2.26}$$

Then, we have

$$P = \frac{L_2}{L_1}, \quad Q = \frac{M_2}{M_1}, \quad \text{and} \quad L_2 = M_2. \tag{2.27}$$

Now by applying (2.25) and (2.27) in [**6**, page 204], we obtain

$$L_1^8 M_2^4 - 9L_1^4 M_2^2 = M_2^6 - L_1^{12}. \tag{2.28}$$

Replacing L_1 in the above equation using (2.27), and simplifying using the result $L_2 = M_2$, we find that

$$M_2^6 = \frac{P^{12} + 9P^8}{P^4 + 1}. \tag{2.29}$$

Again, from (2.26) in [**6**, page 204], we obtain

$$M_1^2M_2+\frac{9}{M_1^2M_2}=\frac{M_1^3}{M_1^6}+\frac{M_2^6}{M_2^3}. \tag{2.30}$$

Employing (2.27), we deduce that

$$M_2^6=\frac{Q^6(Q^6-9Q^2)}{Q^4-1}. \tag{2.31}$$

From (2.29) and (2.31), we arrive at

$$\frac{P^{12}+9P^8}{P^4+1}=\frac{Q^6(Q^6-9Q^2)}{Q^4-1}. \tag{2.32}$$

Simplifying, we obtain

$$9P^4+P^8-9Q^4-10P^4Q^4-P^8Q^4+Q^4+P^4Q^8=0. \tag{2.33}$$

Dividing the above equation P^4Q^4 and rearranging the terms, we complete the proof.

Theorem 2.12. *If*

$$P=\frac{\psi(q)}{q^{1/2}\psi(q^5)} \quad \textit{and} \quad Q=\frac{\psi(q^2)}{q\psi(q^{10})},$$

then

$$\left(\frac{P}{Q}\right)^2-\frac{5}{P^2}-P^2+\left(\frac{Q}{P}\right)^2+4=0. \tag{2.34}$$

Proof. We employ [**6**, page 206] and proceed as in the proof of Theorem 2.9. □

Theorem 2.13. *If*

$$P=\frac{\psi(-q)}{q^{1/2}\psi(-q^5)} \quad \textit{and} \quad Q=\frac{\psi(q^2)}{q\psi(q^{10})},$$

then

$$\left(\frac{P}{Q}\right)^2 - \frac{5}{P^2} - P^2 + \left(\frac{Q}{P}\right)^2 - 4 = 0. \tag{2.35}$$

Proof. Replacing q by $-q$ in Theorem 2.12, we easily arrive at (2.35). □

Theorem 2.14. *If*

$$P = \frac{\psi(-q)}{q^{1/2}\psi(-q^5)} \quad \text{and} \quad Q = \frac{\psi(q)}{q^{1/2}\psi(q^5)},$$

then

$$\left(\frac{P}{Q}\right)^2 + \left(\frac{Q}{P}\right)^2 + \left(\frac{P}{Q} - \frac{Q}{P}\right)\left(\frac{5}{PQ} - PQ\right) - 6 = 0. \tag{2.36}$$

Proof. We use [**6**, page 206] and proceed as in the proof of Theorem 2.11. □

Theorem 2.15. *If*

$$P = \frac{\psi(q)}{q^{1/8}\psi(q^2)} \quad \text{and} \quad Q = \frac{\psi(q^2)}{q^{1/4}\psi(q^4)},$$

then

$$P^2 - \left(\frac{2}{PQ}\right)^2 - \left(\frac{Q}{P}\right)^2 = 0. \tag{2.37}$$

Proof. From [**17**, page 21], we note that

$$(L_1M_1)^4 + \left(\frac{2}{L_1M_1}\right)^4 = \left(\frac{M_1}{L_1}\right)^{12}, \tag{2.38}$$

where

$$L_1 := \frac{f(-q)}{q^{1/24}f(-q^2)} \quad \text{and} \quad M_1 := \frac{f(-q^2)}{q^{1/12}f(-q^4)}. \tag{2.39}$$

Let

$$L_2 := \frac{f^2(-q^2)}{q^{1/6}f^2(-q^4)} \quad \text{and} \quad M_2 := \frac{f^2(-q^4)}{q^{1/3}f^2(-q^8)}. \tag{2.40}$$

Now, we proceed as in the proof of Theorem 2.9 with applications of (2.38) instead of [**6**, page 204] to complete the proof. □

Theorem 2.16. *If*

$$P = \frac{\phi(q)}{\phi(q^5)} \quad \textit{and} \quad Q = \frac{\phi(-q)}{\phi(-q^5)},$$

then

$$PQ + \frac{5}{PQ} - 4 = \frac{Q}{P} + \frac{P}{Q}. \tag{2.41}$$

Proof. From [**5**, page 39], we note that

$$\phi(q) = \frac{f^2(q)}{f(-q^2)}. \tag{2.42}$$

Thus, P and Q can be written as

$$P = \frac{f^2(q)f(-q^{10})}{f(-q^2)f^2(q^5)} \quad \text{and} \quad Q = \frac{f^2(-q)f(-q^{10})}{f(-q^2)f^2(-q^5)}. \tag{2.43}$$

Setting

$$L_1 := \frac{f(-q^2)}{q^{1/3}f(-q^{10})} \quad \text{and} \quad L_2 := \frac{f^2(q)}{q^{1/3}f^2(q^5)}, \tag{2.44}$$

$$M_1 := \frac{f(-q^2)}{q^{1/3}f(-q^{10})} \quad \text{and} \quad M_2 := \frac{f^2(-q)}{q^{1/3}f^2(-q^5)}, \tag{2.45}$$

we find that

$$P = \frac{L_2}{L_1}, \quad Q = \frac{M_2}{M_1}, \quad \text{and} \quad M_1 = L_1. \tag{2.46}$$

Now, from (2.44) and [**6**, page 207], we have

$$L_1^4L_2^2 - 5L_1^2L_2 = L_1^6 - L_2^3. \tag{2.47}$$

From (2.46) and (2.47), we obtain

$$L_1^3 = \frac{5P - P^3}{P^2 - 1}. \tag{2.48}$$

Again, from (2.45) and [**6**, page 206],

$$M_1^4M_2^2 + 5M_1^2M_2 = M_1^6 + M_2^3. \tag{2.49}$$

From (2.46) and (2.49), we find that

$$M_1^3 = \frac{Q^3 - 5Q}{Q^2 - 1}. \tag{2.50}$$

Since $L_1 = M_1$, so from (2.48) and (2.50), we deduce that

$$\frac{5P - P^3}{P^2 - 1} = \frac{Q^3 - 5Q}{Q^2 - 1}. \tag{2.51}$$

Simplifying (2.51), we arrive at

$$(P + Q)(5 - P^2 - 4PQ - Q^2 + P^2Q^2) = 0. \tag{2.52}$$

Since the first factor is nonzero in a neighborhood of the origin, we deduce that

$$5 - P^2 - 4PQ - Q^2 + P^2Q^2 = 0. \tag{2.53}$$

Dividing the above equation by PQ, we complete the proof. □

Theorem 2.17. *If*

$$P = \frac{\phi(-q)}{\phi(-q^5)} \quad \textit{and} \quad Q = \frac{\phi(-q^2)}{\phi(-q^{10})},$$

then

$$\left(\frac{P}{Q}\right)^2 + \left(\frac{Q}{P}\right)^2 - Q^2 - \frac{5}{Q^2} + 4 = 0. \tag{2.54}$$

Proof. Employing (2.42), we note that

$$P = \frac{f^2(-q)f(-q^{10})}{f(-q^2)f^2(-q^5)} \quad \text{and} \quad Q = \frac{f^2(-q^2)f^2(-q^{20})}{f(-q^4)f^2(-q^{10})}. \tag{2.55}$$

Setting

$$L_1 := \frac{f(-q^2)}{q^{1/3}f(-q^{10})} \quad \text{and} \quad L_2 := \frac{f^2(-q)}{q^{1/3}f^2(-q^5)}, \tag{2.56}$$

$$M_1 := \frac{f(-q^4)}{q^{2/3}f(-q^{20})} \quad \text{and} \quad M_2 := \frac{f^2(-q^2)}{q^{2/3}f^2(-q^{10})}, \tag{2.57}$$

we deduce that

$$P = \frac{L_2}{L_1}, \quad Q = \frac{M_2}{M_1}, \quad \text{and} \quad M_2 = L_1^2. \tag{2.58}$$

Now, from (2.56) and [**6**, page 206], we deduce that

$$L_1M_1 + \frac{5}{L_1M_1} = \left(\frac{M_1}{L_1}\right)^3 + \left(\frac{L_1}{M_1}\right)^3. \tag{2.59}$$

Applying the results in (2.58) and simplifying, we find that

$$L_1^6 = \frac{Q^6 - 9Q^4}{Q^2 - 1}. \tag{2.60}$$

Similarly, from (2.57) and [**6**, page 206], we obtain

$$L_1^3 = \frac{P^3 - 5P}{P^2 - 1}. \tag{2.61}$$

From (2.60) and (2.61), we find that

$$\left(\frac{P^3-5P}{P^2-1}\right)^2=\frac{Q^6-9Q^4}{Q^2-1}. \tag{2.62}$$

Simplifying the above equation, we obtain

$$(5-P^2-Q^2+P^2Q^2)(-5P^2+P^4+4P^2Q^2+Q^4-P^2Q^4)=0. \tag{2.63}$$

Now, proceeding as in Theorem 2.9, it can be shown that the first factor of (2.63) is nonzero in a neighborhood of zero. Thus, we have

$$5P^2-P^4-4P^2Q^2-Q^4+P^2Q^4=0. \tag{2.64}$$

Dividing the above equation by P^2Q^2, we complete the proof. □

Theorem 2.18. *If*

$$P=\frac{\phi(q)}{\phi(q^5)} \quad \textit{and} \quad Q=\frac{\phi(-q^2)}{\phi(-q^{10})},$$

then

$$\left(\frac{P}{Q}\right)^2+\left(\frac{Q}{P}\right)^2-Q^2-\frac{5}{Q^2}+4=0. \tag{2.65}$$

Proof. Replacing q by $-q$ in Theorem 2.17, we readily complete the proof. □

Theorem 2.19. (i) (Adiga et al. [**1**]). *If* $\alpha\beta=\pi^2$, *then*

$$e^{-\alpha/8}\alpha^{1/4}\psi(-e^{-\alpha})=e^{-\beta/8}\beta^{1/4}\psi(-e^{-\beta}). \tag{2.66}$$

(ii) (Berndt [**5**, page 43]). *If* $\alpha\beta=\pi^2$, *then*

$$e^{-\alpha/12}\sqrt[4]{\alpha}f(-e^{-2\alpha})=e^{-\beta/12}\sqrt[4]{\beta}f(-e^{-2\beta}). \tag{2.67}$$

3. Values of $r_{k,n}$ and $r'_{k,n}$. In this section, we list the values of $r_{k,n}$ and $r'_{k,n}$ from [**17**].

Theorem 3.1. *If $r_{k,n}$ and $r'_{k,n}$ are as defined in* (1.1) *and* (1.2), *then*

$$
\begin{aligned}
r_{1,1} &= 1, \\
r_{2,2} &= 2^{1/8}, \\
r_{2,4} &= 2^{1/8}\left(1+\sqrt{2}\right)^{1/8}, \\
r_{2,5} &= \sqrt{\frac{1+\sqrt{5}}{2}}, \\
r_{2,8} &= 2^{1/8}\left(1+\sqrt{2}\right)^{1/4}\left(4+\sqrt{2+10\sqrt{2}}\right)^{1/8}, \\
r_{2,9} &= \left(\sqrt{2}+\sqrt{3}\right)^{1/3}, \\
r_{2,18} &= \frac{\left(1+\sqrt{3}\right)^{1/3}\left(1+\sqrt{3}+\sqrt{2}\cdot 3^{3/4}\right)^{1/3}}{2^{11/24}}, \\
r_{2,16} &= 2^{1/8}\left(1+\sqrt{2}\right)^{1/4}\left(4+\sqrt{2+10\sqrt{2}}\right)^{1/8}, \\
r_{2,32} &= 2^{3/16}\left(1+\sqrt{2}\right)^{1/4}\left(16+15\cdot 2^{1/4}+12\sqrt{2}+9\cdot 2^{3/4}\right)^{1/8}, \\
r_{2,50} &= \frac{2^{5/8}}{5^{1/4}-1}, \\
r_{2,72} &= \frac{\left(\left(\sqrt{2}+\sqrt{3}\right)\left(\sqrt{3}+1\right)\left(1+\sqrt{2}-\sqrt{3}+2\cdot 3^{5/4}\right)\right)^{1/3}}{2^{7/16}\left(\sqrt{2}-1\right)^{5/12}}, \\
r_{2,9/2} &= \frac{\left(1+\sqrt{3}+\sqrt{2}\cdot 3^{3/4}\right)^{1/3}}{2^{13/24}}, \\
r_{2,25/2} &= \frac{5^{1/4}+1}{2^{5/8}}, \\
r_{3,3} &= 3^{1/12}\left(3+2\sqrt{3}\right)^{1/12} = \frac{3^{1/8}\left(1+\sqrt{3}\right)^{1/6}}{2^{1/12}}
\end{aligned}
$$

$$r_{3,4} = \sqrt{\frac{\sqrt{3}+1}{\sqrt{2}}},$$

$$r_{3,7} = \left(\frac{\sqrt{3}+\sqrt{7}}{2\left(2-\sqrt{3}\right)}\right)^{1/4},$$

$$r_{3,25} = \frac{1}{2}\left(1+\sqrt[3]{10}+\sqrt{5+2\sqrt[3]{10}+\sqrt[3]{10^2}}\right),$$

$$r_{3,49} = \frac{3+\sqrt[3]{4}\sqrt[3]{7}+\sqrt[3]{2}\sqrt[3]{49}+\sqrt{49+13\sqrt[3]{4}\sqrt[3]{7}+8\sqrt[3]{2}\sqrt[3]{49}}}{2\sqrt{3}},$$

$$r_{4,1} = 1,$$

$$r_{4,4} = 2^{5/16}\left(1+\sqrt{2}\right)^{1/4},$$

$$r_{4,8} = 2^{1/4}\left(1+\sqrt{2}\right)^{3/8}\left(4+\sqrt{2+10\sqrt{2}}\right)^{1/8},$$

$$r_{4,9} = \frac{1}{2}\left(1+\sqrt{2}\sqrt[4]{3}+\sqrt{3}\right),$$

$$r_{5,4} = \left(\frac{1+\sqrt{5}+\sqrt{2}+\sqrt{1+\sqrt{5}}}{2}\right)^{1/2},$$

$$r_{5,5} = \left(25+10\sqrt{5}\right)^{1/6},$$

$$r_{6,6} = \frac{3^{1/8}\sqrt{\sqrt{3}+1}\left(1+\sqrt{3}+\sqrt{2}\cdot 3^{3/4}\right)^{1/3}}{2^{13/24}},$$

$$r_{13,3} = \frac{\sqrt{11+\sqrt{13}}+\sqrt{3+\sqrt{13}}}{2\sqrt{2}},$$

$$r_{13,9} = \frac{1}{4}\left(\left(\sqrt{3}+1\right)\left(\sqrt{3}+\sqrt{13}\right)+2\sqrt{\left(3+2\sqrt{3}\right)\left(4+\sqrt{13}\right)}\right),$$

$$r_{25,4} = \frac{1}{2}\left(3+\sqrt[4]{5}+\sqrt{5}+\sqrt[4]{5^3}\right) = \frac{\sqrt[4]{5}+1}{\sqrt[4]{5}-1},$$

$$r_{25,9} = \frac{\sqrt[4]{60}+2-\sqrt{3}+\sqrt{5}}{\sqrt[4]{60}-2+\sqrt{3}-\sqrt{5}},$$

$$r_{25,7} = \frac{1}{6}\left(4\sqrt{5}+a_1+b_1+\sqrt{\left(4\sqrt{5}+a_1+b_1\right)^2-36}\right),$$

where

$$a_1 = \left(\frac{1}{2}\left(2251\sqrt{5} + 9\sqrt{105}\right)\right)^{1/3}$$

and

$$b_1 = \left(\frac{1}{2}\left(2251\sqrt{5} - 9\sqrt{105}\right)\right)^{1/3},$$

$$r_{25,49} = \frac{1}{8}\left(a_2 + \sqrt{5b_2} + \sqrt{\left(a_2 + 2\sqrt{5b_2}\right)^2 - 64}\right),$$

where $a_2 = 1497 + 651\sqrt{5} + 565\sqrt{7} + 247\sqrt{35}$, $b_2 = 437430 + 195566\sqrt{5} + 165333\sqrt{7} + 73917\sqrt{35}$.

We also note that $r_{k,1} = 1$, $r_{k,n} = r_{n,k}$ and $r_{k,1/n} = 1/r_{k,n}$.

Theorem 3.2. *For any positive real numbers n and k, we have*

$$r'_{2,2} = 2^{5/16}\left(\sqrt{2} - 1\right)^{1/4},$$

$$r'_{4,4} = \frac{2^{9/16}}{\left(9 \cdot 2^{1/4} + 4\sqrt{2} - 3 \cdot 2^{3/4}\right)^{1/8}},$$

$$r'_{5,5} = \sqrt{\frac{5 - \sqrt{5}}{2}},$$

$$r'_{6,6} = \frac{2^{11/16}3^{1/8}\left(\sqrt{2} - 1\right)^{1/12}\left(\sqrt{3} + 1\right)^{1/6}}{\left(2 - 3\sqrt{2} + 3 \cdot 3^{1/4} + 3^{3/4}\right)^{1/3}},$$

$$r'_{3,7} = \left(\frac{2\left(2 + \sqrt{3}\right)}{\sqrt{3} + \sqrt{7}}\right)^{1/4},$$

$$r'_{3,25} = \frac{1 + \sqrt{5}}{2},$$

$$r'_{3,49} = \frac{\sqrt{3} + \sqrt{7}}{2},$$

$$r'_{13,3} = \frac{\sqrt{5 + \sqrt{13}} + \sqrt{\sqrt{13} - 3}}{2\sqrt{2}},$$

$$r'_{13,9} = \frac{1}{4}\left(\left(\sqrt{3}+1\right)\left(\sqrt{13}-\sqrt{3}\right)+2\sqrt{\left(3+2\sqrt{3}\right)\left(4-\sqrt{13}\right)}\right),$$
$$r'_{25,9} = \frac{\sqrt[4]{60}+2+\sqrt{3}-\sqrt{5}}{\sqrt[4]{60}-2-\sqrt{3}+\sqrt{5}},$$
$$r'_{25,7} = \frac{1}{6}\left(2\sqrt{5}+a_4+b_4+\sqrt{\left(2\sqrt{5}+a_4+b_4\right)^2-36}\right),$$
$$r'_{25,1/7} = \frac{1}{6}\left(2\sqrt{5}+a_4+b_4-\sqrt{\left(2\sqrt{5}+a_4+b_4\right)^2-36}\right),$$

where

$$a_4 = \left(\frac{1}{2}\left(17\sqrt{5}+3\sqrt{105}\right)\right)^{1/3} \quad \textit{and} \quad b_4 = \left(\frac{1}{2}\left(17\sqrt{5}-3\sqrt{105}\right)\right)^{1/3},$$

$$r'_{25,49} = \frac{1}{8}\left(a_5+2\sqrt{5b_5}+\sqrt{\left(a_5+2\sqrt{5b_5}\right)^2-64}\right),$$
$$r'_{25,1/49} = \frac{1}{8}\left(a_5+2\sqrt{5b_5}-\sqrt{\left(a_5+2\sqrt{5b_5}\right)^2-64}\right),$$

where $a_5 = 1497 - 651\sqrt{5} + 565\sqrt{7} - 247\sqrt{35}$ *and* $b_5 = 437430 - 195566\sqrt{5} + 165333\sqrt{7} - 73917\sqrt{35}$.

4. Properties of $g_{k,n}$ and $g'_{k,n}$.

Theorem 4.1. *For all positive real numbers k and n, we have*

(i) $g_{k,1} = 1$,

(ii) $g_{k,1/n} = g_{k,n}^{-1}$,

(iii) $g_{k,n} = g_{n,k}$.

Remark. By using the definitions of $\psi(q)$ and $g_{k,n}$, it can be seen that $g_{k,n}$ increases as n increases when $k > 1$. Thus, by Theorem 4.1 (i), $g_{k,n} > 1$ for all $n > 1$ if $k > 1$.

Proof. Using the definition of $g_{k,n}$ and Theorem 2.19 (i), we easily arrive at $g_{k,1} = 1$. Replacing n by $1/n$ in $g_{k,n}$ and using Theorem 2.19 (i), we find that $g_{k,n}\ g_{k,1/n} = 1$. Interchanging n and k in $g_{k,n}$, we complete the proof of (iii). □

Theorem 4.2. *For all positive real numbers k, m, and n*

$$g_{k,n/m} = g_{mk,n}\ g_{nk,m}^{-1}.$$

Proof. By the definition of $g_{k,n}$, we find that

$$g_{mk,n} \quad g_{nk,m}^{-1} = g_{m/n,1/k}.$$

Employing Theorem 4.1 (ii) and (iii), we complete the proof. □

Theorem 4.3. *For all positive real numbers a, b, c, and d, we have*

$$g_{a/b,c/d} = \frac{g_{ad,bc}}{g_{ac,bd}}. \tag{4.1}$$

Proof. Applying Theorem 4.1 (iii) in Theorem 4.2, we deduce that, for all positive real numbers a, b, and n

$$g_{a/b,n} = g_{a,bn}\ g_{b,an}^{-1}. \tag{4.2}$$

Again employing Theorem 4.2 and Theorem 4.1 (iii) in (4.2), we arrive at (4.1). □

Theorem 4.4. *For all positive real numbers k and n, we have*

$$g_{k^2,n} = g_{k,nk}\ g_{k,n/k}.$$

Proof. Setting $a = k$, $b = 1/k$, $c = n$ and $d = 1$ in Theorem 4.3, we deduce that

$$g_{k^2,n} = \frac{g_{k,n/k}}{g_{1/k,nk}}.$$

Employing Theorem 4.1 (ii) and (iii), we readily complete the proof. □

Theorem 4.5. *For all positive real numbers a and b, we have*

(i) $g_{a/b,a/b} = g_{b,b}\ g_{a,a/b^2}$,

(ii) $g_{a,a}\ g_{a,b^2/a} = g_{b,b}\ g_{b,a^2/b}$,

(iii) $g_{a,a}\ g_{b,a^2b} = g_{b,b}\ g_{a,ab^2}$.

Proof. Let a and b be any positive real numbers. By using Theorem 4.1 (ii) and Theorem 4.3, we find that

$$g_{a/b,a/b} = g_{b,b}\ g_{a,a/b^2}. \tag{4.3}$$

So we complete the proof of (i). Similarly, we find that

$$g_{b/a,b/a} = g_{a,a}\ g^{-1}_{b,a^2/b}. \tag{4.4}$$

From (4.2) and (4.3), we derive (ii). By using Theorem 4.1 (ii) and Theorem 4.2, we find that

$$g_{a/b,a/b} = g_{b,b}\ g_{ab^2,a}\ g^{-1}_{a^2,b^2}. \tag{4.5}$$

Similarly, we find that

$$g_{b/a,b/a} = g_{a,a}\ g_{a^2b,b}\ g^{-1}_{b^2,a^2}. \tag{4.6}$$

From (4.4), (4.5), and Theorem 4.1 (ii) and (iii), we complete the proof of (iii). □

Theorem 4.6. *For all positive real numbers k, a, b, c, and d with $ab = cd$, we have*

$$g_{a,b}\ g_{kc,kd} = g_{ka,kb}\ g_{c,d}.$$

Proof. From the definition of $g_{k,n}$ and using $ab = cd$, we derive that for all positive numbers k, a, b, c, and d,

$$g_{ka,kb}\ g^{-1}_{a,b} = g_{kc,kd}\ g^{-1}_{c,d}.$$

Rearranging the terms, we complete the proof. □

Theorem 4.7. *For all positive real numbers n and p, we have*

$$g_{np,np} = g_{n,np^2}\ g_{p,p}.$$

Proof. The result follows immediately from Theorem 4.1 (i) and (iii) and Theorem 4.6 with $a = p^2$, $b = 1$, $c = d = p$ and $k = n$. □

Now, we give relations between the parameters $g_{k,n}$, $g'_{k,n}$, $r_{k,n}$ and $r'_{k,n}$ and then use these relations to determine the values of $g_{k,n}$ and $g'_{k,n}$ by using known values of $r_{k,n}$ and $r'_{k,n}$, where $r_{k,n}$ and $r'_{k,n}$ are given by (1.1) and (1.2).

Theorem 4.8. *Let k and n be any positive real numbers. Then*

(i) $g_{k,n} = r^2_{k,n}/r'_{k,n}$.

(ii) $g'_{k,n} = (r_{2,nk/2}/r_{2,n/2k})\ r_{k,n}$.

Proof. (i) Let $q = e^{-\pi\sqrt{n/k}}$. Replacing q by $-q$ in (2.10) and using the definitions of $g_{k,n}$ and $r_{k,n}$, we find that

$$g_{k,n} = \frac{G_{nk}}{G_{n/k}} r_{k,n}, \tag{4.7}$$

where the class invariant G_n is given by

$$G_n = 2^{-1/2} q^{-1/24} \chi(q),$$

where $q := e^{-\pi\sqrt{n}}$, n is a positive real number, and $\chi(q) = (-q; q^2)_\infty$.

By [**17**, page 17, Theorem 2.2.1], we note that

$$\frac{G_{n/k}}{G_{nk}} = \frac{r'_{k,n}}{r_{k,n}}. \tag{4.8}$$

Using (4.8) in (4.7), we complete the proof of (i).

(ii) Let $q := e^{-\pi\sqrt{n/k}}$. Employing (2.10) and the definitions of $g'_{k,n}$ and $r_{k,n}$, we find that

$$g'_{k,n} = \frac{g_{nk}}{g_{n/k}} r_{k,n}, \tag{4.9}$$

where the class invariant g_n is given by

$$g_n = 2^{-1/2} q^{-1/24} \chi(-q),$$

where $q := e^{-\pi\sqrt{n}}$, n is a positive real number and $\chi(q) = (-q;q^2)_\infty$.

Also, by [**17**, page 18], we have

$$g_n = r_{2,n/2}. \tag{4.10}$$

Using (4.10) in (4.9), we complete the proof of (ii). □

Theorem 4.9. *For every positive real number n, we have*

$$g'_{n,1} = r_{4,n}. \tag{4.11}$$

Proof. From [**17**, page 13], we note that

$$r_{k,n/m} = \frac{r_{mk,n}}{r_{nk,m}}. \tag{4.12}$$

Employing (4.12), Theorem 4.8 (ii) and Theorem 4.1 (i), we complete the proof. □

Theorem 4.10. *For all positive real numbers k and n, we have*

(i) $g_{k,n} = (G_{nk}/G_{n/k})\ r_{k,n}$.

(ii) $g'_{k,n} = (g_{nk}/g_{n/k})\ r_{k,n}$.

Proof. These are (4.7) and (4.9), respectively. □

Theorem 4.11. *For every positive real number n, we have*

(i) $g_{n,n} = G_{n^2}\ r_{n,n}$.

(ii) $g'_{n,n} = 2^{1/8} r_{2,n^2/2}\ r_{n,n} = 2^{1/8} g_{n^2}\ r_{n,n}$.

Proof. (i) With $k = n$ in Theorem 4.8 (i) and then using [**17**, page 17, Corollary 2.2.2], we complete the proof.

(ii) Setting $k = n$ in Theorem 4.8 (ii) and using the value $r_{2,2} = 2^{1/8}$ in Theorem 3.1, we complete the proof of (ii). □

5. General theorems for explicit evaluations of $g_{k,n}$ and $g'_{k,n}$. In this section, we find some general theorems on $g_{k,n}$ and $g'_{k,n}$, and then use these theorems to find some explicit values of $g_{k,n}$ and $g'_{k,n}$.

Theorem 5.1. *We have*

(i) $(1 + \sqrt{3} g_{3,n} g_{3,9n})^3 = (1 + 3g^4_{3,9n})$,

(ii) $\sqrt{5} g_{5,n} g_{5,9n} + (\sqrt{5}/g_{5,n} g_{5,9n}) = (g_{5,9n}/g_{5,n})^2 - 3(g_{5,9n}/g_{5,n}) - 3(g_{5,n}/g_{5,9n}) - (g_{5,n}/g_{5,9n})^2$,

(iii) $3(g_{3,n} g_{3,25n})^2 + (3/(g_{3,n} g_{3,25n})^2) + 5(g_{3,25n}/g_{3,n})^2 + 5(g_{3,n}/g_{3,25n})^2 = (g_{3,25n}/g_{3,n})^3 - (g_{3,n}/g_{3,25n})^3 + 5((g_{3,25n}/g_{3,n}) - (g_{3,n}/g_{3,25n}))$,

(iv) $k_1(\sqrt{3} g_{3,n} g_{3,49n})^3 + k_2(\sqrt{3} g_{3,n} g_{3,49n}) = k_3(\sqrt{3} g_{3,n} g_{3,49n})^2 + k_4(g_{3,n}/g_{3,49n})^2 - k_5$, *where* $k_1 = (g_{3,n}/g_{3,49n})^8 - 1$, $k_2 = -42 g^4_{3,n} \times ((g_{3,n}/g_{3,49n})^4 - 1)$, $k_3 = -3g^4_{3,n}(7 + 3g^4_{3,n})$, $k_4 = 63 g^4_{3,n}(g^4_{3,n} + 1)$, *and* $k_5 = 27(g_{3,n}/g_{3,49n})^4 - 63 g^4_{3,n}(1 + (g_{3,n}/g_{3,49n})^4 - g^4_{3,n})$,

(v)

$$((\sqrt{3}/g_{9,9n}) + \sqrt{3} g_{9,9n} + 3)((\sqrt{3}/g_{9,n}) + \sqrt{3} g_{9,n} + 3) = (g_{9,9n}/g_{9,n})^2.$$

Proof. Proof of (i) follows from [**5**, page 345] and the definition of $g_{k,n}$. Proofs of (ii)–(v) follow from Theorem 2.1–2.4, respectively, and the definition of $g_{k,n}$.

Theorem 5.2. *For any positive real number n, we have*

(i) $(1 - (\sqrt{3}/g'_{3,n} g'_{3,9n}))^3 = (1 - (3/{g'}^4_{3,9n}))$,

(ii) $\sqrt{5}g'_{5,n}g'_{5,9n} + (\sqrt{5}/g'_{5,n}g'_{5,9n}) = (g'_{5,9n}/g'_{5,n})^2 + 3(g'_{5,9n}/g'_{5,n}) + 3(g'_{5,n}/g'_{5,9n}) - (g'_{5,n}/g'_{5,9n})^2$,

(iii) $3(g'_{3,n}g'_{3,25n})^2+(3/(g'_{3,n}g'_{3,25n})^2)+5(g'_{3,25n}/g'_{3,n})^2-5(g'_{3,n}/g'_{3,25n})^2 = (g'_{3,25n}/g'_{3,n})^3 - (g'_{3,n}/g'_{3,25n})^3 - 5((g'_{3,25n}/g'_{3,n}) - (g'_{3,n}/g'_{3,25n}))$,

(iv) $k_1(\sqrt{3}g'_{3,n}g'_{3,49n})^3 + k_2(\sqrt{3}g'_{3,n}g'_{3,49n}) = k_3(\sqrt{3}g'_{3,n}g'_{3,49n})^2 + k_4(g'_{3,n}/g'_{3,49n})^2 - k_5$,

where $k_1 = (g'_{3,n}/g'_{3,49n})^8 - 1$, $k_2 = 42{g'}^4_{3,n}((g'_{3,n}/g'_{3,49n})^4 - 1)$, $k_3 = 3{g'}^4_{3,n}(7 - 3{g'}^4_{3,n})$, $k_4 = 63{g'}^4_{3,n}({g'}^4_{3,n} - 1)$, *and* $k_5 = 27(g'_{3,n}/g'_{3,49n})^4 + 63{g'}^4_{3,n}(1 + (g'_{3,n}/g'_{3,49n})^4 + {g'}^4_{3,n})$,

(v) $((\sqrt{3}/g'_{9,9n})+\sqrt{3}g'_{9,9n}-3)(\sqrt{3}/g'_{9,n}+\sqrt{3}g'_{9,n}-3) = (g'_{9,9n}/g'_{9,n})^2$.

Proof. Proof of (i) follows easily from [**5**, page 345] and the definition of $g'_{k,n}$. Proofs of (ii)–(v) follow from Theorems 2.1–2.4, respectively, and the definition of $g'_{k,n}$.

Theorem 5.3. *We have*

(i) $(g'_{3,n}/g'_{3,4n})^2 + \sqrt{3}((1/(g'_{3,n})^2) - (g'_{3,n})^2) + (g'_{3,4n}/g'_{3,n})^2 = 0$,

(ii) $(g_{3,n}/g'_{3,4n})^2 + \sqrt{3}(1/(g_{3,n})^2 + (g_{3,n})^2) - (g'_{3,4n}/g_{3,n})^2 = 0$,

(iii) $(g_{3,n}/g'_{3,n})^4 + (g'_{3,n}/g_{3,n})^4 + 3\{(g_{3,n}/g'_{3,n})^2 - (g'_{3,n}/g_{3,n})^2\}\{(1/g_{3,n}g'_{3,n})^2 - (g'_{3,n}g_{3,n})^2\} - 10 = 0$,

(iv) $(g'_{5,n}g'_{5,4n})^2 - \sqrt{5}((1/(g'_{5,n})^2) + (g'_{5,n})^2) + (g'_{5,4n}/g'_{5,n})^2 + 4 = 0$,

(v) $(g_{5,n}/g'_{5,4n})^2 - \sqrt{5}((1/(g_{5,n})^2) + (g_{5,n})^2) + (g'_{5,4n}/g_{5,n})^2 - 4 = 0$,

(vi)

$$\left(\frac{g_{5,n}}{g'_{5,n}}\right)^2 + \left(\frac{g'_{5,n}}{g_{5,n}}\right)^2 + \sqrt{5}\left(\frac{g_{5,n}}{g'_{5,n}} - \frac{g'_{5,n}}{g_{5,n}}\right)\left(\left(\frac{1}{g_{5,n}g'_{5,n}}\right) - g'_{5,n}g_{5,n}\right) - 6 = 0,$$

(vii) $\sqrt{2}((g'_{2,n})^2 - (\sqrt{2}/(g'_{2,n}g'_{2,4n})^2)) - (g'_{2,4n}/g'_{2,n})^2 = 0$.

Proof. Proofs of (i)–(vii) follow from Theorems 2.9–2.15, respectively, and the definitions of $g_{k,n}$ and $g'_{k,n}$. ◻

Theorem 5.4. *We have*

(i) $g_{3,3} = (\psi(-e^{-\pi}))/(3^{1/4}e^{-\pi/4}\psi(-e^{-3\pi})) = (3+2\sqrt{3})^{1/4}$, *and*

(iii) $g_{3,9} = (\psi(-e^{-\pi/\sqrt{3}}))/(3^{1/4}e^{-\pi\sqrt{3}/4}\psi(-e^{-3\sqrt{3}\pi})) = (1+2^{1/3})^2/\sqrt{3}$.

Proof. Setting $n = 1/3$ in Theorem 5.1 (i) and employing Theorem 4.1 (ii), we obtain

$$(1+\sqrt{3})^3 = 1 + 3g_{3,3}^4,$$

which readily gives (i). Again, setting $n = 1$ in Theorem 5.1 (i) and recalling the value $g_{k,1} = 1$ from Theorem 5.1 (i), we find that

$$(1+\sqrt{3}g_{3,9})^3 = 1 + 3g_{3,9}^4. \tag{5.1}$$

Solving (5.1) and using the remark given after Theorem 4.1, we prove (ii). □

Baruah and Saikia [**4**] and Adiga et al. [**1**] also proved the results of the above theorem.

Theorem 5.5. *We have*

(i) $g_{5,9} = (1/2)(3+\sqrt{3}+\sqrt{5}+\sqrt{15})$ *and*

(ii) $g_{5,3} = (17\sqrt{5}+38)^{1/6}$.

Proof. Setting $n = 1$ in Theorem 5.1 (ii) and recalling that $g_{k,1} = 1$ from Theorem 4.1 (i), we find that

$$\sqrt{5}\left(g_{5,9} + \frac{1}{g_{5,9}}\right) = (g_{5,9})^2 - 3\left(g_{5,9} + \frac{1}{g_{5,9}}\right) - \left(\frac{1}{g_{5,9}}\right)^2. \tag{5.2}$$

Solving (5.2) and using the fact that $g_{k,n} > 1$ from the remark after Theorem 3.1, we prove (i). Again, setting $n = 1/3$ in Theorem 5.1 (ii) and recalling $g_{k,1/n} = 1/g_{k,n}$ from Theorem 4.1 (ii), we find that

$$\left(g_{5,3}^4 - \frac{1}{g_{5,3}^4}\right) - 3\left(g_{5,3}^2 + \frac{1}{g_{5,3}^2}\right) = 2\sqrt{5}. \tag{5.3}$$

Solving (5.3) and employing $g_{k,n} > 1$ again, we prove (ii). □

Baruah and Saikia [**4**] and Adiga et al. [**1**] also established the results of the above theorem.

Theorem 5.6. *We have*

(i) $g_{3,25}=\left(1+\sqrt[3]{10}+\sqrt{5+2\sqrt[3]{10}+\sqrt[3]{10^2}}\right)^2/(2(1+\sqrt{5}))$,

(ii) $g_{3,7}=(\sqrt{3}+\sqrt{7})^{3/4}/(2^{3/4}(2-\sqrt{3})^{1/4})$,

(iii)
$$g_{13,3}=\frac{\left(\sqrt{11+\sqrt{13}}+\sqrt{3+\sqrt{13}}\right)^2}{(2\sqrt{2}(\sqrt{5+\sqrt{13}}+\sqrt{\sqrt{13}-3}))},$$

(iv)
$$g_{3,49}=\frac{\left(3+\sqrt[3]{4}\cdot\sqrt[3]{7}+\sqrt[3]{2}\cdot\sqrt[3]{49}+\sqrt{49+13\cdot\sqrt[3]{4}\cdot\sqrt[3]{7}+8\cdot\sqrt[3]{2}\cdot\sqrt[3]{49}}\right)^2}{6(\sqrt{3}+\sqrt{7})},$$

(v)
$$g_{25,9}=\frac{\left(\sqrt[4]{60}+2-\sqrt{3}+\sqrt{5}\right)^2\left(\sqrt[4]{60}-2-\sqrt{3}+\sqrt{5}\right)}{\left(\sqrt[4]{60}-2+\sqrt{3}-\sqrt{5}\right)^2\left(\sqrt[4]{60}+2+\sqrt{3}-\sqrt{5}\right)},$$

(vi)
$$g_{13,9}=\frac{\left((\sqrt{3}+1)(\sqrt{3}+\sqrt{13})+2\sqrt{(3+2\sqrt{3})(4+\sqrt{13})}\right)^2}{4\left((\sqrt{3}+1)(\sqrt{13}-\sqrt{3})+2\sqrt{(3+2\sqrt{3})(4-\sqrt{13})}\right)},$$

(vii)
$$g_{25,7}=\frac{\left(4\sqrt{5}+a+b+\sqrt{(4\sqrt{5}+a+b)^2-36}\right)^2}{6\left(2\sqrt{5}+c+d+\sqrt{(2\sqrt{5}+c+d)^2-36}\right)},$$

where

$$a=\left(\frac{1}{2}(2251\sqrt{5}+9\sqrt{105})\right)^{1/3}, \qquad b=\left(\frac{1}{2}(2251\sqrt{5}-9\sqrt{105})\right)^{1/3},$$
$$c=\left(\frac{1}{2}(17\sqrt{5}+3\sqrt{105})\right)^{1/3}, \quad \textit{and } d=\left(\frac{1}{2}(17\sqrt{5}-3\sqrt{105})\right)^{1/3},$$

(viii)

$$g_{25,49}=\frac{\left(a'+2\sqrt{5b'}+\sqrt{(a'+2\sqrt{5b'})^2-64}\right)^2}{8\left(c'+2\sqrt{5d'}+\sqrt{(c'+2\sqrt{5d'})^2-64}\right)},$$

where $a'=1497+651\sqrt{5}+565\sqrt{7}+247\sqrt{35}$, $b'=437430+195566\sqrt{5}+165333\sqrt{7}+73917\sqrt{35}$, $c'=1497-651\sqrt{5}+565\sqrt{7}-247\sqrt{35}$ *and* $d'=437430-195566\sqrt{5}+165333\sqrt{7}-73917\sqrt{35}$.

Proof. The proof of the theorem follows from Theorem 4.8 (i) and the corresponding values of $r_{k,n}$ and $r'_{k,n}$ from Section 3. □

Baruah and Saikia [**4**] also found the first four values of the above theorem.

Theorem 5.7. *We have*

(i) $g'_{1,1}=1$,

(ii) $g'_{2,1}=2^{1/8}(1+\sqrt{2})^{1/8}$,

(iii) $g'_{3,1}=\sqrt{(\sqrt{3}+1)/\sqrt{2}}$,

(iv) $g'_{4,1}=2^{5/16}(1+\sqrt{2})^{1/4}$,

(v) $g'_{5,1}=\left((1+\sqrt{5}+\sqrt{2}\sqrt{1+\sqrt{5}})/2\right)^{1/2}$,

(vi) $g'_{8,1}=2^{1/4}(1+\sqrt{2})^{3/8}\left(4+\sqrt{2+10\sqrt{2}}\right)^{1/8}$,

(vii) $g'_{9,1} = \left(1+\sqrt{2}\sqrt[4]{3}+\sqrt{3}\right)/2$,

(viii) $g'_{25,1} = (\sqrt[4]{5}+1)/(\sqrt[4]{5}-1)$.

Proof. The proof of the theorem follows from Theorem 4.9 and the corresponding values of $r_{4,n}$ from Section 3. □

Theorem 5.8. *We have*

(i) $g'_{2,4} = \left(\sqrt{\sqrt{2}-1}+\sqrt{\sqrt{2}+1}\right)^{1/2}$,

(ii) $g'_{2,8} = \sqrt{2+\sqrt{2}}$,

(iii) $g'_{3,4} = (\sqrt{3}+1)/\sqrt{2}$,

(iv) $g'_{3,16} = (1+\sqrt{3})(\sqrt{2}+1)/\sqrt{2}$,

(v)

$$g'_{3,64} = \Big(102+72\sqrt{2}+59\sqrt{3}+42\sqrt{6} + \sqrt{41680+29472\sqrt{2}+24064\sqrt{3}+17016\sqrt{6}}\Big)^{1/2},$$

(vi) $g'_{3,12} = \left(3+2\sqrt{3}+\sqrt{24+14\sqrt{3}}\right)^{1/2}$,

(vii)

$$g'_{3,36} = \Big(13\sqrt{3}+10\cdot 2^{1/3}\sqrt{3}+8\cdot 2^{2/3} \cdot\sqrt{3}+2\sqrt{373+296\cdot 2^{1/3}+235\cdot 2^{2/3}}\Big)^{1/2},$$

(viii)

$$g'_{3,20} = (1/2)\Big(\sqrt{3}+\sqrt{3}(38+17\sqrt{5})^{2/3} + \sqrt{3+10(38+17\sqrt{5})^{2/3}+3(38+17\sqrt{5})^{4/3}}\Big)^{1/2},$$

(ix)

$$g'_{3,7} = \frac{\left(12208+7048\sqrt{3}+4614\sqrt{7}+2664\sqrt{21}-\sqrt{6k}\right)^{1/4}}{582+333\sqrt{3}+218\sqrt{7}+127\sqrt{21}},$$

where $k = 9623566 + 55561688\sqrt{3} + 36373663\sqrt{7} + 21000344\sqrt{21}$

(x)

$$g'_{3,9} = \frac{746+592\sqrt{3}+470\sqrt[3]{4}+\sqrt{1641279+1302684\sqrt{3}+1033941\sqrt[3]{4}}}{19+15\sqrt[3]{2}+12\sqrt[3]{4}},$$

(xi) $g'_{5,4} = (1/2)\left(4+2\sqrt{5}+\sqrt{2(1+\sqrt{5})}+\sqrt{10(1+\sqrt{5})}\right)^{1/2}$,

(xii)

$$g'_{5,3} = \frac{\left(\sqrt{5}-6(38+17\sqrt{5})^{1/3}+\sqrt{5}(38+17\sqrt{5})^{1/3}+\sqrt{r}\right)^{1/2}}{-2+2\sqrt{5}(38+17\sqrt{5})^{1/3}},$$

where $r = -675 - 304\sqrt{5} + 19(38+17\sqrt{5})^{1/3} + 77\sqrt{5}(38+17\sqrt{5})^{1/3} + 22(38+17\sqrt{5})^{1/3}$,

(xiii)

$$g'_{5,9} = \frac{\left(132+76\sqrt{3}+59\sqrt{5}+34\sqrt{15}+2\sqrt{16406+9472\sqrt{3}+7337\sqrt{5}+4236\sqrt{15}}\right)^{1/2}}{8+5\sqrt{3}+4\sqrt{5}+2\sqrt{15}}.$$

Proof. To prove (i) and (ii), we set $n = 1$ in Theorem 5.3 (vii) and use the value of $g'_{2,1}$ from Theorem 5.7 (ii) and the value of $g'_{2,2} = 2^{3/8}$ from Theorem 6.7 (ii), respectively.

To prove (iii), we set $n = 1$ in Theorem 5.3 (i) and use the value of $g'_{3,1}$ from Theorem 5.7 (iii). To prove (iv) and (v), we set $n = 4$ and 16, respectively, in Theorem 5.3 (i) and successively use the values of $g'_{3,4}$ and $g'_{3,16}$ from the same theorem.

To prove (vi)–(viii), we set $n = 3, 9$ and 5 in Theorem 5.3 (ii) and use the values of $g_{3,3}$, $g_{3,9}$ and $g_{3,5}$ from Theorem 5.4 (i), (iii), Theorem 5.5 (iii), and Theorem 4.1 (iii), respectively.

We set $n = 7$ and 9 in Theorem 5.3 (iii) and use the values of $g_{3,7}$ and $g_{3,9}$ from Theorem 5.6 (iii) and Theorem 5.4 (iii), respectively, to complete the proof of (ix) and (x).

We set $n = 1$ in Theorem 5.3 (iv) and use the value of $g'_{5,1}$ from Theorem 5.7 (v) to prove (xi).

To prove (xii) and (xiii), we set $n = 3$ and 9 in Theorem 5.3 (vi) and use the values of $g_{5,3}$ and $g_{5,9}$ from Theorem 5.5 (iii) and (i), respectively. $\square$

Theorem 5.9. *We have*

(i) $g'_{3,2} = (1+\sqrt{2})^{1/2}$,

(ii) $g'_{4,2} = 2^{3/8}(1+\sqrt{2})^{3/4}$,

(iii) $g'_{5,2} = ((\sqrt{5}+1)/2)^{3/2}$,

(iv) $g'_{2,8} = 2^{1/4}(1+\sqrt{2})^{1/2}$,

(v) $g'_{9,2} = \sqrt{2}+\sqrt{3}$,

(vi) $g'_{4,8} = 2^{3/8}(1+\sqrt{2})^{5/8}\left(4+\sqrt{2+10\sqrt{2}}\right)^{1/8}$,

(vii) $g'_{2,16} = 2^{3/8}(1+\sqrt{2})^{3/4}\left(4+\sqrt{2+10\sqrt{2}}\right)^{3/8}$,

(viii) $g'_{2,32} = 2^{9/16}(1+\sqrt{2})^{3/4}(16+15\sqrt[4]{2}+12\sqrt{2}+9\cdot 2^{3/4})^{3/4}$.

Proof. The proof of the theorem follows directly from Theorem 4.8 (ii) and the values of $r_{k,n}$ from Section 3. $\square$

Theorem 5.10. *We have*

(i) $g_{7,7} = (1/2)(7^{1/4}+\sqrt{4+\sqrt{7}})\Big((35/2)+7\sqrt{7}+(7/2)\sqrt{21+8\sqrt{7}}$
$+\sqrt{147+56\sqrt{7}}\Big)^{1/4}$,

(ii) $g'_{7,7} = 2^{-1/8}g_{49}\left(35+14\sqrt{7}+2\sqrt{147+56\sqrt{7}}+\sqrt{7(147+56\sqrt{7})}\right)^{1/4}$,

where

$$g_{49} = \left(\frac{G_{49}^{12}+\sqrt{G_{49}^{24}-1}}{2G_{49}^{4}}\right)^{1/8} \quad \textit{and} \quad G_{49} = \frac{7^{1/4}+\sqrt{4+\sqrt{7}}}{2}.$$

Proof. First we find the explicit values of $r_{7,7}$ and $r'_{7,7}$ in the following lemma.

Lemma. *We have*

(i) $r_{7,7}=\left((1/2)\left(35+14\sqrt{7}+2\sqrt{147+56\sqrt{7}}+\sqrt{7(147+56\sqrt{7})}\right)\right)^{1/4}$,

(ii) $r'_{7,7}=\left(2^{3/4}\left(35+14\sqrt{7}+7\sqrt{21+8\sqrt{7}}+2\sqrt{147+56\sqrt{7}}\right)^{1/4}\right)\Big/$ $(7^{1/4}+\sqrt{4+\sqrt{7}})$.

Proof of the lemma. We set $q := e^{-2\pi}$ in Theorem 2.5 and then apply Theorem 2.19 (ii) to obtain

$$\nu = \frac{f(-e^{-2\pi/7})}{e^{-4\pi/7}f(-e^{-14\pi})} = \sqrt{7} \tag{5.4}$$

and

$$\mu = \frac{f(-e^{-2\pi})}{e^{-4\pi}f(-e^{-14\pi})} = 7\ r^4_{7,7}. \tag{5.5}$$

Using (5.5) and (5.6) in (2.5), we obtain
(5.6)

$$r_{7,7} = \left(\frac{1}{2}\left(35+14\sqrt{7}+2\sqrt{147+56\sqrt{7}}+\sqrt{7(147+56\sqrt{7})}\right)\right)^{1/4},$$

to complete the proof of (i).

From [**17**, page 17], we have

$$r_{n,n} = G_{n^2}r'_{n,n}. \tag{5.7}$$

Setting $n = 7$ and using the value of G_{49} [**7**, page 191] and (5.6) in (5.7), we complete the proof of (ii). □

Proof of Theorem 5.10. Using Theorem 4.11 and the above lemma, we easily complete the proof. □

6. Explicit values for $\psi(\pm q)$. In this section, we find explicit formulae for the theta functions $\psi(e^{-n\pi})$, $\psi(-e^{-n\pi})$, $\psi(e^{-\pi/n})$ and $\psi(-e^{-\pi/n})$ for any positive real number n and give some examples.

Lemma 6.1. *Let* $a = \pi^{1/4}/\Gamma(3/4)$. *Then*

(i) $\psi(e^{-\pi}) = a2^{-5/8}e^{\pi/8}$

(ii) $\psi(-e^{-\pi}) = a2^{-3/4}e^{\pi/8}$.

Proof. See [**5**, page 123].

Theorem 6.2. *For every positive real number* n, *we have*

(i)

$$\psi(-e^{-n\pi}) = \frac{a2^{-3/4}e^{n\pi/8}}{n^{1/4}g_{n,n}} = \frac{a2^{-3/4}e^{n\pi/8}}{n^{1/4}G_{n^2}r_{n,n}}$$

(ii)

$$\psi(e^{-n\pi}) = \frac{a2^{-5/8}e^{n\pi/8}}{n^{1/4}g'_{n,n}} = \frac{a2^{-3/4}e^{n\pi/8}}{n^{1/4}r_{2,(n^2/2)}r_{n,n}}.$$

Proof. Using the definitions of $g_{n,n}$, $g'_{n,n}$, Lemma 6.1, and Theorem 4.11, we complete the proofs of (i) and (ii).

Theorem 6.3. *For every positive number* n, *we have*

(i)

$$\psi(-e^{-\pi/n}) = \frac{an^{1/4}2^{-3/4}e^{\pi/8n}}{g_{n,n}} = \frac{an^{1/4}2^{-3/4}e^{\pi/8n}}{G_{n^2}r_{n,n}}$$

(ii)

$$\psi(e^{-\pi/n}) = \frac{an^{1/4}2^{-5/8}e^{\pi/8n}}{g'_{1/n,1/n}} = \frac{an^{1/4}2^{-3/4}r_{2,2n^2}e^{\pi/8n}}{r_{n,n}}.$$

Proofs. Replacing n by $1/n$ in Theorem 6.2 (i) and (ii), and using the fact that $g_{1/n,1/n} = g_{n,n}$ and $r_{k,1/n} = r_{k,n}^{-1}$ [**17**, page 12], we complete the proof of (i) and (ii). □

In Theorem 5.4 (i) and Theorem 5.10 (i) and (ii), we have evaluated $g_{3,3}$, $g_{7,7}$, and $g'_{7,7}$, respectively. Now, we give some more explicit values of $g_{n,n}$ and $g'_{n,n}$ and then use these values to determine some values of theta-function $\psi(q)$.

Theorem 6.4. *We have*

(i) $g_{1,1} = 1$,

(ii) $g_{2,2} = 2^{-1/16}(\sqrt{2}+1)^{1/4}$,

(iii) $g_{4,4} = 2^{1/16}(1+\sqrt{2})^{1/2}(9\cdot 2^{1/4}+4\sqrt{2}-3\cdot 2^{3/4})^{1/8}$,

(iv) $g_{5,5} = (5+\sqrt{5})^{3/2}/(2^{3/2}\sqrt{5})$,

(v)

$$g_{6,6} = \frac{3^{1/4}(\sqrt{3}+1)^{5/6}(1+\sqrt{3}+\sqrt{2}\cdot 3^{3/4})^{2/3}(2-3\sqrt{2}+3\cdot 3^{1/4}+3^{3/4})^{1/3}}{2^{85/48}},$$

(vi)

$$g_{9,9} = 2+\sqrt{3}+\left(\frac{1}{3}\right)\left(1269+729\sqrt{3}-27\sqrt{156+90\sqrt{3}}\right)^{1/3} + \left(47+27\sqrt{3}+\sqrt{156+90\sqrt{3}}\right)^{1/3}.$$

Proof. The value in (i) readily follows from Theorem 3.1. The proofs of (ii)–(v) follow from Theorem 4.8 (i) and the values of $r_{k,n}$ and $r'_{k,n}$ given in Section 3.

Next, we set $n = 1$ in Theorem 5.1 (v) and use the value $g_{k,1} = 1$, to obtain

$$\left(\sqrt{3}\left(g_{9,9}+\frac{1}{g_{9,9}}\right)+3\right)\left(2\sqrt{3}+3\right) = g_{9,9}^2. \tag{6.1}$$

Solving equation (6.1), we easily arrive at (vii).

Baruah and Saikia [**4**] and Adiga et al. [**1**] also found the value of $g_{9,9}$.

Theorem 6.5. *We have*

(i) $\psi(-e^{-\pi}) = a2^{-3/4}e^{\pi/8}$,

(ii) $\psi(-e^{-2\pi}) = a2^{-15/16}(\sqrt{2}-1)^{1/4}e^{\pi/4}$,

(iii) $\psi(-e^{-3\pi}) = (a2^{-3/4}e^{3\pi/8})/(3^{1/4}(3+2\sqrt{3})^{1/4})$,

(iv)

$$\psi(-e^{-4\pi}) = \frac{a2^{-21/16}(\sqrt{2}-1)^{1/2}e^{\pi/2}}{(9\sqrt[4]{2}+4\sqrt{2}-3\cdot 2^{3/4})^{1/8}},$$

(v) $\psi(-e^{-5\pi}) = (ae^{5\pi/8}(5-\sqrt{5})^{3/2})/(2^{9/4}5^{5/4})$,

(vi)

$$\psi(-e^{-6\pi}) = \frac{ae^{3\pi/4}2^{37/48}}{\sqrt{3}(\sqrt{3}+1)^{5/6}(1+\sqrt{3}+\sqrt{2}\cdot 3^{3/4})^{2/3}(2-3\sqrt{2}+3^{5/4}+3^{3/4})^{1/3}},$$

(vii)

$$\psi(-e^{-7\pi}) = \frac{a2^{1/2}e^{7\pi/8}}{7^{1/4}\left(7^{1/4}+\sqrt{4+\sqrt{7}}\right)\left(35+14\sqrt{7}+7\sqrt{21+8\sqrt{7}}+2\sqrt{147+56\sqrt{7}}\right)^{1/4}},$$

(viii) $\psi(-e^{-9\pi}) = (a2^{-3/4}e^{9\pi/8})/(\sqrt{3}g_{9,9})$,

where $g_{9,9}$ *is as given in Theorem* 6.4 (vi).

Proof. The proof of the theorem follows from Theorem 6.2 (i) and the values of $g_{n,n}$ from Theorem 5.4 (i), Theorem 5.10 (i) and Theorem 6.4. □

Theorem 6.5 (iii), (v) and (viii) were also proved by Baruah and Bhattacharyya [**3**].

Theorem 6.6. *We have*

(i) $\psi(-e^{-\pi/2}) = a\ 2^{-7/16}e^{\pi/16}(\sqrt{2}-1)^{1/4}$,

(ii) $\psi(-e^{-\pi/3}) = (a\ 3^{1/4}2^{-3/4}e^{-\pi/24})/((3+2\sqrt{3})^{1/4})$,

(iii)

$$\psi(-e^{-\pi/4}) = \frac{a\ 2^{-5/16}e^{\pi/32}(\sqrt{2}-1)^{1/2}}{(9\cdot 2^{1/4}+4\sqrt{2}-3\cdot 2^{3/4})^{1/8}},$$

(iv) $\psi(-e^{-\pi/5}) = (a\ 2^{3/4}5^{3/4}e^{\pi/40}/(\sqrt{5}+5)^{3/2})$,

(v)

$$\psi(-e^{-\pi/6}) = \frac{a\ 2^{61/48}e^{\pi/48}}{(\sqrt{3}+1)^{5/6}(1+\sqrt{3}+\sqrt{2}\cdot 3^{3/4})^{2/3}(2-3\sqrt{2}+3^{5/4}+3^{3/4})^{1/3}},$$

(vi)

$$\psi(-e^{-\pi/7}) = \frac{a\ 2^{1/2}\ 7^{1/4}e^{\pi/56}}{\left(7^{1/4}+\sqrt{4+\sqrt{7}}\right)\left(35+14\sqrt{7}+7\sqrt{21+8\sqrt{7}}+2\sqrt{147+56\sqrt{7}}\right)^{1/4}},$$

(vii) $\psi(-e^{-\pi/9}) = (\sqrt{3}\ a\ 2^{-3/4}e^{\pi/72})/g_{9,9}$,

where $g_{9,9}$ *is as given in Theorem* 6.4(vi).

Proof. The proofs follow from Theorem 6.3 (i) and the values of $g_{n,n}$ from Theorem 5.4 (i), Theorem 5.10 (i) and Theorem 6.4. □

Theorem 6.7. *We have*

(i) $g'_{1,1} = 1$,

(ii) $g'_{2,2} = 2^{3/8}$,

(iii) $g'_{3,3} = 3^{1/3}(1+\sqrt{3}+\sqrt{2}\cdot 3^{3/4})^{1/3}(1+\sqrt{3})^{1/6}/\sqrt{2}$,

(iv) $g'_{4,4} = 2^{3/8}(1+\sqrt{2})^{1/2}$,

(v) $g'_{5,5} = (5+\sqrt{5})^{1/2}(5^{1/4}+1)/2$,

(vi) $g'_{6,6} = (3^{1/8}(1+\sqrt{3})^{5/6}(1+\sqrt{3}+\sqrt{2}\cdot 3^{3/4})^{2/3})/2^{29/24}$,

(vii) $g'_{9,9} = (a+(2(b-2c))^{1/3}+(2(b+2c))^{1/3})/2$,

where $a = 2+\sqrt{2}\cdot 3^{1/4}+2\sqrt{3}+\sqrt{2}\cdot 3^{3/4}$, $b = 82+45\sqrt{2}+48\sqrt{3}+25\sqrt{2}\cdot 3^{3/4}$ *and* $c = \sqrt{3(88+47\sqrt{2}\cdot 3^{1/4}+50\sqrt{3}+27\sqrt{2}\cdot 3^{3/4})}$.

Proofs. The proofs of (i)–(vi) follow from Theorem 4.11 (ii) and the values of $r_{k,n}$ given in Section 3.

Next, we set $n = 1$ in Theorem 5.2 (v) to obtain

$$\left(\frac{\sqrt{3}}{g'_{9,9}}+\sqrt{3}g'_{9,9}-3\right)\left(\frac{\sqrt{3}}{g'_{9,1}}+\sqrt{3}g'_{9,1}-3\right) = \left(\frac{g'_{9,9}}{g'_{9,1}}\right)^2. \tag{6.2}$$

Substituting the value of $g'_{9,1}$ from Theorem 5.7 (vii) in (6.2) and solving the resulting polynomial equation, we complete the proof of (vii).

Theorem 6.8. *We have*

(i) $\psi(e^{-\pi}) = a\ 2^{-5/8}e^{\pi/8}$,

(ii) $\psi(e^{-2\pi}) = a\ 2^{-5/4}e^{\pi/4}$,

(iii)

$$\psi(e^{-3\pi}) = \frac{a\ 2^{-1/8}e^{3\pi/8}}{3^{1/3}(1+\sqrt{3}+\sqrt{2}\cdot 3^{3/4})^{1/3}(1+\sqrt{3})^{1/6}},$$

(iv) $\psi(e^{-4\pi}) = a\ 2^{-2}(2-\sqrt{2})^{1/2}$,

(v) $\psi(e^{-5\pi}) = (a\ 2^{3/8}e^{5\pi/8})/(5^{1/4}(5+\sqrt{5})^{1/2}(1+5^{1/4}))$,

(vi)

$$\psi(e^{-6\pi}) = \frac{a\ 2^{1/4}e^{3\pi/4}}{3^{3/8}(1+\sqrt{3})^{5/6}(1+\sqrt{3}+\sqrt{2}\cdot 3^{3/4})^{2/3}},$$

(vii) $\psi(e^{-7\pi}) = (a\ 7^{-1/4}2^{-1/2}e^{7\pi/8}/g'_{7,7})$,

(viii) $\psi(e^{-9\pi}) = (a\ 2^{-5/8}e^{9\pi/8}/\sqrt{3}g'_{9,9})$,

where $g'_{7,7}$ *and* $g'_{9,9}$ *are as given in Theorem* 6.7.

Proof. The proof of the theorem follows from Theorem 5.2 (ii) and the values of $g'_{n,n}$ from Theorem 5.10 (ii) and Theorem 6.7. □

Theorem 6.8 (i) and (ii) were also proved by Berndt [**7**, page 325].

Theorem 6.9. *We have*

(i) $\psi(e^{-\pi/2}) = a\ 2^{-7/16}(\sqrt{2}+1)^{1/4}e^{\pi/16}$,

(ii) $\psi(e^{-\pi/3}) = a\ 2^{-27/24}3^{-1/8}(\sqrt{3}+1)^{1/6}(1+\sqrt{3}+\sqrt{2}\cdot 3^{3/4})^{1/3}e^{\pi/24}$,

(iii) $\psi(e^{-\pi/4}) = a\ 2^{-7/8}(16+15\cdot 2^{1/4}+12\sqrt{2}+9\cdot 2^{3/4})^{1/8}$,

(iv) $\psi(e^{-\pi/5}) = (a\ 2^{3/8}e^{\pi/40})/((5+\sqrt{5})^{1/2}(5^{1/4}+1))$,

(v)

$$\psi(e^{-\pi/6}) = \frac{a\ 2^{-11/12}e^{\pi/48}((\sqrt{2}+\sqrt{3})(\sqrt{3}+1)(1+\sqrt{2}-\sqrt{3}+2\cdot 3^{5/4}))^{1/3}}{3^{1/8}(1+\sqrt{3})^{1/2}(1+\sqrt{3}+\sqrt{2}\ 3^{3/4})^{1/3}(\sqrt{2}-1)^{5/12}}.$$

Proof. The proof of the theorem follows from Theorem 6.3 (ii) and the values of $r_{k,n}$ from Section 3. □

Theorem 6.9 (i) and (iii) were also proved by Berndt [**7**, page 325].

7. Explicit values of quotients of the theta-function $\phi(q)$. In this section, we give theorems for the explicit evaluations of quotients of the theta-function ϕ in terms of the parameter $g'_{k,n}$ and then use these theorems to find some new explicit values.

Theorem 7.1. *For any positive real number n, we have*

(i) $\phi(-e^{-\pi\sqrt{n/3}})/\phi(-e^{-\pi\sqrt{3n}}) = ((9-3g'^4_{3,n})/(1-3g'^4_{3,n}))^{1/4}$,

(ii) $\phi(-e^{-\pi\sqrt{n/9}})/\phi(-e^{-\pi\sqrt{9n}}) = ((3-\sqrt{3}g'_{9,n})/(1-\sqrt{3}g'_{9,n}))$,

(iii) $\phi(-e^{-\pi\sqrt{n/5}})/\phi(-e^{-\pi\sqrt{5n}}) = ((5-\sqrt{5}g'^2_{5,n})/(1-\sqrt{5}g'^2_{5,n}))^{1/2}$.

Proof. We set $q = -e^{-\pi\sqrt{n/3}}$, $-e^{-\pi\sqrt{n/9}}$, and $-e^{-\pi\sqrt{n/5}}$ in Theorems 2.6–2.8, respectively, and use the definition of $g'_{k,n}$ to complete the proofs. □

Theorem 7.2. *We have*

(i) $\phi(-e^{-\pi/\sqrt{3}})/\phi(-e^{-\pi\sqrt{3}}) = ((9-3(2+\sqrt{3}))/(1-3(2+\sqrt{3})))^{1/4}$,

(ii)

$$\frac{\phi(-e^{-\pi})}{\phi(-e^{-3\pi})} = \left(\frac{36-3^{9/2}(1+\sqrt{3})^{2/3}(1+\sqrt{3}+\sqrt{2}\cdot 3^{3/4})^{4/3}}{4-3^{9/2}(1+\sqrt{3})^{2/3}(1+\sqrt{3}+\sqrt{2}\cdot 3^{3/4})^{4/3}}\right)^{1/4},$$

(iii) $\phi(-e^{-\pi\sqrt{2/3}})/\phi(-e^{-\pi\sqrt{6}}) = (3\sqrt{2}/(4+3\sqrt{2}))^{1/4}$,

(iv)

$$\frac{\phi(-e^{-\pi/\sqrt{9}})}{\phi(-e^{-\pi\sqrt{9}})} = \frac{\sqrt{3}+\sqrt{6}\sqrt[4]{3}-3}{\sqrt{3}+\sqrt{6}\sqrt[4]{3}+1},$$

(v) $\phi(-e^{-\pi})/\phi(-e^{-9\pi}) = (3-b_2)/(1-b_2)$,

where $b_2 = \sqrt{3}g'_{9,9}$ *and* $g'_{9,9}$ *is given by Theorem* 6.7 (vii),

(vi) $\phi(-e^{-\pi\sqrt{2/9}})/\phi(-e^{-3\pi\sqrt{2}}) = 3-\sqrt{6}$,

(vii)

$$\frac{\phi(-e^{-\pi/\sqrt{5}})}{\phi(-e^{-\pi\sqrt{5}})} = \left(\frac{\sqrt{5}+\sqrt{5}\sqrt{2(1+\sqrt{5})}-5}{\sqrt{5}+\sqrt{5}\sqrt{2(1+\sqrt{5})}+3}\right)^{1/2},$$

(viii)

$$\frac{\phi(-e^{-\pi})}{\phi(-e^{-5\pi})} = \left(\frac{5(1+5^{1/4}+\sqrt{5}+5^{3/4})}{13+5\cdot 5^{1/4}+5\sqrt{5}+5\cdot 5^{3/4}}\right)^{1/2},$$

(ix) $\phi(-e^{-\pi\sqrt{2/5}})/\phi(-e^{-\pi\sqrt{10}}) = (\sqrt{5}/(2+\sqrt{5}))^{1/2}$.

Proof. Proofs of (i)–(iii) directly follow from Theorem 7.1 (i) and the values of $g'_{3,1}$ from Theorem 5.7 (iii), $g'_{3,3}$ from Theorem 6.7 (iii), and $g'_{3,2}$ from Theorem 5.9 (i), respectively.

Similarly, proofs of (iv)–(vi) follow from Theorem 7.1 (ii) and the values of $g'_{9,1}$ from Theorem 5.7 (vii), $g'_{9,9}$ from Theorem 6.7 (viii), and $g'_{9,2}$ from Theorem 5.9 (v), respectively and proofs of (vii)–(ix) follow from Theorem 7.1 (iii) and the values of $g'_{5,1}$ from Theorem 5.7 (v), $g'_{5,5}$ from Theorem 6.7 (v), and $g'_{5,2}$ from Theorem 5.9 (iii), respectively.

Several other quotients of $\psi(q)$ and $\phi(q)$ are also evaluated in [**3**].

Theorem 7.3. *For any positive real number n, we have*

(i) $\sqrt{5}(h_{5,n}h'_{5,n/4}+(1/h_{5,n}h'_{5,n/4}))-4 = (h'_{5,n/4}/h_{5,n})+(h_{5,n}/h'_{5,n/4})$,

(ii) $(h'_{5,n}/h'_{5,4n})^2+(h'_{5,4n}/h'_{5,n})^2-\sqrt{5}(h'^2_{5,4n}+(1/h'^2_{5,4n}))+4=0$,

(iii) $(h_{5,n}/h'_{5,n})^2+(h'_{5,n}/h_{5,n})^2-\sqrt{5}(h'^2_{5,n}+(1/h'^2_{5,n}))+4=0$.

Proof. The proof follows from Theorems 2.16–2.18 and the definitions of $h_{k,n}$ and $h'_{k,n}$ from (1.3) and (1.4), respectively. □

Theorem 7.4. *We have*

(i) $h_{5,1} = 1$,

(ii) $h_{5,3} = \sqrt{5\sqrt{5}-1}/\sqrt{2}$,

(iii) $h_{5,1/3} = \sqrt{5\sqrt{5}+1}/\sqrt{2}$,

(iv) $h_{5,9} = (\sqrt{3}+1)/(\sqrt{3}+\sqrt{5})$,

(v) $h_{5,1/9} = (\sqrt{3}+\sqrt{5})/(\sqrt{3}+1)$.

For proofs see [**17**, pages 134, 146, 148].

Theorem 7.5. *We have*

(i) $h'_{5,1/4} = (2+\sqrt{2\sqrt{5}-2})/(\sqrt{5}-1)$,

(ii) $h'_{5,1} = ((2-\sqrt{2\sqrt{5}-2})/(\sqrt{5}-1))^{1/2}$,

(iii)

$$h'_{5,4} = \left(\frac{2(-2+2\sqrt{5}-\sqrt{10(-1+\sqrt{5})}+\sqrt{-2+2\sqrt{5}}-2\sqrt{1+\sqrt{5}+2\sqrt{-2+2\sqrt{5}}})}{4+\sqrt{10(1+\sqrt{5})}-5\sqrt{-2+2\sqrt{5}}}\right)^{1/2},$$

(iv) $h'_{5,3} = ((-2+2\sqrt{5}+\sqrt{6(3-\sqrt{5})})/(3-\sqrt{5}))^{1/2}$,

(v) $h'_{5,1/3} = ((2+2\sqrt{5}+\sqrt{6(3+\sqrt{5})})/(3+\sqrt{5}))^{1/2}$,

(vi)

$$h'_{5,9} = \left(\frac{8+4\sqrt{3}+3\sqrt{5}+2\sqrt{15}+2\sqrt{46+32\sqrt{3}+25\sqrt{5}+12\sqrt{15}}}{5\sqrt{3}+4\sqrt{5}-2\sqrt{15}-8}\right)^{1/2},$$

(vii)

$$h'_{5,1/9} = \left(\frac{8+4\sqrt{3}+3\sqrt{5}+2\sqrt{15}+2\sqrt{46+32\sqrt{3}+25\sqrt{5}+12\sqrt{15}}}{4+3\sqrt{3}+4\sqrt{5}+2\sqrt{15}}\right)^{1/2}.$$

Proof of (i). Setting $n = 1$ in Theorem 7.3 (i) and then using Theorem 7.4 (i), we find that

$$(\sqrt{5}-1)\left(x+\frac{1}{x}\right)-4=0. \tag{7.1}$$

Solving the above polynomial equation (7.1) for x, we complete the proof. □

Proof of (ii). Setting $n = 1$ in Theorem 7.3 (iii) and then using Theorem 7.4 (i), we deduce that

$$(1-\sqrt{5})\left(x^2+\frac{1}{x^2}\right)+4=0. \tag{7.2}$$

Solving the above polynomial equation (7.2), we prove the value of (ii). □

Proof of (iii). Setting $n = 1$ in Theorem 7.3 (ii), substituting the value of $h'_{5,1}$ from (ii) and solving the resulting polynomial equation for $h'_{5,1}$, we readily complete the proof. □

Proofs of (iv)–(vii). Setting $n = 3$, $1/3$, 9, and $1/9$ in Theorem 7.3 (iii) and employing the values of $h_{5,3}$, $h_{5,1/3}$, $h_{5,9}$ and $h_{5,1/9}$ from Theorem 7.4, respectively, and then solving the corresponding polynomial equations, we complete the proofs. □

8. Explicit evaluations of the Rogers-Ramanujan continued fraction. In this section, we discuss about the applications of the parameters $h_{k,n}$, $h'_{k,n}$, $g_{k,n}$ and $g'_{k,n}$ to the explicit evaluations of the famous Rogers-Ramanujan continued fraction $R(q)$, defined by

$$R(q) := \frac{q^{1/5}}{1}\,\frac{}{+}\,\frac{q}{1}\,\frac{}{+}\,\frac{q^2}{1}\,\frac{}{+}\,\frac{q^3}{1}\,\frac{}{+\cdots}, \quad |q|<1. \tag{8.1}$$

Theorem 8.1 [**3**, page 2157]. *We have*

$$\frac{f^6(q)}{qf^6(q^5)} = \frac{\psi^2(-q)}{q\psi^2(-q^5)} \times \frac{\phi^4(q)}{\phi^4(q^5)} \tag{8.2}$$

and

$$\frac{f^6(-q^2)}{q^2f^6(-q^{10})} = \frac{\phi^2(q)}{\phi^2(q^5)} \times \frac{\psi^4(-q)}{q^2\phi^4(-q^5)}. \tag{8.3}$$

The following relation was stated by Ramanujan [**5**, page 267] and first proved by Watson [**14**]

$$\frac{1}{R^5(q)} - 11 - R^5(q) = \frac{f^6(-q)}{qf^6(-q^5)}. \tag{8.4}$$

Replacing q by q^2 and $-q$, in succession, we find that

$$\frac{1}{R^5(q^2)} - 11 - R^5(q^2) = \frac{f^6(-q^2)}{q^2 f^6(-q^{10})} \tag{8.5}$$

and

$$\frac{1}{S^5(q)} + 11 - S^5(q) = \frac{f^6(q)}{qf^6(q^5)}, \tag{8.6}$$

where $S(q) = -R(-q)$.

Employing (8.2)–(8.6) and the definitions of $h_{k,n}$, $h'_{k,n}$, $g_{k,n}$ and $g'_{k,n}$, we easily find the following theorem.

Theorem 8.2. *We have*

(i) $1/(R^5(e^{-\pi\sqrt{n/5}})) - 11 - R^5(e^{-\pi\sqrt{n/5}}) = 5\sqrt{5}{g'}^2_{5,n}{h'}^4_{5,n/4}$;

(ii) $1/(R^5(e^{-2\pi\sqrt{n/5}})) - 11 - R^5(e^{-2\pi\sqrt{n/5}}) = 5\sqrt{5}g^4_{5,n}h^2_{5,n}$;

(iii) $1/(S^5(e^{-\pi\sqrt{n/5}})) + 11 - S^5(e^{-\pi\sqrt{n/5}}) = 5\sqrt{5}g^2_{5,n}h^4_{5,n}$.

From the above theorem, it is clear that we can find explicit values of $R(e^{-\pi\sqrt{n/5}})$, $R(e^{-2\pi\sqrt{n/5}})$ and $S(e^{-\pi\sqrt{n/5}})$ by using the known values of $h_{k,n}$, $h'_{k,n}$, $g_{k,n}$ and $g'_{k,n}$. For example, setting $n = 4$ in Theorem 8.2 (i) and using Theorem 5.8 (xi) and Theorem 7.5 (ii), or setting $n = 1$ in Theorem 8.2 (ii) and using Theorem 4.1 (i) and Theorem 7.4 (i), we find that

$$\frac{1}{R^5\left(e^{-2\pi/\sqrt{5}}\right)} - 11 - R^5\left(e^{-2\pi/\sqrt{5}}\right) = 5\sqrt{5}. \tag{8.7}$$

Solving (8.7) for $R^5(e^{-2\pi/\sqrt{5}})$, we conclude that

$$R^5\left(e^{-2\pi/\sqrt{5}}\right)=\frac{1}{2}\left\{\sqrt{10\left(25+11\sqrt{5}\right)}-\left(5\sqrt{5}+11\right)\right\}.$$

This was first evaluated by Yi [**16**, Corollary 4.3].

Similarly, setting $n=1$ in Theorem 8.2 (iii) and using Theorem 4.1 (i) and Theorem 7.4 (i), we obtain

$$\frac{1}{S^5\left(e^{-\pi/\sqrt{5}}\right)}+11-S^5\left(e^{-\pi/\sqrt{5}}\right)=5\sqrt{5}. \tag{8.8}$$

Solving (8.8) for $S^5\left(e^{-\pi/\sqrt{5}}\right)$, we deduce that

$$S^5\left(e^{-\pi/\sqrt{5}}\right)=\frac{1}{2}\left\{\sqrt{10\left(25-11\sqrt{5}\right)}-\left(5\sqrt{5}-11\right)\right\}.$$

This was recorded by Ramanujan [**13**, page 210] and the first proof was given by Berndt, Chan and Zhang [**9**]. Kang [**10**] and Yi [**16**] also established this value.

9. Explicit evaluations of Ramanujan's cubic continued fraction. In this section, we discuss the applications of the parameters $h_{k,n}$, $h'_{k,n}$, $g_{k,n}$ and $g'_{k,n}$ to the explicit evaluations of Ramanujan's cubic continued fraction $G(q)$, defined by

$$G(q):=\frac{q^{1/3}}{1}\,{}_{+}\,\frac{q+q^2}{1}\,{}_{+}\,\frac{q^2+q^4}{1}\,{}_{+}\,\frac{q^3+q^6}{1}\,{}_{+\cdots},\quad |q|<1.$$

From Theorem 4.21 [**3**, page 48] and the definition of $g_{k,n}$ and $g'_{k,n}$, the following theorem is apparent.

Theorem 9.1. *We have*

(i) $G^3(-e^{-\pi\sqrt{n/3}})=-1/(1+3g_{3,n}^4)$;

(ii) $G^3(e^{-\pi\sqrt{n/3}})=1/(3{g'}_{3,n}^4-1)$.

Employing the values of $g_{3,n}$ for $n = 1$, 3, 1/3, 9, 1/9, 5, 1/5, 25, 1/25, 7, 1/7, 13, 1/13, 49 and 1/49 from Theorems 5.4–5.6 in Theorem 9.1 (i), the values of $G(-e^{-\pi\sqrt{n/3}})$ can be found by solving a cubic equation.

Baruah and Saikia [**4**], Yi [**17**] and Adiga et al. [**1**] also found the values of $G(-e^{-\pi\sqrt{n/3}})$ for $n = 1$, 3, 1/3, 9, 1/9, 5, 1/5, 25, 1/25, 7 and 1/7.

Employing the values of $g'_{3,n}$ for $n = 1$, 2, 3, 4, 7, 9, 12, 16, 20, 36 and 64 from Theorems 5.7–5.9 and Theorem 6.7 in Theorem 9.1 (ii), the values of $G(e^{-\pi\sqrt{n/3}})$ can be found by solving a cubic equation.

Ramanathan [**11**] and Yi [**17**] also evaluated $G(e^{-\pi\sqrt{n/3}})$ for $n = 1$, 2, 3, 4, 9, and 36.

Remark 9.2. Theorem 5.3 (i)–(iii) imply that if we know $g_{3,n}$, then $g'_{3,n}$, and hence $g'_{3,4n}$ can be evaluated. Thus, by Theorem 9.1, if we know $G(-e^{-\pi\sqrt{n/3}})$, then $G(e^{-\pi\sqrt{n/3}})$ and $G(e^{-2\pi\sqrt{n/3}})$ can also be evaluated.

The next theorem follows easily from [**5**, page 345] and the definitions of $g'_{k,n}$ and $h_{k,n}$.

Theorem 9.1. *We have*

(i) $G(e^{-\pi\sqrt{n}}) = 1/(\sqrt{3}g'_{9,n} - 1)$;

(ii) $G(-e^{-\pi\sqrt{n}}) = (1 - \sqrt{3}h_{9,n})/2$.

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