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K-THEORY OF CREPANT RESOLUTIONS OF COMPLEX ORBIFOLDS WITH $SU(2)$ SINGULARITIES

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ABSTRACT. We show that if Q is a closed, reduced, complex orbifold of dimension n such that every local group acts as a subgroup of $SU(2) < SU(n)$, then the K-theory of the unique crepant resolution of Q is isomorphic to the orbifold K-theory of Q.

1. Introduction. Let Q be a reduced, compact, complex orbifold of dimension n, i.e., a compact Hausdorff space locally modeled on $\mathcal{C}^n/G$ where G is a finite group which acts effectively on $\mathcal{C}^n$ with a fixed-point set of codimension at least 2 (for details of the definition and further background, see [**3**]). Then a crepant resolution of Q is given by a pair (Y, π) where Y is a smooth complex manifold of dimension n and $\pi : Y \to Q$ is a surjective map which is biholomorphic away from the singular set of Q, such that $\pi^* K_Q = K_Y$ where K_Q and K_Y denote the canonical line bundles of Q and Y, respectively (see [**7**] for details). In [**11**], it is conjectured that if $\pi : Y \to Q$ is a crepant resolution of a Gorenstein orbifold Q, i.e., an orbifold such that all groups act as subgroups of $SU(n)$, then the orbifold K-theory of Q is isomorphic to the ordinary K-theory of Y. For the case of a global quotient of $\mathcal{C}^n$, this has been verified for $n = 2$ in [**10**] and, for Abelian groups and a specific choice of crepant resolution for $n = 3$ in [**5**]. Here, we apply the 'local' results in the case $n = 2$ to the case of a general orbifold with such singularities.

The K-theory of an orbifold can be defined in several different ways. First, it can be defined in the usual way in terms of equivalence classes of orbifold vector bundles, see [**1**]. As well, it is well known that a reduced orbifold Q can be expressed as the quotient P/G where P is a smooth manifold and G is a compact Lie group [**8**]. In the case of a real orbifold, P can be taken to be the orthonormal frame bundle

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of Q with respect to a Riemannian metric and $G = O(n)$. Similarly, in the complex case, P can be taken to be the unitary frame bundle and $G = U(n)$. Hence, the orbifold K-theory of Q is defined as the G-equivariant K-theory $K_G(P)$. See [**1, 9**] for more details.

In Section 2, we describe the structure of the singular set Σ of Q in the case in question and state the main result. In Section 3, we interpret this decomposition in terms of ideals of the C^*-algebra of Q and prove the result.

2. The decomposition of Σ and statement of the result. Let Q be a closed, reduced, complex orbifold with $\dim_{\mathcal{C}} Q = n$, and fix a Hermitian metric on TQ throughout. Then each point $p \in Q$ is contained in a neighborhood modeled by $\mathcal{C}^n/G_p$ where p corresponds to the origin in $\mathcal{C}^n$ and $G_p < U(n)$. Suppose that each of the local groups G_p act as a subgroup of $SU(2) < SU(n)$, and then each point p is locally modeled by $\mathcal{C}^n/G_p \cong \mathcal{C}^{n-2} \times (\mathcal{C}^2/G_p)$. Suppose further that Q admits a crepant resolution $\pi : Y \to Q$ so that Y is a closed complex n-manifold. By Proposition 9.1.4 of [**7**], (Y, π) is a *local product resolution*, which in this context means the following, see [**7**, 9.1.2] for the general definition:

Fix $p \in Q$, and then there is a neighborhood $U_p \ni p$ modeled by $\mathcal{C}^n/G_p$. By hypothesis, $U_p \cong V \times W/G_p$ where $V \times \{0\} \cong \mathcal{C}^{n-2}$ is the fixed point set of G_p, $W \cong \mathcal{C}^2$ is the orthogonal complement of V in $\mathcal{C}^n$ (for some choice of G_p-invariant metric on $\mathcal{C}^n$), and we identify $G_p < SU(n)$ with its restriction $G_p < SU(2)$. Then for a resolution (Y_p, π_p) of W/G_p, we let $\phi : V \times W/G_p \to \mathcal{C}^n/G_p$, T the ball of radius $R > 0$ about the origin in $\mathcal{C}^n/G_p$ and $U := (\mathrm{id} \times \pi_p)^{-1}(T) \subset V \times Y_p$. There is a local isomorphism $\psi : (V \times Y_p)\backslash U \to Y$ such that the following diagram commutes

$$\begin{array}{ccc} (V \times Y_p)\backslash U & \xrightarrow{\psi} & Y \\ \Big\downarrow{\scriptstyle \mathrm{id} \times \pi_p} & & \Big\downarrow{\scriptstyle \pi} \\ (V \times W/G_p)\backslash T & \xrightarrow{\phi} & \mathcal{C}^n/G_p. \end{array}$$

Hence, each of the singular points in a neighborhood of p is resolved by $V \times Y_p$. Moreover, as (Y, π) is a crepant resolution of Q, (Y_p, π_p) is a crepant resolution of $\mathcal{C}^2/G_p$ [**7**, Proposition 9.1.5], and hence is the unique crepant resolution of $\mathcal{C}^2/G_p$. It is clear that a crepant resolution of Q can be formed by patching together local products of the unique crepant resolutions of $\mathcal{C}^2/G_p$, but we now see that this is the only crepant resolution of Q. Moreover, if S denotes a connected component of the singular set Σ of Q, then a neighborhood of S can be covered by a finite number of charts as above, so that the isotropy subgroups of any $p, q \in S$ are conjugate in $SU(2)$. Moreover, each such chart $\mathcal{C}^n/G^p \cong V \times W/G_p$ restricts to a complex manifold chart of dimension $n-2$ for S.

We summarize this discussion in the following.

Lemma 2.1. *Let Q be a closed, reduced, complex orbifold of complex dimension n, and suppose each of the local groups G_p acts on Q as a subgroup of $SU(2)$. Then there is a unique crepant resolution (Y, π) of Q. The singular set Σ of Q is given by*

$$\Sigma = \bigsqcup_{i=1}^{k} S_i$$

for some k finite, where each S_i is a connected, closed, complex $(n-2)$-manifold and the (conjugacy class of the) isotropy subgroup $G_p < SU(2)$ of p is constant on S_i. Moreover, if N_i is a sufficiently small tubular neighborhood of S_i in Q, then $N_i \cong S_i \times \mathcal{C}^2/G_p$ and $\pi^{-1}(N_i) \cong S^i \times Y_i$ where Y_i is the unique crepant resolution of $\mathcal{C}^2/G_p$.

Such a decomposition may be possible for orbifolds with $SU(3)$ singularities; in this case, components of the singular set have $(n-2)$- and $(n-3)$-dimensional components. The latter are closed manifolds, but the former may be open. However, the techniques in this paper do not easily extend to this case. For finite subgroups of $SU(3)$, crepant resolutions are not unique. While a local isomorphism has been constructed for abelian subgroups of $SU(3)$, see [**5**], this is for a specific choice of resolution.

Using the decomposition given in this lemma, we will show the following:

Theorem 2.2. *Let Q be a closed, reduced, complex orbifold of complex dimension n, and suppose each of the local groups G_p acts* *on Q as a subgroup of $SU(2)$. Let (Y,π) denote the unique crepant resolution of Q, and then*

$$K^*_{\mathrm{orb}}(Q) \cong K^*(Y)$$

as additive groups.

For any n-dimensional orbifold that admits a crepant resolution, the local groups can be chosen to be subgroups of $SU(n)$, see [**7**]. Therefore, we have as an immediate corollary:

Corollary 2.3. *Let Q be a two-dimensional complex orbifold which admits a crepant resolution (Y,π). Then*

$$K^*_{\mathrm{orb}}(Q) \cong K^*(Y)$$

as additive groups.

3. Proof of Theorem 2.2. In order to prove Theorem 2.2, we will show that $K_*(A) \cong K_*(B)$ where A is the C^*-algebra of Q and B the C^*-algebra of Y. So fix an orbifold Q that satisfies the hypotheses of Theorem 2.2, and let k, S_i, N_i, etc., be as given in Lemma 2.1. We assume that the N_i are chosen small enough so that $N_i \cap N_j = \varnothing$ for $i \neq j$.

For each i, let N_i' be a smaller tubular neighborhood of S_i so that $S_i \subset N_i' \subset \overline{N_i'} \subset N_i$, and let $N_0 := Q \backslash \cup_{i=1}^k \overline{N_i'}$. Then $\{N_i\}_{i=0}^k$ is an open cover of Q such that N_0 contains no singular points. Note that the restriction $\pi_{|\pi^{-1}(N_0)}$ is a biholomorphism onto N_0.

Let P denote the unitary frame bundle of Q, and then $Q = P/U(n)$. Let $A := C^*(Q)$ denote the C^*-algebra $C(P) \rtimes_\alpha U(n)$ of Q where α is the action of $U(n)$ on $C(P)$ induced by the usual action on P, and let A^0 denote the dense subalgebra $L^1(U(n), C(P), \alpha)$ of $C(P) \rtimes_\alpha U(n)$. Let I_1^0 denote the ideal in A^0 consisting of functions ϕ such that $\phi(g)$ vanishes on $P_{|S_1}$ for each $g \in U(n)$, i.e., $I_1^0 = L^1(U(n), C_0(P \backslash P_{|S_1}), \alpha)$; as usual, $P_{|S_1}$ denotes the restriction of P to S_1), and let I_1 be the closure of I_1^0 in A. Similarly, for each j with $1 < j \leq k$, set

$$I_j^0 := L^1\left(U(n), C_0\left(P \backslash \bigcup_{i=1}^{j} P_{|S_i}\right), \alpha\right)$$

to be the ideal of functions ϕ in A^0 such that for each $g \in U(n)$, $\phi(g)$ vanishes on the fibers over $S_1, S_2, \dots, S_j$, and I_j the closure of I_j^0 in A. Then we have the ideals

$$I_k \subset I_{k-1} \subset \cdots \subset I_1 \subset I_0 := A.$$

Note that, for each j with $1 \leq j < k$, $I_j/I_{j+1} \cong C(P_{|S_{j+1}}) \rtimes_\alpha U(n)$, and $I_k \cong C_0(P_{|N_0}) \rtimes_\alpha U(n)$.

Similarly, let $B := C(Y)$ denote the algebra of continuous functions on Y, and let J_j denote the ideal of functions which vanish on $\pi^{-1}(\cup_{i=1}^{j} S_i)$. Then we have

$$J_k \subset J_{k-1} \subset \cdots \subset J_1 \subset J_0 := B,$$

with $J_j/J_{j+1} \cong C(\pi^{-1}(S_{j+1}))$ and $J_k \cong C_0(\pi^{-1}(N_0))$.

Recall that π restricts to a biholomorphism

$$\pi_{|\pi^{-1}(N_0)} : \pi^{-1}(N_0) \xrightarrow{\cong} N_0.$$

Hence, as the action of $U(n)$ is free on $P_{|N_0}$,

$$\begin{aligned} K_*(I_k) &= K_*(C_0(P_{|N_0}) \rtimes_\alpha U(n)) \\ &\cong K^*_{U(n)}(P_{|N_0}) \\ &\quad \text{naturally, by the Green-Julg theorem} \\ &\quad \text{([\mathbf{2}, Theorems 20.2.7 and 11.7.1]),} \\ &\cong K^*(P_{|N_0}/U(n)) \\ &\quad \text{as the } U(n) \text{ action is free on } N_0, \\ &= K^*(N_0) \\ &= K^*(\pi^{-1}(N_0)) \\ &= K_*(J_k). \end{aligned}$$

Therefore, there is a natural isomorphism

$$K_*(I_k) \cong K_*(J_k). \tag{3.1}$$

Hence, Theorem 2.2 holds for orbifolds such that $k = 0$, i.e., manifolds. The next lemma gives an inductive step which, along with the previous result, yields the theorem.

Lemma 3.1. *Suppose*

$$K_*(I_j) \cong K_*(J_j)$$

naturally for some j *with* $1 \leq j \leq k$*. Then*

$$K_*(I_{j-1}) \cong K_*(J_{j-1}).$$

Proof. Note that I_j is an ideal in I_{j-1}, with $I_{j-1}/I_j = C(P_{|S_j}) \rtimes_\alpha U(n)$. Similarly, J_j is an ideal in J_{j-1} with $J_{j-1}/J_j = C(\pi^{-1}(S_j))$. We have the standard exact sequences

$$\begin{array}{ccccc} K_0(I_j) & \longrightarrow & K_0(I_{j-1}) & \longrightarrow & K_0(I_{j-1}/I_j) \\ \partial \big\uparrow & & & & \big\downarrow \partial \\ K_1(I_{j-1}/I_j) & \longleftarrow & K_1(I_{j-1}) & \longleftarrow & K_1(I_j) \end{array}$$

and

$$\begin{array}{ccccc} K_0(J_j) & \longrightarrow & K_0(J_{j-1}) & \longrightarrow & K_0(J_{j-1}/J_j) \\ \partial \big\uparrow & & & & \big\downarrow \partial \\ K_1(J_{j-1}/J_j) & \longleftarrow & K_1(J_{j-1}) & \longleftarrow & K_1(J_j). \end{array}$$

So if we show that $K_*(I_{j-1}/I_j) \cong K_*(J_{j-1}/J_j)$ naturally, by the Five lemma, we are done.

Note that I_{j-1}/I_j is the C^*-algebra of the quotient orbifold $P_{|S_j}/U(n)$, which is given by the smooth manifold S_j with the trivial action of G_j (here, G_j denotes a choice from the conjugacy class of isotropy groups G_p for $p \in S_j$). Hence, $I_{j-1}/I_j \cong C(S_j) \otimes C^*(G_j)$. Similarly, we have

$$\begin{aligned} J_{j-1}/J_j &= C(\pi^{-1}(S_j)) \\ &= C(S_j \times Y_j) \\ &= C(S_j) \otimes C(Y_j), \end{aligned}$$

where Y_j is the preimage of the origin in the unique crepant resolution of $\mathcal{C}^2/G_j$. However, $K_0(C^*(G_j)) = R(G)$ [**2**, Proposition 11.1.1 and Corollary 11.1.2] which is naturally isomorphic to $K^0(Y_j)$ by [**10**, Section 4.3]; see also [**5**], and $K^0(Y_j) \cong K_0(C(Y_j))$, so that $K_0(C^*(G_j))$ and $K_0(C(Y_j))$ are isomorphic. With this, by the Künneth theorem for tensor products [**2**, Theorem 23.1.3],

$$\begin{array}{ccccccc}
0 & \longrightarrow & K_0(C(S_j)) \otimes K_0(C^*(G_j)) & \longrightarrow & K_0(C(S_j) \otimes C^*(G_j)) & \longrightarrow \\
 & & \downarrow & & \downarrow & \\
0 & \longrightarrow & K_0(C(S_j)) \otimes K_0(C(Y_j)) & \longrightarrow & K_0(C(S_j) \otimes C(Y_j)) & \longrightarrow
\end{array}$$

$$\begin{array}{ccccc}
\longrightarrow & \operatorname{Tor}(K_0(C(S_j)), K_0(C^*(G_j))) & \longrightarrow & 0 \\
 & \downarrow & & \\
\longrightarrow & \operatorname{Tor}(K_0(C(S_j)), K_0(C(Y_j))) & \longrightarrow & 0
\end{array}$$

and the Five lemma, we have a natural isomorphism

$$K_0(C(S_j) \otimes C^*(G_j)) \cong K_0(C(S_j) \otimes C(Y_j)).$$

So

$$K_0(I_{j-1}/I_j) \cong K_0(J_{j-1}/J_j).$$

For the K_1 groups, we note that by [**2**, Corollary 11.1.2], $K_1(C^*(G_j)) = 0$. As well, $K_1(C(Y_j)) \cong K^1(Y_j)$, and it is known that Y_j is diffeomorphic to a finite collection of 2-spheres which intersect at most transversally at one point, see [**7**]. Therefore, $K^1(Y_j) = 0$. Here, the hypothesis that all groups act as subgroups of $SU(2)$ is crucial. For subgroups of $SU(3)$, the topology of the resolution is not understood sufficiently to compute the K_1 groups.

With this, we again apply the Künneth theorem and Five lemma

$$\begin{array}{ccccc}
0 & \longrightarrow & K_1(C(S_j)) \otimes K_0(C^*(G_j)) \oplus K_0(C(S_j)) \otimes K_1(C^*(G_j)) & \longrightarrow & K_1(C(S_j) \otimes C^*(G_j)) \\
 & & \downarrow & & \downarrow \\
0 & \longrightarrow & K_1(C(S_j)) \otimes K_0(C(Y_j)) \oplus K_0(C(S_j)) \otimes K_1(C(Y_j)) & \longrightarrow & K_1(C(S_j) \otimes C(Y_j))
\end{array}$$

$$\begin{array}{ccccc}
\cdots & \longrightarrow & \operatorname{Tor}(K_1(C(S_j)), K_0(C^*(G_j))) \oplus \operatorname{Tor}(K_0(C(S_j)), K_1(C^*(G_j))) & \longrightarrow & 0 \\
 & & \downarrow & & \\
\cdots & \longrightarrow & \operatorname{Tor}(K_1(C(S_j)), K_0(C(Y_j))) \oplus \operatorname{Tor}(K_0(C(S_j)), K_1(C(Y_j))) & \longrightarrow & 0
\end{array}$$

Therefore, we have a natural isomorphism

$$K_1(C(S_j) \otimes C^*(G_j)) \cong K_1(C(S_j) \otimes C(Y_j)),$$

and

$$K_1(I_{j-1}/I_j) \cong K_1(J_{j-1}/J_j). \qquad \square$$

Now, as $K_*(I_k) \cong K_*(J_k)$, repeated application of Lemma 3.1 yields that $K_*(A) \cong K_*(B)$, and hence we have proven Theorem 2.2.

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