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CONVOLUTION SUMS OF SOME FUNCTIONS ON DIVISORS

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ABSTRACT. One of the main goals in this paper is to establish convolution sums of functions for the divisor sums $\widetilde{\sigma}_s(n) = \sum_{d|n}(-1)^{d-1}d^s$ and $\widehat{\sigma}_s(n) = \sum_{d|n}(-1)^{(n/d)-1}d^s$, for certain s, which were first defined by Glaisher. We first introduce three functions $\mathcal{P}(q)$, $\mathcal{E}(q)$, and $\mathcal{Q}(q)$ related to $\widetilde{\sigma}(n)$, $\widehat{\sigma}(n)$, and $\widetilde{\sigma}_3(n)$, respectively, and then we evaluate them in terms of two parameters x and z in Ramanujan's theory of elliptic functions. Using these formulas, we derive some identities from which we can deduce convolution sum identities. We discuss some formulae for determining $r_s(n)$ and $\delta_s(n)$, $s = 4$, 8, in terms of $\widetilde{\sigma}(n)$, $\widehat{\sigma}(n)$, and $\widetilde{\sigma}_3(n)$, where $r_s(n)$ denotes the number of representations of n as a sum of s squares and $\delta_s(n)$ denotes the number of representations of n as a sum of s triangular numbers. Finally, we find some partition congruences by using the notion of colored partitions.

1. Introduction. In his famous paper [**21**], [**22**, pages 136–162], Ramanujan introduced the three Eisenstein series $P(q)$, $Q(q)$ and $R(q)$ defined for $|q| < 1$ by

$$P(q) := 1 - 24\sum_{n=1}^{\infty}\sigma(n)q^n, \tag{1.1}$$

$$Q(q) := 1 + 240\sum_{n=1}^{\infty}\sigma_3(n)q^n, \tag{1.2}$$

$$R(q) := 1 - 504\sum_{n=1}^{\infty}\sigma_5(n)q^n, \tag{1.3}$$

where, for $s, n \in \mathbf{N}$,

$$\sigma_s(n) = \sum_{d|n} d^s.$$

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As usual, we set $\sigma_1(n) = \sigma(n)$ and $\sigma_s(n) = 0$ if $n \notin \mathbf{N}$. Ramanujan also proved that (1.1)–(1.3) satisfy the differential equations [**21**, (30)] and [**22**, page 142]

$$q\frac{dP(q)}{dq} = \frac{P^2(q) - Q(q)}{12}, \tag{1.4}$$

$$q\frac{dQ(q)}{dq} = \frac{P(q)Q(q) - R(q)}{3}, \tag{1.5}$$

$$q\frac{dR(q)}{dq} = \frac{P(q)R(q) - Q^2(q)}{2}. \tag{1.6}$$

After rewriting (1.4) as

$$P^2(q) = Q(q) + 12q\,\frac{dP(q)}{dq}, \tag{1.7}$$

and equating the coefficients of q^n on both sides, we obtain the arithmetic identity

$$12\sum_{m<n}\sigma(m)\sigma(n-m) = 5\sigma_3(n) - (6n-1)\sigma(n). \tag{1.8}$$

Likewise, from (1.5), we obtain

$$240\sum_{m<n}\sigma(m)\sigma_3(n-m) = 21\sigma_5(n) - (30n-10)\sigma_3(n) - \sigma(n). \tag{1.9}$$

Ramanujan recorded nine identities of the type (1.8) and (1.9) in his notebooks. The history of the convolution sums involving the divisor function $\sigma_s(n)$ goes back to Glaisher [**8, 9, 10**]. A most comprehensive treatment of these identities is given in the paper [**12**]. In their paper [**12**], Huard, Ou, Spearman and Williams prove many such formulae in an elementary manner by using their generalization of Liouville's classical formula given in [**17**]. Recently, Cheng and Williams [**5**] found further convolution sums of the type

$$\sum_{m<n}\sigma(4m-3)\sigma(4n-(4m-3)) = 4\sigma_3(n) - 4\sigma_3(n/2).$$

Now define two functions on which we focus in this paper by, for $s, n \in \mathbf{N}$,

$$\widetilde{\sigma}_s(n) = \sum_{d|n} (-1)^{d-1} d^s, \tag{1.10}$$

$$\widehat{\sigma}_s(n) = \sum_{d|n} (-1)^{(n/d)-1} d^s, \tag{1.11}$$

where we set $\widetilde{\sigma}_1(n) = \widetilde{\sigma}(n)$, $\widehat{\sigma}_1(n) = \widehat{\sigma}(n)$ and $\widetilde{\sigma}_s(n) = \widehat{\sigma}_s(n) = 0$ if $n \notin \mathbf{N}$. The origin of these functions goes back to Glaisher. In his paper [**9**], Glaisher defined seven quantities which depend on the divisors of n, including (1.10) and (1.11), and studied the relations among them. He also found expressions for all seven functions in terms of $\sigma_s(n)$. For instance, the functions $\widetilde{\sigma}_s(n)$ and $\widehat{\sigma}_s(n)$ have the formulae [**9**]

$$\widetilde{\sigma}_s(n) = \sigma_s(n) - 2^{s+1}\sigma_s(n/2), \tag{1.12}$$

$$\widehat{\sigma}_s(n) = \sigma_s(n) - 2\sigma_s(n/2). \tag{1.13}$$

From the relations (1.12) and (1.13), it is clear that, for all $n \geq 0$,

$$\widetilde{\sigma}_s(2n+1) = \sigma_s(2n+1) = \widehat{\sigma}_s(2n+1). \tag{1.14}$$

One of our goals in the present paper is to establish convolution sums involving $\widetilde{\sigma}_s$ and $\widehat{\sigma}_s$ for certain s. So we need to define three functions related to (1.10) and (1.11) by, for $|q| < 1$,

$$\mathcal{P}(q) := 1 + 8\sum_{n=1}^{\infty} \widetilde{\sigma}(n) q^n, \tag{1.15}$$

$$\mathcal{E}(q) := 1 + 24\sum_{n=1}^{\infty} \widehat{\sigma}(n) q^n, \tag{1.16}$$

$$\mathcal{Q}(q) := 1 - 16\sum_{n=1}^{\infty} \widetilde{\sigma}_3(n) q^n. \tag{1.17}$$

Analogous to (1.4)–(1.6), our three functions (1.15)–(1.17) satisfy the differential equations [**11, 19, 20**]

$$q\frac{d\mathcal{P}(q)}{dq} = \frac{\mathcal{P}^2(q) - \mathcal{Q}(q)}{4}, \tag{1.18}$$

$$q\frac{d\mathcal{E}(q)}{dq} = \frac{\mathcal{E}(q)\mathcal{P}(q) - \mathcal{Q}(q)}{2}, \tag{1.19}$$

$$q\frac{d\mathcal{Q}(q)}{dq} = \mathcal{P}(q)\mathcal{Q}(q) - \mathcal{E}(q)\mathcal{Q}(q). \tag{1.20}$$

If we define the related series analogues to [**5**]

$$\mathcal{P}_{r,2}(q) = \sum_{n=0}^{\infty} \widetilde{\sigma}(2n+r) q^{2n+r}, \quad r = 0, 1, \tag{1.21}$$

$$\mathcal{E}_{r,2}(q) = \sum_{n=0}^{\infty} \widehat{\sigma}(2n+r) q^{2n+r}, \quad r = 0, 1, \tag{1.22}$$

$$\mathcal{Q}_{r,2}(q) = \sum_{n=0}^{\infty} \widetilde{\sigma}_3(2n+r) q^{2n+r}, \quad r = 0, 1, \tag{1.23}$$

then we find many identities involving the series $\mathcal{P}_{r,2}(q)$, $\mathcal{E}_{r,2}(q)$, $\mathcal{Q}_{r,2}(q)$, and the functions $\mathcal{P}(q)$, $\mathcal{E}(q)$, and $\mathcal{Q}(q)$.

In Section 2, we evaluate (1.15), (1.16), (1.17), (1.21), (1.22) and (1.23) in terms of two parameters x and z in Ramanujan's theory of elliptic functions. Using these formulas, we derive some identities involving Ramanujan's theta functions. In Section 3, we find representations for certain infinite series related to $\mathcal{P}(q)$, $\mathcal{E}(q)$, and $\mathcal{Q}(q)$. In Section 4, using the evaluations we obtained in Section 2, we derive convolution sums of (1.10) and (1.11). In Section 5, we discuss some formulae for determining $r_s(n)$ and $\delta_s(n)$ in terms of $\widetilde{\sigma}_s(n)$ and $\widehat{\sigma}_s(n)$, where $r_s(n)$ denotes the number of representations of n as a sum of s squares and $\delta_s(n)$ denotes the number of representations of n as a sum of s triangular numbers. Finally, we find some partition congruences connected with $\widetilde{\sigma}_s(n)$ and $\widehat{\sigma}_s(n)$ by using the notion of colored partitions.

2. Evaluations and identities involving Ramanujan's theta functions. To derive the desired identities, we need to use evaluations of theta functions [**3**, pages 122–138] to determine the quantities $\mathcal{P}(q^r)$, $\mathcal{E}(q^r)$, $\mathcal{Q}(q^r)$, $\mathcal{P}(-q)$, $\mathcal{E}(-q)$, and $\mathcal{Q}(-q)$, $r = 1, 2$.

If

$$y = \pi \frac{{}_2F_1\left((1/2),(1/2);1;1-x\right)}{{}_2F_1\left((1/2),(1/2);1;x\right)}, \quad |x| < 1, \tag{2.1}$$

where ${}_2F_1$ denotes the Gaussian hypergeometric function, the evaluations are given in terms of, in Ramanujan's notation,

$$z := {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; x\right) \tag{2.2}$$

and x. The derivative y' is given by

$$\frac{dy}{dx} = -\frac{1}{x(1-x)z^2}; \tag{2.3}$$

see, for example, Berndt's book [**2**, page 87]. The function $z := {}_2F_1\left((1/2),(1/2);1;x\right)$ satisfies the differential equation [**3**, page 120]

$$\frac{d^2z}{dz^2} = \frac{z}{4x(1-x)} - \frac{(1-2x)}{x(1-x)}\frac{dz}{dx}. \tag{2.4}$$

From now on, we will denote

$$q := e^{-y}.$$

Ramanujan's theta functions $\varphi(q)$, $\psi(q)$, and $f(-q)$ [**3**, Entry 22, page 36] are defined, for $|q| < 1$, by

$$\varphi(q) := \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{(-q;q^2)_\infty(q^2;q^2)_\infty}{(q;q^2)_\infty(-q^2;q^2)_\infty}, \tag{2.5}$$

$$\psi(q) := \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}, \tag{2.6}$$

$$f(-q) := \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n+1)/2} = (q;q)_\infty, \tag{2.7}$$

where, as usual, for any complex number a, we write

$$(a;q)_\infty := \prod_{n=0}^{\infty}(1-aq^n).$$

Here, the product representations arise from the Jacobi triple product identity [**3**, Entry 19, page 35]. In the following lemma, we list the evaluations of the theta functions in terms of x and z [**3**, Entries 10–12, pages 122–124], which we will employ in a majority of our proofs.

Lemma 2.1. *If y and z are defined by* (2.1) *and* (2.2), *respectively, and $\psi(q)$, $\varphi(q)$, and $f(-q)$ are defined by* (2.5), (2.6) *and* (2.7), *respectively, then*

$$\varphi(q) = \sqrt{z}, \tag{2.8}$$

$$\varphi(-q) = (1-x)^{1/4}\sqrt{z}, \tag{2.9}$$

$$q^{1/8}\psi(q) = 2^{-1/2}x^{1/8}\sqrt{z}, \tag{2.10}$$

$$q^{1/4}\psi(q^2) = 2^{-1}x^{1/4}\sqrt{z}, \tag{2.11}$$

$$q^{1/24}f(-q) = 2^{-1/6}(1-x)^{1/6}x^{1/24}\sqrt{z}. \tag{2.12}$$

Using these evaluations, we obtain formulas for $\mathcal{P}(q)$, $\mathcal{E}(q)$, and $\mathcal{Q}(q)$.

Theorem 2.2. *If y and z are defined as in* (2.1) *and* (2.2)*, respectively, and $q := e^{-y}$, then*

$$\mathcal{P}(q) = z^2(1-x) + 4x(1-x)z\frac{dz}{dx}, \tag{2.13}$$

$$\mathcal{E}(q) = z^2(1+x), \tag{2.14}$$

$$\mathcal{Q}(q) = z^4(1-x)^2. \tag{2.15}$$

Proof of (2.13). In the derivation below, we find that, by using (2.10),

$$\begin{aligned}\mathcal{P}(q) &= 1 + 8\sum_{n=1}^{\infty}\frac{(-1)^{n-1}n}{e^{ny}-1}\\ &= 1 - 8\frac{d}{dy}\sum_{n=1}^{\infty}(-1)^n\operatorname{Log}(1-e^{-ny})\\ &= 1 - 8\frac{d}{dy}\operatorname{Log}\prod_{n=1}^{\infty}\frac{1-e^{-2ny}}{1-e^{-(2n-1)y}}\\ &= -8\frac{d}{dy}\operatorname{Log}\{e^{-y/8}\psi(e^{-y})\},\end{aligned}$$

where we use the infinite product representation of $\psi(e^{-y})$ in (2.6). If we employ (2.10) and (2.3), then we find that

$$\begin{aligned}\mathcal{P}(q) &= 8x(1-x)z^2\frac{d}{dx}\operatorname{Log}\{2^{-1/2}\sqrt{z}x^{1/8}\}\\ &= z^2(1-x) + 4x(1-x)z\frac{dz}{dx}.\end{aligned}$$

Proof of (2.14). In the derivation below, we employ (2.12) and (2.9) to find that

$$\begin{aligned}\mathcal{E}(q) &= 1 + 24\sum_{n=1}^{\infty}\frac{n}{e^{ny}+1}\\ &= 1 - 24\frac{d}{dy}\sum_{n=1}^{\infty}\operatorname{Log}(1+e^{-ny})\\ &= -24\frac{d}{dy}\operatorname{Log}\left\{e^{-y/24}\frac{f(-e^{-y})}{\varphi(-e^{-y})}\right\}.\end{aligned}$$

Again using the evaluations (2.9) and (2.12) and applying (2.3), we find that

$$\begin{aligned}\mathcal{E}(q) &= 24x(1-x)z^2\frac{d}{dx}\operatorname{Log}\{2^{-1/6}(1-x)^{-1/12}x^{1/24}\}\\ &= z^2(1+x),\end{aligned}$$

which completes our proof. □

Proof of (2.15). From (1.18), we have

$$q\frac{d\mathcal{P}(q)}{dq} = \frac{\mathcal{P}^2(q)-\mathcal{Q}(q)}{4}.$$

Thus, by the chain rule, we deduce that

$$\frac{d\mathcal{P}(e^{-y})}{dy} = \frac{\mathcal{Q}(e^{-y})-\mathcal{P}^2(e^{-y})}{4}.$$

Moreover, by (2.3), we derive that

$$\frac{d\mathcal{P}(e^{-y})}{dx} = -\frac{1}{x(1-x)z^2}\frac{d\mathcal{P}(e^{-y})}{dy}.$$

Hence,

$$-x(1-x)z^2\frac{d\mathcal{P}(e^{-y})}{dx} = \frac{\mathcal{Q}(e^{-y})-\mathcal{P}^2(e^{-y})}{4}. \tag{2.16}$$

Thus we see that we can determine $\mathcal{Q}(e^{-y})$ from (2.13) and (2.16). Using (2.13) and the hypergeometric differential equation (2.4), we find, upon direct calculation, that

$$\frac{d\mathcal{P}(e^{-y})}{dx} = 2(1-x)z\frac{dz}{dx} + 4x(1-x)\left(\frac{dz}{dx}\right)^2. \tag{2.17}$$

Thus from (2.13), (2.16) and (2.17), we see that

$$\mathcal{Q}(q) = \mathcal{Q}(e^{-y}) = \left\{(1-x)z^2 + 4x(1-x)z\frac{dz}{dx}\right\}^2 - 4x(1-x)z^2\left\{2(1-x)z\frac{dz}{dx} + 4x(1-x)\left(\frac{dz}{dx}\right)^2\right\}.$$

Upon simplifying, we reach the desired conclusion. □

Before proceeding further, we briefly mention the procedure [**3**, page 125], called *duplication*, in the theory of elliptic functions. If

$$\Omega(x, e^{-y}, z) = 0, \tag{2.18}$$

and x', y', and z' is another set of parameters such that

$$\Omega(x', e^{-y'}, z') = 0$$

and

$$x = \frac{4\sqrt{x'}}{(1+\sqrt{x'})^2},$$

then we can deduce the "new" formula

$$\Omega\left(\left(\frac{1-\sqrt{1-x}}{1+\sqrt{1-x}}\right)^2, e^{-2y}, \frac{1}{2}z(1+\sqrt{1-x})\right) = 0, \tag{2.19}$$

from the "old" formula (2.18). This process is called *obtaining a formula by duplication*. We will use this procedure in many proofs.

Applying the process of duplication to (2.13), (2.14) and (2.15), we obtain

$$\mathcal{P}(q^2) = z^2(1-x) + 2x(1-x)z\,\frac{dz}{dx}, \tag{2.20}$$

$$\mathcal{E}(q^2) = z^2(1-\frac{1}{2}x), \tag{2.21}$$

$$\mathcal{Q}(q^2) = z^4(1-x). \tag{2.22}$$

Berndt [**3**, page 126] has also described the process of obtaining a new formula from (2.18) by changing the sign of q. If (2.18) holds, then the formula

$$\Omega\left(\frac{x}{x-1}, -q, z\sqrt{1-x}\right) = 0 \tag{2.23}$$

also holds. This result is attributed to Jacobi by Berndt [**3**, page 126].

Applying Jacobi's change of sign procedure to (2.13), (2.14) and (2.15), we deduce that

$$\mathcal{P}(-q) = z^2(1-2x) + 4x(1-x)z\frac{dz}{dx}, \tag{2.24}$$

$$\mathcal{E}(-q) = z^2(1-2x), \tag{2.25}$$

$$\mathcal{Q}(-q) = z^4. \tag{2.26}$$

Simple calculations analogous to [**5**] show that

$$\mathcal{P}_{0,2}(q) = \frac{1}{16}\big(-2 + \mathcal{P}(q) + \mathcal{P}(-q)\big), \tag{2.27}$$

$$\mathcal{P}_{1,2}(q) = \frac{1}{16}\big(\mathcal{P}(q) - \mathcal{P}(-q)\big), \tag{2.28}$$

$$\mathcal{E}_{0,2}(q) = \frac{1}{48}\big(-2 + \mathcal{E}(q) + \mathcal{E}(-q)\big), \tag{2.29}$$

$$\mathcal{E}_{1,2}(q) = \frac{1}{48}\big(\mathcal{E}(q) - \mathcal{E}(-q)\big), \tag{2.30}$$

$$\mathcal{Q}_{0,2}(q) = \frac{1}{32}\big(-2 - \mathcal{Q}(q) - \mathcal{Q}(-q)\big), \tag{2.31}$$

$$\mathcal{Q}_{1,2}(q) = \frac{1}{32}\big(-\mathcal{Q}(q) + \mathcal{Q}(-q)\big). \tag{2.32}$$

Using (2.13)–(2.15) and (2.24)–(2.26), we obtain the evaluations of the series $\mathcal{P}_{r,2}(q)$, $\mathcal{E}_{r,2}(q)$ and $\mathcal{Q}_{r,2}(q)$ as follows:

Theorem 2.3. *We have that*

(2.33)
$$\mathcal{P}_{0,2}(q) = \frac{1}{16}\left(-2 + (2-3x)z^2 + 8x(1-x)z\frac{dz}{dx}\right),$$

(2.34)
$$\mathcal{P}_{1,2}(q) = \frac{1}{16}xz^2,$$

(2.35)
$$\mathcal{E}_{0,2}(q) = \frac{1}{48}\big(-2 + (2-x)z^2\big),$$

(2.36)
$$\mathcal{E}_{1,2}(q) = \frac{1}{16}xz^2,$$

(2.37)
$$\mathcal{Q}_{0,2}(q) = \frac{1}{32}\big(2 - (2-2x+x^2)z^4\big),$$

(2.38)
$$\mathcal{Q}_{1,2}(q) = \frac{1}{32}x(2-x)z^4.$$

We note a few results which are used in the next section. Using (2.3) and $q := e^{-y}$, we have

$$\frac{1}{q}\frac{dq}{dx} = -\frac{dy}{dx} = \frac{1}{x(1-x)z^2}$$

so that

$$\frac{dq}{dx} = \frac{q}{x(1-x)z^2}. \tag{2.39}$$

From (2.4), (2.13) and (2.39), we obtain

$$\begin{aligned}\frac{d\mathcal{P}(q)}{dq} &= \frac{(d\mathcal{P}(q)/dx)}{(dq/dx)} \\ &= \frac{(d/dx)\big(z^2(1-x) + 4x(1-x)z(dz/dx)\big)}{q/(x(1-x)z^2)}\end{aligned}$$

$$= \frac{-z^2 + (6-10x)z(dz/dx) + 4x(1-x)(dz/dx)^2}{q/(x(1-x)z^2)}$$
$$\times \frac{+4x(1-x)z(d^2z/dx^2)}{q/(x(1-x)z^2)}$$
$$= \frac{(2-2x)z(dz/dx) + 4x(1-x)(dz/dx)^2}{q/(x(1-x)z^2)},$$

so that

$$q\frac{d\mathcal{P}(q)}{dq} = 2x(1-x)^2z^3\frac{dz}{dx} + 4x^2(1-x)^2z^2\left(\frac{dz}{dx}\right)^2. \tag{2.40}$$

Similarly, from (2.4), (2.20) and (2.39), we obtain

$$q\frac{d\mathcal{P}(q^2)}{dq} = -\frac{x(1-x)z^4}{2} + 2x(1-x)^2z^3\frac{dz}{dx} + 2x^2(1-x)^2z^2\left(\frac{dz}{dx}\right)^2. \tag{2.41}$$

In a similar manner, we find that

$$q\frac{d\mathcal{E}(q)}{dq} = x(1-x)z^4 + 2x(1-x)(1+x)z^3\frac{dz}{dx}, \tag{2.42}$$

$$q\frac{d\mathcal{E}(q^2)}{dq} = -\frac{x(1-x)z^4}{2} + x(1-x)(2-x)z^3\frac{dz}{dx}. \tag{2.43}$$

Next, using Lemma 2.1 and using (2.13), (2.14), (2.15), (2.20), (2.21), (2.22), (2.24), (2.25) and (2.26), we obtain the following identities.

Theorem 2.4. *Recall that $\mathcal{P}$, $\mathcal{E}$, and $\mathcal{Q}$ are defined by* (1.15), (1.16) *and* (1.17), *respectively, and that $\varphi(q)$ and $\psi(q)$ are defined in* (2.5) *and* (2.6), *respectively. Then*

$$\mathcal{Q}(q) = \varphi^8(-q), \tag{2.44}$$
$$16\psi^4(q^2) + \varphi^4(q) = \mathcal{E}(q), \tag{2.45}$$
$$2\mathcal{E}(q^2) + \mathcal{E}(q) = 3\varphi^4(q), \tag{2.46}$$
$$\varphi^4(q)\mathcal{E}(q) + \mathcal{Q}(q^2) = 2\varphi^8(q), \tag{2.47}$$
$$\mathcal{E}(q) - \mathcal{E}(q^2) = 24q\psi^4(q^2), \tag{2.48}$$
$$\mathcal{P}(q) - \mathcal{P}(-q) = 16q\psi^4(q^2), \tag{2.49}$$
$$\mathcal{Q}(q) + \mathcal{Q}(-q) = 32q(8\psi^8(q^2) - \psi^8(q)), \tag{2.50}$$
$$\mathcal{E}^2(q) - \mathcal{Q}(q) = 64q\psi^8(q). \tag{2.51}$$

Proof of (2.44). The result is clear from (2.9) and (2.15).

Proof of (2.45). The equality

$$16\psi^4(q^2) + \varphi^4(q) = xz^2 + z^2 = (1+x)z^2 = \mathcal{E}(q)$$

follows from (2.8), (2.11) and (2.14).

Proof of (2.46). Employing (2.14) and (2.21), we have

$$2\mathcal{E}(q^2) + \mathcal{E}(q) = 3z^2.$$

So the proof is completed by using (2.8).

Proof of (2.47). By (2.8), (2.14) and (2.15), we find that

$$\varphi^4(q)\mathcal{E}(q) + \mathcal{Q}(q^2) = z^4(1+x) + z^4(1-x) = 2z^4 = 2\varphi^8(q).$$

Proof of (2.48). By using (2.14), (2.21) and (2.11), we obtain

$$\mathcal{E}(q) - \mathcal{E}(q^2) = \frac{3}{2}xz^2 = 24q\psi^4(q^2).$$

Proof of (2.49). From (2.13) and (2.24), we find that

$$\mathcal{P}(q) - \mathcal{P}(-q) = (1-x)z^2 - (1-2x)z^2 = xz^2 = 16q\psi^4(q^2).$$

Proof of (2.50). By the definition of $\mathcal{Q}$, we obtain

$$\begin{aligned}\mathcal{Q}(q) + \mathcal{Q}(-q) &= -16\sum_{n=1}^{\infty}(2n-1)^3 q^{2n-1}\left(\frac{1}{1-q^{2n-1}} + \frac{1}{1+q^{2n-1}}\right)\\ &= -32\sum_{n=1}^{\infty}\frac{(2n-1)^3 q^{2n-1}}{1-q^{4n-2}} = 32q(8\psi^8(q^2) - \psi^8(q)),\end{aligned}$$

where we use Example(ii) in [**3**, page 139].

Proof of (2.51). From (2.14) and (2.15), we see that

$$\begin{aligned}\mathcal{E}^2(q) - \mathcal{Q}(q) =& 4xz^4 = \left(\frac{1}{16}xz^2\right)\left(64z^2\right)\\ =& (q\psi^4(q^2))\cdot(64\varphi^4(q)),\end{aligned}$$

where the last equality follows from (2.8) and (2.10). After employing the fact [**3**, Entry 25, page 40],

$$\varphi(q)\psi(q^2) = \psi^2(q),$$

we achieve the desired result.

3. Representations of certain infinite series. In this section, we derive some representations of the infinite series connected with the functions $\mathcal{P}(q)$, $\mathcal{E}(q)$ and $\mathcal{Q}(q)$.

Theorem 3.1. *We have*

$$1 - 24\sum_{n=1}^{\infty}\frac{2n-1}{e^{(2n-1)y}+1} = (1-2x)z^2. \tag{3.1}$$

Proof. From (2.14) and (2.21), we find that

$$2\mathcal{E}(q^2) - \mathcal{E}(q) = 2(1-\frac{x}{2})z^2 - (1+x)z^2 = (1-2x)z^2. \tag{3.2}$$

On the other hand, by the definition of $\mathcal{E}(q)$ in (1.16), we know that

$$\begin{aligned}2\mathcal{E}(q^2) - \mathcal{E}(q) &= 2\left(1+24\sum_{n=1}^{\infty}\frac{n}{e^{2ny}+1}\right) - \left(1+24\sum_{n=1}^{\infty}\frac{n}{e^{ny}+1}\right)\\ &= 1+24\sum_{n=1}^{\infty}\frac{2n}{e^{2ny}+1} - 24\sum_{n=1}^{\infty}\left(\frac{2n}{e^{2ny}+1}+\frac{2n-1}{e^{(2n-1)y}+1}\right)\\ &= 1-24\sum_{n=1}^{\infty}\frac{2n-1}{e^{(2n-1)y}+1}. \qquad \square\end{aligned}$$

Remark. We can compare this result with some of the results in [**3**, Entry 13, page 127]. For example, [**3**, (viii)], we have

$$1+24\sum_{n=1}^{\infty}\frac{n}{e^{ny}+1}=(1+x)z^2.$$

By using the representations for $P(q)$, $Q(q)$ and $R(q)$ and their algebraic relations, Berndt [**3**] also lists further representations, such as

$$1+8\sum_{n=1}^{\infty}\frac{(-1)^{n-1}n^5}{e^{ny}-1}=(1-x)(1-x^2)z^6, \tag{3.3}$$

$$17-32\sum_{n=1}^{\infty}\frac{(-1)^{n-1}n^7}{e^{ny}-1}=(1-x)^2(17-2x+17x^2)z^8, \tag{3.4}$$

$$1+8\sum_{n=1}^{\infty}\frac{(-1)^{n}n^5}{e^{2ny}-1}=(1-x)\left(1-\frac{1}{2}x\right)z^6, \tag{3.5}$$

$$17-32\sum_{n=1}^{\infty}\frac{(-1)^{n}n^7}{e^{2ny}-1}=(1-x)(17-17x+2x^2)z^8. \tag{3.6}$$

Theorem 3.2. *We have*

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}n^3}{\sinh(ny)}=\frac{1}{8}x(1-x)z^4, \tag{3.7}$$

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}n^5}{\sinh(ny)}=\frac{1}{8}x(1-x)(1-2x)z^6, \tag{3.8}$$

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}n^7}{\sinh(ny)}=\frac{1}{16}x(1-x)(2-17x+17x^2)z^8. \tag{3.9}$$

Proof. We use the elementary fact

$$\frac{1}{x-1}-\frac{1}{x^2-1}=\frac{x}{x^2-1}=\frac{1}{x-x^{-1}}. \tag{3.10}$$

To prove (3.7), we simply use the definition of $\mathcal{Q}$ and (3.10) to obtain

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}n^3}{e^{ny}-e^{-ny}} = -\frac{1}{16}\left\{\mathcal{Q}(e^{-y}) - \mathcal{Q}(e^{-2y})\right\} = \frac{1}{16}x(1-x)z^4,$$

where we used (2.15) and (2.22) in the last equality. For (3.8), by (3.10), the sum to be evaluated is equal to

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{(-1)^{n-1}n^5}{e^{ny}-e^{-ny}} &= \frac{1}{8}\left\{\left(1+8\sum_{n=1}^{\infty}\frac{(-1)^n n^5}{e^{ny}-1}\right) - \left(1+8\sum_{n=1}^{\infty}\frac{(-1)^n n^5}{e^{2ny}-1}\right)\right\} \\ &= \frac{1}{8}\left\{(1-x)(1-x^2)z^6 - (1-x)\left(1-\frac{1}{2}x\right)z^6\right\} \\ &= \frac{1}{8}x(1-x)(1-2x)z^6,\end{aligned}$$

where we employ (3.3) and (3.5) to derive (3.8). In a similar manner, we can deduce (3.9) by using (3.10) and applying (3.4), (3.6). □

Applying the *duplication* process to (3.7)–(3.9), respectively, gives

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}n^3}{\sinh(2ny)} = \frac{1}{32}\sqrt{1-x}(1-\sqrt{1-x})^2 z^4, \tag{3.11}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}n^5}{\sinh(2ny)} = \frac{1}{64}\sqrt{1-x}(1-\sqrt{1-x})^2(x-2+6\sqrt{1-x})z^6, \tag{3.12}$$

$$\begin{aligned}&\sum_{n=1}^{\infty} \frac{(-1)^{n-1}n^7}{\sinh(2ny)} \\ &\quad = \frac{1}{512}(1-x)(1-\sqrt{1-x})^2(76\sqrt{1-x}-30(2-x)+x^2)z^8.\end{aligned} \tag{3.13}$$

Remark. We can compare the above results with some results in [**3**, Entry 15, page 132]. For example, Berndt proved that

$$\sum_{n=1}^{\infty} \frac{n^3}{\sinh(ny)} = \frac{1}{8}xz^4.$$

4. Some convolution sums of $\widetilde{\sigma}_s(n)$ and $\widehat{\sigma}_s(n)$. We begin this section by recalling again the three differential equations satisfied by $\mathcal{P}(q)$, $\mathcal{E}(q)$ and $\mathcal{Q}(q)$:

$$q\frac{d\mathcal{P}(q)}{dq} = \frac{\mathcal{P}^2(q) - \mathcal{Q}(q)}{4}, \tag{4.1}$$

$$q\frac{d\mathcal{E}(q)}{dq} = \frac{\mathcal{E}(q)\mathcal{P}(q) - \mathcal{Q}(q)}{2}, \tag{4.2}$$

$$q\frac{d\mathcal{Q}(q)}{dq} = \mathcal{P}(q)\mathcal{Q}(q) - \mathcal{E}(q)\mathcal{Q}(q). \tag{4.3}$$

It is then easy to show that the following convolution sums follow from (4.1)–(4.3).

Theorem 4.1.

$$4\sum_{m<n} \widetilde{\sigma}(m)\widetilde{\sigma}(n-m) = -\widetilde{\sigma}_3(n) + (2n-1)\widetilde{\sigma}(n), \tag{4.4}$$

$$24\sum_{m<n} \widehat{\sigma}(m)\widetilde{\sigma}(n-m) = -2\widetilde{\sigma}_3(n) + (6n-3)\widehat{\sigma}(n) - \widetilde{\sigma}(n), \tag{4.5}$$

$$16\sum_{m<n} (\widetilde{\sigma}(m) - 3\widehat{\sigma}(m))\widetilde{\sigma}_3(n-m) = 2n\widetilde{\sigma}_3(n) + \widetilde{\sigma}(n) - 3\widetilde{\sigma}(n). \tag{4.6}$$

Proof. We can rewrite (4.1) as

$$\mathcal{P}^2(q) = \mathcal{Q}(q) + 4q\frac{d\mathcal{P}(q)}{dq}.$$

Then we have

$$\left(1 + 8\sum_{n=1}^{\infty} \widetilde{\sigma}(n)q^n\right)^2 = \left(1 - 16\sum_{n=1}^{\infty} \widetilde{\sigma}_3(n)q^n\right) + 32\sum_{n=1}^{\infty} n\widetilde{\sigma}(n)q^n. \tag{4.7}$$

Equating the coefficients of q^n on both sides of (4.7), we obtain (4.4). In a similar manner, the remaining two convolution sums (4.5) and (4.6) can be derived from (4.2) and (4.3), respectively. □

It naturally arises to question the evaluation of the sum

$$\sum_{m<n} \widehat{\sigma}(m)\widehat{\sigma}(n-m),$$

which will be mentioned in the following theorem.

Theorem 4.2. *We have*

(4.8)

$$36\sum_{m<n} \widehat{\sigma}(m)\widehat{\sigma}(n-m) = \begin{cases} -3\widehat{\sigma}(n) + 3\widetilde{\sigma}_3(n) & \textit{if } n \textit{ is odd,} \\ -3\widehat{\sigma}(n) - 5\widetilde{\sigma}_3(n) + 4\widetilde{\sigma}_3(n/2) & \textit{if } n \textit{ is even.} \end{cases}$$

Proof. By using (2.14), (2.15), (2.22) and (2.28), we can easily derive the identity

$$\begin{aligned} \mathcal{E}^2(q) &= z^4(1+x)^2 \\ &= z^4(5(1-x)^2 - 4(1-x) + 4x(2-x)) \\ &= 5\mathcal{Q}(q) - 4\mathcal{Q}(q^2) + 128\mathcal{Q}_{1,2}(q). \end{aligned}$$

Equating coefficients of q^n gives the desired evaluation. □

Remark. We point out that certain of the convolution sums considered here can be evaluated from known results in an elementary manner. For example, by using the relation (1.13), we have that

$$\begin{aligned} \sum_{m<n} \widehat{\sigma}(m)\widehat{\sigma}(n-m) &= \sum_{m<n} (\sigma(m)-2\sigma(m/2))(\sigma(n-m)-2\sigma((n-m)/2)) \\ &= \sum_{m<n} \sigma(m)\sigma(n-m) - 2\sum_{m<n} \sigma(m/2)\sigma(n-m) \\ &\quad - 2\sum_{m<n} \sigma((n-m)/2)\sigma(m) \\ &\quad + 4\sum_{m<n} \sigma(m/2)\sigma((n-m)/2) \\ &= A(n) - 4B(n) + 4A(n/2), \end{aligned}$$

where

$$A(n) = \sum_{m<n} \sigma(m)\sigma(n-m)$$

and

$$B(n) = \sum_{m<n/2} \sigma(m)\sigma(n-2m).$$

The values of $A(n)$ and $B(n)$ are given in [**12**].

Theorem 4.3. *We have*

$$16 \sum_{m<n} \widetilde{\sigma}(m)\widetilde{\sigma}_3(n-m) = -\widetilde{\sigma}_5(n) + 2(n-1)\widetilde{\sigma}_3(n) + \widetilde{\sigma}(n). \tag{4.9}$$

Proof. From the differential equation (4.3), we find that

$$1 + 8\sum_{n=1}^{\infty} \widetilde{\sigma}_5(n)q^n = \mathcal{E}(q)\mathcal{Q}(q) = \mathcal{P}(q)\mathcal{Q}(q) - q\frac{d\mathcal{Q}(q)}{dq}, \tag{4.10}$$

where the second equality comes from [**11**, (2.2.8)]. So we complete the proof by equating the coefficients of q^n on both sides of (4.10). □

Remark. Note that the identities (4.4) and (4.9) are analogues of the identities (1.8) and (1.9), respectively, which we mentioned in Section 1. The identity (4.4) was also proved by Glaisher [**9**] by theory of the elliptic functions.

Using the formulas given in Section 2, for $r \neq s$ and $r, s \in \{1, 2\}$, we determine the products $\mathcal{P}(q^r)\mathcal{P}(q^s)$ and $\mathcal{P}(q^r)\mathcal{E}(q^s)$ as linear combinations of $\mathcal{Q}(q)$, $\mathcal{Q}(q^2)$ and the derivatives of $\mathcal{P}(q)$, $\mathcal{P}(q^2)$, $\mathcal{E}(q)$, and $\mathcal{E}(q^2)$.

Theorem 4.4. *We have*

$$\mathcal{P}(q)\mathcal{P}(q^2) = \mathcal{Q}(q^2) + q\frac{d\mathcal{P}(q)}{dq} + 2q\frac{d\mathcal{P}(q^2)}{dq}, \tag{4.11}$$

$$\mathcal{P}(q^2)\mathcal{E}(q) = \mathcal{Q}(q^2) + \frac{1}{3}\left(q\frac{d\mathcal{E}(q)}{dq} + 2q\frac{d\mathcal{E}(q^2)}{dq}\right) + \left(q\frac{d\mathcal{P}(q)}{dq} - 2q\frac{\mathcal{P}(q^2)}{dq}\right), \tag{4.12}$$

$$\mathcal{P}(q)\mathcal{E}(q^2) = \frac{1}{2}(3\mathcal{Q}(q^2) - \mathcal{Q}(q)) + 2q\frac{d\mathcal{E}(q^2)}{dq}. \tag{4.13}$$

We just give the proof of (4.11), since the remaining proofs are similar.

Proof of (4.11). By (2.22), (2.40) and (2.41), we have

$$\begin{aligned}
&\mathcal{Q}(q^2) + q\frac{d\mathcal{P}(q)}{dq} + 2q\frac{d\mathcal{P}(q^2)}{dq} \\
&\qquad = (1-x)z^4 + 2x(1-x)^2z^3\frac{dz}{dx} + 4x^2(1-x)^2z^2\left(\frac{dz}{dx}\right)^2 \\
&\qquad\quad - x(1-x)z^4 + 4x(1-x)^2z^3\frac{dz}{dx} + 4x^2(1-x)^2z^2\left(\frac{dz}{dx}\right)^2 \\
&\qquad = (1-x)^2z^4 + 6x(1-x)^2z^3\frac{dz}{dx} + 8x^2(1-x)^2z^2\left(\frac{dz}{dx}\right)^2 \\
&\qquad = \mathcal{P}(q)\mathcal{P}(q^2),
\end{aligned}$$

where we simply calculate the product of (2.13) and (2.20). This completes the proof of (4.11). The remaining formulas can be proved similarly.

Equating the coefficients of q^n on both sides in the three identities in Theorem 4.4, we obtain the next theorem.

Theorem 4.5. *We have*

(4.14)
$$8\sum_{m<n/2}\widetilde{\sigma}(m)\widetilde{\sigma}(n-2m) = -\widetilde{\sigma}_3(n/2) + (n-1)\widetilde{\sigma}(n) + (2n-1)\widetilde{\sigma}(n/2),$$

(4.15)
$$\begin{aligned}24\sum_{m<n/2}\widetilde{\sigma}(m)\widehat{\sigma}(n-2m) = &-2\widetilde{\sigma}_3(n/2) + (2n-3)\widehat{\sigma}(n) + 4n\widehat{\sigma}(n/2) \\ &+ n\widetilde{\sigma}(n) - (2n+1)\widetilde{\sigma}(n/2),\end{aligned}$$

(4.16)
$$24\sum_{m<n/2}\widehat{\sigma}(m)\widetilde{\sigma}(n-2m) = \widetilde{\sigma}_3(n) - 3\widetilde{\sigma}_3(n/2) + (6n-3)\widehat{\sigma}(n/2) - \widetilde{\sigma}(n).$$

The next theorem shows that for $r \in \{0,1\}$ and $s \in \{1,2\}$, the products of the form $\mathcal{P}_{r,2}(q)\big(-1+\mathcal{P}(q^s)\big)$ and $\mathcal{E}_{r,2}(q)\big(-1+\mathcal{P}(q^s)\big)$ can be expressed as linear combinations of $\mathcal{P}(q)$, $\mathcal{P}(q^2)$, $\mathcal{E}(q)$, $\mathcal{E}(q^2)$, $\mathcal{Q}(q)$, $\mathcal{Q}(q^2)$ and the derivatives of $\mathcal{P}(q)$, $\mathcal{P}(q^2)$, $\mathcal{E}(q)$ and $\mathcal{E}(q^2)$. A MAPLE program was run to determine the identities.

Theorem 4.6. *We have*

(4.17)
$$\mathcal{P}_{0,2}(q)\big(-1+\mathcal{P}(q)\big) = \frac{1}{8}+\frac{1}{16}(\mathcal{Q}(q)+\mathcal{Q}(q^2))-\frac{1}{24}(\mathcal{E}(q)-7\mathcal{E}(q^2)) - \frac{1}{2}\mathcal{P}(q^2)+\frac{1}{2}q\frac{d\mathcal{P}(q)}{dq}-\frac{1}{12}\left(q\frac{\mathcal{E}(q)}{dq}+q\frac{\mathcal{E}(q^2)}{dq}\right),$$

(4.18)
$$\mathcal{P}_{0,2}(q)\big(-1+\mathcal{P}(q^2)\big) = \frac{1}{8}+\frac{1}{8}(\mathcal{Q}(q^2)-3\mathcal{P}(q^2)+\mathcal{E}(q^2)) + \frac{1}{16}\left(q\frac{\mathcal{P}(q)}{dq}+6q\frac{\mathcal{P}(q^2)}{dq}\right)-\frac{1}{48}\left(q\frac{\mathcal{E}(q)}{dq}+2q\frac{\mathcal{E}(q^2)}{dq}\right),$$

(4.19)
$$\mathcal{P}_{1,2}(q)\big(-1+\mathcal{P}(q)\big) = \frac{1}{16}(\mathcal{Q}(q)-\mathcal{Q}(q^2))-\frac{1}{24}(\mathcal{E}(q)-\mathcal{E}(q^2)) + \frac{1}{12}\left(q\frac{\mathcal{E}(q)}{dq}-q\frac{\mathcal{E}(q^2)}{dq}\right),$$

(4.20)
$$\mathcal{P}_{1,2}(q)\big(-1+\mathcal{P}(q^2)\big) = \frac{1}{24}(\mathcal{E}(q^2)-\mathcal{E}(q))+\frac{1}{24}\left(q\frac{\mathcal{E}(q)}{dq}-q\frac{\mathcal{E}(q^2)}{dq}\right),$$

(4.21)
$$\mathcal{E}_{0,2}(q)\big(-1+\mathcal{P}(q)\big) = \frac{1}{24}+\frac{1}{16}(\mathcal{Q}(q^2)-3\mathcal{Q}(q))-\frac{1}{24}\mathcal{P}(q) - \frac{1}{24}\mathcal{E}(q^2)-\frac{1}{12}q\frac{\mathcal{E}(q^2)}{dq},$$

(4.22)
$$\mathcal{E}_{0,2}(q)\big(-1+\mathcal{P}(q^2)\big) = \frac{1}{24}+\frac{1}{24}(\mathcal{Q}(q^2)-\mathcal{P}(q^2)-\mathcal{E}(q^2)) + \frac{1}{24}q\frac{\mathcal{E}(q^2)}{dq}.$$

Again we just give the proof of (4.17), since the remaining proofs are similar.

Proof of (4.17). By (2.15) and (2.22), we have

$$\mathcal{Q}(q) + \mathcal{Q}(q^2) = 2z^4 - 3xz^4 + x^2z^4,$$

and from (2.14) and (2.21),

$$\mathcal{E}(q) - 7\mathcal{E}(q^2) = -6z^2 + \frac{9}{2}xz^2,$$

and from (2.42) and (2.43)

$$q\frac{\mathcal{E}(q)}{dq} + q\frac{\mathcal{E}(q^2)}{dq} = \frac{1}{2}xz^4 - \frac{1}{2}x^2z^4 + 4x(1-x)z^3\frac{dz}{dx} + x^2(1-x)z^3\frac{dz}{dx}.$$

Therefore, by (2.20), (2.40) and the previous three equalities, we finally obtain

$$\begin{aligned}
&\frac{1}{8} + \frac{1}{16}(\mathcal{Q}(q) + \mathcal{Q}(q^2)) - \frac{1}{24}(\mathcal{E}(q) - 7\mathcal{E}(q^2)) - \frac{1}{2}\mathcal{P}(q^2) \\
&\qquad + \frac{1}{2}q\frac{d\mathcal{P}(q)}{dq} - \frac{1}{12}\bigg(q\frac{\mathcal{E}(q)}{dq} + q\frac{\mathcal{E}(q^2)}{dq}\bigg) \\
&\qquad = \frac{1}{8} + \frac{1}{16}(2z^4 - 3xz^4 + x^2z^4) - \frac{1}{24}\bigg(-6z^2 + \frac{9}{2}xz^2\bigg) \\
&\qquad - \frac{1}{2}\bigg(z^2(1-x) + 2x(1-x)z\frac{dz}{dx}\bigg) \\
&\qquad + \frac{1}{2}\bigg(2x(1-x)^2z^3\frac{dz}{dx} + 4x^2(1-x)^2z^2\bigg(\frac{dz}{dx}\bigg)^2\bigg) \\
&\qquad - \frac{1}{12}\bigg(\frac{1}{2}xz^4 - \frac{1}{2}x^2z^4 + 4x(1-x)z^3\frac{dz}{dx} + x^2(1-x)z^3\frac{dz}{dx}\bigg) \\
&\qquad = \frac{1}{8} - \frac{1}{4}z^2 + \frac{5}{16}xz^2 - x(1-x)z\frac{dz}{dx} \\
&\qquad + \frac{1}{8}z^4 - \frac{5}{16}xz^4 + x(1-x)z^3\frac{dz}{dx} \\
&\qquad + \frac{3}{16}x^2z^4 - \frac{5}{4}x^2(1-x)z^3\frac{dz}{dx} + 2\bigg(x(1-x)z\frac{dz}{dx}\bigg)^2 \\
&\qquad = \mathcal{P}_{0,2}(q)\big(-1 + \mathcal{P}(q)\big).
\end{aligned}$$

Equating the coefficients of q^n on both sides of the six formulas in Theorem 4.6, we obtain the following convolution sums.

Theorem 4.7. *We have*

$$8\sum_{m<n/2}\widetilde{\sigma}(2m)\widetilde{\sigma}(n-2m)=-\widetilde{\sigma}_3(n)-\widetilde{\sigma}_3(n/2)+4n\widetilde{\sigma}(n)-4\widetilde{\sigma}(n/2) -(2n+1)\widehat{\sigma}(n)+(2n+7)\widehat{\sigma}(n/2), \tag{4.23}$$

$$8\sum_{m<n/2}\widetilde{\sigma}(2m)\widetilde{\sigma}(n/2-m)=-2\widetilde{\sigma}_3(n/2)+n/2\widetilde{\sigma}(n)+(3n-3)\widetilde{\sigma}(n/2) -n/2\widehat{\sigma}(n)-(n-3)\widehat{\sigma}(n/2), \tag{4.24}$$

$$8\sum_{m<(n+1)/2}\widetilde{\sigma}(2m-1)\widetilde{\sigma}(n-(2m-1)) =-\widetilde{\sigma}_3(n)+\widetilde{\sigma}_3(n/2)+(2n-1)\widehat{\sigma}(n)-(2n-1)\widehat{\sigma}(n/2), \tag{4.25}$$

$$8\sum_{m<(n+1)/2}\widetilde{\sigma}(2m-1)\widetilde{\sigma}((n+1)/2-m) =(n-1)\widehat{\sigma}(n)-(n-1)\widehat{\sigma}(n/2), \tag{4.26}$$

$$8\sum_{m<n/2}\widehat{\sigma}(2m)\widetilde{\sigma}(n-2m)=\frac{1}{3}\widetilde{\sigma}_3(n)-\widetilde{\sigma}_3(n/2)+(2n-1)\widehat{\sigma}(n/2), \tag{4.27}$$

$$8\sum_{m<n/2}\widehat{\sigma}(2m)\widetilde{\sigma}(n/2-m) =-\frac{2}{3}\widetilde{\sigma}_3(n/2)-\frac{1}{3}\widetilde{\sigma}(n/2)+(n-1)\widehat{\sigma}(n/2). \tag{4.28}$$

5. On the representations of integers as sums of squares and triangular numbers. It is immediate from the definitions of $\varphi(q)$ and $\psi(q)$ in (2.5) and (2.6), respectively, that if

$$\varphi^s(q):=\sum_{n=0}^{\infty}r_s(n)q^n \tag{5.1}$$

and

$$\psi^s(q) := \sum_{n=0}^{\infty} \delta_s(n) q^n, \tag{5.2}$$

then $r_s(n)$ and $\delta_s(n)$ are the number of representations of n as a sum of s squares and s triangular numbers, respectively. Clearly, $r_s(0) = \delta_s(0) = 1$. Here, for each nonnegative integer n, the triangular number T_n is defined by

$$T_n := \frac{n(n+1)}{2}.$$

By using the representations and identities derived in Section 2, we find expressions for $r_s(n)$ and $\delta_s(n)$, $s = 4, 8$, as sums of our functions $\widetilde{\sigma}(n)$, $\widehat{\sigma}(n)$, and $\widetilde{\sigma}_3(n)$.

Theorem 5.1. *For each positive integer n, we have*

$$r_4(n) = 16\widehat{\sigma}(n/2) + 8\widehat{\sigma}(n), \tag{5.3}$$

$$\delta_4(n) = \widetilde{\sigma}(2n+1), \tag{5.4}$$

$$r_8(n) = 16(-1)^{n-1}\widetilde{\sigma}_3(n) \tag{5.5}$$

$$8\delta_8(n) = \widetilde{\sigma}_3(n+1) - \widetilde{\sigma}_3(2(n+1)). \tag{5.6}$$

Proof of (5.3). The identity (2.46) is equivalent to the identity

$$3\sum_{n=1}^{\infty} r_4(n) q^n = 48\sum_{n=1}^{\infty} \widehat{\sigma}(n) q^{2n} + 24\sum_{n=1}^{\infty} \widehat{\sigma}(n) q^n. \tag{5.7}$$

The identity (5.7) follows after equating the coefficients of q^n on both sides of (5.7).

Proof of (5.4). By (2.28) and (2.49), we have

$$q\psi^4(q^2) = \mathcal{P}_{1,2}(q). \tag{5.8}$$

Hence, we have

$$q\sum_{n=0}^{\infty}\delta_4(n)q^{2n}=\sum_{n=0}^{\infty}\widetilde{\sigma}(2n+1)q^{2n+1},$$

which is the identity (5.4).

Proof of (5.5). It is clear from (2.26) that

$$\sum_{n=1}^{\infty}r_8(n)q^n=-16\sum_{n=1}^{\infty}\widetilde{\sigma}_3(n)(-q)^n. \tag{5.9}$$

Proof of (5.6). From (2.11), (2.22) and (2.37), we have

$$8q^2\psi^8(q^2)=\frac{1}{16}-\frac{1}{16}\mathcal{Q}(q^2)-\mathcal{Q}_{0,2}(q).$$

Hence, we derive

$$8\sum_{n=1}^{\infty}\delta_8(n-1)q^{2n}=\sum_{n=1}^{\infty}\widetilde{\sigma}_3(n)q^{2n}-\sum_{n=1}^{\infty}\widetilde{\sigma}_3(2n)q^{2n}. \tag{5.10}$$

Equating the coefficients of q^n on both sides of (5.10), we obtain the desired result.

Remarks. Jacobi [**13, 14, 15**] showed that $r_4(n)$ is 8 times the sum of the divisors of n that are not multiples of 4, that is,

$$r_4(n)=8(\sigma(n)-4\sigma(n/4)). \tag{5.11}$$

Many proofs of (5.11) have been given; see for example [**1**], [**4**, page 15]. Spearman and Williams [**24**] gave the simplest arithmetic proof of this formula. If we use $\widehat{\sigma}(n)=\sigma(n)-2\sigma(n/2)$ from (1.13), then we note that our expression for $r_4(n)$ in (5.3) is the same as (5.11). By the fact (1.14), we can express (5.4) as

$$\delta_4(n)=\sigma(2n+1). \tag{5.12}$$

The formula (5.12) is proved in an elementary way [**12**, Theorem 10], and in using modular forms [**18**, Theorem 3]. The evaluation of $\delta_4(n)$ goes back to Legendre [**6, 16**]. The formula (5.5) first appeared implicitly in the work of Jacobi [**14**] and explicitly in the work of Eisenstein [**7**]. Williams [**25**] gave an arithmetic proof of this formula by showing that

$$r_8(n) = 16\sigma_3(n) - 32\sigma_3(n/2) + 256\sigma_3(n/4).$$

Using the theory of modular forms, Ono, Robins, and Wahl [**18**, Theorem 5] derive a formula for $\delta_8(n)$, namely

$$\delta_8(n) = \sigma_3(n+1) - \sigma_3((n+1)/2). \tag{5.13}$$

Formula (5.13) is also proved in an elementary way in [**12**, Theorem 12]. It is not hard to show that (5.13) is the same expression as (5.6). From (1.12), we deduce that

$$\widetilde{\sigma}_3(n) = \sigma_3(n) - 16\sigma_3(n/2). \tag{5.14}$$

Then we have

(5.15)

$$\begin{aligned} 8\delta_8(n) &= \widetilde{\sigma}_3(n+1) - \widetilde{\sigma}_3(2(n+1)) \\ &= \sigma_3(n+1) - 16\sigma_3((n+1)/2) - (\sigma_3(2(n+1)) - 16\sigma_3(n+1)) \\ &= 8(\sigma_3(n+1) - \sigma_3((n+1)/2)), \end{aligned}$$

where, in the last equality, we use the identity

$$\sigma_3(2n) = 9\sigma_3(n) - 8\sigma_3(n/2). \tag{5.16}$$

The identity (5.16) can be proved by letting $n := 2^a N$, N is odd, and then by considering the cases $a = 0$ and $a > 0$. After dividing both sides of (5.15) by 8, we have the desired identity (5.13).

6. Some partition congruences. If r is a nonzero integer, we define the function $p_r(n)$ by

$$\sum_{n=0}^{\infty} p_r(n)q^n := \prod_{n=1}^{\infty}(1-q^n)^r. \tag{6.1}$$

Note that $p_{-1}(n) = p(n)$ is the ordinary partition function. A positive integer n has k *colors* if there are k copies of n available and all of them are viewed as distinct objects. Partitions of positive integers into parts with colors are called *colored partitions*. For example, if 1 is allowed to have 2 colors, say r (*red*), and g (*green*), then all colored partitions of 2 are 2, $1_r + 1_r$, $1_g + 1_g$, $1_r + 1_g$. Letting $p_{e,r}(n)$ and $p_{o,r}(n)$ denote the number of r-colored partitions into an even (respectively, odd) number of distinct parts, it is easy to see that

$$p_r(n) = p_{e,r}(n) - p_{o,r}(n), \tag{6.2}$$

when r is a positive integer.

We prove a congruence for the function $\mu(n)$ which is defined by

$$\sum_{n=0}^{\infty} \mu(n) q^n := \prod_{n=1}^{\infty} (1-q^n)^8 (1-q^{2n})^8. \tag{6.3}$$

It follows that

$$\mu(n) = \mu_e(n) - \mu_o(n), \tag{6.4}$$

where $\mu_e(n)$ and $\mu_o(n)$ are the number of 16-colored partitions into an even (respectively, odd) number of distinct parts, where all the parts of the latter eight colors are even.

Theorem 6.1. *If $\mu(n)$ is defined by* (6.4),

$$\mu(3n-1) \equiv 0 \pmod 3.$$

We generally denote by J an integral power series in q whose coefficients are integers.

Proof. It is obvious from (1.16) that

$$\mathcal{E}(q) = 1 + 3J.$$

Also $n^3 - n \equiv 0 \pmod 3$, and so, from (1.15) and (1.17), we obtain

$$\mathcal{Q}(q) = \mathcal{P}(q) + 3J.$$

Hence,

$$(6.5)\qquad \begin{aligned}(\mathcal{E}^2(q)-\mathcal{Q}(q))\mathcal{Q}(q) &= (\mathcal{E}(q)(1+3J)-(\mathcal{P}(q)+3J))\mathcal{Q}(q)\\ &= \mathcal{E}(q)\mathcal{Q}(q)-\mathcal{P}(q)\mathcal{Q}(q)+3J.\end{aligned}$$

By (2.44) and (2.51), we find that

$$(6.6)\qquad \begin{aligned}(\mathcal{E}^2(q)-\mathcal{Q}(q))\mathcal{Q}(q) &= 64q\psi^8(q)\varphi^8(-q)\\ &= 64q\prod_{n=1}^{\infty}(1-q^n)^8(1-q^{2n})^8,\end{aligned}$$

where the last equality comes from the fact [**3**, page 39]

$$\varphi(-q)\psi(q)=f(-q)f(-q^2),$$

where $f(-q)$ is defined by (2.7). On the other hand, observe that, from (1.17) and (1.20),

$$(6.7)\qquad 16\sum_{n=1}^{\infty}n\widetilde{\sigma}_3(n)q^n=-q\frac{d\mathcal{Q}(q)}{dq}=\mathcal{E}(q)\mathcal{Q}(q)-\mathcal{P}(q)\mathcal{Q}(q).$$

In summary, by (6.5), (6.6) and (6.7), we conclude that

$$(6.8)\qquad 64\sum_{n=0}^{\infty}\mu(n)q^{n+1}=16\sum_{n=1}^{\infty}n\widetilde{\sigma}_3(n)q^n+3J.$$

But the coefficient of q^{3n} on the right side of (6.8) is a multiple of 3. So we obtain

$$\mu(3n-1)\equiv 0 \pmod 3. \qquad \square$$

Secondly, we prove a congruence for the function $\nu(n)$ which is defined by

$$(6.9)\qquad \sum_{n=0}^{\infty}\nu(n)q^n:=\prod_{n=1}^{\infty}(1-q^{2n})^8(1+q^n)^8.$$

Thus $\nu(n)$ is the number of partitions of n into 16 colors, 8 appear at most once (say S_1), and 8 are even and appear at most once (say S_2), weighted by the parity of colors from the set S_2.

Theorem 6.2. *If $\nu(n)$ is defined by* (6.9), *then*

$$\nu(n-1) \equiv \widetilde{\sigma}_3(n) \pmod{3}.$$

Proof. Recall from (2.51) of Theorem 2.4 that

$$\begin{aligned} \mathcal{E}^2(q) - \mathcal{Q}(q) &= 64q \prod_{n=1}^{\infty} \frac{(1-q^{2n})^8}{(1-q^{2n-1})^8} \\ &= 64q \prod_{n=1}^{\infty} (1-q^{2n})^8 (1+q^n)^8, \end{aligned} \tag{6.10}$$

where, in the last equality, we used the fact [**3**, (22.3)]

$$\prod_{n=1}^{\infty} (1+q^n) = \prod_{n=1}^{\infty} (1-q^{2n-1})^{-1}.$$

Then, by (6.9) and (6.10), we deduce that

$$\begin{aligned} 64 \sum_{n=0}^{\infty} \nu(n) q^{n+1} = 48 \sum_{n=1}^{\infty} \widehat{\sigma}(n) q^n &+ 576 \sum_{n=2}^{\infty} \sum_{m=1}^{n-1} \widehat{\sigma}(m) \widehat{\sigma}(n-m) q^n \\ &+ 16 \sum_{n=1}^{\infty} \widetilde{\sigma}_3(n) q^n. \end{aligned}$$

Comparing the coefficients of q^n on both sides of the above equation, we obtain the identity

$$4\nu(n-1) = 3\widehat{\sigma}(n) + \widetilde{\sigma}_3(n) + 36 \sum_{m=1}^{n-1} \widehat{\sigma}(m) \widehat{\sigma}(n-m).$$

We then deduce that

$$\nu(n-1) \equiv \widetilde{\sigma}_3(n) \pmod{3}. \qquad \square$$

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