ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 37, Number 5, 2007

UPPER BOUNDS FOR UNITARY PERFECT NUMBERS AND UNITARY HARMONIC NUMBERS

TAKESHI GOTO

ABSTRACT. We prove the following two theorems: (1) If N is a unitary perfect number with k distinct prime factors, then $N < 2^{2^k}$. (2) If N is a unitary harmonic number with k distinct prime factors, then $N < (2^{2^k})^k$.

1. Introduction. Let σ_j be the divisor function defined by

$$\sigma_j(N) = \sum_{d \mid N} d^j.$$

This function is multiplicative, that is, $\sigma_j(ab) = \sigma_j(a)\sigma_j(b)$ if $(a,b)=1$. A positive integer N is said to be a *perfect number* if $\sigma_1(N) = 2N$. It is well known that an even perfect number has a form $2^{p-1}(2^p-1)$ with $2^p - 1$ prime. As of October, 2006, 44 even perfect numbers are known (for the newest information, see the web site of GIMPS: `http://www.mersenne.org/prime.htm`). It is still open whether or not odd perfect numbers (OPNs) exist; however, many conditions for their existence are known. For example, Brent, Cohen and te Riele [**1**] showed that OPNs must be greater than 10^{300}. Suppose that N is an OPN with k distinct prime factors. Dickson [**5**] showed that, for a fixed positive integer k, there exist only finitely many such N. Moreover, it was shown by Hagis [**7**] and Chein [**2**] independently that k must be greater than 7. Pomerance [**13**] showed that $N < (4k)^{(4k)2^{k^2}}$, and this bound was improved by Heath-Brown [**9**] to 4^{4^k}, by Cook [**4**] to D^{4^k} with $D = (195)^{1/7} \approx 2.12$, by Nielsen [**10**] to 2^{4^k}.

Subbarao and Warren [**15**] introduced the concept of *unitary perfect numbers* (UPNs). A positive integer d is said to be a unitary divisor of N if $d \mid N$ and $(d, N/d) = 1$. So we define the unitary divisor function

2000 AMS *Mathematics subject classification.* Primary 11A25, 11Y70.
Received by the editors on May 19, 2005.

by

$$\sigma_j^*(N) = \sum_{\substack{d|N \\ (d,N/d)=1}} d^j.$$

This function is also multiplicative. A positive integer N is said to be a UPN if $\sigma_1^*(N) = 2N$. Subbarao and Warren listed four UPNs: $6, 60, 90$ and 87360. They showed that every UPN is even and conjectured that there exist only four UPNs; however, Wall [**16**] discovered the fifth one:

$$2^{18} \cdot 3 \cdot 5^4 \cdot 7 \cdot 11 \cdot 13 \cdot 19 \cdot 37 \cdot 79 \cdot 109 \cdot 157 \cdot 313,$$

an integer with 24 digits. He showed that this is the exact fifth one, that is, there exist no unknown UPNs less than the number above. It is still open whether or not there exist other UPNs. Suppose that N is an unknown UPN with k distinct prime factors. Subbarao and Warren showed that, for a fixed positive integer k, there exist only finitely many such N, Subbarao [**14**] that k must be greater than 7 and Wall [**17**] improved to $k > 9$. One of the aims of this paper is to give a bound for UPNs which is analogous to Nielsen's bound for OPNs. In Section 2, we give an upper bound for components of UPNs (a prime power p^e is said to be a component of N if $p^e \mid N$ and $p^{e+1} \nmid N$).

Theorem 1.1. *Suppose that N is a* UPN *with k distinct prime factors. Let N_i be the ith smallest component of N. Then it follows that $N_i \le (2^{2^{i-1}} - 1)(k - i + 1)$ with equality if and only if $N = 6$, $N_1 = 2$, $N_2 = 3$ or $N = 60$, $N_1 = 3$.*

From Theorem 1.1 it is immediate that $N \le 2^{2^k - 1} k!$; however, we give a better bound in Section 3.

Theorem 1.2. *If N is a* UPN *with k distinct prime factors, then it follows that $N \le 2^{2^{k-1}-1}(2^{2^{k-1}} - 1)$ with equality if and only if $N = 6$. In particular, $N < 2^{2^k}$.*

Ore [**11**] introduced some object named after *harmonic numbers* by Pomerance [**12**]. Because this term also means another object, $1 + 1/2 + \cdots + 1/n$, we may use a term *Ore's* (*harmonic*) *numbers* to avoid confusion. A positive integer N is said to be a harmonic number if the harmonic mean of its divisors

$$H(N) = \frac{N\sigma_0(N)}{\sigma_1(N)}$$

is integral. Ore proved that every perfect number is harmonic and conjectured that no nontrivial odd harmonic numbers exist (1 is called a trivial harmonic number). In fact, no nontrivial odd harmonic numbers are known, though thousands of harmonic numbers are known. If Ore's conjecture is true, it follows that no OPNs exist.

Hagis and Load [**7**] introduced the concept of *unitary harmonic numbers* (UHNs). A positive integer N is said to be a UHN if the harmonic mean of its unitary divisors

$$H^*(N) = \frac{N\sigma_0^*(N)}{\sigma_1^*(N)}$$

is integral. Many examples of UHNs are known, (see [**8**] or Section 5 in this paper); however, it is still open whether or not there exist infinitely many UHNs. Hagis and Load showed that every UPN is a UHN. They also proved the following two facts:

(a) For a given positive integer c, there exist only finitely many UHNs N with $H^*(N) = c$.

(b) For a given positive integer k, there exist only finitely many UHNs with k distinct prime factors.

In Section 4, we prove the following theorem.

Theorem 1.3. *Suppose that N is a* UHN *with k distinct prime factors and $H^*(N) = c$. Then it follows that*

(a) $N \le c^{c^2}$,

(b) $N \le (2^{2^k})^k$

with each equality if and only if $N = 1$.

2. Upper bound for components of UPNs. In this section, $N_1, \dots, N_k$ would denote prime powers satisfying $N_i < N_j$, $(N_i, N_j) = 1$ for $i < j$. We often use the following lemma.

Lemma 2.1. *If* $\sigma^*_{-1}(N_1 \cdots N_k) = s$*, then*

$$N_1 \leq \frac{k}{s-1}$$

with equality if and only if $N_i = N_{i-1} + 1$, $i = 2, \dots, k$.

Proof. Since

$$\begin{aligned} s = \sigma^*_{-1}(N_1 \cdots N_k) &= \frac{N_1+1}{N_1} \cdot \frac{N_2+1}{N_2} \cdots \frac{N_k+1}{N_k} \\ &\leq \frac{N_1+1}{N_1} \cdot \frac{N_1+2}{N_1+1} \cdots \frac{N_1+k}{N_1+k-1} = \frac{N_1+k}{N_1}, \end{aligned}$$

the lemma holds. □

Before proving Theorem 1.1, we give some bounds for components of UPNs.

Proposition 2.2. *Suppose that* $N = N_1 \cdots N_k$ *is a* UPN*. Then the following inequalities hold.*

$$\begin{aligned} N_1 &\leq k, \\ N_2 &\leq 3(k-1), \\ N_3 &\leq 9(k-2), \\ N_4 &\leq 54(k-3), \\ N_5 &\leq 648(k-4), \\ N_6 &\leq 321408(k-5), \\ N_7 &\leq \frac{103305030912}{5}(k-6). \end{aligned}$$

Proof. Since $\sigma^*_{-1}(N) = 2$, we immediately have $N_1 \leq k$ from Lemma 2.1. Next, since $\sigma^*_{-1}(N_2 \cdots N_k) = 2/\sigma^*_{-1}(N_1)$, Lemma 2.1 implies

$$N_2 \leq \frac{k-1}{2/\sigma^*_{-1}(N_1) - 1} \leq \frac{k-1}{2/\sigma^*_{-1}(2) - 1} = 3(k-1).$$

Similarly, we have

$$N_3 \le \frac{k-2}{2/\sigma^*_{-1}(N_1N_2)-1}.$$

It is necessary that $\sigma^*_{-1}(N_1N_2) < 2$. Note that $\sigma^*_{-1}(2\cdot 3) = 2$, $\sigma^*_{-1}(2\cdot 5) = 1.8$, $\sigma^*_{-1}(3\cdot 4) \approx 1.6$. Hence, $\sigma^*_{-1}(N_1N_2) \le \sigma^*_{-1}(2\cdot 5)$, and

$$N_3 \le \frac{k-2}{2/\sigma^*_{-1}(2\cdot 5)-1} = 9(k-2).$$

For a bound of N_4, we must investigate when $\sigma^*_{-1}(N_1N_2N_3)$ is maximal under the condition $\sigma^*_{-1}(N_1N_2N_3) < 2$. By a direct calculation, we have

$$\sigma^*_{-1}(2\cdot 5\cdot 9) = 2,\quad \sigma^*_{-1}(2\cdot 5\cdot 11) \approx 1.96,\quad \sigma^*_{-1}(2\cdot 7\cdot 9) \approx 1.90,$$
$$\sigma^*_{-1}(3\cdot 4\cdot 5) = 2,\quad \sigma^*_{-1}(3\cdot 4\cdot 7) \approx 1.90.$$

Hence it follows that

$$N_4 \le \frac{k-3}{2/\sigma^*_{-1}(2\cdot 5\cdot 11)-1} = 54(k-3).$$

By a computer search, we have

$$\sigma^*_{-1}(N_1N_2N_3N_4) < 2 \Longrightarrow \sigma^*_{-1}(N_1N_2N_3N_4) \le \sigma^*_{-1}(2\cdot 5\cdot 11\cdot 59) = \frac{1296}{649},$$

hence

$$N_5 \le \frac{k-4}{2/\sigma^*_{-1}(2\cdot 5\cdot 11\cdot 59)-1} = 648(k-4).$$

Similarly, we have

$$N_6 \le \frac{k-5}{2/\sigma^*_{-1}(2\cdot 5\cdot 11\cdot 61\cdot 479)-1} = 321408(k-5),$$
$$N_7 \le \frac{k-6}{2/\sigma^*_{-1}(2\cdot 5\cdot 11\cdot 61\cdot 479\cdot 321413)-1} = \frac{103305030912}{5}(k-6),$$

as required. □

Note that $\sigma^*_{-1}(2\cdot 5\cdot 11\cdot 59) > \sigma^*_{-1}(2\cdot 5\cdot 11\cdot 61)$; however, $\sigma^*_{-1}(2\cdot 5\cdot 11\cdot 59\cdot 653) < \sigma^*_{-1}(2\cdot 5\cdot 11\cdot 61\cdot 479)$. So the computer search is very heavy. For a rough estimate of $\sigma^*_{-1}(N_1 \cdots N_r)$, we show the following lemma.

Lemma 2.3. *Let $x_1, \dots, x_r$ be positive integers such that $1 \le x_1 \le \cdots \le x_r$. If*

$$\prod_{i=1}^{r} \left(1 + \frac{1}{x_i}\right) < 2,$$

then

$$\prod_{i=1}^{r} \left(1 + \frac{1}{x_i}\right) \le \frac{2^{2^r} - 1}{2^{2^r - 1}}$$

with equality if and only if $x_i = 2^{2^{i-1}}$, $i = 1, \dots, r$.

Using this lemma, we can easily prove Theorem 1.1.

Proof of Theorem 1.1. Suppose that $N = N_1 \cdots N_k$ is a UPN. From Lemma 2.1 it follows that

$$N_i \le \frac{k - i + 1}{2/\sigma^*_{-1}(N_1 \cdots N_{i-1}) - 1}. \tag{1}$$

From Lemma 2.3,

$$\sigma^*_{-1}(N_1 \cdots N_{i-1}) \le \frac{2^{2^{i-1}} - 1}{2^{2^{i-1} - 1}}. \tag{2}$$

So we have the required inequality. Suppose that the equality holds. Then both equalities in (1) and (2) hold; hence, it is necessary that $k - i + 1 \le 3$ and $i - 1 \le 1$. Therefore, $k \le 4$. Subbarao and Warren [**15**] showed that all such UPNs are 6, 60 and 90, so we have the cases which are mentioned in the statement of the lemma. □

In the rest of this section, we prove Lemma 2.3. Cook [**4**] essentially showed Lemma 2.4. By a similar argument, we can also prove Lemma 2.5.

Lemma 2.4 [**4**]. *Let $m_1, \dots, m_r$ be positive integers such that $2 \le m_1 < \cdots < m_r$. If real numbers $x_1, \dots, x_r$ satisfy $2 \le x_1 \le \cdots \le x_r$*

and

$$\prod_{i=1}^{u} x_i \geq \prod_{i=1}^{u} m_i, \quad u = 1, \dots, r, \tag{3}$$

then it follows that

$$\prod_{i=1}^{r} \left(1 - \frac{1}{x_i}\right) \geq \prod_{i=1}^{r} \left(1 - \frac{1}{m_i}\right)$$

with equality if and only if $x_i = m_i$, $i = 1, \dots, r$.

Lemma 2.5. *Let* $m_1, \dots, m_r$ *be positive integers such that* $1 \leq m_1 < \cdots < m_r$. *If real numbers* $x_1, \dots, x_r$ *satisfy* $1 \leq x_1 \leq \cdots \leq x_r$ *and* (3), *then it follows that*

$$\prod_{i=1}^{r} \left(1 + \frac{1}{x_i}\right) \leq \prod_{i=1}^{r} \left(1 + \frac{1}{m_i}\right)$$

with equality if and only if $x_i = m_i$, $i = 1, \dots, r$.

Proof of Lemma 2.3. For a contradiction, assume that

$$\frac{2^{2^r} - 1}{2^{2^r - 1}} < \prod_{i=1}^{r} \left(1 + \frac{1}{x_i}\right) < 2. \tag{4}$$

Then it is easily verified that

$$\prod_{i=1}^{r} x_i > 2^{2^r - 1}. \tag{5}$$

Put $m_i = 2^{2^{i-1}}$. From Lemma 2.5, if

$$\prod_{i=1}^{u} x_i \geq \prod_{i=1}^{u} m_i, \quad u = 1, \dots, r,$$

then

$$\prod_{i=1}^{r} \left(1 + \frac{1}{x_i}\right) \leq \prod_{i=1}^{r} \left(1 + \frac{1}{m_i}\right) = \frac{2^{2^r} - 1}{2^{2^r - 1}},$$

a contradiction. Therefore, we may assume that there exists an integer s such that $1 \le s \le r$ and

$$\prod_{i=1}^{s} x_i < \prod_{i=1}^{s} m_i. \tag{6}$$

Suppose that s is the maximal one. From (5) we have $s < r$. We easily show that

$$\prod_{i=1}^{s}\left(1+\frac{1}{x_i}\right) < \prod_{i=1}^{s}\left(1+\frac{1}{m_i}\right).$$

From (4) it follows that

$$\prod_{i=s+1}^{r}\left(1+\frac{1}{x_i}\right) > \prod_{i=s+1}^{r}\left(1+\frac{1}{m_i}\right).$$

Using again Lemma 2.5, we see that there exists an integer t such that $s < t \le r$ and $\prod_{i=s+1}^{t} x_i < \prod_{i=s+1}^{t} m_i$. From (6) we have $\prod_{i=1}^{t} x_i < \prod_{i=1}^{t} m_i$, which is contradictory to maximality of s. □

3. Upper bound for UPNs. In this section, we prove Theorem 1.2, so N would denote a UPN, that is, $\sigma_{-1}^{*}(N) = 2$. Let $N = N_1 \cdots N_k$ with $N_i < N_j$, $N_i, N_j = 1$ for $i < j$. Using Lemma 2.4, Cook [**4**] showed the following proposition to give an upper bound of OPNs (for a detailed proof, see also [**10**]).

Proposition 3.1 [**4**]. *Let r, a, b be positive integers. If integers $x_1, \ldots, x_r$ satisfy $2 \le x_1 < \cdots < x_r$ and*

$$\prod_{i=1}^{r}\left(1-\frac{1}{x_i}\right) \le \frac{a}{b} < \prod_{i=1}^{r-1}\left(1-\frac{1}{x_i}\right) \le 1,$$

then it follows that

$$a\prod_{i=1}^{r} x_i \le (a+1)^{2^r} - (a+1)^{2^{r-1}}$$

with equality if and only if $x_i = n_i$, $1 \le i \le r$, *where*

$$n_i = \begin{cases} (a+1)^{2^{i-1}} + 1 & i = 1, \dots, r-1, \\ (a+1)^{2^{i-1}} & i = r. \end{cases}$$

Using this proposition, we immediately have the weak upper bound 2^{2^k} in the statement of Theorem 1.2. In fact, since

$$\prod_{i=1}^{k} \left(1 - \frac{1}{N_i + 1}\right) = \frac{1}{2},$$

it follows that

$$N = \prod_{i=1}^{k} N_i < \prod_{i=1}^{k} (N_i + 1) \le 2^{2^k} - 2^{2^{k-1}} < 2^{2^k}.$$

In order to take the strong upper bound $2^{2^{k-1}-1}(2^{2^{k-1}} - 1)$, we need the following proposition.

Proposition 3.2. *Let* r, a, b *be positive integers. If integers* $x_1, \dots, x_r$ *satisfy* $1 \le x_1 < \cdots < x_r$ *and*

$$1 \le \prod_{i=1}^{r-1} \left(1 + \frac{1}{x_i}\right) < \frac{a}{b} \le \prod_{i=1}^{r} \left(1 + \frac{1}{x_i}\right), \tag{7}$$

then it follows that

$$\prod_{i=1}^{r} x_i \le (b+1)^{2^r - 1} - (b+1)^{2^{r-1} - 1}$$

with equality if and only if $x_i = m_i$, $1 \le i \le r$, *where*

$$m_i = \begin{cases} (b+1)^{2^{i-1}} & i = 1, \dots, r-1, \\ (b+1)^{2^{i-1}} - 1 & i = r. \end{cases}$$

Using Proposition 3.2, we immediately have the following proposition, which is an extension of Theorem 1.2.

Proposition 3.3. *Let a, b, k be positive integers. Suppose that N is a positive integer with k distinct prime factors such that $\sigma^*_{-1}(N) = a/b$. Then it follows that $N \le (b+1)^{2^{k-1}-1}((b+1)^{2^{k-1}} - 1)$. Furthermore, if the equality holds, then $k \le 2$.*

Proof. Let $N = N_1 \cdots N_k$ as usual. Since

$$\prod_{i=1}^{k} \left(1 + \frac{1}{N_i}\right) = \frac{a}{b},$$

it follows that

$$N = \prod_{i=1}^{k} N_i \le (b+1)^{2^k-1} - (b+1)^{2^{k-1}-1} = (b+1)^{2^{k-1}-1}((b+1)^{2^{k-1}} - 1)$$

by use of Proposition3.2. Suppose that the equality holds. Then $N_i = m_i$ (m_is are integers given in the statement of Proposition 3.2). If $k \ge 3$, then $(N_1, N_2) > 1$, a contradiction. Hence, we have $k \le 2$. □

In the rest of this section, we prove Proposition 3.2, using Lemmas 2.5 and 3.4.

Lemma 3.4. *Let $m_1, \ldots, m_r$ be positive integers such that $1 \le m_1 < \cdots < m_r$. If real numbers $x_1, \ldots, x_r$ satisfy*

$$1 \le x_1 \le \cdots \le x_r \tag{8}$$

and

$$\prod_{i=1}^{u} x_i \ge \prod_{i=1}^{u} m_i, \quad u = 1, \ldots, r-1, \qquad \prod_{i=1}^{r} x_i \le \prod_{i=1}^{r} m_i, \tag{9}$$

then

$$\prod_{i=1}^{r}(x_i+1) \le \prod_{i=1}^{r}(m_i+1)$$

with equality if and only if $x_i = m_i$, $i = 1, \dots, r$.

Proof. The proof is similar to one of Lemma 2.4 (see [**4**] for details). The set of points $(x_1, \dots, x_r) \in \mathbf{R}^r$ satisfying (8) and (9) is a compact region. Hence, there exists a point $(x_1, \dots, x_r)$ such that the value $\prod_{i=1}^{r}(x_i+1)$ would be maximal in this region. The goal is to show that $(x_1, \dots, x_r) = (m_1, \dots, m_r)$. If not so, then we can make the value $\prod_{i=1}^{r}(x_i+1)$ increase.

We now show this fact. Suppose that $x_i = m_i$, $i = 1, \dots, t-1$, $x_t > m_t$, $x_t = \cdots = x_s < x_{s+1}$. Take a real number $K > 1$, and change x_t to $K^{-1}x_t$, x_s to Kx_s. Then the value $(x_t+1)(x_s+1)$ increases. If K is small enough, then (8) still holds. We can see that (9) also holds by some discussion. □

Proof of Proposition 3.2. The proof is by induction on r. When $r = 1$, the assumption is

$$1 < \frac{a}{b} \le 1 + \frac{1}{x_1}.$$

Since $b < a$ and $a, b \in \mathbf{N}$, we have $b + 1 \le a$ and

$$\frac{b+1}{b} \le 1 + \frac{1}{x_1},$$

hence $x_1 \le b$. Assume now that $r \ge 2$ and that the result holds for each positive integers less than r.

If $x_1 < m_1 = b + 1$, then $b(x_1+1) + 1 < (b+1)^2$. From (7) we have

$$\prod_{i=2}^{r-1}\left(1 + \frac{1}{x_i}\right) < \frac{ax_1}{b(x_1+1)} \le \prod_{i=2}^{r}\left(1 + \frac{1}{x_i}\right);$$

therefore, the induction hypothesis implies

$$\begin{aligned} x_1 \prod_{i=2}^{r} x_i &\le m_1 (b(x_1+1)+1)^{2^{r-2}-1}((b(x_1+1)+1)^{2^{r-2}}-1) \\ &< (b+1)^{2^{r-1}-1}((b+1)^{2^{r-1}}-1), \end{aligned}$$

as required. So we may assume that $x_1 \ge m_1$.

If $x_1 x_2 < m_1 m_2$, then Lemma 3.4 implies

$$b(x_1+1)(x_2+1)+1 < b(m_1+1)(m_2+1)+1 = (b+1)^4.$$

From (7) we have

$$\prod_{i=3}^{r-1}\left(1+\frac{1}{x_i}\right) < \frac{a x_1 x_2}{b(x_1+1)(x_2+1)} \le \prod_{i=2}^{r}\left(1+\frac{1}{x_i}\right).$$

By the induction hypothesis, it follows that

$$\begin{aligned} x_1 x_2 \prod_{i=3}^{r} x_i &\le m_1 m_2 (b(x_1+1)(x_2+1)+1)^{2^{r-3}-1} \\ &\quad \times ((b(x_1+1)(x_2+1)+1)^{2^{r-3}}-1) \\ &< (b+1)^{2^{r-1}-1}((b+1)^{2^{r-1}}-1), \end{aligned}$$

as required. So we may assume that $x_1 x_2 \ge m_1 m_2$. Since

$$b\prod_{i=1}^{u}(m_i+1)+1 = ((b+1)-1)((b+1)+1)\cdots((b+1)^{2^{u-1}}+1)+1 = (b+1)^{2^u},$$

we can repeat this discussion. So we can assume that

$$\prod_{i=1}^{u} x_i \ge \prod_{i=1}^{u} m_i, \quad u = 1, \dots, r-1.$$

If this inequality is false when $u = r$, then the required inequality automatically holds. Hence we can apply Lemma 2.5 to our case, and we have

$$\prod_{i=1}^{r}\left(1+\frac{1}{x_i}\right) \le \prod_{i=1}^{r}\left(1+\frac{1}{m_i}\right) = \frac{b+1}{b}.$$

On the other hand, the assumption means

$$\prod_{i=1}^{r}\left(1+\frac{1}{x_i}\right) \geq \frac{a}{b} \geq \frac{b+1}{b};$$

therefore, we have equality, and $x_i = m_i$, $i = 1, \dots, r$. □

4. Upper bound for UHNs. In this section, we prove Theorem1.3. First, recall that the results which Hagis and Load [**8**] produced (they did not mention Lemma 4.2 (b); however, they essentially showed it in the proof of (a)).

Lemma 4.1 [**8**]. *Let N be a* UHN *with k distinct prime factors, and let $H^*(N) = c$. Then it follows that*

$$k \leq \frac{2^{k+1}}{k+2} \leq c \leq 2^k.$$

Furthermore, we have first equality only when $k = 2$, second equality only when $N = 2$ or 6, third equality only when $N = 1$.

Lemma 4.2 [**8**]. *Let N be a* UHN *with k distinct prime factors, and let $H^*(N) = c$. Then the following facts hold.*

(a) *If $k \leq 3$, then $N \in \{1, 6, 45, 60, 90, 1512, 15925, 55125\}$.*

(b) *If $c \leq 5$, then $N \in \{1, 6, 45, 60, 90\}$.*

Proof of Theorem 1.3. Since

$$\sigma^*_{-1}(N) = \frac{\sigma^*_0(N)}{H^*(N)} = \frac{2^k}{c},$$

Proposition 3.3 implies that

$$N \leq (c+1)^{2^{k-1}-1}((c+1)^{2^{k-1}} - 1) < (c+1)^{2^k}. \tag{10}$$

From Lemma 4.2, if $k \leq 3$ or $c \leq 5$, then the required inequalities hold. So we may assume that $k \geq 4$ and $c \geq 6$. From Lemma 4.1 we have

$c+1 \le 2^k$. Hence, (10) implies (b). We turn now to (a). From $k \ge 4$, we easily see $k+2 < 2^{(2k+3)/4}$. Hence, Lemma 4.1 implies that

$$c \ge \frac{2^{k+1}}{k+2} > 2^{(2k+1)/4}.$$

Therefore, $2^k < c^2/\sqrt{2}$. From $c \ge 6$ we easily see $c+1 < c^{\sqrt{2}}$. Hence, it follows that

$$n < (c+1)^{2^k} < (c+1)^{c^2/\sqrt{2}} < c^{c^2},$$

as required. □

Acknowledgments. I would like to thank Professor Graeme Cohen for introducing me to the topic of unitary harmonic numbers.

APPENDIX

5. Table of UHNs. Hagis and Load [**8**] listed all UHNs less than 10^6 and essentially all UHNs N with $H^*(N) \le 5$. The table below is the list of all UHNs N with $H^*(N) \le 50$. Shibata and the author [**6**] gave an algorithm to get original harmonic numbers. In order to give the table below, we use a similar algorithm. It takes about 15 hours to compute with a computer of Pentium IV, 3GHz and Mathematica program. By another computer search, it becomes clear that the table contains all UHNs less than 10^7. From Lemma 4.1 the table contains all UHNs N with $\omega(N) \le 5$ ($\omega(N)$ denotes the number of distinct prime factors of N). Hence, the table is an extension of Lemma 4.2. In the table, there are 124 UHNs, 9 odd UHNs, and 41 "seeds" (seeds are marked with asterisk). A UHN is said to be a seed if it does not have a smaller proper unitary divisor which is a UHN (see [**3**] for original harmonic seeds).

TABLE 1. All unitary harmonic numbers N with $H^*(N) \leq 50$.

$H^*(N)$	N	Factorization of N
1	*1	
2	*6	$2{\cdot}3$
3	*45	$3^2 5$
4	*60	$2^2 3{\cdot}5$
	90	$2{\cdot}3^2 5$
6	*1512	$2^3 3^3 7$
7	420	$2^2 3{\cdot}5{\cdot}7$
	630	$2{\cdot}3^2 5{\cdot}7$
	*15925	$5^2 7^2 13$
	*55125	$3^2 5^3 7^2$
9	*3780	$2^2 3^3 5{\cdot}7$
	*46494	$2{\cdot}3^4 7{\cdot}41$
10	7560	$2^3 3^3 5{\cdot}7$
	*9100	$2^2 5^2 7{\cdot}13$
	*31500	$2^2 3^2 5^3 7$
	*330750	$2{\cdot}3^3 5^3 7^2$
11	16632	$2^3 3^3 7{\cdot}11$
12	*51408	$2^4 3^3 7{\cdot}17$
	*66528	$2^5 3^3 7{\cdot}17$
	*185976	$2^3 3^4 7{\cdot}41$
	*661500	$2^2 3^3 5^3 7^2$
13	5460	$2^2 3{\cdot}5{\cdot}7{\cdot}13$
	8190	$2{\cdot}3^2 5{\cdot}7{\cdot}13$
	*646425	$3^2 5^2 13^2 17$
	716625	$3^2 5^3 7^2 13$
14	95550	$2{\cdot}3{\cdot}5^2 7^2 13$

TABLE 1. (Continued).

$H^*(N)$	N	Factorization of N
15	27300	$2^2 3 \cdot 5^2 7 \cdot 13$
	*40950	$2 \cdot 3^2 5^2 7 \cdot 13$
	232470	$2 \cdot 3^4 5 \cdot 7 \cdot 41$
	*20341125	$3^4 5^3 7^2 41$
16	*87360	$2^6 3 \cdot 5 \cdot 7 \cdot 13$
17	64260	$2^2 3^3 5 \cdot 7 \cdot 17$
	790398	$2 \cdot 3^4 7 \cdot 17 \cdot 41$
18	81900	$2^2 3^2 5^2 7 \cdot 13$
	*464940	$2^2 3^4 5 \cdot 7 \cdot 41$
	859950	$2 \cdot 3^3 5^2 7^2 13$
19	143640	$2^3 3^3 5 \cdot 7 \cdot 19$
	172900	$2^2 5^2 7 \cdot 13 \cdot 19$
	598500	$2^2 3^2 5^3 7 \cdot 19$
	6284250	$2 \cdot 3^3 5^3 7^2 19$
20	*163800	$2^3 3^2 5^2 7 \cdot 13$
	257040	$2^4 3^3 5 \cdot 7 \cdot 17$
	332640	$2^5 3^3 5 \cdot 7 \cdot 11$
	929880	$2^3 3^4 5 \cdot 7 \cdot 41$
	40682250	$2 \cdot 3^4 5^3 7^2 41$
22	565488	$2^4 3^3 7 \cdot 11 \cdot 17$
	2045736	$2^3 3^4 7 \cdot 11 \cdot 41$
	7276500	$2^2 3^3 5^3 7^2 11$
	*21965856	$2^5 3 \cdot 11^2 31 \cdot 61$
23	1182384	$2^4 3^3 7 \cdot 17 \cdot 23$
	1530144	$2^5 3^3 7 \cdot 11 \cdot 23$
	4277448	$2^3 3^4 7 \cdot 23 \cdot 41$
	15214500	$2^2 3^3 5^3 7^2 23$

TABLE 1. (Continued).

$H^*(N)$	N	Factorization of N
24	3439800	$2^3 3^3 5^2 7^2 13$
	*6323184	$2^4 3^4 7 \cdot 17 \cdot 41$
	*8182944	$2^5 3^4 7 \cdot 11 \cdot 41$
	*11442816	$2^7 3^3 7 \cdot 11 \cdot 43$
	81364500	$2^2 3^4 5^3 7^2 41$
25	*9705347500	$2^2 5^4 79 \cdot 157 \cdot 313$
27	52886925	$3^4 5^2 7^2 13 \cdot 41$
	*124987536	$2^4 3^5 17 \cdot 31 \cdot 61$
	*161748576	$2^5 3^5 11 \cdot 31 \cdot 61$
	*30156053112	$2^3 3^7 23 \cdot 137 \cdot 547$
29	791700	$2^2 3 \cdot 5^2 7 \cdot 13 \cdot 29$
	1187550	$2 \cdot 3^2 5^2 7 \cdot 13 \cdot 29$
	6741630	$2 \cdot 3^4 5 \cdot 7 \cdot 29 \cdot 41$
	589892625	$3^4 5^3 7^2 29 \cdot 41$
31	2708160	$2^6 3 \cdot 5 \cdot 7 \cdot 13 \cdot 31$
33	900900	$2^2 3^2 5^2 7 \cdot 11 \cdot 13$
	5114340	$2^2 3^4 5 \cdot 7 \cdot 11 \cdot 41$
	9459450	$2 \cdot 3^3 5^2 7^2 11 \cdot 13$
34	1392300	$2^2 3^2 5^2 7 \cdot 13 \cdot 17$
	7903980	$2^2 3^4 5 \cdot 7 \cdot 17 \cdot 41$
	14619150	$2 \cdot 3^3 5^2 7^2 13 \cdot 17$
36	105773850	$2 \cdot 3^4 5^2 7^2 13 \cdot 41$
37	5314680	$2^3 3^3 5 \cdot 7 \cdot 19 \cdot 37$
	6397300	$2^2 5^2 7 \cdot 13 \cdot 19 \cdot 37$
	22144500	$2^2 3^2 5^3 7 \cdot 19 \cdot 37$
	232517250	$2 \cdot 3^3 5^3 7^2 19 \cdot 37$

TABLE 1. (Continued).

$H^*(N)$	N	Factorization of N
38	3112200	$2^3 3^2 5^2 7{\cdot}13{\cdot}19$
	4883760	$2^4 3^3 5{\cdot}7{\cdot}17{\cdot}19$
	6320160	$2^5 3^3 5{\cdot}7{\cdot}11{\cdot}19$
	17667720	$2^3 3^4 5{\cdot}7{\cdot}19{\cdot}41$
	772962750	$2{\cdot}3^4 5^3 7^2 19{\cdot}41$
39	*54299700	$2^2 3^3 5^2 7{\cdot}13^2 17$
40	*5569200	$2^4 3^2 5^2 7{\cdot}13{\cdot}17$
	*7202700	$2^5 3^2 5^2 7{\cdot}11{\cdot}13$
	31615920	$2^4 3^4 5{\cdot}7{\cdot}17{\cdot}41$
	40914720	$2^5 3^4 5{\cdot}7{\cdot}11{\cdot}41$
	57214080	$2^7 3^3 5{\cdot}7{\cdot}11{\cdot}43$
43	24315984	$2^4 3^3 7{\cdot}11{\cdot}17{\cdot}43$
	87966648	$2^3 3^4 7{\cdot}11{\cdot}41{\cdot}43$
	312889500	$2^2 3^3 5^3 7^2 11{\cdot}43$
	944531808	$2^5 3{\cdot}11^2 31{\cdot}43{\cdot}61$
44	37837800	$2^3 3^3 5^2 7^2 11{\cdot}13$
	69555024	$2^4 3^4 7{\cdot}11{\cdot}17{\cdot}41$
	329487840	$2^5 3^2 5{\cdot}11^2 31{\cdot}61$
	895009500	$2^2 3^4 5^3 7^2 11{\cdot}41$
	*3778127232	$2^7 3{\cdot}11^2 31{\cdot}43{\cdot}61$
45	624937680	$2^4 3^5 5{\cdot}17{\cdot}31{\cdot}61$
	808742880	$2^5 3^5 5{\cdot}11{\cdot}31{\cdot}61$
	87348127500	$2^2 3^2 5^4 79{\cdot}157{\cdot}313$
	150780265560	$2^3 3^7 5{\cdot}23{\cdot}137{\cdot}547$
46	79115400	$2^3 3^3 5^2 7^2 13{\cdot}23$
	145433232	$2^4 3^4 7{\cdot}17{\cdot}23{\cdot}41$
	188207712	$2^5 3^4 7{\cdot}11{\cdot}23{\cdot}41$
	263184768	$2^7 3^3 7{\cdot}11{\cdot}23{\cdot}43$
	1871383500	$2^2 3^4 5^3 7^2 23{\cdot}41$

TABLE 1. (Continued).

$H^*(N)$	N	Factorization of N
47	161670600	$2^3 3^3 5^2 7^2 13{\cdot}47$
	297189648	$2^4 3^4 7{\cdot}17{\cdot}41{\cdot}47$
	384598368	$2^5 3^4 7{\cdot}11{\cdot}41{\cdot}47$
	537812352	$2^7 3^3 7{\cdot}11{\cdot}43{\cdot}47$
	3824131500	$2^2 3^4 5^3 7^2 41{\cdot}47$
48	116953200	$2^4 3^3 5^2 7^2 13{\cdot}17$
	151351200	$2^5 3^3 5^2 7^2 11{\cdot}13$
	423095400	$2^3 3^4 5^2 7^2 13{\cdot}41$
	*868795200	$2^6 3^3 5^2 7{\cdot}13^2 17$
	*1407466368	$2^7 3^4 7{\cdot}11{\cdot}41{\cdot}43$
	*5881607424	$2^8 3^3 7{\cdot}11{\cdot}43{\cdot}257$
	*22350712320	$2^9 3^5 5{\cdot}19{\cdot}31{\cdot}61$
49	*1191483216	$2^4 3^3 7^3 11{\cdot}17{\cdot}43$
	*4310365752	$2^3 3^4 7^3 11{\cdot}41{\cdot}43$
	475562027500	$2^2 5^4 7^2 79{\cdot}157{\cdot}313$
50	*174696255000	$2^3 3^2 5^4 79{\cdot}157{\cdot}313$

REFERENCES

1. R.P. Brent, G.L. Cohen and H.J.J. te Riele, *Improved techniques for lower bounds for odd perfect numbers*, Math. Comp. **57** (1991), 857–868.

2. J.E.Z. Chein, *An odd perfect number has at least* 8 *prime factors*, Ph.D. thesis, Pennsylvania State University, 1979.

3. G.L. Cohen and R.M. Sorli, *Harmonic seeds*, Fibonacci Quart. **36** (1998), 386–390; Errata, Fibonacci Quart. **39** (2001), 4.

4. R.J. Cook, *Bounds for odd perfect numbers*, in *Number theory*, CRM Proc. Lecture Notes **19** (1999), 67–71.

5. L.E. Dickson, *Finiteness of odd perfect and primitive abundant numbers with n distinct prime factors*, Amer. J. Math. **35** (1913), 413–422.

6. T. Goto and S. Shibata, *All numbers whose positive divisors have integral harmonic mean up to* 300, Math. Comp. **73** (2004), 475–491.

7. P. Hagis, Jr., *Outline of a proof that every odd perfect number has at least eight prime factors*, Math. Comp. **35** (1980), 1027–1032.

8. P. Hagis, Jr. and G. Lord, *Unitary harmonic numbers*, Proc. Amer. Math. Soc. **51** (1975), 1–7.

9. D.R. Heath-Brown, *Odd perfect numbers*, Math. Proc. Cambridge Philos. Soc. **115** (1994), 191–196.

10. Pace P. Nielsen, *An upper bound for odd perfect numbers*, Integers **3** (2003), A14, 9 pp. (electronic).

11. O. Ore, *On the averages of the divisors of a number*, Amer. Math. Monthly **55** (1948), 615–619.

12. C. Pomerance, *On a problem of Ore*: *Harmonic numbers*, Notices Amer. Math. Soc. **20** (1973), A–648.

13. ———, *Multiply perfect numbers, Mersenne primes, and effective computability*, Math. Ann. **226** (1977), 195–206.

14. M.V. Subbarao, *Are there an infinity of unitary perfect numbers?* Amer. Math. Monthly **77** (1970), 389–390.

15. M.V. Subbarao and L.J. Warren, *Unitary perfect numbers*, Canad. Math. Bull. **9** (1966), 147–153.

16. C.R. Wall, *The fifth unitary perfect number*, Canad. Math. Bull. **18** (1975), 115–122.

17. ———, *New unitary perfect numbers have at least nine odd components*, Fibonacci Quart. **26** (1988), 312–317.

Department of Mathematics, Faculty of Science and Technology, Tokyo University of Science, Noda, Chiba, 278-8510, Japan
Email address: goto_takeshi@ma.noda.tus.ac.jp