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H-CONTACT UNIT TANGENT SPHERE BUNDLES

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ABSTRACT. We study how the geometry of a Riemannian manifold (M, g) is influenced by the property that its unit tangent sphere bundle $(T_1M, \eta, \bar{g})$ is H-*contact*, that is, the characteristic vector field ξ of T_1M is harmonic.

1. Introduction. The study of the geometric properties of a Riemannian manifold (M, g) via the investigation of its unit tangent sphere bundle T_1M, is a well known and interesting research field in Riemannian geometry. T_1M can be equipped with its "natural" metric g_S (the one induced by the Sasaki metric of the tangent bundle), as well as with the contact metric $\bar{g}$ of its standard contact metric structure $(\eta, \bar{g})$. In both cases, geometrical properties of T_1M influence those of the base manifold M itself, and conversely.

For example, all the information about the geodesics of (M, g) is encoded in the geodesic flow on T_1M, which is precisely the characteristic vector field ξ of its standard contact metric structure $(\eta, \bar{g})$. Riemannian manifolds whose unit tangent sphere bundle is either K-contact or (strongly) φ-symmetric or a (k, μ)-space, were completely classified, see [**3, 8, 23**]. We can refer to [**12**] for a survey about the contact metric geometry of T_1M.

Recently, many authors have studied the harmonicity of unit vector fields in several geometric situations, see, for example, [**14**] for a survey. If (M, g) is a compact and orientable Riemannian manifold, a unit vector field V of M is called *harmonic* if it is a critical point for the energy functional restricted to the set of all unit vector fields of M, [**24, 25**].

An interesting geometrical situation, in which a distinguished vector field appears in a natural way, is given by a contact manifold (M, η) where we have the characteristic vector field ξ. On the other hand, ξ

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plays a fundamental role in the geometry of a contact metric manifold (M, η, g) [**2**]. Moreover, the first examples of harmonic vector fields, Hopfs vector fields, are in fact the characteristic vector fields of the standard Sasakian structure on odd-dimensional spheres. So, it is natural to study the harmonicity of the characteristic vector field of a contact metric manifold, see [**20**] for a survey. A contact metric manifold (M, η, g) whose characteristic vector field ξ is a harmonic vector field is called an *H-contact manifold* [**19**]. In the same paper, the second author proved that (M, η, g) is H-contact if and only if ξ is an eigenvector for the Ricci operator. Such a characterization also makes clear that the class of H-contact metric manifolds extends several interesting classes of contact metric manifolds, like Sasakian, K-contact, (strongly) locally φ-symmetric and (k, μ)-spaces.

Boeckx and Vanhecke [**10**] proved that the geodesic flow of a two-point homogeneous space is harmonic, that is, *the unit tangent sphere bundle over a two-point homogeneous space is H-contact*. In the same paper, the converse was proved to hold when the base manifold M is either two- or three-dimensional. Up to our knowledge, the general problem of characterizing H-contact unit tangent sphere bundles is still open. More explicitly, in this paper we shall deal with the following question, which was first asked in [**10**]:

Question 1.1. *Are the two-point homogeneous spaces the only Riemannian manifolds whose unit tangent sphere bundles are H-contact, that is, have a harmonic geodesic flow?*

The paper is organized in the following way. Section 2 and Section 3 will be devoted to recall some basic facts and results about unit tangent sphere bundles and H-contact spaces, respectively.

In Section 4, we assume that the base manifold (M, g) is locally reducible, and we prove that T_1M is H-contact if and only if (M, g) is locally flat.

In Section 5, as a consequence of a more general result, we prove that the unit tangent sphere bundle T_1M of a *conformally flat* Riemannian manifold (M, g) is H-contact if and only if (M, g) has constant sectional curvature.

In Section 6, we assume that M itself carries a contact metric structure, and we characterize the property that T_1M is H-contact in some interesting classes of contact metric manifolds. In Section 7, the base manifold is supposed to carry a Kähler structure, and we get that a four-dimensional Kähler manifold (M, g, J) which is not Ricci-flat, has constant holomorphic sectional curvature if and only if T_1M is H-contact. Moreover, we consider Bochner-Kähler manifolds with H-contact unit tangent sphere bundle.

2. The unit tangent sphere bundle and its natural contact metric structure. A *contact manifold* is a $(2n+1)$-dimensional manifold M equipped with a global 1-form η such that $\eta \wedge (d\eta)^n \neq 0$ everywhere on M. It has an underlying almost contact structure (η, φ, ξ) where ξ is a global vector field, called the *characteristic vector field*, and φ a global tensor of type (1.1) such that

$$\eta(\xi) = 1, \quad \varphi\xi = 0, \quad \eta\varphi = 0, \quad \varphi^2 = -I + \eta \otimes \xi.$$

A Riemannian metric g can be found such that

$$\eta = g(\xi, \cdot), \quad d\eta = g(\cdot, \varphi\cdot), \quad g(\cdot, \varphi\cdot) = -g(\varphi\cdot, \cdot).$$

We refer to (M, η, g) or to $(M, \eta, g, \xi, \varphi)$ as a contact metric (or Riemannian) manifold. If L denotes the Lie differentiation, we denote by h and l the operators defined by

$$h = \frac{1}{2} L_\xi \varphi, \quad lX = R(X, \xi)\xi.$$

The tensor h is symmetric and satisfies

$$\nabla\xi = -\varphi - \varphi h, \quad \nabla_\xi \varphi = 0, \quad h\varphi = -\varphi h, \quad h\xi = 0. \tag{2.1}$$

A K-*contact* manifold is a contact metric manifold (M, η, g) such that ξ is a Killing vector field with respect to g. Clearly, M is K-contact if and only if $h = 0$. Moreover, M is K-contact if and only if

$$Q\xi = 2n\xi,$$

where Q is the Ricci operator of (M, g).

A contact metric manifold (M,η,g) is a *Sasakian manifold* if its curvature tensor satisfies

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y,$$

for all vector fields X and Y. Any Sasakian manifold is K-contact, and the converse also holds for three-dimensional spaces. It is easy to prove that, if M is a contact metric three-manifold of constant sectional curvature 1, then M is necessarily Sasakian. We refer to [**2**] for more information about contact metric manifolds.

A contact metric manifold (M,η,g) is said to be an *H-contact manifold* if ξ is a harmonic vector field. The following characterization was proved in [**19**].

Theorem 2.1 [**19**]. *A contact metric manifold (M,η,g) is H-contact if and only if ξ is an eigenvector of Q, and hence*

$$Q\xi = (2n - \operatorname{tr} h^2)\xi.$$

It should be noted that the class of H-contact metric manifolds is very large. In particular, K-contact spaces (and hence, Sasakian manifolds), (k,μ)-spaces, (strongly) locally φ-symmetric spaces are all examples of H-contact manifolds. We refer to [**18, 19**] and the survey [**20**] for more details on H-contact spaces.

Next, let $\bar{\pi} : TM \to M$ be the tangent bundle of an n-dimensional Riemannian manifold (M,g). The tangent space to TM at a point (x,u) splits into the direct sum of the vertical subspace $VTM_{(x,u)} = \ker \pi_{*|(x,u)}$ and the horizontal subspace $HTM_{(x,u)}$ with respect to the Levi Civita connection ∇ of M. If X is a vector field on M, X^h and X^v will denote respectively the horizontal and the vertical lift of X on TM. The map $X \mapsto X^h$, respectively $X \mapsto X^v$, is an isomorphism between T_xM and $HTM_{(x,u)}$, respectively, T_xM and $VTM_{(x,u)}$. The Sasaki metric g_S on TM is defined by

$$g_S(A,B) = g(\bar{\pi}_* A, \bar{\pi}_* B) + g(KA, KB),$$

where A, B are the vector field on TM and K is the connection map corresponding to the Levi Civita connection of M. TM admits an almost complex structure J defined by $JX^h = X^v$ and $JX^v = -X^h$.

The *unit tangent sphere bundle* $\pi : T_1M \to M$ is the hypersurface of TM defined by $T_1M = \{(x,u) \in TM : g_x(u,u) = 1\}$. We shall denote again by g_S the metric induced on T_1M by the Sasaki metric of TM.

The *geodesic flow* of (M,g) is the horizontal vector field of TM defined by

$$\xi'_u = -JN = u^i \left(\frac{\partial}{\partial x^i} \right)^h ,$$

where $(x,u) \in TM$, N is the unit vector normal to T_1M and $u = u^i(\partial/\partial x^i)$ in local coordinates. If $(x,z) \in T_1M$, then ${\xi'}_z$ is tangent to T_1M. Hence, ξ' can be considered as a vector field on T_1M. Let η' be the 1-form on T_1M dual to ξ' with respect to g_S, and φ' the (1,1) tensor given by $\varphi' X = JX - \eta'(X)N$. Then

$$(\xi, \eta, \varphi, \bar{g}) = \left(\frac{1}{2}\eta', 2\xi', \varphi', \frac{1}{4} g_S \right)$$

is the standard contact metric structure on T_1M.

We now describe the Ricci tensor of $(T_1M, \eta, \bar{g})$. In general, the vertical lift of a vector (field) is not tangent to T_1M. For this reason, the *tangential lift* of $X \in T_xM$ is defined by

$$X^t_{(x,u)} = (X - g(X,u)u)^v = \overline{X}^v ,$$

where we put $\overline{X} = X - g(X,u)$. Since $g_S = 4\bar{g}$, the Riemannian connection, the curvature tensor of type (1,3) and the Ricci tensor of (T_1M, g_S) coincide with the corresponding ones of $(T_1M, \bar{g})$. Consider $x \in M$, and let $\{e_1, \dots, e_n = u\}$ be an orthonormal basis of T_xM. Then, $\{2e^t_1, \dots, 2e^t_{n-1}, 2e^h_1, \dots, 2e^h_{n-1}, \xi = 2u^h\}$ is an orthonormal basis of T_zT_1M, where $z = (x,u)$. Computing explicitly the Ricci tensor $\bar{\varrho}$ of $(T_1M, \eta, \bar{g})$, we have (see for example [**9**])

$$\begin{aligned} \bar{\varrho}_z(X^t, Y^t) &= (n-2)(g_x(X,Y) - g_x(X,u)g_x(Y,u)) \\ &\quad + \frac{1}{4} \sum_{i=1}^{n} g_x(R(u,X)e_i, R(u,Y)e_i), \end{aligned} \tag{2.2}$$

$$\bar{\varrho}_z(X^t, Y^h) = \frac{1}{2}((\nabla_u \varrho)_x(X,Y) - (\nabla_X \varrho)_x(u,Y)), \tag{2.3}$$

$$(2.4) \qquad \bar{\varrho}_z(X^h, Y^h) = \varrho_x(X, Y) - \frac{1}{2}\sum_{i=1}^{n} g_x(R(u, e_i)X, R(u, e_i)Y),$$

where ∇, $R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$ and ϱ are respectively the Levi-Civita connection, the curvature tensor and the Ricci tensor of M. By $\overline{Q}$ we shall denote the Ricci operator of T_1M.

3. Harmonicity of the geodesic flow on T_1M. In this section, we shall recall some basic facts about harmonic vector fields and H-contact metric manifolds.

A unit vector field V on a Riemannian manifold (M, g) determines a map between (M, g) itself and its unit tangent sphere bundle (T_1M, g_S). If M^n is compact and orientable, the *energy* of V is defined as the energy of the corresponding map:

$$E(V) = \frac{1}{2}\int_M \|dV\|^2 \, dv = \frac{n}{2}\mathrm{vol}\,(M, g) + \frac{1}{2}\int_M \|\nabla V\|^2 \, dv.$$

V is called *harmonic* if it is a critical point for E in the set of all unit vector fields of M.

The corresponding critical point condition

$$(3.1) \qquad \text{“}\bar{\Delta}V \text{ is collinear to } V,\text{”}$$

where $\bar{\Delta}V = -\mathrm{tr}\,\nabla^2 V$ is the *rough Laplacian* of (M, g), has been determined in [**24, 25**].

Note that (3.1) also makes sense when M is non-orientable or non-compact. For this reason, (3.1) has been taken as definition of a harmonic vector field on an arbitrary Riemannian manifold [**13**]. For further details and references about harmonic vector fields, we can refer to [**14**].

Consider now the unit tangent sphere bundle $(T_1M, \eta, \bar{g})$ of an n-dimensional Riemannian manifold (M, g). Let $z = (x, u)$ be a point of T_1M and $\{e_1, \dots, e_n = u\}$ an orthonormal basis of T_xM. Then, the corresponding orthonormal basis of T_zT_1M is given by $\{2e_1^t, \dots, 2e_{n-1}^t, 2e_1^h, \dots, 2e_{n-1}^h, \xi = 2u^h\}$. From Theorem 2.1 it follows that $(T_1M, \eta, \bar{g})$ is H-contact, that is, its characteristic vector field $\xi = 2u^h$ is an eigenvector for the Ricci operator, if and only if

$$\bar{\varrho}_z(\xi, X^t) = 0 \quad \text{for all} \quad X \in T_xM,$$
$$\bar{\varrho}_z(\xi, X^h) = 0 \quad \text{for all} \quad X \perp \xi.$$

Then, from (2.3) and (2.4) we respectively get

(3.2)
$$(\nabla_u \varrho)_x(u, X) = (\nabla_X \varrho)_x(u, u) \quad \text{for all} \quad X \in T_xM,$$
(3.3)
$$2\varrho_x(X, u) = \sum_{i=1}^{n} g_x(R(u, e_i)X, R(u, e_i)u) \quad \text{for all} \quad X \perp u.$$

Since u is an arbitrary unit vector tangent to M at x, it is easy to show that (3.2) is equivalent to requiring that the Ricci tensor is a *Codazzi tensor*, that is,

(3.4)
$$(\nabla_X \varrho)(Y, Z) = (\nabla_Y \varrho)(X, Z),$$

for all tangent vector fields X, Y, Z. So, we get the following characterization of H-contact unit tangent sphere bundles:

Proposition 3.1. *The unit tangent sphere bundle* $(T_1M, \eta, \bar{g})$ *of a Riemannian manifold* (M, g) *is* H*-contact if and only if*

(a) *the Ricci tensor* ϱ *of* (M, g) *is a Codazzi tensor, and*

(b) (3.3) *holds, for all* $x \in M$ *and* $\{e_1, \dots, e_n = u\}$ *orthonormal basis of* T_xM.

According to Proposition 3.1, the harmonicity of the characteristic vector field of T_1M is reflected by curvature conditions (3.3) and (3.4) on the base manifold (M, g). Condition (3.4) is well known and has been investigated by several authors. It is interesting to note that the curvature of (M, g) is said to be *harmonic* when (3.4) holds. On the other hand, (3.3) is a quite new curvature condition, to which a clear geometric meaning is not associated yet. As we showed, (3.3) expresses the fact that $\overline{Q}\xi$ *is orthogonal to the horizontal vectors of the contact distribution* $\operatorname{Ker}\eta$. Some results of this paper, more precisely the results of Sections 4 and 5, help to understand which kind of Riemannian manifolds can satisfy (3.3). In particular, we get that the condition (3.3) characterizes the harmonicity of the geodesic flow of T_1M when (M,g) is a locally reducible Riemannian manifold or its Weyl conformal tensor vanishes.

4. *H*-contact unit tangent sphere bundle over a reducible manifold. We start by proving the main result of this section:

Theorem 4.1. *A Riemannian manifold (M,g) is (locally) flat if and only if it is locally reducible and $\overline{Q}\xi$ is orthogonal to the horizontal vectors of* $\operatorname{Ker}\eta$.

Proof. The "if" part is obvious, since a locally flat manifold trivially satisfies condition (b) of Proposition 3.1.

In order to prove the converse, assume that M is locally isometric to a Riemannian product $M'\times M''$. Let $x=(x',x'')$ be a point of M, $v'\in T_{x'}M'$ and $v''\in T_{x''}M''$ arbitrary unit vectors. Consider $\{e'_1,\dots,e'_r=v'\}$ and $\{e''_1,\dots,e''_s=v''\}$ orthonormal bases of $T_{x'}M'$ and $T_{x''}M''$, respectively, where $r=\dim M'$, $s=\dim M''$ and $n=r+s=\dim M$. Then, $\{e'_1,\dots,e'_r,e''_1,\dots,e''_s\}$ is an orthonormal basis of T_xM. We now build new orthonormal bases of T_xM, considering, for all $a\in\mathbf{R}$, the vectors

$$\begin{aligned}&\{E_1,\dots,E_{n-1},U=E_n\}\\ &=\Big\{e'_1,\dots,e'_{r-1},e''_1,\dots,e''_{s-1},\frac{1}{\sqrt{1+a^2}}(-ae'_r+e''_s),\frac{1}{\sqrt{1+a^2}}(e'_r+ae''_s)\Big\}.\end{aligned}$$

Applying (3.3) to $\varrho_x(E_{n-1},U)$, we get

$$2\varrho_x(E_{n-1},U)=\sum_{i=1}^{n-1}g_x(R(U,E_i)E_{n-1},R(U,E_i)U). \tag{4.1}$$

Since M is locally isometric to $M'\times M''$, it is easy to check that

$$\varrho_x(E_{n-1},U)=\frac{a}{1+a^2}(-\varrho'_{x'}(e'_r,e'_r)+\varrho''_{x''}(e''_s,e''_s)), \tag{4.2}$$

where ϱ' and ϱ'' denote the Ricci tensor of M' and M'', respectively, and we used the previous definition of E_{n-1} and U. Using again the definition of $\{E_1, \dots, E_{n-1}, U = E_n\}$ in order to compute the second term of the formula (4.1), some routine calculations give

(4.3)
$$\sum_{i=1}^{n-1} g_x(R(U,E_i)E_{n-1}, R(U,E_i)U) = -\frac{a}{(1+a^2)^2}\sum_{i=1}^{r-1}\|R'(e'_r,e'_i)e'_r\|_1^2 + \frac{a^3}{(1+a^2)^2}\sum_{i=1}^{s-1}\|R''(e''_s,e''_i)e''_s\|_2^2,$$

where R' and R'' are respectively the curvature tensor of M' and M'', and the lengths of the vectors are calculated with respect to the Riemannian metrics g' of M' and g'' of M'', respectively.

Using (4.2) and (4.3) in (4.1), we obtain

$$-R_1 + a^2R_2 + 2(1+a^2)(\varrho'_{x'}(e'_r,e'_r) - \varrho''_{x''}(e''_s,e''_s)) = 0, \tag{4.4}$$

where we put $R_1 = \sum_i \|R'(e'_r,e'_i)e'_r\|_1^2$ and $R_2 = \sum_i \|R''(e''_s,e''_i)e''_s\|_2^2$. Note that $R_1 \geq 0$ and $R_2 \geq 0$.

Since $\{E_1, \dots, E_{n-1}, U = E_n\}$ is an orthonormal basis of T_xM for all $a \in \mathbf{R}$, and, by hypothesis, (3.4) holds with respect to an arbitrary orthonormal basis of T_xM, (4.4) must hold for all $a \in \mathbf{R}$. In particular, from (4.4) it follows that

(4.5)
$$(2\varrho'_{x'}(e'_r,e'_r) - 2\varrho''_{x''}(e''_s,e''_s) + R_2)a^2 + (2\varrho'_{x'}(e'_r,e'_r) - 2\varrho''_{x''}(e''_s,e''_s) - R_1) = 0$$

for all $a \neq 0$. So, the coefficients of the polynomial in (4.5) must vanish, that is,

$$2\varrho'_{x'}(e'_r,e'_r) - 2\varrho''_{x''}(e''_s,e''_s) + R_2 = 2\varrho'_{x'}(e'_r,e'_r) - 2\varrho''_{x''}(e''_s,e''_s) - R_1 = 0,$$

from which it follows at once that $R_1 = -R_2$. But $R_1 \geq 0$ and $R_2 \geq 0$. Therefore, $R_1 = R_2 = 0$ and so, $R'(e'_r,e'_i)e'_r = 0$ for all $i = 1, \dots, r$ and $R''(e''_s,e''_i)e''_s = 0$ for all $i = 1, \dots, s$. Hence, $R'_{e'_r} = R''_{e''_s} = 0$, where $R_V = R(\cdot, V)V$ denotes the *Jacobi operator*. So, $R'_{v'} = 0$ and $R''_{v''} = 0$, for all $v' \in T_{x'}M'$ and $v'' \in T_{x''}M''$ respectively. This implies at once that M' and M'' are flat, from which it follows that M itself is locally flat. $\square$

From Theorem 4.1 it follows a positive answer to Question 1.1 when the base manifold is locally reducible. In fact, taking into account Proposition 3.1, we obtain at once the following

Theorem 4.2. *Let (M, g) be a locally reducible Riemannian manifold. Then, the unit tangent sphere bundle T_1M is H-contact if and only if (M, g) is locally flat.*

Recently, Boeckx [**7**] characterized locally reducible unit tangent sphere bundles in the following way.

Theorem 4.3 [**7**]. *The unit tangent sphere bundle T_1M of a Riemannian manifold M, of dimension greater than two, is locally reducible if and only if the base manifold has a flat factor.*

Theorem 4.3 was proved equipping T_1M with its natural metric g_S. Since the contact metric $\bar{g}$ of T_1M is homothetic to g_S, the same result holds for $(T_1M, \bar{g})$. According to Theorem 4.3, the local reducibility of T_1M implies the local reducibility of the base manifold M, since M must have a flat factor. So, from Theorem 4.2 we obtain at once the following characterization of flat Riemannian manifolds via some curvature conditions on their unit tangent sphere bundles:

Theorem 4.4. *A Riemannian manifold (M, g) is locally flat if and only if its unit tangent sphere bundle T_1M is locally reducible and H-contact.*

We can also use Theorem 4.1 to study semi-symmetric spaces whose unit tangent sphere bundle is H-contact. A *semi-symmetric space* is a Riemannian manifold (M, g) such that its curvature tensor R satisfies the condition

$$R(X, Y) \cdot R = 0,$$

for all vector fields X, Y on M, where $R(X, Y)$ acts as a derivation on R. This is equivalent to requiring that the curvature tensor of (M, g) at a point $p \in M$, R_p, is the same as the curvature tensor of a symmetric space (which may change with the point p). So, locally symmetric spaces are obviously semi-symmetric. In any dimension greater than two, there do exist examples of semi-symmetric spaces which are not locally symmetric. Szabó [**22**] proved a local structure theorem. This theorem states that for every n-dimensional semi-symmetric space, there exists an everywhere dense open subset U such that, around every point of U, the space is locally isometric to a direct product of symmetric spaces, two-dimensional Riemannian spaces, elliptic, hyperbolic, Euclidean and Kählerian cones, and spaces foliated by $(n-2)$-dimensional Euclidean spaces.

Assume now that (M,g) is a semi-symmetric space with an H-contact unit tangent sphere bundle. Then, the Ricci tensor of (M,g) is a Codazzi tensor and so, by a result of [**4**], (M,g) must be locally symmetric. Moreover, if (M,g) is locally reducible, then Theorem 4.1 implies that (M,g) is flat. So, we get the following

Theorem 4.5. *Let (M,g) be a semi-symmetric space. If T_1M is an H-contact manifold, then either (M,g) is locally flat or it is an irreducible locally symmetric space.*

5. H-contact unit tangent sphere bundle over a conformally flat manifold. As is well known, a Riemannian manifold (M,g) is said to be (*locally*) *conformally flat* if for any point $p\in M$ there exist a neighborhood U of p and a positive smooth function $f:U\to\mathbf{R}$ such that fg is a flat metric. The study of conformally flat Riemannian manifolds is a classical field of research in Riemannian geometry.

Let p be a point of M and $\{e_1,\dots,e_n\}$ any orthonormal basis of the tangent space T_pM. If (M,g) is conformally flat, we have

$$\begin{aligned} R_{ijkh} = {} & \frac{1}{n-2}(g_{ik}\varrho_{jh}+g_{jh}\varrho_{ik}-g_{ih}\varrho_{jk}-g_{jk}\varrho_{ih}) \\ & -\frac{\tau}{(n-1)(n-2)}(g_{ik}g_{jh}-g_{ih}g_{jk}). \end{aligned} \tag{5.1}$$

Formula (5.1) expresses in local coordinates the property that the *Weyl conformal curvature tensor* W vanishes. $W=0$ characterizes conformally flat Riemannian manifolds of dimension $n\geq 4$, while it is satisfied by any three-dimensional Riemannian manifold.

In [**10**, Proposition 2], it was proved that if ξ is harmonic on T_1M and $\dim M = 2$ or 3, then (M, g) has constant curvature. We now extend this result, by proving the following

Theorem 5.1. *Let (M, g) be a Riemannian manifold with vanishing Weyl conformal curvature tensor. M has constant sectional curvature if and only if $\overline{Q}\xi$ is orthogonal to the horizontal vectors of* $\operatorname{Ker}\eta$.

Proof. If (M, g) has constant sectional curvature, routine calculations show that (b) of Proposition 3.4 holds, since both terms of (3.4) vanish, for all $x \in M$ and $\{e_1, \dots, e_n\}$ orthonormal basis of T_xM. In other words, $\overline{Q}\xi$ of $(T_1M, \eta, \bar{g})$ is orthogonal to the horizontal vectors of $\operatorname{Ker}\eta$.

Conversely, suppose now that (M, g) is a Riemannian manifold such that $W = 0$ and (b) of Proposition 3.4 holds. Let x be a point of M and $\{e'_1, \dots, e'_n\}$ an orthonormal basis of T_xM, of eigenvectors of the Ricci operator Q of (M, g), that is, $\varrho_x(e'_k, e'_h) = \varrho_k\delta_{kh}$, for all $k, h = 1, \dots, n$, where by $\varrho_1, \dots, \varrho_n$ we denote the Ricci eigenvalues. Fix arbitrarily two indices $i \neq j$. Then, for any $\theta \in \mathbf{R}$, the set

$$\begin{aligned}\{e_1, \dots, e_{n-1}, e_n\} = \{e'_k/k \neq i, j\} &\cup \{v = -\sin\theta e'_i + \cos\theta e'_j\} \\ &\cup \{u = \cos\theta e'_i + \sin\theta e'_j\}\end{aligned}$$

is a new orthonormal basis of T_xM.

Since $v \perp u$ and $\{e_1, \dots, e_n = u\}$ is an orthonormal basis of T_xM, applying (3.4) to compute $\varrho_x(v, u)$, we get

$$2\varrho_x(v, u) = \sum_{r=1}^{n} g_x(R(u, e_r)v, R(u, e_r)u). \tag{5.2}$$

We can use (5.1) in order to compute the second term of (5.2). We first note that, since $\{e'_1, \dots, e'_n\}$ is a basis of Ricci eigenvectors, from (5.1) it easily follows that

$$R(e'_r, e'_s)e'_s = K_{rs}e'_r$$

for all $r \neq s = 1, \dots, n$, where we put

$$K_{rs} = \frac{\varrho_r + \varrho_s}{n-2} - \frac{\tau}{(n-1)(n-2)}. \tag{5.3}$$

Note that $K_{rs} = K_{sr}$, for all $r \neq s$.

Next, we compute $g_x(R(u,e_r)v, R(u,e_r)u)$, for all $r = 1,\dots,n$. Taking into account the fact that

$$e_r = \begin{cases} e_k & \text{for some } k \neq i,j, \text{ if } r < n-1, \\ v & \text{if } r = n-1, \\ u & \text{if } r = n, \end{cases}$$

we get

$$(5.4)\quad g(R(u,e_r)v, R(u,e_r)u) \\ = \begin{cases} (K_{jk} - K_{ik})\sin\theta\cos\theta(K_{ik}\cos^2\theta + K_{jk}\sin^2\theta), \\ \text{if } r < n-1, \text{ where } \ e_r = e'_k, \\ 0 \qquad\qquad \text{if } r = n-1, \\ 0 \qquad\qquad \text{if } r = n. \end{cases}$$

Using (5.4) in (5.2), we obtain

$$(5.5)\quad 2\varrho_x(v,u) = \sum_{k\neq i,j} (K_{jk} - K_{ik})\sin\theta\cos\theta(K_{ik}\cos^2\theta + K_{jk}\sin^2\theta).$$

On the other hand,

$$(5.6)\qquad \varrho_x(v,u) = (\varrho_j - \varrho_i)\sin\theta\cos\theta.$$

Using (5.6) in (5.5), we get

$$(5.7)\qquad 2(\varrho_i - \varrho_j) - \sum_{k\neq i,j}(K_{jk} - K_{ik})(K_{ik}\cos^2\theta + K_{jk}\sin^2\theta) = 0$$

for all $\theta \in \mathbf{R}$ such that $\sin\theta\cos\theta \neq 0$. From (5.3) it follows that

$$K_{jk} - K_{ik} = \frac{\varrho_i - \varrho_j}{n-2},$$

for all $k \neq i,j$. So, (5.7) becomes

$$(5.8)\qquad (\varrho_i - \varrho_j)\Big\{2 + \frac{1}{n-2}\sum_{k\neq i,j}(K_{ik}\cos^2\theta + K_{jk}\sin^2\theta)\Big\} = 0.$$

Since (5.8) holds for all $\theta \in \mathbf{R}$ such that $\sin\theta\cos\theta \neq 0$, from (5.8) it follows that $\varrho_i = \varrho_j$. Hence, all the Ricci eigenvalues coincide, and so (M, g) is an Einstein manifold. Since (M, g) is conformally flat, we can conclude that (M, g) has constant sectional curvature. □

If $(T_1M, \eta, \bar{g})$ is H-contact, then $\overline{Q}\xi$ is orthogonal to the horizontal vectors of $\operatorname{Ker}\eta$, by Proposition 3.1. Therefore, from Theorem 5.1, we get the following

Theorem 5.2. *Let (M, g) be a conformally flat Riemannian manifold. Then, $(T_1M, \eta, \bar{g})$ is H-contact if and only if (M, g) has constant sectional curvature. In this case, T_1M is (strongly) φ-symmetric and a (k, μ)-space.*

6. H-contact unit tangent sphere bundles over contact metric manifolds. In this section, we give a positive answer to Question 1.1 when M belongs to the class of Sasakian manifolds or, more generally, it is a (k, μ)-space. We first deal with the Sasakian case, by proving the following

Theorem 6.1. *Let (M, η, g) be a Sasakian manifold. Then, T_1M is H-contact if and only if (M, g) has constant sectional curvature $+1$. In this case, T_1M is K-contact.*

Proof. If (M, g) has constant sectional curvature $+1$, then T_1M is K-contact [**23**], and so it is H-contact.

Assume now that T_1M is H-contact. Then, Proposition 3.1 holds. In particular, the Ricci tensor ϱ of (M, g) is a Codazzi tensor. Since (M, η, g) is Sasakian, this implies that (M, g) is Einstein [**1**].

Next, let ξ be the characteristic vector field of (M, η, g) at a point $x \in M$ and $u \in T_xM$ a unit vector different from $\pm\xi$ and not orthogonal to ξ. Put $v = g_x(\xi, u)u - \xi$. Hence, v is orthogonal to u but not to ξ. Consider an orthonormal basis $\{e_1, \dots, e_n = u\}$ of T_xM, where $n = \dim M$. Since (M, g) is Einstein, $\varrho_x(u, v) = 0$. Hence, from (3.3) we have

$$\sum_{r=1}^{n} g_x(R(u,e_r)v, R(u,e_r)u) = 0. \tag{6.1}$$

Since M is Sasakian, its curvature tensor satisfies

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y, \tag{6.2}$$

for all $X, Y \in T_xM$. Taking into account (6.2) and $v = g_x(\xi,u)u - \xi$, (6.1) becomes

$$g_x(\xi,u)\left(\sum_r \|R(u,e_r)u\|^2 - \varrho(u,u)\right) = 0,$$

that is, since u is not orthogonal to ξ,

$$\varrho(u,u) = \sum_r \|R(u,e_r)u\|^2.$$

Next, by definition we have

$$\varrho(u,u) = \sum_r R(u,e_r,u,e_r) = -\sum_r g_x(R(u,e_r)u,e_r).$$

Hence, (6.3) yields

$$\sum_r g_x(R(u,e_r)u, R(u,e_r)u + e_r) = 0, \tag{6.4}$$

for all u not orthogonal to ξ.

Note that $\{e_1, \dots, e_n = u\}$ is an arbitrary orthonormal basis of T_xM including u. Suppose now that $\{e_1, \dots, e_n = u\}$ is an orthonormal basis of eigenvectors for the symmetric endomorphism $R_u = R(\cdot,u)u$ (Jacobi operator), that is, $R_u(e_i) = \lambda_i e_i$ for all $i = 1, \dots, n$. Then from (6.4) we get

$$\sum_r \lambda_r(\lambda_r - 1),$$

that is,

$$\sum_r \lambda_r^2 = \sum \lambda_r. \tag{6.5}$$

For $r = n$, we have $e_n = u$ and $R_u u = 0$. So, in (6.5), r runs from 1 to $n-1$.

Since (M, g) is Sasakian and Einstein, we also have

$$\varrho_x(u,u) = \frac{\tau}{n} g_x(u,u) = \frac{\tau}{n} = n-1.$$

Hence, from (6.3) we also get

$$\sum_r \lambda_r^2 = n-1,$$

that is, $\lambda_1, \dots, \lambda_{n-1}$ satisfy the system

$$\begin{cases} \sum_r \lambda_r = n-1 \\ \sum_r \lambda_r^2 = n-1. \end{cases} \tag{6.6}$$

It is easy to show that $\lambda_1 = \cdots = \lambda_{n-1} = 1$ is the only solution of (6.6). In fact, the hyperplane $\sum_r \lambda_r = n-1$ is tangent to the Euclidean sphere $\sum_r \lambda_r^2 = n-1$ at the point $(\lambda_1, \dots, \lambda_{n-1}) = (1, \dots, 1)$.

We can now conclude that (M, g) has constant sectional curvature $+1$. In fact, we first note that, since $\lambda_r = 1$ for all $r = 1, \dots, n-1$, we have

$$R_u e_r = e_r \tag{6.7}$$

and so, $K(u, e_r) = R(u, e_r, u, e_r) = 1$ for all $r = 1, \dots, n-1$. Next, let $v \in T_x M$ be any unit vector orthogonal to u. Using (6.7), easy calculations show that $K(u,v) = 1$. Because u is an arbitrary unit tangent vector not orthogonal to ξ, so far we have proved that

$$K(u,v) = 1 \quad \text{whenever either } u \text{ or } v \text{ is not orthogonal to } \xi, \tag{6.8}$$

that is, either u or v does not belong to $\operatorname{Ker} \eta$. Thus, we are left with the case when $u, v \in \operatorname{Ker} \eta$. In this case, consider a new orthonormal basis $\{e_1 = u, e_2 = v, e_3, \dots, e_n\}$, where $e_3, \dots, e_n$ are chosen so that none of them is orthogonal to ξ. Using (6.8) and the fact that (M, g) is an Einstein Sasakian manifold, we have

$$n(n-1) = \tau = \sum_{i \neq j} R_{ijij} = 2R_{1212} + n(n-1) - 2,$$

from which it follows that also $K(u, v) = R_{1212} = 1$. Therefore, (M, g) has constant sectional curvature 1 and $T_1 M$ is K-contact [**23**]. □

We now consider the case of a (k, μ)-space. A contact metric manifold (M, η, g) is said to be a (k, μ)*-space* if its characteristic vector field ξ belongs to the (k, μ)*-nullity distribution*, that is, it satisfies

$$R(X, Y)\xi = k(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY), \tag{6.9}$$

for all tangent vectors X and Y. (k, μ)-spaces are a well-known class of contact metric manifolds which generalizes the class of Sasakian manifolds. They were introduced and first studied in [**3**]. A full classification of non-Sasakian (k, μ)-spaces can be found in [**6**].

In [**3**] the curvature tensor of a non Sasakian (k, μ)-space was determined. In particular, the Ricci operator Q is completely described by

$$QX = [2(n-1) - n\mu]X + [2(n-1) + \mu]hX + [2(1-n) + n(2k+\mu)]\eta(X)\xi. \tag{6.10}$$

It was also remarked in [**3**] that, for $\mu = 2(1-n)$, Q is of the form

$$QX = aX + b\eta(X)\xi \tag{6.11}$$

and so, the (k, μ)-space (M, η, g) is an η*-Einstein space.*

Differentiating (6.10) by a tangent vector field Y, one can express the covariant derivative of Q, and so, of ϱ, in terms of η and h. Explicitly, we obtain

$$\begin{aligned}(\nabla_X \varrho)(Y, Z) = {} & [2(n-1) + \mu] g((\nabla_X h)Y, Z) \\ & + [2(1-n) + n(2k+\mu)] \\ & \times \{g(Y, (\nabla_X \xi)\eta(Z)) + g(Z, (\nabla_X \xi)\eta(Y))\}.\end{aligned} \tag{6.12}$$

(k, μ)-spaces whose Ricci tensor is a Codazzi tensor were studied in [**17**], where the following result was proved.

Theorem 6.2 [**17**]. *Let (M, η, g) be a (k, μ)-space with harmonic curvature. Then, M is either an Einstein Sasakian manifold, or an η-Einstein manifold, or locally isometric to a Riemannian product $\mathbf{R}^{n+1} \times S^n(4)$ (including a flat contact metric structure for $n = 1$).*

We are now ready to prove the following

Theorem 6.3. *Let (M, η, g) be a (k, μ)-space. Then, T_1M is H-contact if and only if either M is flat (and three-dimensional) or M has constant sectional curvature $+1$.*

Proof. As we already remarked, if M has constant sectional curvature, then T_1M is H-contact. Conversely, assume that (M, g) is a (k, μ)-space with H-contact unit tangent sphere bundle. According to Proposition 3.1, the Ricci tensor ϱ of (M, g) is a Codazzi tensor. Hence, (M, g) must be one of the manifolds listed in Theorem 6.2.

If M is an Einstein Sasakian manifold, from Theorem 6.1 it follows that M has constant sectional curvature $+1$. In the sequel, we shall assume (M, η, g) is non Sasakian. If (M, g) is locally isometric to $\mathbf{R}^{n+1} \times S^n(4)$, Theorem 4.1 implies that M is flat, and so $n = 1$. Therefore, we are left with the case when (M, g) is a non Sasakian η-Einstein (k, μ)-space whose unit tangent sphere bundle T_1M is H-contact. Taking into account Theorem 4.1, we can assume that (M, g) is locally irreducible and we shall prove that this case cannot occur.

Since (M, g) is a non Sasakian (k, μ)-space, it is (*strongly*) *locally φ-symmetric* [**5, 8**]. We do not need here a detailed description of (strongly) locally φ-symmetric spaces and we can refer to [**6**] for more information and results about them. We just point out that a (strongly) locally φ-symmetric space satisfies an infinity number of curvature conditions. Among the others, we have

$$(\nabla_u \rho)(u, u) = 0,$$
$$(\nabla_u \rho)(\xi, \xi) = 0,$$

for all $u \in \operatorname{Ker} \eta$. Taking into account the fact that ϱ is also a Codazzi tensor, we obtain that

$$\nabla\varrho = 0 \iff (\nabla_\xi \rho)(u,v) = 0 \quad \text{for all } u, v \in \operatorname{Ker}\eta.$$

Consider $u, v \in \operatorname{Ker}\eta$. According to (6.12), we have

$$(\nabla_\xi \rho)(u,v) = [2(n-1)+\mu]g((\nabla_\xi h)(u,v) = 0,$$

since (M,g) is η-Einstein, and so $\mu = 2(1-n)$. Thus, (M,g) is Ricci-parallel. Being irreducible, (M,g) must be an Einstein manifold [**16**]. We remark that from (6.10) it follows that

$$\begin{aligned} Q\xi &= 2nk\xi, \\ Qu &= (2(n-1)-n\mu)u, \end{aligned}$$

where $u \in \operatorname{Ker}\eta$. Since M is Einstein, we must have

$$2nk = 2(n-1) - n\mu,$$

that is, taking into account $\mu = 2(1-n)$,

$$k = \frac{n^2-1}{n}. \tag{6.13}$$

We know from Theorem 6.2 that in the non Sasakian case we have $k < 1$. According to (6.13), this can occur only if $n = 1$, in which case $k = 0$, again by (6.13), and $\mu = 2(1-n) = 0$. Therefore, (6.10) gives $Q = 0$, that is, being three-dimensional, (M,g) should be flat, which contradicts the assumption that (M,g) is irreducible and this ends the proof. □

Remark 6.4. If (M,η,g) is a non Sasakian (k,μ)-space, then the property "T_1M is H-contact" is not equivalent to "T_1M is K-contact". In fact, according to Theorems 6.1 and 6.3, a three-dimensional flat manifold is a (non-Sasakian) (k,μ)-space whose unit tangent sphere bundle is H-contact but not K-contact.

7. H-contact unit tangent sphere bundles over Kähler manifolds. We first prove the following.

Proposition 7.1. *Let (M, g, J) be a Kähler manifold. If T_1M is H-contact, then M is a Kähler-Einstein space.*

Proof. Let (M, g, J) be a Kähler manifold whose unit tangent sphere bundle is H-contact. If M is locally reducible, Theorem 4.1 implies at once that M is flat, in particular, it is an Einstein manifold. Assume now that M is locally irreducible. According to Proposition 3.1, ϱ is a Codazzi tensor. Therefore, M is Ricci-parallel, as was proved in [**16**]. Since M is locally irreducible, it again must be an Einstein manifold, see for example [**16**]. □

Note that a compact Kähler-Einstein space with nonnegative sectional curvature is locally symmetric [**15**]. Therefore, as a consequence of Proposition 7.1, we have

Proposition 7.2. *Let (M, g, J) be a compact Kähler manifold with nonnegative sectional curvature. If T_1M is H-contact, then M is a Kähler-Einstein locally symmetric space.*

When $\dim M = 4$, we can improve Proposition 7.2 by proving the following

Theorem 7.3. *Let (M, g, J) be a four-dimensional Kähler manifold. Suppose that M satisfies one of the following properties*:

a) *M has either nonnegative or non positive sectional curvature, or*

b) *M is not Ricci-flat.*

Then, T_1M is H-contact if and only if M has constant holomorphic sectional curvature.

Proof. Let (M, g, J) be a four-dimensional Kähler manifold, satisfying either a) or b). If (M, g, J) has constant holomorphic sectional curvature, then T_1M is H-contact [**10**]. Conversely, assume now that T_1M is H-contact. From Proposition 7.1 it follows that M is Einstein. So, at any point $x \in M$ there exists an *adapted Singer-Thorpe basis*, that is, an orthonormal basis $\{e_1, e_2 = Je_1, e_3, e_4 = Je_3\}$ of T_xM, such that the components of the curvature tensor with respect to $\{e_1, e_2, e_3, e_4\}$

are given by

$$(7.1)\qquad \begin{array}{ll} R_{1212}=R_{3434}=a, & R_{1234}=\alpha, \\ R_{1313}=R_{2424}=b, & R_{1342}=\beta, \\ R_{1414}=R_{2323}=c, & R_{1423}=\gamma, \\ R_{ijkk}=0 \qquad \text{whenever three indices differ} & \end{array}$$

and moreover,

$$(7.2)\qquad \alpha=b+c=\tau/4-a,\ \ \beta=-b,\ \ \gamma=-c,$$

see for example [**21**]. Starting from $\{e_1,e_2,e_3,e_4\}$, we now build new orthonormal bases of T_xM, considering, for all $\theta\in\mathbf{R}$,

$$\{e_1',e_2',e_3',e_4'\}=\{e_1,e_2,v=-\sin\theta e_3+\cos\theta e_4,u=\cos\theta e_3+\sin\theta e_4\}.$$

Applying (3.4) to the orthogonal vectors u and v, standard calculations give

$$\begin{aligned} 0=2\varrho_x(v,u)&=\sum_{r=1}^{3} g_x(R(u,e_r')v,R(u,e_r')u) \\ &=(\sin^2\theta-\cos^2\theta)\{(b-c)^2-(\beta-\gamma)^2\}. \end{aligned}$$

So, it suffices to consider $\theta\in\mathbf{R}$ such that $\sin^2\theta\neq\cos^2\theta$, to conclude that $(b-c)^2=(\beta-\gamma)^2$. Proceeding in a similar way, starting from orthonormal bases of type $\{e_1,e_3,v=-\sin\theta e_2+\cos\theta e_4,u=\cos\theta e_2+\sin\theta e_4\}$ and $\{e_1,e_4,v=-\sin\theta e_2+\cos\theta e_3,u=\cos\theta e_2+\sin\theta e_3\}$, we also obtain that $(a-b)^2=(\alpha-\beta)^2$ and $(a-c)^2=(\alpha-\gamma)^2$. So, the curvature components with respect to the adapted Singer-Thorpe basis $\{e_1,e_2,e_3,e_4\}$ satisfy

$$(7.3)\qquad \begin{cases} \alpha=b+c=\tau/4-a, \\ \beta=-b, \\ \gamma=-c, \\ (a-b)^2=(\alpha-\beta)^2, \\ (b-c)^2=(\beta-\gamma)^2, \\ (a-c)^2=(\alpha-\gamma)^2. \end{cases}$$

If the sectional curvature of M is either nonnegative or non positive, we only have to consider the solutions of (7.3) such that a, b and c

have the same sign. It is easy to prove that the only solutions of (7.3), compatible with this condition, are

$$\begin{cases} a = b = c = 0, \\ \alpha = \beta = \gamma = 0 \end{cases} \tag{7.4}$$

and

$$\begin{cases} c = b, \\ a = 4b, \\ \alpha = 2b, \\ \beta = \gamma = -b. \end{cases} \tag{7.5}$$

Clearly, (7.4) corresponds to a flat metric, that is, M is locally isometric to C^2. As concerns (7.5), note that the scalar curvature τ of (M, g) is constant and $\tau/4 = a + b + c = 6b$ (with $b \neq 0$, otherwise we have again a flat metric). So, b is constant and (7.4) implies that all the curvature components with respect to the adapted Singer-Thorpe basis $\{e_1, e_2, e_3, e_4\}$ are constant. In other words, (M, g) is a four-dimensional *curvature homogeneous* Einstein space. It follows from an unpublished theorem by Derdzinski that such a space is locally symmetric [**9**, page 409]. So, (M, g) must be locally isometric to one of the two-point homogeneous spaces $\mathbf{R}^4$, S^4, H^4, CP^2 or CH^2, or to one of the product manifolds $S^2 \times S^2$ or $H^2 \times H^2$. Taking into account Theorem 4.1 and the fact that M is Kähler, we can conclude that M has constant holomorphic sectional curvature.

Suppose now that the scalar curvature τ of (M, g) is different from zero at some point. Since M is Einstein, $\tau \neq 0$ at any point and this is equivalent to requiring that M is not Ricci-flat. It is easy to prove that the only solution of (7.3) for which $\tau \neq 0$ is given by (7.5). As we proved already, (7.5) corresponds to a Kähler manifold of constant (non-vanishing) holomorphic sectional curvature. □

A Kähler manifold is said to be *Bochner-Kähler* if its conformal curvature Bochner tensor vanishes. Such manifolds are also called *Bochner-flat*. Bochner-Kähler manifolds are the Kähler analogous of conformally flat Riemannian manifolds. At first, one might expect the theory of Bochner-Kähler manifolds to parallel the theory of conformally flat Riemannian manifolds, but one soon finds apparent differences. Bochner-Kähler manifolds have nontrivial local geometry and they do not have a global isometric uniformization. Bryant's paper [**11**] provides an explicit local classification of Bochner-Kähler metrics and an in-depth study of their global geometry, including a classification of the compact and complete Bochner-Kähler manifolds. In particular, a Bochner-Kähler manifold is Einstein if and only if it has constant holomorphic sectional curvature. Taking into account that a Kähler manifold with H-contact unit tangent sphere bundle is Einstein (Proposition 7.1), we obtain at once the following new result on Bochner-Kähler manifolds.

Theorem 7.4. *Let (M, g, J) be a Bochner-flat Kähler manifold. Then, T_1M is H-contact if and only if (M, g, J) has constant holomorphic sectional curvature.*

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