ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 37, Number 4, 2007

OSCILLATION THEOREMS RELATED TO AVERAGING TECHNIQUE FOR DAMPED PDE WITH p–LAPLACIAN

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ABSTRACT. We present some oscillation theorems related to integral averaging technique for damped PDEs with p-Laplacian

$$(E) \qquad \sum_{i,\,j=1}^{N} D_i[a_{ij}(x)\|Dy\|^{p-2}D_jy] + \langle b(x), \|Dy\|^{p-2}Dy\rangle + c(x)f(y) = 0.$$

The results obtained extend the criteria for the Sturm-Liouville linear equation due to Kamenev, Kong, Philos and Wong to equation (E).

1. Introduction. We consider the second order damped partial differential equation (PDE) with p-Laplacian

$$(1.1) \quad \sum_{i,\,j=1}^{N} D_i[a_{ij}(x)\|Dy\|^{p-2}D_jy]+\langle b(x), \|Dy\|^{p-2}\,Dy\rangle+c(x)f(y)=0$$

in an exterior domain $\Omega(r_0) := \{x \in \mathbf{R}^N : \|x\| \geq r_0\}$, where $r_0 > 0$, $x = (x_i)_{i=1}^N \in \mathbf{R}^N$, $N \geq 2$, $p > 1$, $D_iy = \partial y/\partial x_i$, $Dy = (D_iy)_{i=1}^N$, $\|\cdot\|$ and $\langle\,\cdot,\cdot\,\rangle$ denote the usual Euclidean norm and the usual scalar product in $\mathbf{R}^N$, respectively.

Throughout this paper, we assume that the following conditions hold.

(A1) $A = (a_{ij}(x))_{N\times N}$ is a real symmetric positive define matrix function with $a_{ij} \in C_{\rm loc}^{1+\mu}(\Omega(r_0), \mathbf{R})$, $0 < \mu < 1$.

2000 AMS *Mathematics Subject Classification.* Primary 34C10, 35B05, 35J60.
Key words and phrases. Oscillation, PDE with p-Laplacian, damped, Riccati inequality, integral averaging technique.
Received by the editors on Nov. 26, 2004, and in revised form on April 15, 2005.

Denote by $\lambda_{\min}(x)$ the smallest eigenvalue of the matrix A. We assume that there exists a function $\lambda \in C([r_0, \infty), \mathbf{R}^+)$ such that

$$\min_{\|x\|=r} \frac{\lambda_{\min}(x)}{\|A\|^{p/(p-1)}} \geq \lambda(r) \quad \text{for} \quad r \geq r_0,$$

where $\|A\|$ means the norm of the matrix A, i.e.,

$$\|A\| = \left[\sum_{i,j=1}^{N} a_{ij}^2(x) \right]^{1/2};$$

(A2) $b(x) = (b_i(x))_{i=1}^N$, b_i, $c \in C_{\text{loc}}^{\mu}(\Omega(r_0), \mathbf{R})$, $0 < \mu < 1$;

(A3) $f \in C(\mathbf{R}, \mathbf{R}) \cup C^1(\mathbf{R} - \{0\}, \mathbf{R})$ with $yf(y) > 0$ and

$$\frac{f'(y)}{|f(y)|^{(p-2)/(p-1)}} \geq \varepsilon > 0 \quad \text{for} \quad y \neq 0.$$

By a solution of (1.1) is meant a function $y \in C^{2+\mu}(\Omega(r_0), \mathbf{R})$, $0 < \mu < 1$, which has the property $a_{ij}(x)\|Dy\|^{p-2}D_j y \in C^{1+\mu}(\Omega(r_0), \mathbf{R})$ and satisfies (1.1) at each $x \in \Omega(r_0)$. Regarding the question of existence of solution of (1.1), we refer the reader to the monograph [**1**]. In what follows, our attention is restricted to these solutions which don't vanish identically in any neighborhood of ∞. The oscillation is considered in the usual sense, that is, a solution $y(x)$ of (1.1) is said to be oscillatory if it has arbitrarily large zeros, i.e., the set $\{x \in \mathbf{R}^N : y(x) = 0\}$ is unbounded, otherwise it is called nonoscillatory. Equation (1.1) is called oscillatory if all its solutions are oscillatory. Conversely, (1.1) is nonoscillatory if there exists a solution which is not oscillatory.

The PDEs with p-Laplacian have applications in various physical and biological problems, in the study of non-Newtonian fluids, in glaciology and slow diffusion problems. For a more detailed discussion about applications of PDE with p-Laplacian, the reader is referred to [**1**] and the references cited therein. In the qualitative theory of nonlinear PDE, one of the important themes is to determine whether or not solutions of the equation under consideration are oscillatory. In the last decade, oscillation or nonoscillation of solutions of the half-linear PDE with p-Laplacian

$$\text{div}\,(\|Dy\|^{p-2}\,Dy) + c(x)|y|^{p-2}y = 0 \tag{1.2}$$

has received much attention and been extensively studied by many authors, see, e.g., [**2, 4, 6–9, 12, 14–16**]. These investigations were mostly based on the so-called Riccati technique, which was developed by Noussair and Swanson [**10**], consisting in the fact that if $y = y(x)$ is a nonoscillatory solution of (1.2) then the $N-$dimensional vector function w defined by

$$w(x) = \frac{\|Dy\|^{p-2}Dy}{|y|^{p-2}y} \tag{1.3}$$

satisfies the partial Riccati-type differential equation

$$\operatorname{div} w(x) = -\, c(x) - (p-1)\|w\|^q, \quad q = \frac{p}{p-1}. \tag{1.4}$$

Recently, Mařík [**9**], by using Riccati-type inequality and integral averages, has generalized Kamenev's criteria [**3**] to the half-linear PDE with damping

$$\operatorname{div}(\|Dy\|^{p-2}\, Dy) + \langle b(x), \|Dy\|^{p-2}\, Dy\rangle + c(x)|y|^{p-2}y = 0, \tag{1.5}$$

which seems to be the first paper to study the oscillation of (1.5). However, his result is not very sharp, because the two-parametric weighting function $H(t,x)$ introduced by Philos [**11**], which is used in the proof, must satisfy some harsh conditions (see [**9**, Theorem 3.10]), and the Kamenev-type theorem has not been well-developed for (1.5) in [**9**]. Our aim here is motivated by the recent papers [**5, 11, 13**] dealing with oscillatory properties of the Sturm-Liouville linear equation

$$(r(t)y'(t))' + p(t)y(t) = 0, \tag{1.6}$$

and is to extend the results of Kamenev [**3**], Kong [**5**], Philos [**11**] and Wong [**13**] to general equation (1.1), thereby improving the main results in [**12, 14–16**]. Our methodology is somewhat different from that of previous authors; we believe that our approach is simpler and also provides a more unified account for study of Kamenev-type oscillation theorems for (1.1).

2. Notations and lemmas. The following notations will be used throughout this paper. Set

$$\begin{aligned} S_r &= \{x \in \mathbf{R}^N : \|x\| = r\}, \\ \Omega(r_1, r_2) &= \{x \in \mathbf{R}^N : r_1 \le \|x\| \le r_2\}, \\ \Omega_1(r_1, r_2) &= \{x \in \mathbf{R}^N : r_1 \le \|x\| < r_2\}, \\ \Omega_2(r_1, r_2) &= \{x \in \mathbf{R}^N : r_1 < \|x\| \le r_2\}, \end{aligned}$$

and

$$D_0 = \{(r, s) : r > s \ge r_0\} \quad \text{and} \quad D = \{(r, s) : r \ge s \ge r_0\}.$$

For $l > 1$, we define

$$\begin{aligned} C(x) &= c(x) - \frac{1}{p}\left(\frac{l}{\varepsilon q}\right)^{p-1} \lambda_{\min}^{1-p}(x)\|A\|^p \|b(x)A^{-1}\|^p, \\ C_M(r) &= \int_{S_r} C(x)\, d\sigma, \end{aligned}$$

and

$$g(r) = \omega_N r^{N-1} (l^*)^{p-1} \lambda^{1-p}(r), \quad k = \varepsilon^{1-p}(p-1)^{p-1} p^{-p},$$

where $\omega_N = \int_{S_1} d\sigma = 2\pi^{N/2}/\Gamma(N/2)$ denotes the surface measure of the unit sphere, q and l^* are conjugate numbers to p and l, respectively, i.e., $1/p + 1/q = 1$ and $1/l + 1/l^* = 1$.

Definition 2.1. A function $H \in C(D, \mathbf{R}^+)$ is said to belong to a function set $\Re$, denoted by $H \in \Re$, if

(1) $H(r, r) = 0$ for $r \ge r_0$, $H(r, s) > 0$ on D_0;

(2) $H(r, s)$ has continuous and nonpositive partial derivative $\partial H/\partial s$ on D_0;

(3) there exists a function $h \in C(D_0, \mathbf{R})$ such that

$$-\frac{\partial H}{\partial s}(r, s) = h(r, s) H(r, s) \quad \text{on} \quad D_0.$$

Definition 2.2. A function $H \in C(D, \mathbf{R}^+)$ is said to belong to a function set $\Im$, denoted by $H \in \Im$, if

(1) $H(r,r) = 0$ for $r \geq r_0$, $H(r,s) > 0$ on D_0;

(2) $H(r,s)$ has continuous partial derivatives $\partial H(r,s)/\partial r$ and $\partial H(r,s)/\partial s$ on D_0;

(3) there exist two functions h_1, $h_2 \in C(D_0, \mathbf{R})$ such that

$$\frac{\partial H}{\partial r}(r,s) = h_1(r,s)H(r,s) \quad \text{and} \quad \frac{\partial H}{\partial s}(r,s) = -\,h_2(r,s)H(r,s) \text{ on } D_0.$$

Letting $\rho \in C^1([r_0,\infty), \mathbf{R}^+)$, we take two operators $\Gamma_\tau^\rho(\cdot;r)$ and $\Theta_\tau^\rho(\cdot;r)$, which are defined in [**13**] in terms of H and ρ, as follows:

$$\Gamma_\tau^\rho(\phi;r) = \int_\tau^r H(r,s)\phi(s)\rho(s)\,ds, \quad r \geq \tau \tag{2.1}$$

and

$$\Theta_\tau^\rho(\phi;r) = \int_\tau^r H(s,\tau)\phi(s)\rho(s)\,ds, \quad r \geq \tau, \tag{2.2}$$

where $\phi \in C([r_0,\infty), \mathbf{R})$. It is easy to verify that $\Gamma_\tau^\rho(\,\cdot\,;r)$ and $\Theta_\tau^\rho(\,\cdot\,;r)$ are linear operators and satisfy

$$\begin{aligned}\Gamma_\tau^\rho(\psi';r) &= -\,H(r,\tau)\rho(\tau)\psi(\tau) - \Gamma_\tau^\rho\left(\left[-\,h_2 + \frac{\rho'}{\rho}\right]\psi;r\right)\\ &\geq -\,H(r,\tau)\rho(\tau)\psi(\tau) - \Gamma_\tau^\rho\left(\left|h_2 - \frac{\rho'}{\rho}\right||\psi|;r\right), \quad r \geq \tau\end{aligned} \tag{2.3}$$

and

$$\begin{aligned}\Theta_\tau^\rho(\psi';r) &= H(r,\tau)\rho(r)\psi(r) - \Theta_\tau^\rho\left(\left[h_1 + \frac{\rho'}{\rho}\right]\psi;r\right)\\ &\geq H(r,\tau)\rho(r)\psi(\tau) - \Theta_\tau^\rho\left(\left|h_1 + \frac{\rho'}{\rho}\right||\psi|;r\right), \quad r \geq \tau,\end{aligned} \tag{2.4}$$

where $\psi \in C^1([r_0,\infty), \mathbf{R})$, $h_1 = h(s,\tau)$ and $h_2 = h(r,s)$.

The following Lemma 2.1 plays an important role in our proof, which is a modified version of Lemma 1 in [**10**] for the semi-liner elliptic equation.

Lemma 2.1. *Let $l > 1$. Suppose that* (1.1) *has a nonoscillatory solution $y = y(x) \neq 0$ for all $x \in \Omega(r_1)$, $r_1 \geq r_0$. Then the N-dimensional vector function $\mathbf{W}(x)$ is well defined on $\Omega(r_1)$ by*

$$\mathbf{W} = \mathbf{W}(\mathbf{x}) = (W_i(x))_{i=1}^N, \quad W_i(x) = \frac{1}{f(y)}\bigg(\sum_{j=1}^N a_{ij}(x)\|Dy\|^{p-2}D_j y\bigg) \tag{2.5}$$

and satisfies the following partial Riccati-type inequality

$$\operatorname{div}\mathbf{W} \leq -C(x) - \frac{\varepsilon}{l^*}\,\frac{\lambda_{\min}(x)}{\|A\|^q}\,\|\mathbf{W}\|^q. \tag{2.6}$$

Proof. Without loss of generality, let us consider that $y = y(x) > 0$ on $\Omega(r_1)$. Differentiation of $W_i(x)$ with respect to x_i gives

$$\begin{aligned} D_i W_i(x) = &- \frac{f'(y)}{f^2(y)}\, D_i y\bigg(\sum_{j=1}^N a_{ij}(x)\|Dy\|^{p-2}D_j y\bigg) \\ &+ \frac{1}{f(y)}\, D_i\bigg(\sum_{j=1}^N a_{ij}(x)\|Dy\|^{p-2}D_j y\bigg), \end{aligned}$$

for $i = 1, \dots, N$. Summation over i and use of (1.1) lead to

$$\operatorname{div}\mathbf{W} = -c(x) - \frac{f'(y)}{f^2(y)}\,\|Dy\|^{p-2}(Dy)^T A(Dy) - \bigg\langle b(x), \frac{\|Dy\|^{p-2}Dy}{f(y)}\bigg\rangle. \tag{2.7}$$

Note that

$$(Dy)^T A(Dy) \geq \lambda_{\min}(x)\|Dy\|^2$$

and

$$\|\mathbf{W}\| \leq \frac{1}{f(y)}\,\|A\|\,\|Dy\|^{p-1}.$$

This, along with (2.7) as well as (A3), implies that

$$\begin{aligned} \operatorname{div}\mathbf{W} \leq &- c(x) - \frac{f'(y)}{|f(y)|^{(p-2)/(p-1)}}\,\frac{\lambda_{\min}(x)}{\|A\|^q}\,\|\mathbf{W}\|^q \\ &- \langle\, b(x), A^{-1}\mathbf{W}\rangle \\ \leq &- c(x) - \frac{\varepsilon\lambda_{\min}(x)}{\|A\|^q}\,\|\mathbf{W}\|^q - \langle\, b(x)A^{-1}, \mathbf{W}\rangle. \end{aligned} \tag{2.8}$$

Application of Young's inequality yields

$$
\begin{aligned}
(2.9)\quad & \frac{\varepsilon\lambda_{\min}(x)}{\|A\|^q}\,\|\mathbf{W}\|^q + \langle b(x)A^{-1},\mathbf{W}\rangle \\
&= \frac{\varepsilon q}{l}\,\frac{\lambda_{\min}(x)}{\|A\|^q}\left[\frac{1}{q}\,\|\mathbf{W}\|^q + \frac{l}{\varepsilon q}\,\frac{\|A\|^q}{\lambda_{\min}(x)}\,\langle b(x)A^{-1},\mathbf{W}\rangle + \frac{l}{l^* q}\,\|\mathbf{W}\|^q\right] \\
&\geq -\frac{1}{p}\left(\frac{l}{\varepsilon q}\right)^{p-1}\|A\|^p\lambda_{\min}^{1-p}(x)\|b(x)A^{-1}\|^p + \frac{\varepsilon}{l^*}\,\frac{\lambda_{\min}(x)}{\|A\|^q}\,\|\mathbf{W}\|^q.
\end{aligned}
$$

Combining (2.8) and (2.9), we get that (2.6) holds. This completes the proof. □

Lemma 2.2. *Let $H \in \Im$, $\rho \in C^1([r_0,\infty),\mathbf{R}^+)$ and $l > 1$. Assume that $y = y(x)$ is a solution of* (1.1) *such that $y(x) > 0$ for $x \in \Omega_1(t,v)$. Put*

$$
(2.10)\qquad Z(r) = \int_{S_r}\langle \mathbf{W}(x),\nu(x)\rangle\,d\sigma.
$$

Then

$$
(2.11)\qquad \frac{1}{H(v,t)}\,\Gamma_t^{\rho}\left(C_M - kg\left|h_2 - \frac{\rho'}{\rho}\right|^p;v\right) \leq \rho(t)Z(t).
$$

where $\mathbf{W}(\mathbf{x})$ is defined by (2.5), *$\boldsymbol{\nu}(x)$ is the normal unit vector and $h_2 = h(v,s)$.*

Proof. In view of Lemma 2.1, then (2.6) holds. By the Green formula in (2.10), observing that (2.6), we have

$$
(2.12)\quad Z'(r) = \int_{S_r}\operatorname{div}\mathbf{W}\,d\sigma \leq -\int_{S_r}C(x)\,d\sigma - \frac{\varepsilon}{l^*}\,\lambda(r)\int_{S_r}\|\mathbf{W}\|^q\,d\sigma.
$$

Hölder's inequality shows that

$$
\begin{aligned}
|Z(r)| &\leq \int_{S_r}\|\mathbf{W}(x)\|\,\|\nu(x)\|\,d\sigma \leq \left(\int_{S_r}d\sigma\right)^{1/p}\left(\int_{S_r}\|\mathbf{W}\|^q\,d\sigma\right)^{1/q} \\
&= \left(\omega_N r^{N-1}\right)^{1/p}\left(\int_{S_r}\|\mathbf{W}\|^q\,d\sigma\right)^{1/q},
\end{aligned}
$$

and equivalently,

$$\left(\int_{S_r} \|\mathbf{W}\|^q \, d\sigma\right) \geq (\omega_N r^{N-1})^{1/(1-p)} |Z(r)|^q,$$

which, together with (2.12), implies that

$$Z'(r) + C_M(r) + \varepsilon g^{1-q}(r)|Z(r)|^q \leq 0. \tag{2.13}$$

Applying the operator $\Gamma_t^\rho(\cdot, r)$ to (2.13) for $r \in [t, v)$ and using (2.3), we find

$$\begin{aligned} \Gamma_t^\rho(C_M; r) \leq H(r,t)\rho(t)Z(t) + \Gamma_t^\rho\left(\left|h_2 - \frac{\rho'}{\rho}\right| |Z|; r\right) \\ - \varepsilon \, \Gamma_t^\rho(g^{1-q}|Z|^q; \, r). \end{aligned} \tag{2.14}$$

The Young's inequality follows

$$\left|h_2 - \frac{\rho'}{\rho}\right| |Z| \leq kg\left|h_2 - \frac{\rho'}{\rho}\right|^p + \varepsilon g^{1-q}|Z|^q. \tag{2.15}$$

Substituting (2.15) into (2.14), we conclude

$$\frac{1}{H(r,t)} \, \Gamma_t^\rho\left(C_M - kg\left|h_2 - \frac{\rho'}{\rho}\right|^p; r\right) \leq \rho(t)Z(t),$$

using $r \to v^-$ in the above we obtain (2.11). □

Similar to the proof of Lemma 2.2, we have

Lemma 2.3. *Let $H \in \Im$, $\rho \in C^1([r_0, \infty), \mathbf{R}^+)$ and $l > 1$. Assume that $y = y(x)$ is a solution of (1.1) such that $y(x) > 0$ for $x \in \Omega_2(u, t)$. Then*

$$\frac{1}{H(t,u)} \, \Theta_u^\rho\left(C_M - kg\left|h_1 + \frac{\rho'}{\rho}\right|^p; t\right) \leq -\rho(t)Z(t), \tag{2.16}$$

where $Z(r)$ is defined by (2.10) and $h_1 = h_1(s, u)$.

3. Main results. In this section, we will establish some oscillation theorems for (1.1). The first one is an analogue of Philos's criteria [**11**] for (1.6).

Theorem 3.1. *Let* $H \in \Re$, $\rho \in C^1([r_0, \infty), \mathbf{R}^+)$ *and* $l > 1$. *If*

$$\limsup_{r\to\infty} \frac{1}{H(r, r_0)}\, \Gamma^{\rho}_{r_0}\left(C_M - kg\left|h - \frac{\rho'}{\rho}\right|^p; r\right) = \infty, \tag{3.1}$$

where $h = h(r, s)$, *then* (1.1) *is oscillatory.*

Proof. Let $y = y(x)$ be a nonoscillatory solution of (1.1); without loss of generality, we may assume that $y = y(x) > 0$ for $\Omega(r_1)$ for some sufficiently large $r_1 \geq r_0$. Replacing r_1, r, h by t, v, h_2, respectively, then by Lemma 2.2, we have that for all $r > r_1 \geq r_0$,

$$\Gamma^{\rho}_{r_1}\left(C_M - kg\left|h - \frac{\rho'}{\rho}\right|^p; r\right) \leq \rho(r_1)Z(r_1)H(r, r_1).$$

Thus, we get that for all $r \geq r_0$,

$$\begin{aligned}
&\Gamma^{\rho}_{r_0}\left(C_M - kg\left|h - \frac{\rho'}{\rho}\right|^p; r\right)\\
&\quad = \Gamma^{\rho}_{r_0}\left(C_M - kg\left|h - \frac{\rho'}{\rho}\right|^p; r_1\right) + \Gamma^{\rho}_{r_1}\left(C_M - kg\left|h - \frac{\rho'}{\rho}\right|^p; r\right)\\
&\quad \leq H(r, r_0)\left[\int_{r_0}^{r_1} \rho(s)|C_M(s)|\, ds + \rho(r_1)|Z(r_1)|\right].
\end{aligned}$$

Dividing both sides of the above inequality by $H(r, r_0)$ and taking the superior limit as $r \to \infty$, we obtain a contradiction with (3.1). The proof is complete. □

Remark 3.1. Theorem 3.1 improves Theorem 1 in [**16**] for (1.2).

Remark 3.2. Comparing Theorem 3.1 with Theorem 3.10 in [**9**], the parametric weighting function $H(r, s)$ introduced in this paper seems more reasonable. Furthermore, we also point out that the technical assumptions imposed for the function $H(r, s)$ are minimal.

The next theorems provide some extensions of Kong-type interval criteria [**5**] to (1.1).

Theorem 3.2. *Let $H \in \Im$, $\rho \in C^1([r_0,\infty),\mathbf{R}^+)$ and $l > 1$. If, for each $T \geq r_0$, there exist increasing divergent sequences of positive numbers $\{u_n\}$, $\{t_n\}$, $\{v_n\}$ with $T \leq u_n < t_n < v_n$ such that*

$$\frac{1}{H(t_n,u_n)}\,\Theta^{\rho}_{u_n}\left(C_M - kg\left|h_1 + \frac{\rho'}{\rho}\right|^p; t_n\right) + \frac{1}{H(v_n,t_n)}\,\Gamma^{\rho}_{t_n}\left(C_M - kg\left|h_2 - \frac{\rho'}{\rho}\right|^p; v_n\right) > 0, \tag{3.2}$$

where $h_1 = h_1(s,u_n)$ and $h_2 = h(v_n,s)$, then (1.1) *is oscillatory.*

Proof. Let $y = y(x)$ be a nonoscillatory solution of (1.1); without loss of generality, we assume that $y = y(x) > 0$ for $\Omega(T)$ for some sufficiently large $T \geq r_0$. For any interval sequence $[u_n,v_n] \subset [T,\infty)$, by Lemmas 2.2 and 2.3, we have that for any $t_n \in [u_n,v_n]$,

$$\frac{1}{H(t_n,u_n)}\,\Theta^{\rho}_{u_n}\left(C_M - kg\left|h_1 + \frac{\rho'}{\rho}\right|^p; t_n\right) \leq -\rho(t_n)Z(t_n) \tag{3.3}$$

and

$$\frac{1}{H(v_n,t_n)}\,\Gamma^{\rho}_{t_n}\left(C_M - kg\left|h_2 - \frac{\rho'}{\rho}\right|^p; v_n\right) \leq \rho(t_n)Z(t_n). \tag{3.4}$$

But (3.2) implies that both (3.3) and (3.4) do not hold for the given t_n, and hence $y(x)$ must have a zero either in $\Omega_1(u_n,t_n)$ or $\Omega_2(t_n,v_n)$. Thus, $y = y(x)$ has at least one zero in $[u_n,v_n]$. Note that $\lim_{n\to\infty} u_n = \infty$. We can see that $y(x)$ have arbitrary large zeros. Therefore, (1.1) is oscillatory. The theorem is proved. □

Theorem 3.3. *Let $H \in \Im$, $\rho \in C^1([r_0,\infty),\mathbf{R}^+)$ and $l > 1$. If, for each $\tau \geq r_0$,*

$$\limsup_{r\to\infty} \Theta^{\rho}_{\tau}\left(C_M - kg\left|h_1 + \frac{\rho'}{\rho}\right|^p; r\right) > 0 \tag{3.5}$$

and

$$\limsup_{r\to\infty} \Gamma_\tau^\rho\bigg(C_M - kg\bigg|h_2 - \frac{\rho'}{\rho}\bigg|^p; r\bigg) > 0, \tag{3.6}$$

where $h_1 = h_1(s,\tau)$ *and* $h_2 = h_2(r,s)$, *then* (1.1) *is oscillatory.*

Proof. For any $T \geq r_0$, let $u_n = T$. In (3.5), we choose $\tau = u_n$. Then there exists $t_n > u_n$ such that

$$\Theta_{u_n}^\rho\bigg(C_M - kg\bigg|h_1 + \frac{\rho'}{\rho}\bigg|^p; t_n\bigg) > 0. \tag{3.7}$$

In (3.6) we choose $\tau = t_n$. Then there exists $v_n > t_n$ such that

$$\Gamma_{t_n}^\rho\bigg(C_M - kg\bigg|h_2 - \frac{\rho'}{\rho}\bigg|^p; v_n\bigg) > 0. \tag{3.8}$$

Combining (3.7) and (3.8), we obtain (3.2). The conclusion is thus from Theorem 3.2. The proof is complete. □

For the case when $H := H(r-s) \in \Im$, we have that $h_1(r-s) = h_2(r-s)$ and denote them by $h(r-s)$. The subclass of $\Im$ containing such $H(r-s)$ is denoted by $\Im_0$. Applying Theorem 3.2 to $\Im_0$, we obtain

Theorem 3.4. *Let* $H \in \Im_0$, $\rho \in C^1([r_0,\infty), \mathbf{R}^+)$ *and* $l > 1$. *If, for each* $T \geq r_0$, *there exist increasing divergent sequences of positive numbers* $\{u_n\}$, $\{t_n\}$ *with* $T < u_n < t_n$, *such that*

$$\begin{aligned} \int_{u_n}^{t_n} & H(s-u_n)[\rho(s)C_M(s) + \rho(2t_n - s)C_M(2t_n - s)]\,ds \\ & > k\bigg[\int_{u_n}^{t_n} H(s-u_n)\bigg[g(s)\rho(s)\bigg|h(s-u_n) + \frac{\rho'(s)}{\rho(s)}\bigg|^p \\ & \quad + g(2t_n - s)\rho(2t_n - s)\bigg|h(s-u_n) - \frac{\rho'(2t_n - s)}{\rho(2t_n - s)}\bigg|^p\bigg]\,ds\bigg], \end{aligned} \tag{3.9}$$

then (1.1) *is oscillatory.*

Proof. Let $v_n = 2t_n - u_n$. Then $H(v_n - t_n) = H(t_n - u_n) = H((v_n - u_n)/2)$, and for any $\xi \in L[u_n, v_n]$, we have

$$\int_{t_n}^{v_n} H(v_n - s)\xi(s)\,ds = \int_{u_n}^{t_n} H(s - u_n)\xi(2t_n - s)\,ds.$$

Thus that (3.9) holds implies that (3.2) holds for $H \in \Im_0$ and $\rho \in C^1([r_0,\infty), \mathbf{R}^+)$. Therefore (1.1) is oscillatory by Theorem 3.2. The theorem is proved. $\square$

Next, we define

$$G(r) = \int_{r_0}^{r} g^{1/(1-p)}(s)\,ds, \quad \rho(r) = 1, \quad r \geq r_0,$$

and let

$$H(r,s) = [G(r) - G(s)]^{\alpha}, \quad (r,s) \in D,$$

where $\alpha > p-1$ is a constant; then

$$h_1(r,s) = \frac{\alpha\, g^{1/(1-p)}(r)}{G(r) - G(s)}, \quad h_2(r,s) = \frac{\alpha\, g^{1/(1-p)}(s)}{G(r) - G(s)}, \quad (r,s) \in D_0.$$

Note that

$$\begin{aligned}\Gamma_\tau^\rho\left(g\left|h_2 - \frac{\rho'}{\rho}\right|^p; r\right) &= \alpha^p \int_\tau^r [G(r) - G(s)]^{\alpha-p}\,dG(s)\\ &= \frac{\alpha^p}{\alpha - p + 1}\,[G(r) - G(\tau)]^{\alpha-p+1}\end{aligned}$$

and

$$\begin{aligned}\Theta_\tau^\rho\left(g\left|h_1 + \frac{\rho'}{\rho}\right|^p; r\right) &= \alpha^p \int_\tau^r [G(s) - G(\tau)]^{\alpha-p}\,dG(s)\\ &= \frac{\alpha^p}{\alpha - p + 1}\,[G(r) - G(\tau)]^{\alpha-p+1}.\end{aligned}$$

Therefore, by Theorems 3.1 and 3.3, we can easily show that the following theorems hold; here we omit the details.

Theorem 3.5. *Let* $\lim_{r\to\infty} G(r) = \infty$ *and* $l > 1$. *If, for some* $\alpha > p-1$,

$$\limsup_{r\to\infty} \frac{1}{G^{\alpha}(r)} \int_{r_0}^{r} [G(r) - G(s)]^{\alpha} C_M(s)\, ds = \infty, \tag{3.10}$$

then (1.1) *is oscillatory.*

Remark 3.3. Theorem 3.5 improves Theorem 4 in [**12**] and Theorem 3.1 in [**14**] for (1.2).

Theorem 3.6. *Let* $\lim_{r\to\infty} G(r) = \infty$ *and* $l > 1$. *If, for some* $\alpha > p-1$,

$$\limsup_{r\to\infty} \frac{1}{G^{\alpha-p+1}(r)} \int_{\tau}^{r} [G(s) - G(\tau)]^{\alpha} C_M(s)\, ds \geq \frac{k\alpha^p}{\alpha - p + 1} \tag{3.11}$$

and

$$\limsup_{r\to\infty} \frac{1}{G^{\alpha-p+1}(r)} \int_{\tau}^{r} [G(r) - G(s)]^{\alpha} C_M(s)\, ds \geq \frac{k\alpha^p}{\alpha - p + 1}, \tag{3.12}$$

then (1.1) *is oscillatory.*

Now we give two examples to illustrate our results. To the best of our knowledge, no previous oscillation criteria can be applied to these examples.

Example 3.1. Consider equation (1.1) with

$$\begin{aligned} A(x) &= \operatorname{diag}\left(\frac{1}{\|x\|}, \frac{1}{\|x\|}\right), \quad b(x) = \left(\frac{\sin\|x\|}{\|x\|^3}, \frac{\cos\|x\|}{\|x\|^3}\right), \\ c(x) &= \frac{\|x\|\sin\|x\| + 2 - \cos\|x\|}{\|x\|}, \end{aligned} \tag{3.13}$$

where $x \in \Omega(r_0)$, $N = 2$, $p > 3/2$. For $l > 1$, a direct computation shows

$$\lambda(r) = 2^{\,-q/2} r^{1/(p-1)}, \quad g(r) = \pi(l^*)^{\,p-1} 2^{1+p/2}$$

and

$$C(x)=\frac{\|x\|\sin\|x\|+2-\cos\|x\|}{\|x\|}-\frac{1}{p}\left(\frac{l}{\varepsilon\, q}\right)^{p-1}2^{p/2}\|x\|^{-1-2p}.$$

Then

$$G(r)=c_1(r-r_0)\quad\text{and}\quad C_M(r)=2\pi(\,r\sin r+2-\cos r\,)-c_2r^{-2p},$$

where

$$c_1=(l^*)^{-1}\pi^{1/(1-p)}2^{(p+2)/(2(1-p))}\quad\text{and}\quad c_2=\frac{\pi}{p}\left(\frac{l}{\varepsilon\, q}\right)^{p-1}2^{1+p/2}.$$

Now, condition (3.10) leads to

$$\begin{aligned}&\frac{1}{G^2(r)}\int_{r_0}^{r}[G(r)-G(s)]^2C_M(s)\,ds\\&\quad\geq\frac{2\pi}{r^2}\int_{r_0}^{r}(r-s)^2d[\,s(2-\cos s)]-\frac{c_2}{r^2}\int_{r_0}^{r}(r-s)^2s^{-2p}\,ds\\&\quad=\frac{2\pi}{r^2}\left\{-r_0(2-\cos r_0)(r-r_0)^2+2\int_{r_0}^{r}s(2-\cos s)(r-s)\,ds\right\}\\&\qquad-\frac{c_2}{r^2}\int_{r_0}^{r}(r-s)^2s^{-2p}\,ds\longrightarrow\infty\quad\text{as}\quad r\to\infty;\end{aligned}$$

hence, (3.13) is oscillatory by Theorem 3.5.

Example 3.2. Consider equation (1.1) with

$$\begin{aligned}&A(x)=\operatorname{diag}\left(\frac{1}{\|x\|^6},\frac{1}{\|x\|^6},\frac{1}{\|x\|^6},\frac{1}{\|x\|^6}\right),\\&b_i(x)=0,\quad i=1,\dots,4,\qquad c(x)=\frac{162\,\pi^{3/2}u}{\|x\|^{10}},\end{aligned}\tag{3.14}$$

where $x\in\Omega(1)$, $N=p=4$, $u\geq(3/4)^3$. Let $\varepsilon=1$ and $l^*=\pi^{1/2}/2^{-5/3}$; then

$$q=\frac{4}{3},\quad k=\frac{1}{4}\left(\frac{3}{4}\right)^3,\quad \lambda(r)=2^{-4/3}r^2,\quad \omega_4=2\pi^{3/2},$$

and

$$g(r) = \frac{\pi^3}{r^3}, \quad G(r) = \frac{1}{2\pi}(r^2 - 1), \quad C_M(r) = \frac{324\,\pi^3 u}{r^7}.$$

Now, for $\alpha > 3$, we have that for all $\tau \ge 1$,

$$\begin{aligned}\limsup_{r\to\infty} \frac{1}{G^{\alpha-3}(r)} \int_\tau^r [G(s) - G(\tau)]^\alpha C_M(s)\, ds \\ &= \frac{81u}{2} \limsup_{r\to\infty} \frac{1}{r^{2(\alpha-3)}} \int_\tau^r \frac{(s^2-\tau^2)^\alpha}{s^7}\, ds \\ &= \frac{81u}{4(\alpha-3)} \limsup_{r\to\infty} \frac{(r^2-\tau^2)^\alpha}{r^{2\alpha}} = \frac{81u}{4(\alpha-3)}.\end{aligned}$$

For any $u \ge (3/4)^3$, there exists $\alpha > 3$ such that $81u/(4(\alpha-3)) > k\alpha^4/(\alpha-3)$. This means (3.11) holds. By Lemma 3.1 in [**5**], we find that (3.12) holds for the same α. Thus, all conditions of Theorem 3.6 are satisfied, so (3.14) is oscillatory.

Remark 3.4. The theorems above are presented in the form of a high degree of generality. They extend, improve, and complement a number of existing results in [**12, 14**–**16**] and handle some cases not covered by known criteria even for (1.2). They also give rather wide possibilities of deriving different explicit oscillation criteria for (1.1) with appropriate choices of the functions $H(r,s)$ and $\rho(r)$. Though throughout the paper we have always chosen $H \in \Im_0$, there are interesting possibilities to apply our results, for instance, with

$$H(r,s) = \left[\int_s^r \frac{dz}{\theta(z)}\right]^\alpha, \quad r \ge s \ge r_0,$$

where $\alpha > p - 1$ is a constant, $\theta \in C([r_0,\infty), \mathbf{R}^+)$ satisfying $\int_{r_0}^\infty 1/\theta(z)\, ds = \infty$. In fact, one of the important cases to be considered is $\theta(z) = z^\gamma$ with $\gamma \le 1$.

Acknowledgments. I am grateful to the referee for her/his suggestions and comments which led to an important improvement of my original manuscript.

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