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SOME MIXED-TYPE REVERSE-ORDER LAWS FOR THE MOORE-PENROSE INVERSE OF A TRIPLE MATRIX PRODUCT

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ABSTRACT. Using some rank formulas for partitioned matrices and outer inverses of a matrix, we derive necessary and sufficient conditions for a group of mixed-type reverse-order laws to hold for the Moore-Penrose inverse of a triple matrix product.

1. Introduction. Throughout this paper, A^*, $r(A)$ and $\mathcal{R}(A)$ denote the conjugate transpose, rank and range (column space) of a complex matrix A, respectively; $[\,A,\, B\,]$ denotes a row block matrix consisting of A and B.

Suppose A and B are two nonsingular matrices of the same size. Then the product AB is nonsingular, too, and the inverse of AB satisfies the ordinary reverse-order law $(AB)^{-1} = B^{-1}A^{-1}$. This law can be used to simplify various matrix expressions that involve inverses of matrix products. This formula, however, cannot trivially be extended to generalized inverses of matrix products. For an $m \times n$ matrix A, the Moore-Penrose inverse $A^{\dagger}$ of A is defined to be the unique solution of the following four Penrose equations

(i) $AXA = A$,

(ii) $XAX = X$,

(iii) $(AX)^* = AX$,

(iv) $(XA)^* = XA$.

For simplicity, let $E_A = I - AA^{\dagger}$ and $F_A = I - A^{\dagger}A$, which are two orthogonal projectors induced by A. A matrix X is called a generalized inverse of A, denoted by A^-, if it satisfies $AXA = A$, an outer inverse of A if it satisfies $XAX = X$, and a reflexive generalized inverse of A,

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denoted by A_r^-, if it satisfies both $AXA = A$ and $XAX = X$. General properties of the Moore-Penrose inverse can be found in [**1–3, 7**].

Let A, B and C be three matrices such that ABC is defined. One of the basic topics in the theory of generalized inverses is to investigate various reverse-order laws related to generalized inverses matrix products. Because both $AA^\dagger$ and $A^\dagger A$ are not necessarily identity matrices, the reverse-order laws $(AB)^\dagger = B^\dagger A^\dagger$ and $(ABC)^\dagger = C^\dagger B^\dagger A^\dagger$ do not always hold. In other words, they hold for some A, B and C, and for others they do not. Hence, it is of interest to seek necessary and sufficient conditions for

$$(AB)^\dagger = B^\dagger A^\dagger \quad \text{and} \quad (ABC)^\dagger = C^\dagger B^\dagger A^\dagger$$

to hold. In addition to these two reverse-order laws, $(AB)^\dagger$ and $(ABC)^\dagger$ may be expressed as

$$\begin{aligned} (AB)^\dagger &= B^\dagger A^\dagger + X_1, & (AB)^\dagger &= B^\dagger X_2 A^\dagger, \\ (ABC)^\dagger &= C^\dagger B^\dagger A^\dagger + Y_1, & (ABC)^\dagger &= C^\dagger Y_1 B^\dagger Y_2 A^\dagger, \end{aligned}$$

or other forms, for example,

$$\begin{aligned} (AB)^\dagger &= (A^\dagger AB)^\dagger (ABB^\dagger)^\dagger, & (AB)^\dagger &= B^\dagger (A^\dagger ABB^\dagger)^\dagger A^\dagger, \\ (ABC)^\dagger &= C^\dagger (A^\dagger ABCC^\dagger)^\dagger A^\dagger, & (ABC)^\dagger &= (BC)^\dagger B (AB)^\dagger. \end{aligned}$$

Due to the importance of reverse-order laws in dealing with generalized inverses of matrix products, various reverse-order laws have been widely investigated in the literature since 1960s. Some previous work on the reverse-order laws

$$(ABC)^\dagger = C^\dagger B^\dagger A^\dagger, \quad (ABC)^\dagger = (BC)^\dagger B (AB)^\dagger$$

for a triple matrix product can be found in [**4, 5, 8, 9**]. The law $(ABC)^\dagger = (BC)^\dagger B(AB)^\dagger$ arises from rewriting

$$ABC = ABB^\dagger BC = (AB)B^\dagger (BC) \stackrel{\text{def}}{=} PNQ$$

and considering the reverse-order law

$$(PNQ)^\dagger = Q^\dagger N^\dagger P^\dagger.$$

In this paper, we consider the following mixed-type reverse-order laws for $(ABC)^\dagger$:

(1.1) $(ABC)^\dagger = C^\dagger(A^\dagger ABCC^\dagger)^\dagger A^\dagger$,
(1.2) $(ABC)^\dagger = C^*(A^*ABCC^*)^\dagger A^*$,
(1.3) $(ABC)^\dagger = (C^*C)^\dagger[(C^\dagger B^*A^\dagger)^\dagger]^*(AA^*)^\dagger$,
(1.4) $(ABC)^\dagger = C^*C(AA^*ABCC^*C)^\dagger AA^*$,
(1.5) $(ABC)^\dagger = (BC)^\dagger[(AB)^\dagger ABC(BC)^\dagger]^\dagger(AB)^\dagger$,
(1.6) $(ABC)^\dagger = (BC)^*[(AB)^*ABC(BC)^*]^\dagger(AB)^*$,
(1.7)
$$(ABC)^\dagger = [(BC)^*(BC)]^\dagger\{[(BC)^\dagger(B^*)^\dagger(AB)^\dagger]^\dagger\}^*[(AB)(AB)^*]^\dagger,$$
(1.8)
$$(ABC)^\dagger = [I_q-(C^\dagger F_B)(C^\dagger F_B)^\dagger]C^\dagger B^\dagger A^\dagger[I_m-(E_BA^\dagger)^\dagger(E_BA^\dagger)].$$

These mixed-type reverse-order laws in fact are all reasonable expressions of $(ABC)^\dagger$ under different decompositions of ABC. It is easy to verify that

$$A = AA^\dagger A = AA^*(A^\dagger)^* = (A^\dagger)^*A^*A = (AA^*A)(A^*A)^\dagger = (AA^*)^\dagger(AA^*A). \tag{1.9}$$

From (1.9), ABC can be written as

$$ABC = AA^\dagger ABCC^\dagger C = A(A^\dagger ABCC^\dagger)C \overset{\text{def}}{=} P_1N_1Q_1.$$

Then, (1.1) arises from considering the reverse-order law $(P_1N_1Q_1)^\dagger = Q_1^\dagger N_1^\dagger P_1^\dagger$. Write

$$ABC = (A^\dagger)^*A^*ABCC^*(C^\dagger)^* = (A^\dagger)^*(A^*ABCC^*)(C^\dagger)^* \overset{\text{def}}{=} P_2N_2Q_2.$$

Then, (1.2) is from $(P_2N_2Q_2)^\dagger = Q_2^\dagger N_2^\dagger P_2^\dagger$. Write

$$ABC = AA^*(A^\dagger)^*B(C^\dagger)^*C^*C = AA^*[(A^\dagger)^*B(C^\dagger)^*]C^*C \overset{\text{def}}{=} P_3N_3Q_3.$$

Then, $(P_3N_3Q_3)^\dagger = Q_3^\dagger N_3^\dagger P_3^\dagger$ is (1.3). Write

$$ABC = (AA^*)^\dagger AA^*ABCC^*C(C^*C)^\dagger = (AA^*)^\dagger(AA^*ABCC^*C)(C^*C)^\dagger \overset{\text{def}}{=} P_4N_4Q_4.$$

Then, $(P_4N_4Q_4)^\dagger = Q_4^\dagger N_4^\dagger P_4^\dagger$ is (1.4). Write

$$\begin{aligned} ABC &= AB(AB)^\dagger ABC(BC)^\dagger BC = AB[\,(AB)^\dagger ABC(BC)^\dagger\,]BC \\ &\overset{\text{def}}{=} P_5N_5Q_5. \end{aligned}$$

Then, $(P_5N_5Q_5)^\dagger = Q_5^\dagger N_5^\dagger P_5^\dagger$ is (1.5). Write

$$\begin{aligned} ABC &= [\,(AB)^\dagger\,]^*(AB)^*ABC(BC)[\,(BC)^\dagger\,]^* \\ &= [\,(AB)^\dagger\,]^*[\,(AB)^*ABC(BC)^*\,][\,(BC)^\dagger\,]^* \overset{\text{def}}{=} P_6N_6Q_6. \end{aligned}$$

Then, $(P_6N_6Q_6)^\dagger = Q_6^\dagger N_6^\dagger P_6^\dagger$ becomes (1.6). Write

$$\begin{aligned} ABC &= AB(AB)^*[\,(AB)^\dagger\,]^*B^\dagger[\,(BC)^\dagger\,]^*(BC)^*BC \\ &= [\,AB(AB)^*\,]\{[\,(AB)^\dagger\,]^*B^\dagger[\,(BC)^\dagger\,]^*\}[\,(BC)^*BC\,] \overset{\text{def}}{=} P_7N_7Q_7. \end{aligned}$$

Then, $(P_7N_7Q_7)^\dagger = Q_7^\dagger N_7^\dagger P_7^\dagger$ becomes (1.7).

It has been shown that the rank of matrix is a simple and powerful method for investigating the relations between any two matrix expressions involving generalized inverses. In fact, any two matrices A and B of the same size are equal if and only if $r(\,A - B\,) = 0$. If one can find some nontrivial formulas for the rank of $A - B$, then necessary and sufficient conditions for $A = B$ to hold can be derived from these rank formulas. As examples, several simple rank formulas for the differences of matrices found by the present author are given below

$$r(\,A^kA^\dagger - A^\dagger A^k\,) = r\begin{bmatrix} A^k \\ A^* \end{bmatrix} + r[\,A^k,\, A^*\,] - 2r(A),$$

$$r(\,A^*A^\dagger - A^\dagger A^*\,) = r(\,AA^*A^2 - A^2A^*A\,),$$

$$r(\,AB - ABB^\dagger A^\dagger AB\,) = r[\,A^*,\, B\,] + r(AB) - r(A) - r(B),$$

$$r\left([\,A,\, B\,]^\dagger - \begin{bmatrix} A^\dagger \\ B^\dagger \end{bmatrix}\right) = r[\,AA^*B,\, BB^*A\,],$$

$$r\left([\,A,\, B\,]^\dagger[\,A,\, B\,] - \begin{bmatrix} A^\dagger A & 0 \\ 0 & B^\dagger B \end{bmatrix}\right) = r(A) + r(B) - r[\,A,\, B\,],$$

$$\min_{A^-,B^-} r(\,A^- - B^-\,) = r(\,A - B\,) - r\begin{bmatrix} A \\ B \end{bmatrix} - r[\,A,\, B\,] + r(A) + r(B),$$

see Tian [**10–14**]. For $(AB)^\dagger$, it is shown in Tian [**14, 15**] that

$$\begin{aligned} &r[\,(AB)^\dagger - B^\dagger(A^\dagger ABB^\dagger)^\dagger A^\dagger\,] \\ &\qquad = r\begin{bmatrix} M \\ MB^*B \end{bmatrix} + r[\,M,\ AA^*M\,] - 2r(M), \end{aligned} \tag{1.10}$$

$$\begin{aligned} &r[\,(AB)^\dagger - B^*(A^*ABB^*)^\dagger A^*\,] \\ &\qquad = r\begin{bmatrix} M \\ MB^*B \end{bmatrix} + r[\,M,\ AA^*M\,] - 2r(M), \end{aligned} \tag{1.11}$$

$$\begin{aligned} &r[\,(AB)^\dagger - B^\dagger A^\dagger - B^\dagger(E_BF_A)^\dagger A^\dagger\,] \\ &\qquad = r\begin{bmatrix} M \\ MB^*B \end{bmatrix} + r[\,M,\ AA^*M\,] - 2r(M), \end{aligned} \tag{1.12}$$

where $M = AB$. Many consequences can be derived from these rank equalities. For instance, letting the right-hand sides of these rank equalities be zero and simplifying by some elementary methods, one can immediately obtain necessary and sufficient conditions for the matrices on the left-hand sides to be zero.

In order to simplify ranks of block matrices, we need to use the following formulas due to Marsaglia and Styan [**6**]:

$$r[\,A,\ B\,] = r(A) + r(\,B - AA^\dagger B\,), \tag{1.13}$$

$$r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} = r(B) + r(C) + r[\,(I - BB^\dagger)A(I - C^\dagger C)\,]; \tag{1.14}$$

if $\mathcal{R}(B) \subseteq \mathcal{R}(A)$ and $\mathcal{R}(C^*) \subseteq \mathcal{R}(A^*)$, then

$$r\begin{bmatrix} A & B \\ C & D \end{bmatrix} = r(A) + r(\,D - CA^\dagger B\,). \tag{1.15}$$

In general, the rank of the Schur complement $D - CA^\dagger B$ is

$$r(\,D - CA^\dagger B\,) = r\begin{bmatrix} A^*AA^* & A^*B \\ CA^* & D \end{bmatrix} - r(A), \tag{1.16}$$

which is derived from (1.15) and $A^*(A^*AA^*)^\dagger A^* = A^\dagger$, see [**17**]. Another rank formula widely used in this paper is given below.

Lemma 1.1 [**11, 15**]. *Suppose $X_1, X_2 \in \mathbf{C}^{m\times n}$. Then they are two outer inverses of some $n \times m$ matrix, i.e., there is an M such that $X_1MX_1 = X_1$ and $X_2MX_2 = X_2$, if and only if*

$$r(X_1 - X_2) = r\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + r[X_1,\ X_2] - r(X_1) - r(X_2). \tag{1.17}$$

The "only if" part is proved in [**11**]; the "if" part is given in [**15**]. In addition, we use the following properties, see [**1, 3, 7**] when simplifying various rank equalities:

$$\mathcal{R}(B) \subseteq \mathcal{R}(A) \Longleftrightarrow r[A,\ B] = r(A), \tag{1.18}$$

$$\mathcal{R}(A) \subseteq \mathcal{R}(B) \quad \text{and} \quad r(A) = r(B) \Longrightarrow \mathcal{R}(A) = \mathcal{R}(B), \tag{1.19}$$

$$\mathcal{R}(A) = \mathcal{R}(AA^*) = \mathcal{R}(AA^*A) = \mathcal{R}(AA^\dagger) = \mathcal{R}[(A^\dagger)^*], \tag{1.20}$$

$$\mathcal{R}(A^*) = \mathcal{R}(A^*A) = \mathcal{R}(A^*AA^*) = \mathcal{R}(A^\dagger) = \mathcal{R}(A^\dagger A), \tag{1.21}$$

$$r(AB^\dagger) = r(AB^*), \quad \mathcal{R}(AB^\dagger) = \mathcal{R}(AB^*), \tag{1.22}$$

$$\mathcal{R}(A_1) = \mathcal{R}(A_2) \quad \text{and} \quad \mathcal{R}(B_1) = \mathcal{R}(B_2) \Longrightarrow r[A_1,\ B_1] = r[A_2,\ B_2]. \tag{1.23}$$

2. Main results. In this section, we shall establish a set of rank formulas associated with (1.1)–(1.8), and then derive from these rank formulas necessary and sufficient conditions for (1.1)–(1.8) to hold.

Theorem 2.1. *Let $A \in \mathbf{C}^{m\times n}$, $B \in \mathbf{C}^{n\times p}$, $C \in \mathbf{C}^{p\times q}$ and let $M = ABC$. Then*

$$r[M^\dagger - C^\dagger(A^\dagger MC^\dagger)^\dagger A^\dagger] = r\begin{bmatrix} M \\ MC^*C \end{bmatrix} + r[M,\ AA^*M] - 2r(M). \tag{2.1}$$

In particular, the reverse-order law (1.1) *holds if and only if M satisfies the following two range equalities*

$$\mathcal{R}(AA^*M) = \mathcal{R}(M) \quad \textit{and} \quad \mathcal{R}(C^*CM^*) = \mathcal{R}(M^*). \tag{2.2}$$

Proof. Let $X_1 = C^\dagger(A^\dagger MC^\dagger)^\dagger A^\dagger$. It is easy to verify that

$$\begin{aligned} MX_1M &= M[\,C^\dagger(A^\dagger MC^\dagger)^\dagger A^\dagger\,]M \\ &= A(A^\dagger MC^\dagger)(A^\dagger MC^\dagger)^\dagger(A^\dagger MC^\dagger)C \\ &= A(A^\dagger MC^\dagger)C = M \end{aligned}$$

and

$$\begin{aligned} X_1MX_1 &= [\,C^\dagger(A^\dagger MC^\dagger)^\dagger A^\dagger\,]M[\,C^\dagger(A^\dagger MC^\dagger)^\dagger A^\dagger\,] \\ &= C^\dagger(A^\dagger MC^\dagger)^\dagger(A^\dagger MC^\dagger)(A^\dagger MC^\dagger)^\dagger A^\dagger \\ &= C^\dagger(A^\dagger MC^\dagger)^\dagger A^\dagger = X_1. \end{aligned}$$

Hence, the matrix X_1 is a reflexive generalized inverse of M with $r(X_1) = r(M)$. Applying (1.17) to $M^\dagger - X_1$ gives

$$r(\,M^\dagger - X_1\,) = r\begin{bmatrix} M^\dagger \\ X_1 \end{bmatrix} + r[\,M^\dagger,\ X_1\,] - 2r(M). \tag{2.3}$$

Note that

$$\begin{aligned} \mathcal{R}(X_1) &\subseteq \mathcal{R}[\,C^\dagger(A^\dagger MC^\dagger)^\dagger\,] = \mathcal{R}[\,C^\dagger(A^\dagger MC^\dagger)^*\,] = \mathcal{R}[\,C^\dagger B^*A^*(A^\dagger)^*\,] \\ &\subseteq \mathcal{R}(C^\dagger B^*A^*), \end{aligned}$$

and also note that $r(X_1) = r(C^\dagger B^*A^*) = r(M)$. Hence, $\mathcal{R}(X_1) = \mathcal{R}(C^\dagger B^*A^*)$ by (1.19). Then we see by (1.23) that

$$r[\,M^\dagger,\ X_1\,] = r[\,M^*,\ C^\dagger B^*A^*\,].$$

From the following two equalities

$$C^*C[\,M^*,\ C^\dagger B^*A^*\,] = [\,C^*CM^*,\ C^*CC^\dagger B^*A^*\,] = [\,C^*CM^*,\ M^*\,]$$

and

$$\begin{aligned} C^\dagger(C^\dagger)^*[\,C^*CM^*,\ M^*\,] &= [\,C^\dagger(C^\dagger)^*C^*CC^*B^*A^*,\ C^\dagger(C^\dagger)^*C^*B^*A^*\,] \\ &= [\,M^*,\ C^\dagger B^*A^*\,], \end{aligned}$$

we also obtain $r[\,M^*,\ C^\dagger B^*A^*\,] = r[\,C^*CM^*,\ M^*\,]$. Hence,

$$r[\,M^\dagger,\ X_1\,] = r[\,C^*CM^*,\ M^*\,] = r\begin{bmatrix} M \\ MC^*C \end{bmatrix}.$$

Similarly, we obtain

$$r\begin{bmatrix} M^\dagger \\ X_1 \end{bmatrix} = r\begin{bmatrix} M^* \\ M^*AA^* \end{bmatrix} = r[\, M,\ AA^*M\,].$$

Thus, (2.3) is reduced to (2.1). Letting the right-hand side of (2.1) be zero, we obtain (2.2) by (1.18) and (1.19). □

As a special case, if $\mathcal{R}(B) \subseteq \mathcal{R}(A^*)$ and $\mathcal{R}(B^*) \subseteq \mathcal{R}(C)$, then (2.1) becomes

$$(2.4)\qquad r(\, M^\dagger - C^\dagger B^\dagger A^\dagger\,) = r\begin{bmatrix} M \\ MC^*C \end{bmatrix} + r[\, M,\ AA^*M\,] - 2r(M).$$

In particular, the reverse-order law $(ABC)^\dagger = C^\dagger B^\dagger A^\dagger$ holds if and only if ABC satisfies (2.2). Moreover, if both A and C are nonsingular, then

$$(2.5)\quad r(\, M^\dagger - C^{-1} B^\dagger A^{-1}\,) = r\begin{bmatrix} M \\ MC^*C \end{bmatrix} + r[\, M,\ AA^*M\,] - 2r(M),$$

and the reverse-order law $(ABC)^\dagger = C^{-1}B^\dagger A^{-1}$ holds if and only if ABC satisfies (2.2). Results (2.4) and (2.5) were given in Tian [**11**].

Theorem 2.2. *Let $A \in \mathbf{C}^{m\times n}$, $B \in \mathbf{C}^{n\times p}$, $C \in \mathbf{C}^{p\times q}$ and let $M = ABC$. Then*

$$(2.6)\qquad r[\, M^\dagger - C^*(A^*MC^*)^\dagger A^*\,] = r\begin{bmatrix} M \\ MC^*C \end{bmatrix} + r[\, M,\ AA^*M\,] - 2r(M).$$

In particular, the reverse-order law (1.2) *holds if and only if M satisfies* (2.2).

Proof. Let $X_2 = C^*(A^*MC^*)^\dagger A^*$. Then it is easy to verify that

$$\begin{aligned} MX_2M &= MC^*(A^*MC^*)^\dagger A^*M \\ &= (A^\dagger)^*(A^*MC^*)(A^*MC^*)^\dagger(A^*MC^*)(C^\dagger)^* \\ &= (A^\dagger)^*A^*MC^*(C^\dagger)^* = M \end{aligned}$$

and

$$\begin{aligned} X_2MX_2 &= C^*(A^*MC^*)^\dagger A^*MC^*(A^*MC^*)^\dagger A^* \\ &= C^*(A^*MC^*)^\dagger (A^*MC^*)(A^*MC^*)^\dagger A^* \\ &= C^*(A^*MC^*)^\dagger A^* = X_2. \end{aligned}$$

These two results imply that $C^*(A^*MC^*)^\dagger A^*$ is a reflexive generalized inverse of M. Hence by (1.17)

$$r(\, M^\dagger - X_2\,) = r\begin{bmatrix} M^\dagger \\ X_2 \end{bmatrix} + r[\, M^\dagger,\ X_2\,] - 2r(M). \tag{2.7}$$

From (1.18)–(1.23) we also find that

$$r\begin{bmatrix} M^\dagger \\ X_2 \end{bmatrix} = r\begin{bmatrix} M^* \\ M^*AA^* \end{bmatrix} = r[\, M,\ AA^*M\,]$$

and

$$r[\, M^\dagger,\, X_2\,] = r[\, M^*,\ C^*CM^*\,] = r\begin{bmatrix} M \\ MC^*C \end{bmatrix}.$$

Thus, (2.6) follows from (2.7). □

Theorem 2.3. *Let $A \in \mathbf{C}^{m\times n}$, $B \in \mathbf{C}^{n\times p}$, $C \in \mathbf{C}^{p\times q}$ and let $M = ABC$. Then*

$$\begin{aligned} &r\{\, M^\dagger - (C^*C)^\dagger[\,(C^\dagger B^*A^\dagger)^\dagger\,]^*(AA^*)^\dagger\,\} \\ &\qquad = r\begin{bmatrix} M \\ M(C^*C)^2 \end{bmatrix} + r[\, M,\ (AA^*)^2M\,] - \ 2r(M). \end{aligned} \tag{2.8}$$

In particular, the reverse-order law (1.3) *holds if and only if*

$$\mathcal{R}[\,(AA^*)^2M\,] = \mathcal{R}(M) \quad \text{and} \quad \mathcal{R}[\,(C^*C)^2M^*\,] = \mathcal{R}(M^*). \tag{2.9}$$

Proof. Let $X_3 = (C^*C)^\dagger[\,(C^\dagger B^*A^\dagger)^\dagger\,]^*(AA^*)^\dagger$. Then it is easy to verify that

$$\begin{aligned} MX_3M &= ABC(C^*C)^\dagger[\,(C^\dagger B^*A^\dagger)^\dagger\,]^*(AA^*)^\dagger ABC \\ &= AB(C^\dagger)^*[\,(C^\dagger B^*A^\dagger)^\dagger\,]^*(A^\dagger)^*BC \\ &= AA^*[\,(A^\dagger)^*B(C^\dagger)^*\,][\,(C^\dagger B^*A^\dagger)^\dagger\,]^*[\,(A^\dagger)^*B(C^\dagger)^*\,]C^*C \\ &= AA^*[\,(A^\dagger)^*B(C^\dagger)^*\,]C^*C = M \end{aligned}$$

and

$$X_3MX_3 = [\,(C^\dagger B^*A^\dagger)^\dagger\,]^*(AA^*)^\dagger ABC(C^*C)^\dagger[\,(C^\dagger B^*A^\dagger)^\dagger\,]^*(AA^*)^\dagger$$
$$= [\,(C^\dagger B^*A^\dagger)^\dagger\,]^*(A^\dagger)^*B(C^\dagger)^*[\,(C^\dagger B^*A^\dagger)^\dagger\,]^*(AA^*)^\dagger = X_3.$$

Hence, X_3 is a reflexive generalized inverse of M, and $r(X_3) = r(M)$. Applying (1.17)–(1.23) to $M^\dagger - X_3$ gives

$$r(\,M^\dagger - X_3\,) = r\begin{bmatrix} M^\dagger \\ X_3 \end{bmatrix} + r[\,M^\dagger,\ X_3\,] - 2r(M), \tag{2.10}$$

where

$$r\begin{bmatrix} M^\dagger \\ X_3 \end{bmatrix} = r\begin{bmatrix} M^* \\ M^*(AA^*)^2 \end{bmatrix} = r[\,M,\ (AA^*)^2M\,]$$

and

$$r[\,M^\dagger,\ X_3\,] = r[\,M^*,\ (C^*C)^2M^*\,] = r\begin{bmatrix} M \\ M(C^*C)^2 \end{bmatrix}.$$

Hence, (2.10) is reduced to (2.8). Let the right-hand side of (2.8) be zero, and notice $r[\,(AA^*)^2M\,] = r[\,M(C^*C)^2\,] = r(M)$. Then we obtain (2.9) by (1.18) and (1.19). □

Moreover, the right-hand sides of (1.4)–(1.7) are all reflexive generalized inverses of ABC. We leave the verification of these results for the reader. In these cases, we are able to find by (1.17)–(1.23) the following three theorems.

Theorem 2.4. *Let* $A \in \mathbf{C}^{m\times n}$, $B \in \mathbf{C}^{n\times p}$, $C \in \mathbf{C}^{p\times q}$, *and let* $M = ABC$. *Then*

$$\begin{aligned} &r[\,M^\dagger - C^*C(AA^*MC^*C)^\dagger AA^*\,] \\ &\qquad = r\begin{bmatrix} M \\ M(C^*C)^2 \end{bmatrix} + r[\,M,\ (AA^*)^2M\,] - 2r(M). \end{aligned} \tag{2.11}$$

In particular, the reverse-order law (1.4) *holds if and only if* M *satisfies* (2.9); *i.e.,* (1.3) *and* (1.4) *are equivalent.*

Theorem 2.5. *Let $A \in \mathbf{C}^{m\times n}$, $B \in \mathbf{C}^{n\times p}$, $C \in \mathbf{C}^{p\times q}$, and let $M = ABC$. Then*

$$r\{ M^{\dagger} - (BC)^{\dagger}[(AB)^{\dagger}ABC(BC)^{\dagger}]^{\dagger}(AB)^{\dagger} \} = r\begin{bmatrix} M \\ M(BC)^*(BC) \end{bmatrix} + r[M, \ (AB)(AB)^*M] - 2r(M) \tag{2.12}$$

$$r\{ M^{\dagger} - (BC)^{*}[(AB)^{*}ABC(BC)^{*}]^{\dagger}(AB)^{*} \} = r\begin{bmatrix} M \\ M(BC)^*(BC) \end{bmatrix} + r[M, \ (AB)(AB)^*M] - 2r(M). \tag{2.13}$$

Hence, the reverse-order laws in (1.5) *and* (1.6) *are equivalent, and they hold if and only if M satisfies*

$$\mathcal{R}[(AB)(AB)^*M] = \mathcal{R}(M) \quad \textit{and} \quad \mathcal{R}[(BC)^*(BC)M^*] = \mathcal{R}(M^*). \tag{2.14}$$

If $r(ABC) = r(B)$, then $r(AB) = r(BC) = r(B)$, and then $(AB)^{\dagger}AB = B^{\dagger}B$ and $BC(BC)^{\dagger} = BB^{\dagger}$. Hence, (2.14) is satisfied and (1.5) is reduced to $(ABC)^{\dagger} = (BC)^{\dagger}B(AB)^{\dagger}$. This was proved in Tian [**8**].

Theorem 2.6. *Let $A \in \mathbf{C}^{m\times n}$, $B \in \mathbf{C}^{n\times p}$, $C \in \mathbf{C}^{p\times q}$, and let $M = ABC$. Then*

$$r\{ M^{\dagger} - [(BC)^*(BC)]^{\dagger}\{[(BC)^{\dagger}(B^*)^{\dagger}(AB)^{\dagger}]^{\dagger}\}^*[(AB)(AB)^*]^{\dagger} \} = r\begin{bmatrix} M \\ M\,[(BC)^*(BC)]^2 \end{bmatrix} + r[M, \ [(AB)(AB)^*]^2\, M] - 2r(M). \tag{2.15}$$

In particular, the reverse-order law (7) *holds if and only if M satisfies*

$$\mathcal{R}\{ [(AB)(AB)^*]^2 M \} = \mathcal{R}(M)$$

and

$$\mathcal{R}\{[(BC)^*(BC)]^2 M^*\} = \mathcal{R}(M^*). \tag{2.16}$$

Note that the right-hand sides of (1.1)–(1.7) are all outer inverses of ABC. Some rank equalities for the differences of these outer inverses can also be derived from by (1.17).

Theorem 2.7. *Let $A \in \mathbf{C}^{m\times n}$, $B \in \mathbf{C}^{n\times p}$, $C \in \mathbf{C}^{p\times q}$, and let $M = ABC$. Then*

$$\begin{aligned} & r[\, C^{\dagger}(A^{\dagger}MC^{\dagger})^{\dagger}A^{\dagger} - C^{*}(A^{*}MC^{*})^{\dagger}A^{*}\,] \\ & \qquad = r\begin{bmatrix} M \\ M(C^{*}C)^{2} \end{bmatrix} + r[\, M,\, (AA^{*})^{2}M\,] - 2r(M), \end{aligned} \tag{2.17}$$

$$\begin{aligned} & r\{\, C^{\dagger}(A^{\dagger}MC^{\dagger})^{\dagger}A^{\dagger} - (C^{*}C)^{\dagger}[\,(C^{\dagger}B^{*}A^{\dagger})^{\dagger}\,]^{*}(AA^{*})^{\dagger}\,\} \\ & \qquad = r\begin{bmatrix} M \\ MC^{*}C \end{bmatrix} + r[\, M, \mathring{A}^{*}M\,] - 2r(M), \end{aligned} \tag{2.18}$$

$$\begin{aligned} & r[\, C^{*}(A^{*}MC^{*})^{\dagger}A^{*} - C^{*}C(AA^{*}ABCC^{*}C)^{\dagger}AA^{*}\,] \\ & \qquad = r\begin{bmatrix} M \\ MC^{*}C \end{bmatrix} + r[\, M, \mathring{A}^{*}M\,] - 2r(M). \end{aligned} \tag{2.19}$$

Observe that the right-hand sides of (2.1), (2.6), (2.18) and (2.19) are all identical. We obtain the following theorem.

Theorem 2.8. *Let $A \in \mathbf{C}^{m\times n}$, $B \in \mathbf{C}^{n\times p}$, $C \in \mathbf{C}^{p\times q}$, and let $M = ABC$. Then the following statements are equivalent*:

(a) $(ABC)^{\dagger} = C^{\dagger}(A^{\dagger}ABCC^{\dagger})^{\dagger}A^{\dagger}$.

(b) $(ABC)^{\dagger} = C^{*}(A^{*}ABCC^{*})^{\dagger}A^{*}$.

(c) $C^{\dagger}(A^{\dagger}ABCC^{\dagger})^{\dagger}A^{\dagger} = (C^{*}C)^{\dagger}[\,(C^{\dagger}B^{*}A^{\dagger})^{\dagger}\,]^{*}(AA^{*})^{\dagger}$.

(d) $A(CC^{\dagger}B^{*}A^{\dagger}A)^{\dagger}C = AA^{*}[\,(ABC)^{\dagger}\,]^{*}C^{*}C$.

(e) $C^{*}(A^{*}ABCC^{*})^{\dagger}A^{*} = C^{*}C(AA^{*}ABCC^{*}C)^{\dagger}AA^{*}$.

(f) $\mathcal{R}(AA^{*}M) = \mathcal{R}(M)$ *and* $\mathcal{R}(C^{*}CM^{*}) = \mathcal{R}(M^{*})$.

From (2.8), (2.11) and (2.17) we obtain the following result.

Theorem 2.9. *Let $A \in \mathbf{C}^{m\times n}$, $B \in \mathbf{C}^{n\times p}$, $C \in \mathbf{C}^{p\times q}$, and let $M = ABC$. Then the following statements are equivalent*:

(a) $(ABC)^{\dagger} = (C^*C)^{\dagger}[\,(C^{\dagger}B^*A^{\dagger})^{\dagger}\,]^*(AA^*)^{\dagger}$.

(b) $(ABC)^{\dagger} = C^*C(AA^*ABCC^*C)^{\dagger}AA^*$.

(c) $C^{\dagger}(A^{\dagger}MC^{\dagger})^{\dagger}A^{\dagger} = C^*(A^*MC^*)^{\dagger}A^*$.

(d) $\mathcal{R}[\,(AA^*)^2M\,] = \mathcal{R}(M)$ *and* $\mathcal{R}[\,(C^*C)^2M^*\,] = \mathcal{R}(M^*)$.

In addition, we are also able to establish by (1.17) the following rank equalities

$$r\{\,M^{\dagger} - (CC^*C)^{\dagger}[\,(A^*A)^{\dagger}B(CC^*)^{\dagger}\,]^{\dagger}(AA^*A)^{\dagger}\,]\,\} = r\begin{bmatrix} M \\ M(C^*C)^3 \end{bmatrix} + r[\,M,\,(AA^*)^3M\,] - 2r(M), \tag{2.20}$$

$$r\{\,M^{\dagger} - C^*CC^*[\,(A^*A)^2B(CC^*)^2\,]^{\dagger}A^*AA^*\} = r\begin{bmatrix} M \\ M(C^*C)^3 \end{bmatrix} + r[\,M,\,(AA^*)^3M\,] - 2r(M), \tag{2.21}$$

$$r\{\,M^{\dagger} - [\,(C^*C)^{\dagger}\,]^2[\,(A^*AA^*)^{\dagger}B(C^*CC^*)^{\dagger}\,]^{\dagger}[\,(AA^*)^{\dagger}\,]^2\} = r\begin{bmatrix} M \\ M(C^*C)^4 \end{bmatrix} + r[\,M,\,(AA^*)^4M\,] - 2r(M), \tag{2.22}$$

$$r\{\,M^{\dagger} - (B^*B)^2[\,(AA^*)^2M(C^*C)^2\,]^{\dagger}(AA^*)^2\} = r\begin{bmatrix} M \\ M(C^*C)^4 \end{bmatrix} + r[\,M,\,(AA^*)^4M\,] - 2r(M), \tag{2.23}$$

$$r[\,C^{\dagger}(A^{\dagger}MC^{\dagger})^{\dagger}A^{\dagger} - C^*C(AA^*MC^*C)^{\dagger}AA^*\,] = r\begin{bmatrix} M \\ M(C^*C)^3 \end{bmatrix} + r[\,M,\,(AA^*)^3M\,] - 2r(M). \tag{2.24}$$

Equalities (2.20), (2.21) and (2.24) imply the following result.

Theorem 2.10. *Let* $A \in \mathbf{C}^{m\times n}$, $B \in \mathbf{C}^{n\times p}$ *and* $C \in \mathbf{C}^{p\times q}$, *and let* $M = ABC$. *Then the following statements are equivalent*:

(a) $(ABC)^{\dagger} = (CC^*C)^{\dagger}[\,(A^*A)^{\dagger}B(CC^*)^{\dagger}\,]^{\dagger}(AA^*A)^{\dagger}$.

(b) $(ABC)^{\dagger} = C^*CC^*[\,(A^*A)^2B(CC^*)^2\,]^{\dagger}A^*AA^*$.

(c) $C^{\dagger}(A^{\dagger}MC^{\dagger})^{\dagger}A^{\dagger} = C^*C(AA^*MC^*C)^{\dagger}AA^*$.

(d) $(A^{\dagger}MC^{\dagger})^{\dagger} = CC^*C(AA^*MC^*C)^{\dagger}AA^*A$.

(e) $\mathcal{R}[\,(AA^*)^3M\,] = \mathcal{R}(M)$ *and* $\mathcal{R}[\,(C^*C)^3M^*\,] = \mathcal{R}(M^*)$.

The following consequence is derived from the two formulas in (2.22) and (2.23).

Theorem 2.11. *Let* $A \in \mathbf{C}^{m\times n}$, $B \in \mathbf{C}^{n\times p}$, $C \in \mathbf{C}^{p\times q}$, *and let* $M = ABC$. *Then the following statements are equivalent*:

(a) $(ABC)^{\dagger} = [\,(C^*C)^{\dagger}\,]^2[\,(A^*AA^*)^{\dagger}B(C^*CC^*)^{\dagger}\,]^{\dagger}[\,(AA^*)^{\dagger}\,]^2$.

(b) $(ABC)^{\dagger} = (B^*B)^2[\,(AA^*)^2M(C^*C)^2\,]^{\dagger}(AA^*)^2$.

(c) $\mathcal{R}[\,(AA^*)^4M\,] = \mathcal{R}(M)$ *and* $\mathcal{R}[\,(C^*C)^4M^*\,] = \mathcal{R}(M^*)$.

Some more rank equalities related to the right-hand sides of (1.5), (1.6) and (1.7) can also be established. For instance,

(2.25)
$$r\{\,(BC)^{\dagger}[\,(AB)^{\dagger}M(BC)^{\dagger}\,]^{\dagger}(AB)^{\dagger} - (BC)^*[\,(AB)^*M(BC)^*\,]^{\dagger}(AB)^*\,\} = r\begin{bmatrix} M \\ M\,[\,(BC)^*(BC)\,]^2 \end{bmatrix} + r[\,M,\ [\,(AB)(AB)^*\,]^2M\,] - 2r(M),$$

(2.26)
$$r\{\,(BC)^{\dagger}[\,(AB)^{\dagger}M(BC)^{\dagger}\,]^{\dagger}(AB)^{\dagger} - [\,(BC)^*(BC)\,]^{\dagger}\{[\,(BC)^{\dagger}(B^*)^{\dagger}(AB)^{\dagger}\,]^{\dagger}\}^*[\,(AB)(AB)^*\,]^{\dagger}\,\} = r\begin{bmatrix} M \\ M(BC)^*BC \end{bmatrix} + r[\,M,\ AB(AB)^*M\,] - 2r(M),$$

(2.27)

$$r\{\,(ABC)^{\dagger} - (BC)^{*}(BC)[\,(AB)(AB)^{*}M(BC)^{*}(BC)\,]^{\dagger}(AB)(AB)^{*}\,\}$$
$$= r\begin{bmatrix} M \\ M\,[\,(BC)^{*}(BC)\,]^{2} \end{bmatrix} + r[\,M,\ [\,(AB)(AB)^{*}\,]^{2}M\,] - 2r(M).$$

The following consequence is derived from (2.12), (2.13) and (2.26).

Theorem 2.12. *Let $A \in \mathbf{C}^{m\times n}$, $B \in \mathbf{C}^{n\times p}$ and $C \in \mathbf{C}^{p\times q}$, and let $M = ABC$. Then the following statements are equivalent*:

(a) $(ABC)^{\dagger} = (BC)^{\dagger}[\,(AB)^{\dagger}ABC(BC)^{\dagger}\,]^{\dagger}(AB)^{\dagger}$.

(b) $(ABC)^{\dagger} = (BC)^{*}[\,(AB)^{*}ABC(BC)^{*}\,]^{\dagger}(AB)^{*}$.

(c) $M_2^{\dagger}(M_1^{\dagger}MM_2^{\dagger})^{\dagger}M_1^{\dagger} = (M_2^{*}M_2)^{\dagger}\{[\,M_2^{\dagger}(B^{*})^{\dagger}M_1^{\dagger}\,]^{\dagger}\}^{*}(\,M_1M_1^{*})^{\dagger}$, *where $M_1 = AB$ and $M_2 = BC$.*

(d) $\mathcal{R}[\,(AB)(AB)^{*}M\,] = \mathcal{R}(M)$ *and* $\mathcal{R}[\,(BC)^{*}(BC)M^{*}\,] = \mathcal{R}(M^{*})$.

It can also be derived from (2.15), (2.25) and (2.27) that

Theorem 2.13. *Let $A \in \mathbf{C}^{m\times n}$, $B \in \mathbf{C}^{n\times p}$, $C \in \mathbf{C}^{p\times q}$, and let $M = ABC$. Then the following four statements are equivalent*:

(a) $(ABC)^{\dagger} = [\,(BC)^{*}(BC)\,]^{\dagger}\{[\,(BC)^{\dagger}(B^{*})^{\dagger}(AB)^{\dagger}\,]^{\dagger}\}^{*}[\,(AB)(AB)^{*}\,]^{\dagger}$.

(b) $(ABC)^{\dagger} = (BC)^{*}(BC)[\,(AB)(AB)^{*}M(BC)^{*}(BC)\,]^{\dagger}(AB)(AB)^{*}$.

(c) $(BC)^{\dagger}[\,(AB)^{\dagger}M(BC)^{\dagger}\,]^{\dagger}(AB)^{\dagger} = (BC)^{*}[\,(AB)^{*}M(BC)^{*}\,]^{\dagger}(AB)^{*}$.

(d) $\mathcal{R}[\,((AB)(AB)^{*})^{2}M\,] = \mathcal{R}(M)$ *and* $\mathcal{R}[\,((BC)^{*}(BC))^{2}M^{*}\,] = \mathcal{R}(M^{*})$.

Some other rank equalities derived from the right-hand sides of (1.1)–(1.7) are given below

(2.28) $$r\{\,C^{\dagger}(A^{\dagger}ABCC^{\dagger})^{\dagger}A^{\dagger} - (BC)^{*}[\,(AB)^{*}ABC(BC)^{*}\,]^{\dagger}(AB)^{*}\,\}$$
$$= r\begin{bmatrix} M \\ M(BC)^{*}(BC)C^{*}C \end{bmatrix} + r[\,M,\ AA^{*}(AB)(AB)^{*}M\,] - 2r(M),$$

$$r\{ C^*(A^*ABCC^*)^\dagger A^* - (BC)^*[\,(AB)^*ABC(BC)^*\,]^\dagger (AB)^* \} = r\begin{bmatrix} MC^*C \\ M(BC)^*BC \end{bmatrix} + r[\,AA^*M,\ AB(AB)^*M\,] - 2r(M), \tag{2.29}$$

$$r\{ (C^*C)^\dagger[\,(C^\dagger B^*A^\dagger)^\dagger\,]^*(AA^*)^\dagger - (BC)^*[\,(AB)^*ABC(BC)^*\,]^\dagger(AB)^* \} = r\begin{bmatrix} M \\ M(BC)^*(BC)(C^*C)^2 \end{bmatrix} + r[\,M,\ (AA^*)^2(AB)(AB)^*M\,] - 2r(M). \tag{2.30}$$

From (2.28), (2.29) and (2.30), we see that

$$C^\dagger(A^\dagger MC^\dagger)^\dagger A^\dagger = (BC)^*[\,(AB)^*ABC(BC)^*\,]^\dagger(AB)^*$$

holds if and only if

$$\mathcal{R}[\,AA^*(AB)(AB)^*M\,] = \mathcal{R}(M)$$

and

$$\mathcal{R}[\,C^*C(BC)^*(BC)M^*\,] = \mathcal{R}(M^*);$$

the equality

$$C^*(A^*ABCC^*)^\dagger A^* = (BC)^*[\,(AB)^*ABC(BC)^*\,]^\dagger(AB)^*$$

holds if and only if

$$\mathcal{R}(AA^*M) = \mathcal{R}[\,(AB)(AB)^*M\,]$$

and

$$\mathcal{R}(C^*CM^*) = \mathcal{R}[\,(BC)^*(BC)M^*\,];$$

the equality

$$(C^*C)^\dagger[\,(C^\dagger B^*A^\dagger)^\dagger\,]^*(AA^*)^\dagger = (BC)^*[\,(AB)^*ABC(BC)^*\,]^\dagger(AB)^*$$

holds if and only if

$$\mathcal{R}[\,(AA^*)^2(AB)(AB)^*M\,] = \mathcal{R}(M)$$

and

$$\mathcal{R}[\,(C^*C)^2(BC)^*(BC)M^*\,] = \mathcal{R}(M^*).$$

Moreover, some mixed forms of the reverse-order laws in (1.1)–(1.7) can be derived. For example, applying (1.5) to the product $A^\dagger ABCC^\dagger = (A^\dagger A)B(CC^\dagger)$ in (1.1) gives the following reverse-order law for $(ABC)^\dagger$:

$$(ABC)^\dagger = C^\dagger(BCC^\dagger)^\dagger[\,(AB)^\dagger ABC(BC)^\dagger\,]^\dagger(A^\dagger AB)^\dagger A^\dagger. \tag{2.31}$$

It is easy to verify that the right-hand side of (2.31) is a reflexive generalized inverse of $M = ABC$. Hence, we can find by (1.16)–(1.23) the following rank equality

$$\begin{aligned} &r\{\,(ABC)^\dagger - C^\dagger(BCC^\dagger)^\dagger[\,(AB)^\dagger M(BC)^\dagger\,]^\dagger(A^\dagger AB)^\dagger A^\dagger\,\} \\ &= r\begin{bmatrix} M & 0 \\ 0 & BC \\ M(BC)^*BC & MC^*C \end{bmatrix} + r\begin{bmatrix} M & 0 & AB(AB)^*M \\ 0 & AB & AA^*M \end{bmatrix} \\ &\quad - r(AB) - r(BC) - 2r(M). \end{aligned} \tag{2.32}$$

In particular, (2.31) holds if and only if A, B and C satisfy the following four conditions

$$\mathcal{R}[\,(AB)(AB)^*M\,] = \mathcal{R}(M), \qquad \mathcal{R}(AA^*M) \subseteq \mathcal{R}(AB),$$
$$\mathcal{R}[\,(BC)^*(BC)M^*\,] = \mathcal{R}(M^*), \qquad \mathcal{R}(C^*CM^*) \subseteq \mathcal{R}[\,(BC)^*\,].$$

Applying (1.5) to the product $A^*ABCC^* = (A^*A)B(CC^*)$ in (1.2) gives the following reverse-order law for $(ABC)^\dagger$:

$$(ABC)^\dagger = C^*(BCC^*)^\dagger[\,(AB)^\dagger ABC(BC)^\dagger\,]^\dagger(A^*AB)^\dagger A^*. \tag{2.33}$$

It is easy to verify that the right-hand side of (2.33) is a reflexive generalized inverse of $M = ABC$. In this case, the following rank equality

$$\begin{aligned} &r\{\,(ABC)^\dagger - C^*(BCC^*)^\dagger[\,(AB)^\dagger M(BC)^\dagger\,]^\dagger(A^*AB)^\dagger A^*\,\} \\ &= r\begin{bmatrix} M & 0 \\ 0 & BCC^*C \\ M(BC)^*BC & M \end{bmatrix} + r\begin{bmatrix} M & 0 & AB(AB)^*M \\ 0 & AA^*AB & M \end{bmatrix} \\ &\quad - r(AB) - r(BC) - 2r(M) \end{aligned} \tag{2.34}$$

is derived from (1.16)–(1.23). In particular, (2.33) holds if and only if A, B and C satisfy the following four conditions

$$\mathcal{R}[\,(AB)(AB)^*M\,] = \mathcal{R}(M), \qquad \mathcal{R}(M) \subseteq \mathcal{R}(AA^*AB),$$
$$\mathcal{R}[\,(BC)^*(BC)M^*\,] = \mathcal{R}(M^*), \qquad \mathcal{R}(M^*) \subseteq \mathcal{R}[\,C^*C(BC)^*\,].$$

Finally, we show a rank equality related to the reverse-order law in (1.8).

Theorem 2.14. *Let $A \in \mathbf{C}^{m\times n}$, $B \in \mathbf{C}^{n\times p}$, $C \in \mathbf{C}^{p\times q}$, and suppose that $\mathcal{R}(B) \subseteq \mathcal{R}(A^*)$ and $\mathcal{R}(B^*) \subseteq \mathcal{R}(C)$. Then*

(2.35)

$$r[\,(ABC)^\dagger - (\,I_q - (C^\dagger F_B)(C^\dagger F_B)^\dagger\,)C^\dagger B^\dagger A^\dagger(\,I_m - (E_BA^\dagger)^\dagger(E_BA^\dagger)\,)\,] = m + q - r(A) - r(C).$$

Hence, the equality in (1.8) *holds if and only if* $r(A) = m$ *and* $r(C) = q$. *In particular, if both A and C are nonsingular matrices, then ABC satisfies the identity*

$$(ABC)^\dagger = [\,I_q - (C^{-1}F_B)(C^{-1}F_B)^\dagger\,]C^{-1}B^\dagger A^{-1}[\,I_m - (E_BA^{-1})^\dagger(E_BA^{-1})\,].$$

Proof. Let $M = ABC$ and

$$N = [\,I_q - (C^\dagger F_B)(C^\dagger F_B)^\dagger\,]C^\dagger B^\dagger A^\dagger[\,I_m - (E_BA^\dagger)^\dagger(E_BA^\dagger)\,].$$

It is easy to verify that under $\mathcal{R}(B) \subseteq \mathcal{R}(A^*)$ and $\mathcal{R}(B^*) \subseteq \mathcal{R}(C)$, the matrix N is an outer inverse of M. Hence by (1.17)

$$r(\,M^\dagger - N\,) = r\begin{bmatrix} M^\dagger \\ N \end{bmatrix} + r[\,M^\dagger,\, N\,] - r(M) - r(N). \tag{2.36}$$

Simplifying the ranks of the matrices in this expression by (1.13)–(1.16) gives $r(M) = r(B)$, $r(N) = r(B)$ and

$$r\begin{bmatrix} M^\dagger \\ N \end{bmatrix} = m + r(B) - r(C), \qquad r[\,M^\dagger,\, N\,] = q + r(A) - r(C).$$

The process is tedious and therefore is omitted here. Substituting these results into (2.36) yields (2.35). □

Remark 2.15. It can be seen from (1.10)–(1.12) that the following three reverse-order laws

$$(AB)^{\dagger} = B^{\dagger}A^{\dagger} - B^{\dagger}(E_BF_A)^{\dagger}A^{\dagger}, \tag{2.37}$$

$$(AB)^{\dagger} = B^{\dagger}(A^{\dagger}ABB^{\dagger})^{\dagger}A^{\dagger}, \tag{2.38}$$

$$(AB)^{\dagger} = B^{*}(A^{*}ABB^{*})^{\dagger}A^{*} \tag{2.39}$$

are equivalent. Equality (2.37) is noticed by the author when comparing different reflexive generalized inverses of the block matrix $W = \begin{bmatrix} I_n & B \\ A & 0 \end{bmatrix}$. Two reasonable extensions of (2.37) to a triple matrix product ABC are given as follows

$$(ABC)^{\dagger} = (BC)^{\dagger}B(AB)^{\dagger} - (BC)^{\dagger}B(E_{BC}BF_{AB})^{\dagger}B(AB)^{\dagger}, \tag{2.40}$$

$$(ABC)^{\dagger} = C^{\dagger}B^{\dagger}A^{\dagger} - C^{\dagger}(E_{BC}BF_{AB})^{\dagger}A^{\dagger}, \tag{2.41}$$

both of which are derived from decompositions of the block matrix

$$W = \begin{bmatrix} B & BC \\ AB & 0 \end{bmatrix} \tag{2.42}$$

and its reflexive generalized inverses. In fact, W can be decomposed as

$$W = \begin{bmatrix} I_n & 0 \\ A & I_m \end{bmatrix} \begin{bmatrix} B & 0 \\ 0 & -ABC \end{bmatrix} \begin{bmatrix} I_p & C \\ 0 & I_q \end{bmatrix} \overset{\text{def}}{=} U_1J_1V_1$$

and

$$W = \begin{bmatrix} I_n & \left[I_n - (BC)(BC)^{\dagger} \right] B(AB)^{\dagger} \\ 0 & I_m \end{bmatrix} \begin{bmatrix} T & BC \\ AB & 0 \end{bmatrix} \begin{bmatrix} I_p & 0 \\ (BC)^{\dagger}B & I_q \end{bmatrix} \overset{\text{def}}{=} U_2J_2V_2,$$

where $T = [\, I_n - (BC)(BC)^{\dagger}\,]B[\, I_q - (AB)^{\dagger}(AB)\,]$. From these two decompositions of W, we obtain two reflexive generalized inverses of W as follows

$$W_r^{-} = V_1^{-1}J_1^{\dagger}U_1^{-1} = \begin{bmatrix} I_p & -C \\ 0 & I_q \end{bmatrix} \begin{bmatrix} B^{\dagger} & 0 \\ 0 & -(ABC)^{\dagger} \end{bmatrix} \begin{bmatrix} I_n & 0 \\ -A & I_m \end{bmatrix} = \begin{bmatrix} B^{\dagger} - C(ABC)^{\dagger}A & C(ABC)^{\dagger} \\ (ABC)^{\dagger}A & -(ABC)^{\dagger} \end{bmatrix} \tag{2.43}$$

and

(2.44)

$$
\begin{aligned}
W_r^- &= V_2^{-1}J_2^{\dagger}U_2^{-1} \\
&= \begin{bmatrix} I_p & 0 \\ -(BC)^{\dagger}B & I_q \end{bmatrix} \begin{bmatrix} T^{\dagger} & (AB)^{\dagger} \\ (BC)^{\dagger} & 0 \end{bmatrix} \\
&\quad \times \begin{bmatrix} I_n & -\left[I_n - (BC)(BC)^{\dagger} \right] B(AB)^{\dagger} \\ 0 & I_m \end{bmatrix} \\
&= \begin{bmatrix} T^{\dagger} & (AB)^{\dagger} - T^{\dagger}B(AB)^{\dagger} \\ (BC)^{\dagger} - (BC)^{\dagger}BT^{\dagger} & (BC)^{\dagger}BT^{\dagger}B(AB)^{\dagger} - (BC)^{\dagger}B(AB)^{\dagger} \end{bmatrix}.
\end{aligned}
$$

Comparing the lower-right blocks of (2.43) and (2.44) leads to the mixed-type reverse-order law (2.40). Letting the upper left blocks of (2.43) and (2.44) be equal gives

$$
B^{\dagger} - C(ABC)^{\dagger}A = [\, (I_n - P_{BC})B(I_p - P_{(AB)^*})\,]^{\dagger}, \tag{2.45}
$$

which suggests the reverse-order law for $(ABC)^{\dagger}$ in (2.41). As of this writing, the author has not yet found satisfactory necessary and sufficient conditions for (2.40), (2.41) and (2.45) to hold.

Remark 2.16. In addition to the reverse-order laws investigated in the paper, the Moore-Penrose inverses of AB and ABC may satisfy some identities. The following identities are shown in Tian and Cheng [**16**]

$$
\begin{aligned}
(AB)^{\dagger} &= (A^{\dagger}AB)^{\dagger}(ABB^{\dagger})^{\dagger}, \\
(AB)^{\dagger} &= [\,(A^{\dagger})^*B\,]^{\dagger}(B^{\dagger}A^{\dagger})^*[\,A(B^{\dagger})^*\,]^{\dagger}, \\
(ABC)^{\dagger} &= (A^{\dagger}ABC)^{\dagger}B(ABCC^{\dagger})^{\dagger}, \\
(ABC)^{\dagger} &= [\,(AB)^{\dagger}ABC\,]^{\dagger}B^{\dagger}[\,ABC(BC)^{\dagger}\,]^{\dagger}, \\
(ABC)^{\dagger} &= [\,(ABB^{\dagger})^{\dagger}ABC\,]^{\dagger}B[\,ABC(B^{\dagger}BC)^{\dagger}\,]^{\dagger}, \\
(ABC)^{\dagger} &= [\,(A^{\dagger})^*BC\,]^{\dagger}(A^{\dagger})^*B(C^{\dagger})^*[\,AB(C^{\dagger})^*\,]^{\dagger}, \\
(ABC)^{\dagger} &= \{\,[\,A(B^{\dagger})^*\,]^{\dagger}ABC\,\}^{\dagger}B\{\,ABC[\,(B^{\dagger})^*C\,]^{\dagger}\,\}^{\dagger}, \\
(ABC)^{\dagger} &= \{\,[\,(AB)^{\dagger}\,]^*C\,\}^{\dagger}[\,(AB)^{\dagger}\,]^*B^{\dagger}[\,(BC)^{\dagger}\,]^*\{\,A[\,(BC)^{\dagger}\,]^*\,\}^{\dagger}.
\end{aligned}
$$

It is expected that more reverse-order laws for the triple product ABC can be constructed, and necessary and sufficient conditions for them to hold can be determined by the matrix rank method.

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