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GENERALIZED S-TYPE LIE ALGEBRAS

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ABSTRACT. The generalized W-type Lie algebra $W(e^{\pm x_1}, \dots, e^{\pm x_m}, m)$ is introduced in the paper [**5**] using exponential functions. We define generalized S-type Lie algebras $S(e^{\pm x_1}, \dots, e^{\pm x_m}, m)$ over $\mathbf{F}$ and $S_p(e^{\pm x_1}, \dots, e^{\pm x_m}, m)$ over $\mathbf{F_p}$. We show that the Lie algebras $S(e^{\pm x_1}, \dots, e^{\pm x_m}, m)$ and $S_p(e^{\pm x_1}, \dots, e^{\pm x_m}, m)$ are simple.

1. Preliminaries. Let $\mathbf{F}$ be a field of characteristic zero (not necessarily algebraically closed) and $\mathbf{F_p}$ a field of characteristic p (not necessarily algebraically closed). Throughout this paper, $\mathbf{N}$ and $\mathbf{Z}$ will denote the nonnegative integers and the integers, respectively. Let $\mathbf{F}^\bullet$ be the multiplicative group of nonzero elements of $\mathbf{F}$. Let $\mathbf{F}[x_1, \dots, x_m]$ be the polynomial ring in indeterminates $x_1, \dots, x_m$. Throughout this paper, let us assume that $m > 1$. Let us define the $\mathbf{F}$-algebra $V(e^{\pm x_1}, \dots, e^{\pm x_m}, m)$ spanned by

$$\{e^{a_1 x_1} \cdots e^{a_m x_m} x_1^{i_1} \cdots x_m^{i_m} \mid a_1, \dots, a_m \in \mathbf{Z}, i_1, \dots, i_m \in \mathbf{N}\} \tag{1}$$

where m is a fixed nonnegative integer and $e^{a_w x_w}$, $1 \leq w \leq m$, denotes the exponential function. We define the Lie algebra $W(e^{\pm x_1}, \dots, e^{\pm x_m}, m)$ over $\mathbf{F}$ which holds the following two conditions:

(i) $W(e^{\pm x_1}, \dots, e^{\pm x_m}, m)$ is the set $\{g\partial_u \mid g \in V(e^{\pm x_1}, \dots, e^{\pm x_m}, m), 1 \leq u \leq m\}$ with the obvious addition,

(ii) the Lie bracket on $W(e^{\pm x_1}, \dots, e^{\pm x_m}, m)$ is given as follows: $[g_1\partial_u, g_2\partial_v] = g_1\partial_u(g_2)\partial_v - g_2\partial_v(g_1)\partial_u$, for g_1, $g_2 \in V(e^{\pm x_1}, \dots, e^{\pm x_m}, m)$, $1 \leq u \leq m$, where ∂_u, $1 \leq u \leq m$, denotes the partial derivative with respect to x_u.

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The Lie algebra $W(e^{\pm x_1},\dots,e^{\pm x_m},m)$ has the standard basis

(2)

$$B_{W(e^{\pm x_1},\dots,e^{\pm x_m},m)} = \{e^{a_1x_1}\cdots e^{a_mx_m}x_1^{i_1}\cdots x_m^{i_m}\partial_u \mid a_1,\dots,a_m\in\mathbf{Z},\; i_1,\dots,i_m\in\mathbf{N},\ 1\le u\le m\}.$$

For each basis term $e^{a_1x_1}\cdots e^{a_mx_m}x_1^{i_1}\cdots x_m^{i_m}\partial_u$ of $W(e^{\pm x_1},\dots,e^{\pm x_m},m)$, we call $e^{a_1x_1}\cdots e^{a_mx_m}$ the exponential part, $x_1^{i_1}\cdots x_m^{i_m}$ the polynomial part, a_v the exponent of x_v, and i_v the degree of x_v, $1\le v\le m$. The Lie algebra $W(e^{\pm x_1},\dots,e^{\pm x_m},m)$ is $\mathbf{Z}^m$-graded as follows:

$$W(e^{\pm x_1},\dots,e^{\pm x_m},m) = \bigoplus_{(a_1,\dots,a_m)\in\mathbf{Z}^m} W_{(a_1,\dots,a_m)} \tag{3}$$

where $W_{(a_1,\dots,a_m)}$ is the vector subspace of $W(e^{\pm x_1},\dots,e^{\pm x_m},m)$ spanned by

$$\{e^{a_1x_1}\cdots e^{a_mx_m}x_1^{i_1}\cdots x_m^{i_m}\partial_u \mid i_1,\dots,i_m\in\mathbf{N},\ 1\le u\le m\}.$$

We call $W_{(a_1,\dots,a_m)}$ the $(a_1,\dots,a_m)$-homogeneous component. The $(0,\dots,0)$-homogeneous component $W_{(0,\dots,0)}$ is the well-known Witt algebra $W^+(m)$ which is simple [**6**]. Every homogeneous component $W_{(a_1,\dots,a_m)}$ is a $W_{(0,\dots,0)}$-module [**1**]. The generalized special type Lie algebra $S(e^{\pm x_1},\dots,e^{\pm x_m},m)$ is a Lie subalgebra of $W(e^{\pm x_1},\dots,e^{\pm x_m},m)$ with elements

$$\left\{ \sum_{k\in I,1\le t\le m} g_{k,t}\partial_t \;\middle|\; \sum_{k\in I,1\le t\le m}\partial_t(g_{k,t})=0,\ g_{k,t}\in V(e^{\pm x_1},\dots,e^{\pm x_m},m)\right\}$$

where $\sum_{k,1\le t\le m} g_{k,t}\partial_t$ has only finitely many nonzero terms [**6**] and I is an index set. Note that $x_i\partial_i\notin S(e^{\pm x_1},\dots,e^{\pm x_m},m)$ and $(x_i\partial_i - x_j\partial_j)\in S(e^{\pm x_1},\dots,e^{\pm x_m},m)$ for $1\le i,j\le m$. For the element $(x_i\partial_i - x_j\partial_j)\in S(e^{\pm x_1},\dots,e^{\pm x_m},m)$, $1\le i,j\le m$, it is convenient to use the parenthesis ( ) of $(x_i\partial_i - x_j\partial_j)$, because $x_i\partial_i$ and $x_j\partial_j$ are not in $S(e^{\pm x_1},\dots,e^{\pm x_m},m)$. The Lie subalgebra $\overline{S(e^{\pm x_1},\dots,e^{\pm x_m},m)}$ of $S(e^{x_1},\dots,e^{x_m},m)$ is generated by

$$G_{\overline{S(e^{\pm x_1},\dots,e^{\pm x_m},m)}} = \big\{e^{a_1x_1}\cdots\widehat{e^{a_tx_t}}e^{a_{t+1}x_{t+1}}\cdots e^{a_mx_m}x_1^{i_1}\cdots\widehat{x_t^{i_t}}x_{t+1}^{i_{t+1}}\cdots x_m^{i_m}\partial_t \mid a_1,\dots,a_m\in\mathbf{Z},\ i_1,\dots,i_m\in\mathbf{N},\ 1\le t\le m\big\}$$

where $\widehat{e^{a_tx_t}}$ and $\widehat{x_t^{i_t}}$ mean that those factors are omitted. In the Lie algebra $W(e^{\pm x_1},\dots,e^{\pm x_m},m)$, if $n=0$, then we have the Witt algebra $W^+(m)$ [**6**]. Similarly, we have the special type Lie algebra $S^+(m)$ in the paper [**6**] with the set of elements

$$\bigg\{\sum_{k\in J,1\le v\le m} f_{k,v}\partial_v \,\bigg|\, \sum_{k\in J,1\le v\le m}\partial_v(f_{k,v})=0,\ f_k\in\mathbf{F}[x_1,\dots,x_m]\bigg\}$$

where J is an index set. The Lie subalgebra $\overline{S^+(m)}$ of $S^+(m)$ is generated by

$$G_{\overline{S^+(m)}} = \big\{x_1^{i_1}\cdots\widehat{x_t^{i_t}}x_{t+1}^{i_{t+1}}\cdots x_m^{i_m}\partial_t \mid i_1,\dots,i_m\in\mathbf{N},\ 1\le t\le m\big\}.$$

The Lie algebra $\overline{S^+(m)}$ has the standard basis

$$(4)\quad B_{\overline{S^+(m)}} = \big\{\big[x_1^{i_1}\cdots\widehat{x_t^{i_t}}x_{t+1}^{i_{t+1}}\cdots x_m^{i_m}\partial_t, x_1^{i_1}\cdots\widehat{x_v^{i_v}}x_{v+1}^{i_{v+1}}\cdots x_m^{i_m}\partial_v\big] \mid i_1,\dots,j_m\in\mathbf{N},\ 1\le t,v\le m\big\}.$$

We may find a basis $B_{\overline{S(e^{\pm x_1},\dots,e^{\pm x_m},m)}}$ of $\overline{S(e^{\pm x_1},\dots,e^{\pm x_m},m)}$ as $B_{\overline{S^+(m)}}$ in (4). Since the Lie algebra $W(e^{\pm x_1},\dots,e^{\pm x_m},m)$ is $\mathbf{Z}^m$-graded, $S(e^{\pm x_1},\dots,e^{\pm x_m},m)$ is $\mathbf{Z}^m$-graded naturally as follows:

$$(5)\qquad S(e^{\pm x_1},\dots,e^{\pm x_m},m) = \bigoplus_{(a_1,\dots,a_m)\in\mathbf{Z}^m} S_{(a_1,\dots,a_m)}.$$

The $(0,\dots,0)$-homogeneous component $S_{(0,\dots,0)}$ is the well-known special type Lie algebra $S^+(m)$ in the paper [**6**] and $S_{(a_1,\dots,a_m)}$ is a vector subspace of $W_{(a_1,\dots,a_m)}$.

For any basis elements $e^{a_1x_1}\cdots e^{a_mx_m}x_1^{i_1}\cdots x_m^{i_m}\partial_t$ and $e^{b_1x_1}\cdots e^{b_mx_m}\times x_1^{j_1}\cdots x_m^{j_m}\partial_v$ in $W(e^{\pm x_1},\dots,e^{\pm x_m},m)$, we may define the order $>_L$ as follows:

$$c_1e^{a_1x_1}\cdots e^{a_mx_m}x_1^{i_1}\cdots x_m^{i_m}\partial_t >_L c_2e^{b_1x_1}\cdots e^{b_mx_m}x_1^{j_1}\cdots x_m^{j_m}\partial_v$$

if and only if $a_1 > b_1$, or $a_1 = b_1$ and $i_1 > i_2$, or $\dots$, or $a_1 = b_1,\dots,$ $i_m = j_m$, and $v < t$ for any c_1, $c_2 \in \mathbf{F}^\bullet$. Thus we may consider that a Lie subalgebra of $W(e^{\pm x_1},\dots,e^{\pm x_m},m)$ has the order $>_L$. Naturally, the Lie algebra $S(e^{\pm x_1},\dots,e^{\pm x_m},m)$ has the order $>_L$. Let S be a subset of a Lie algebra L. An element $l\in L$ is ad-diagonal with respect to S, if $[l,s] = cs$ holds for any $s\in S$ where c is a fixed scalar which depends on l and s. The Lie algebra $\overline{S^+(m)}$ has the ad-diagonals $\{\sum_{u,v\in K} c_{u,v}(x_u\partial_u - x_v\partial_v) \mid 1\le u,\ v\le m,\ c_{u,v}\in\mathbf{F},\ K\subset \{1,\dots,m\}\}$ with respect to $B_{\overline{S^+(m)}}$ in $\overline{S^+(m)}$. Let $\mathbf{F_p}$ be a field of characteristic p (not necessarily algebraically closed) and $\mathbf{Z_p}$ denote the prime field where p is a prime number. Let us assume that m is a fixed positive integer such that $m\ge 1$. Let us define the $\mathbf{F_p}$-algebra $V_p(e^{\pm x_1},\dots,e^{\pm x_m},m)$ spanned by

(6)

$$\{e^{a_1x_1}\cdots e^{a_mx_m}x_1^{i_1}\cdots x_m^{i_m}\partial_u \mid a_1,\dots,a_m,i_1,\dots,i_m\in\mathbf{Z_p},\ 1\le u\le m\}$$

where m is a fixed nonnegative integer, $e^{a_wx_w}$, $1\le w\le m$, denotes the exponential function (formally), and ∂_u, $1\le u\le m$, denotes the partial derivative with respect to x_u. We may define the W-type Lie algebra $W_p(e^{\pm x_1},\dots,e^{\pm x_m},m)$ over $\mathbf{F_p}$ as $W(e^{\pm x_1},\dots,e^{\pm x_m},m)$ and S-type Lie algebra $S_p(e^{\pm x_1},\dots,e^{\pm x_m},m)$ over $\mathbf{F_p}$ as $S(e^{\pm x_1},\dots,e^{\pm x_m},m)$. The Lie algebra $W_p(e^{\pm x_1},\dots,e^{\pm x_m},m)$ is simple [**5**]. The Lie algebra $S_p(e^{\pm x_1},\dots,e^{\pm x_m},m)$ has a similar ${\mathbf{Z_p}}^m$-gradation in (5).

2. Generalized S-type Lie algebra over $\mathbf{F}$ or $\mathbf{F_p}$.

Proposition 1. *The Lie algebra $S^+(m)$ and the Lie algebra $\overline{S^+(m)}$ are the same.*

Proof. Since the Lie algebra $\overline{S^+(m)}$ is a subalgebra of $S^+(m)$, it is enough to show that $S^+(m)\subset\overline{S^+(m)}$. Let l be any element of $S^+(m)$.

It is enough to show that the element l is the sum of basis elements in $B_{\overline{S^+(m)}}$ of $\overline{S^+(m)}$. Let us prove this proposition by induction on the number of basis terms of l which are in $B_{W^+(m)} \cap S^+(m)$. If l has only one basis term in $B_{W^+(m)} \cap S^+(m)$, then l is a generator of $G_{\overline{S^+(m)}}$ in (1). Thus there is nothing to prove. Let us assume that we have proven the proposition when l has k basis terms in $B_{W^+(m)} \cap S^+(m)$. Let us assume that l has $k+1$ basis terms in $B_{W^+(m)} \cap S^+(m)$. If l has a basis term l_1 in $G_{\overline{S^+(m)}}$, then $l - cl_1$ has at most k basis terms in $B_{W^+(m)} \cap S^+(m)$ by taking an appropriate scalar c. By induction, $l - cl_1 \in \overline{S^+(m)}$, i.e., $l \in \overline{S^+(m)}$. Without loss of generality, we may assume that l has the following form $c_1 x_1^{i_1+1} x_2^{i_2} \cdots x_m^{i_m} \partial_1 + c_2 x_1^{i_1} \cdots x_t^{i_t+1} x_{t+1}^{i_{t+1}} \cdots x_m^{i_m} \partial_t + *$ where $i_1 \neq 0$, $*$ is the sum of remaining terms of l and $c_1 \in \mathbf{F}^{\bullet}$. We have that

$$l + \frac{c_1}{i_1+1}\,[x_1^{i_1+1}\partial_t, \quad x_2^{i_2} \cdots x_t^{i_t+1} x_{t+1}^{i_{t+1}} \cdots x_m^{i_m}\partial_1] \in \overline{S^+(m)}$$

by induction. Since

$$\frac{c_1}{i_1+1}\,[x_1^{i_1+1}\partial_t, \quad x_2^{i_2} \cdots x_t^{i_t+1} x_{t+1}^{i_{t+1}} \cdots x_m^{i_m}\partial_1] \in \overline{S^+(m)},$$

we have that $l \in \overline{S^+(m)}$ by induction. Therefore, we have proven the proposition. □

Proposition 2. *The Lie algebra* $\overline{S(e^{\pm x_1}, \ldots, e^{\pm x_m}, m)}$ *and the Lie algebra* $S(e^{\pm x_1}, \ldots, e^{\pm x_m}, m)$ *are the same.*

Proof. Since the Lie algebra $\overline{S(e^{\pm x_1}, \ldots, e^{\pm x_m}, m)}$ is a subalgebra of $S(e^{\pm x_1}, \ldots, e^{\pm x_m}, m)$, it is enough to show that $S(e^{\pm x_1}, \ldots, e^{\pm x_m}, m) \subset \overline{S(e^{\pm x_1}, \ldots, e^{\pm x_m}, m)}$. Let l be any element of $S(e^{\pm x_1}, \ldots, e^{\pm x_m}, m)$. It is enough to show that the element l is the sum of basis elements in $B_{\overline{S(e^{\pm x_1}, \ldots, e^{\pm x_m}, m)}}$. If l is in the $(0, \ldots, 0)$-homogeneous component $S_{(0,\ldots,0)}$ of $\overline{S(e^{\pm x_1}, \ldots, e^{\pm x_m}, m)}$, then there is nothing to prove by Proposition 1. Let us prove this proposition by induction on the number of terms of l in $B_{W(e^{\pm x_1}, \ldots, e^{\pm x_m}, m)} \cap S(e^{\pm x_1}, \ldots, e^{\pm x_m}, m)$ and the number of exponents of basis terms of l in $B_{W(e^{\pm x_1}, \ldots, e^{\pm x_m}, m)}$. If l has only one basis term, then l is a generator in $G_{\overline{S(e^{\pm x_1}, \ldots, e^{\pm x_m}, m)}}$

in (1). Thus there is nothing to prove. Let us assume that we have proven the proposition when l has k terms in $B_{W(e^{\pm x_1},\dots,e^{\pm x_m},m)} \cap S(e^{\pm x_1},\dots,e^{\pm x_m},m)$. Let us assume that l has $k+1$ basis terms in $B_{W(e^{\pm x_1},\dots,e^{\pm x_m},m)} \cap S(e^{\pm x_1},\dots,e^{\pm x_m},m)$. By Proposition 1, without loss of generality, we may assume that l has the form as follows:

$$\begin{aligned} l &= c_1 e^{a_1x_1}\cdots e^{a_ux_u}\cdots e^{a_mx_m}x_1^{i_1}\cdots x_m^{i_m}\partial_1 \\ &\quad + c_2 e^{a_1x_1}\cdots e^{a_ux_u}\cdots e^{a_mx_m}x_1^{i_1}\cdots x_m^{i_m}\partial_u + * \end{aligned}$$

where $c_1,\ c_2 \in \mathbf{F}$, $*$ is the sum of the remaining terms of l, and $a_1, a_u \neq 0$. Let us prove the proposition by the degree i_1 of x_1. Since $a_u \neq 0$, we have that

$$l_1 = l - \frac{c_1}{a_u}\,[e^{a_1x_1}x_1^{i_1}\partial_u, e^{a_2x_2}\cdots e^{a_mx_m}x_2^{i_2}\cdots x_m^{i_m}\partial_1]. \tag{7}$$

If $i_1 = 0$, then l_1 has $k+1$ terms or k terms in $B_{W(e^{\pm x_1},\dots,e^{\pm x_m},m)} \cap S(e^{\pm x_1},\dots,e^{\pm x_m},m)$. If l_1 has k terms, then there is nothing to prove by induction. Let us assume that l_1 has $k+1$ terms. Without loss of generality, we may assume that $u = m$ by (7), i.e.,

$$l_1 = c_3 e^{a_1x_1}\cdots e^{a_mx_m}x_1^{i_1}\cdots x_m^{i_m}\partial_m + **$$

where $c_3 \in \mathbf{F}$, $**$ is the sum of the remaining terms of l_1. Since $e^{a_1x_1}\cdots e^{a_mx_m}x_1^{i_1}\dots x_m^{i_m}\partial_n$ is the maximal term of l_1, if $c_3 \neq 0$, then $l_1 \notin S(e^{\pm x_1},\dots,e^{\pm x_m},m)$, i.e., $l \notin S(e^{\pm x_1},\dots,e^{\pm x_m},m)$ by (7). This contradiction shows that l_1 has at most k terms. This implies that $l_1 \in S(e^{\pm x_1},\dots,e^{\pm x_m},m)$ by induction. Thus we may assume that

$$l = l_1 + \frac{c_1}{a_u}\,[e^{a_1x_1}x_1^{i_1}\partial_u, e^{a_2x_2}\cdots e^{a_mx_m}x_2^{i_2}\cdots x_m^{i_m}\partial_1]. \tag{8}$$

This implies that l is the sum of elements in $B_{W(e^{\pm x_1},\dots,e^{\pm x_m},m)} \cap S(e^{\pm x_1},\dots,e^{\pm x_m},m)$. Therefore we have proved the proposition. □

By Proposition 1 and Proposition 2, the basis $B_{\overline{S^+(m)}}$ is the standard basis of $S^+(m)$ and $B_{\overline{S(e^{\pm x_1},\dots,e^{\pm x_m},m)}}$ is the standard basis of $S(e^{\pm x_1},\dots,e^{\pm x_m},m)$. A similar result of Proposition 2 for the Lie algebra $S_p(e^{\pm x_1},\dots,e^{\pm x_m},m)$ holds.

Note 1. The $(0,\dots,0)$-homogeneous component $S_{(0,\dots,0)}$ of

$$\overline{S(e^{\pm x_1},\dots,e^{\pm x_m},m)},$$

respectively

$$\overline{S_p(e^{\pm x_1},\dots,e^{\pm x_m},m)},$$

in (5) is the simple Lie algebra $S^+(m)$ in the paper [**6**], respectively [**5**]. $\square$

Lemma 1. *The only Lie ideal of $S(e^{\pm x_1},\dots,e^{\pm x_m},m)$, respectively $S_p(e^{\pm x_1},\dots,e^{\pm x_m},m)$, which contains a nonzero element in $S_{(0,\dots,0)}$ is $S(e^{\pm x_1},\dots,e^{\pm x_m},m)$, respectively $S_p(e^{\pm x_1},\dots,e^{\pm x_m},m)$, where $S_{(0,\dots,0)}$ is the $(0,\dots,0)$-homogeneous component of $S(e^{\pm x_1},\dots,e^{\pm x_m},m)$, respectively $S_p(e^{\pm x_1},\dots,e^{\pm x_m},m)$, in* (5).

Proof. Let I be a nonzero ideal of $S(e^{\pm x_1},\dots,e^{\pm x_m},m)$, respectively $S_p(e^{\pm x_1},\dots,e^{\pm x_m},m)$, which contains an element in $S_{(0,\dots,0)}$. Since $S_{(0,\dots,0)}$ is simple [**6**], $S_{(0,\dots,0)}\subset I$. For any element $e^{a_1x_1}\cdots e^{a_mx_m}x_1^{i_1}\cdots \widehat{x_t^{i_t}}x_{t+1}^{i_{t+1}}\cdots x_m^{i_m}\partial_t\in S(e^{\pm x_1},\dots,e^{\pm x_m},m)$, $1\le t\le m$, we have that

$$\begin{aligned}[]
[\partial_1, e^{a_1x_1}&\cdots e^{a_mx_m}x_1^{i_1}\cdots \widehat{x_t^{i_t}}x_{t+1}^{i_{t+1}}\cdots x_m^{i_m}\partial_t]\\
&= a_1e^{a_1x_1}\cdots e^{a_mx_m}x_1^{i_1}\cdots \widehat{x_t^{i_t}}x_{t+1}^{i_{t+1}}\cdots x_m^{i_m}\partial_t\\
&\quad + i_1e^{a_1x_1}\cdots e^{a_mx_m}x_1^{i_1-1}x_2^{i_2}\cdots \hat{x}_t^{i_t}x_{t+1}^{i_{t+1}}\cdots x_m^{i_m}\partial_t\in I
\end{aligned}$$

where $a_1\neq 0$. By induction on i_1 in (9), we know that $e^{a_1x_1}\cdots e^{a_mx_m}x_1^{i_1}\cdots \widehat{x_t^{i_t}}x_{t+1}^{i_{t+1}}\cdots x_m^{i_m}\partial_t\in I$. For any element $e^{a_1x_1}\cdots\widehat{e^{a_tx_t}}\cdots e^{a_mx_m}x_1^{i_1}\cdots \widehat{x_t^{i_t}}x_{t+1}^{i_{t+1}}\cdots x_m^{i_m}\partial_t\in S(e^{\pm x_1},\dots,e^{\pm x_m},m)$, $1\le t\le m$, without loss of generality, we may assume that $i_k\neq 0$, $n\le k\le m$. By

$$\begin{aligned}[]
[\partial_k, e^{a_1x_1}&\cdots\widehat{e^{a_tx_t}}\cdots e^{a_mx_m}x_1^{i_1}\cdots \widehat{x_t^{i_t}}x_{t+1}^{i_{t+1}}\cdots x_m^{i_m}\partial_t]\\
&= i_ke^{a_1x_1}\cdots\widehat{e^{a_tx_t}}\cdots e^{a_mx_m}x_1^{i_1}\cdots \widehat{x_t^{i_t-1}}x_{t+1}^{i_{t+1}}\cdots x_m^{i_m}\partial_t\in I.
\end{aligned}$$

By induction on i_k of $e^{a_1x_1}\cdots\widehat{e^{a_tx_t}}\cdots e^{a_mx_m}x_1^{i_1}\cdots \widehat{x_t^{i_t}}x_{t+1}^{i_{t+1}}\cdots x_m^{i_m}\partial_t$, we have that $e^{a_1x_1}\cdots\widehat{e^{a_tx_t}}\cdots e^{a_mx_m}x_1^{i_1}\cdots \widehat{x_t^{i_t}}x_{t+1}^{i_{t+1}}\cdots x_m^{i_m}\partial_t\in I$. This implies that $I = S(e^{\pm x_1}, \ldots, e^{\pm x_m}, m)$, respectively $I = S_p(e^{\pm x_1}, \ldots, e^{\pm x_m}, m)$. Therefore we have proven the lemma. $\square$

Theorem 1. *The Lie algebra* $S(e^{\pm x_1}, \ldots, e^{\pm x_m}, m)$*, respectively* $S_p(e^{\pm x_1}, \ldots, e^{\pm x_n}, m)$*, is simple.*

Proof. Let I be a nonzero ideal of $S(e^{\pm x_1}, \ldots, e^{\pm x_m}, m)$, respectively $S_p(e^{\pm x_1}, \ldots, e^{\pm x_n}, m)$, and l a nonzero element in I. Let us prove the theorem by induction on the number of different homogeneous components of l which contains a basis term of l. If l has one homogeneous component and $l \in S_{(0,\ldots,0)}$, then there is nothing to prove by Lemma 1. Let us assume that l has one homogeneous component which is not in $S_{(0,\ldots,0)}$. Let us prove that l is in $S_{(0,\ldots,0)}$ by induction on the number of basis terms of l in $B_{\overline{S(e^{\pm x_1},\ldots,e^{\pm x_m},m)}}$. If l has one basis term, then l has the form $e^{a_1 x_1} \cdots \widehat{e^{a_t x_t}} \cdots e^{a_m x_m} x_1^{i_1} \cdots \widehat{x_t^{i_t}} x_{t+1}^{i_{t+1}} \cdots x_m^{i_m} \partial_t$, $c \in \mathbf{F}$, such that at least one of $a_1, \ldots, a_{k-1}, a_k, \ldots$ and a_m is not zero. Otherwise, there is nothing to prove by Lemma 1. Let us assume that $k < m$; by taking $e^{-a_1 x_1} \cdots e^{-a_m x_m} \partial_t$, we have that

$$[e^{-a_1 x_1} \cdots e^{-a_m x_m} \partial_t, e^{a_1 x_1} \cdots \widehat{e^{a_t x_t}} \cdots e^{a_m x_m} x_1^{i_1} \cdots \widehat{x_t^{i_t}} x_{t+1}^{i_{t+1}} \cdots x_m^{i_m} \partial_t] \in I.$$

Let us assume that $k \leq n$, $l = e^{a_1 x_1} \cdots \widehat{e^{a_t x_t}} \cdots e^{a_m x_m} x_1^{i_1} \cdots \widehat{x_t^{i_t}} x_{t+1}^{i_{t+1}} \cdots x_m^{i_m} \partial_t$, a_1. Then we have that $0 \neq [e^{-a_1 x_1} \cdots e^{-a_m x_m} \partial_t, e^{a_1 x_1} \cdots \widehat{e^{a_t x_t}} \cdots e^{a_m x_m} x_1^{i_1} \cdots \widehat{x_t^{i_t}} x_{t+1}^{i_{t+1}} \cdots x_m^{i_m} \partial_t] \in S_0$. This implies that the ideal $S_{(0,\ldots,0)} \subset I = S(e^{\pm x_1}, \ldots, e^{\pm x_m}, m)$, respectively $I = S_p(e^{\pm x_1}, \ldots, e^{\pm x_m}, m)$, by Lemma 1. By induction, we may assume that if l has k homogeneous components, then the ideal $I = S(e^{\pm x_1}, \ldots, e^{\pm x_m}, m)$, respectively $S_p(e^{\pm x_1}, \ldots, e^{\pm x_m}, m)$. Let us assume that l has $k+1$ homogeneous components which contains a basis term of l. If l has a term in $S_{(0,\ldots,0)}$, then there is nothing to prove by taking an appropriate ∂_v, $v \in I$, since

$$0 \neq [\partial_v, [\partial_v, [\ldots, [\partial_v, l] \ldots] \in I \tag{10}$$

where we have applied the Lie bracket appropriate times in (10) so that $[\partial_v, [\partial_v, [\ldots, [\partial_v, l] \ldots]$ has at most k homogeneous components.

This implies that $[e^{-a_1x_1}\cdots e^{-a_mx_m}\partial_v, l]$ has a nonzero basis term in $S_{(0,\dots,0)}$. Thus we have proven the theorem by Lemma 1. Similarly to (10), we can find an element in I such that it is the sum of terms in at most k different homogeneous components of $S(e^{\pm x_1},\dots,e^{\pm x_m},m)$, respectively $S_p(e^{\pm x_1},\dots,e^{\pm x_m},m)$. By induction, we can prove that $I = S(e^{\pm x_1},\dots,e^{\pm x_m},m)$, respectively $I = S_p(e^{\pm x_1},\dots,e^{\pm x_m},m)$. Therefore we have proven the theorem. □

3. Conjectures and questions. This is a good place to pose the following questions. The Lie algebra $S(e^{\pm x_1},\dots,e^{\pm x_m},m)$ has the Lie subalgebra S_m spanned by $\{(x_u\partial_u - x_v\partial_v), x_u\partial_v \mid 1 \le u,v \le m\}$ which is isomorphic to $sl_m(\mathbf{F})$ as Lie algebras [**1**].

Question 1. Is there a Lie subalgebra A of $S^+(m)$ which is isomorphic to the Lie algebra $sl_m(\mathbf{F})$ such that $A \neq S_m$?

Question 2. For any Lie algebra automorphism θ of $S^+(m)$, does the equality $\theta((x_u\partial_u - x_v\partial_v)) = c(x_w\partial_w - x_p\partial_p)$ hold for $c \in \mathbf{F}^\bullet$ where $1 \le u,v,w,p \le m$?

Question 3. For any Lie algebra automorphism θ of $S^+(2)$, does the equality $\theta(S_m) = S_m$ hold?

Thus we have the following interesting conjecture.

Conjecture. *For any Lie algebra automorphism θ of $S(e^{\pm x_1},\dots,e^{\pm x_m},m)$, $\theta((x_u\partial_u - x_v\partial_v)) = c(x_w\partial_w - x_p\partial_p)$ and $\theta(S_m) = S_m$ hold where $1 \le u,v,w,p \le m$ and $c \in \mathbf{F}$.*

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