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DISTRIBUTION OF THE ZEROS OF THE SOLUTIONS OF HYPERBOLIC DIFFERENTIAL EQUATIONS WITH MAXIMA

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ABSTRACT. In this paper, the hyperbolic differential equation with maxima of the form

$$u_{tt} - [\Delta u + \mu(t)\Delta u(x, t-\tau)] + c\Big(x, t, u(x,t), \max_{s\in[t-\sigma,t]} u(x,s)\Big) = f(x,t),$$

where Ω is a bounded domain in R^n and $\tau, \sigma = \text{const} > 0$, are considered. Sufficient conditions for existence of zeros of the solutions of the problems considered in bounded domains are obtained.

1. Introduction. In the last few decades, great attention has been paid to automatic control systems and their applications to computational mathematics and modeling [**15**]. Today, scientists around the globe are showing an increasing interest in differential equations which contains the maxima operator. In many applications the maxima can arise when the control theory corresponds to the maximal deviation of the regulated quantity. For example, neutral hyperbolic and parabolic equations with maxima were investigated in [**18**–**20**]. Ladas, Gyori, Bainov and Mishev developed and worked on oscillation theory for the differential equation with delay as in [**4, 14**] that led to maxima operator applications. Differential equations with maxima have appeared in various systems such as the system

$$u'(t) = -\delta u(t) + p \max_{t-h\le s\le t} u(s) + f(t), \tag{i}$$

where δ and p are positive constants. This system has appeared in the theory of automatic control in [**15**] and the references therein. The

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last equation (i) has been used for studying the stability of differential systems with delay as in [**5, 6, 8**]. Other systems such as the Hausrath equation indicated a more complicated dynamic system as in [**10**]:

$$\text{(ii)} \qquad u'(t) = -\,\delta u(t) + \delta \max_{t-h\le s\le t} |u(s)|, \quad \delta > 0, \quad t \ge 0,$$

Pinto and Trofimchuk investigated the stability and existence of multiple periodic solutions of the scalar delay differential equation as in [**22**]:

$$\text{(iii)} \qquad x' = -\,\delta x(t) + p \max_{u\in[t-h,t]} x(u) + f(t),$$

where $f(t)$ is a periodic forcing term and δ, p are positive constants. Other examples could be found in [**2, 5, 7, 9, 11, 12, 24**]. Hadeler showed a model describing the vision process in the compound eye using maxima as in [**7**]:

$$\text{(iv)} \qquad u'(t) = -\,\delta u(t) + p \max\{u(\tau(t)), c\},$$

where δ and p are positive constants and $c < 0$.

The distribution of the zeros of the solutions of some classes of neutral type hyperbolic differential equations with maxima is under consideration. The problems about ordinary differential equations with maxima application are found in the theory of automatic control of various real systems [**16, 17, 23**]. The necessity of study of differential equations with "maxima" is also emphasized in the survey of Myshkis [**21**]. Theorems of existence and uniqueness of the solution of ordinary differential equations with maxima are obtained in [**1, 21**]. Oscillation and asymptotic properties of the solutions of various classes of functional differential equations with maxima are investigated in [**3, 13**]. This paper presents distribution of the zeros of the solutions of hyperbolic differential equations with maxima. The sufficient conditions for existence of zeros of the solutions of the problems considered in bounded domains are obtained. In Section 2, we present the formulation of the problem and main results.

2. Formulation of the problem and main results. We consider neutral type hyperbolic differential equations with maxima of the form

$$
\begin{aligned}
&u_{tt}(x,t) - [\Delta u(x,t) + \mu(t)\Delta u(x,t-\tau)] \\
&\qquad + c\left(x,t,u(x,t), \max_{s\in[t-\sigma,t]} u(x,s)\right) = f(x,t), \\
&\qquad (x,t) \in \Omega \times (0,\infty) \equiv G,
\end{aligned} \tag{1}
$$

where $\Delta u(x,t) = \sum_{i=1}^{n} u_{x_i x_i}(x,t)$ and Ω is a bounded domain in R^n with a piecewise smooth boundary. The boundary conditions are considered of the form:

$$u(x,t) = g(x,t), \quad (x,t) \in \partial\Omega \times [0,\infty), \tag{2}$$

$$\frac{\partial u}{\partial n} + \gamma(x,t) = 0, \qquad (x,t) \in \partial\Omega \times [0,\infty). \tag{3}$$

We shall say that the conditions (H) are met if the following conditions hold:

H1. $\mu(t) \in C([0,\infty);[0,\infty))$.

H2. $\tau = \text{const} > 0$, $\sigma = \text{const} > 0$.

H3. $c(x,t,\xi,\eta) \in C(\overline{G} \times R^2; R)$.

H4. $c(x,t,\xi,\eta) \geq K_1^2\,\xi + K_2^2\,\eta$, for $(x,t) \in G$, $\xi \geq 0$, $\eta \geq 0$; $c(x,t,\xi,\eta) \leq K_1^2\,\xi + K_2^2\,\eta$, for $(x,t) \in G$, $\xi \leq 0$, $\eta \leq 0$;

where K_i, $i = 1,2$, are nonnegative constants.

H5. $f(x,t) \in C(\overline{G};R)$, $f(x,t) \not\equiv 0$.

H6. $g(x,t) \in C(\partial\Omega \times [0,\infty);R)$.

H7. $\gamma(x,t) \in C(\partial\Omega \times [0,\infty);[0,\infty))$.

Definition 1. The solution $u(x,t) \in C^2(G) \cap C^1(\overline{G})$ of problem (1), (2), ((1), (3)) is said to oscillate in the domain G if for any positive number ρ there exists a point $(x',t') \in \Omega \times [\rho,\infty)$ such that the equality $u(x',t') = 0$ should hold.

We shall note that theorems on distribution of the zeros of the solutions of hyperbolic differential equations without "maxima" are obtained in the papers of Yoshida [**25, 26**].

The following Dirichlet problem is considered in the domain Ω

$$\Delta U(x) + \alpha U(x) = 0, \quad x \in \Omega, \tag{4}$$

$$U(x) = 0, \quad x \in \partial\Omega, \tag{5}$$

where $\alpha =$ const. It is well known that the smallest eigenvalue α_0 of problem (4), (5), is positive and the corresponding eigenfunction $\varphi(x)$ can be chosen so that $\varphi(x) > 0$ for $x \in \Omega$.

The following notation is introduced

$$T = \max(\tau, \sigma), \qquad L = \sqrt{\alpha_0 + K_1^2 + K_2^2}. \tag{6}$$

With each solution $u(x,t) \in C^2(G) \cap C^1(\overline{G})$ of problem (1), (2), we associate the function

$$w(t) = \int_\Omega u(x,t)\varphi(x)\,dx, \quad t \geq 0. \tag{7}$$

Lemma 1. *Let conditions* (H) *hold, and let* $u(x,t) \in C^2(G) \cap C^1(\overline{G})$ *be a solution of problem* (1), (2)*, satisfying the condition*

$$u(x,t) > 0, \quad \text{for} \quad (x,t) \in \Omega \times (\lambda - T,\, \lambda + \pi/L), \quad \lambda \geq T. \tag{8}$$

Then the function $w(t)$ *is a positive solution in the interval* $[\lambda,\, \lambda + \pi/L)$ *of the inequality*

$$w''(t) + L^2\, w(t) \leq \Phi(t), \quad t \in [\lambda,\, \lambda + \pi/L), \tag{9}$$

where

$$\Phi(t) = \int_\Omega f(x,t)\varphi(x)\,dx - \int_{\partial\Omega} g(x,t)\,\frac{\partial\varphi}{\partial n}\,ds - \mu(t)\int_{\partial\Omega} g(x,t-\tau)\,\frac{\partial\varphi}{\partial n}\,ds. \tag{10}$$

Proof. From condition (8) it follows that $u(x,\, t-\tau) \geq 0$ and $\max_{s\in[t-\sigma,t]} u(x,s) \geq 0$ for $(x,t) \in \Omega \times [\lambda,\, \lambda + \pi/L)$. Multiply both sides of equation (1) with the function $\varphi(x)$ and integrate with respect to x over the domain Ω. For $t \in [\lambda, \lambda + \pi/L)$, we obtain

(11)
$$\frac{d^2}{dt^2}\int_\Omega u(x,t)\varphi(x)\,dx - \left[\int_\Omega \Delta u(x,t)\varphi(x)\,dx + \mu(t)\int_\Omega \Delta u(x,t-\tau)\varphi(x)\,dx\right] + \int_\Omega c\big(x,t,u(x,t), \max_{s\in[t-\sigma,t]} u(x,s)\big)\varphi(x)\,dx = \int_\Omega f(x,t)\varphi(x)\,dx.$$

From Green's formula it follows that

(12)
$$\begin{aligned}\int_\Omega \Delta u(x,t)\varphi(x)\,dx &= -\int_{\partial\Omega} u(x,t)\,\frac{\partial\varphi}{\partial n}\,dS + \int_\Omega u(x,t)\Delta\varphi(x)\,dx\\ &= -\int_{\partial\Omega} g(x,t)\,\frac{\partial\varphi}{\partial n}\,dS - \alpha_0\int_\Omega u(x,t)\varphi(x)\,dx,\end{aligned}$$

(13)
$$\begin{aligned}\int_\Omega \Delta u(x,t-\tau)\varphi(x)\,dx &= -\int_{\partial\Omega} u(x,t-\tau)\,\frac{\partial\varphi}{\partial n}\,ds + \int_\Omega u(x,t-\tau)\Delta\varphi(x)\,dx\\ &= -\int_{\partial\Omega} g(x,t-\tau)\,\frac{\partial\varphi}{\partial n}\,ds - \alpha_0\int_\Omega u(x,t-\tau)\varphi(x)\,dx.\end{aligned}$$

From condition H4 and the inequality, $\max_{s\in[t-\sigma,t]} u(x,s) \ge u(x,t)$, it follows that

$$\begin{aligned}\int_\Omega c\big(x,t,u(x,t), \max_{s\in[t-\sigma,t]} u(x,s)\big)\varphi(x)\,dx\\ \ge \int_\Omega [K_1^2\,u(x,t) + K_2^2\,u(x,t)]\varphi(x)\,dx.\end{aligned} \tag{14}$$

Using (12)–(14), we obtain that

$$\begin{aligned}&w''(t) + \alpha_0[w(t) + \mu(t)\,w(t-\tau)] + (K_1^2 + K_2^2)\,w(t)\\ &\quad\le \int_\Omega f(x,t)\varphi(x)\,dx - \int_{\partial\Omega} g(x,t-\tau)\,\frac{\partial\varphi}{\partial n}\,ds - \mu(t)\int_{\partial\Omega} g(x,t-\tau)\,\frac{\partial\varphi}{\partial n}\,ds.\end{aligned}$$

In view of

$$\alpha_0\,\mu(t)\,w(t-\tau) \ge 0 \quad \text{for} \quad t \in [\lambda, \lambda + \pi/L),$$

the last inequality implies (9). Analogously to the proof of Lemma 1, the following lemma is proved.

Lemma 2. *Let conditions* (H) *hold, and let* $u(x,t) \in C^2(G) \cap C^1(\overline{G})$ *be a solution of problem* (1), (2), *satisfying the condition*

$$u(x,t) < 0 \quad \text{for} \quad (x,t) \in \Omega \times (\lambda - T, \lambda + \pi/L), \quad \lambda \geq T. \tag{15}$$

Then the function $w_1(t) = -\,w(t)$ *is a positive solution of the inequality*

$$w_1''(t) + L^2\, w_1(t) \leq -\,\Phi(t), \quad t \in [\lambda,\, \lambda + \pi/L), \tag{16}$$

where the function $\Phi(t)$ *is defined by* (10). *In the proof of the subsequent theorem we shall use the following result of Yoshida* [**25**].

Lemma 3. *Suppose that there exists a real number* $\lambda \geq T$ *such that the following condition holds*

$$\int_\lambda^{\lambda+\pi/L} \sin L(t-\lambda) \cdot F(t)\, dt \leq 0.$$

Then the differential inequality

$$z''(t) + L^2\, z(t) \leq F(t), \quad t \in [\lambda,\, \lambda + \pi/L)$$

has no positive solutions in the interval $[\lambda,\, \lambda + \pi/L)$.

Theorem 1. *Let conditions* (H) *hold and a number* $\lambda \geq T$ *exist such that the following condition should hold*

$$F(\lambda) \equiv \int_\lambda^{\lambda+\pi/L} \Phi(t) \sin L(t-\lambda)\, dt \equiv 0. \tag{17}$$

Then each solution $u(x,t) \in C^2(G) \cap C^1(\overline{G})$ *of problem* (1), (2), *has a zero in the domain* $\Omega \times (\lambda - T,\, \lambda + \pi/L)$.

Proof. Suppose that this is not true, and let $u(x,t) \in C^2(G) \cap C^1(\overline{G})$ be a solution of problem (1), (2), which has no zeros in the domain

$\Omega \times (\lambda - T,\, \lambda + \pi/L)$. If $u(x,t) > 0$ for $(x,t) \in \Omega \times (\lambda - T,\, \lambda + \pi/L)$, then from Lemma 1 it follows that the function $w(t)$, defined by (7), is a positive solution in the interval $[\lambda,\, \lambda + \pi/L)$ of inequality (9). But, from condition (17) and Lemma 3, it follows that inequality (9) has no positive solutions in this interval. The case when $u(x,t) < 0$ for $(x,t) \in \Omega \times (\lambda - T,\, \lambda + \pi/L)$ is considered analogously. This completes the proof of Theorem 1. $\square$

Definition 2. The function $F(\lambda) \in C([t, \infty); R)$ is said to oscillate if there exists a sequence of zeros $\{\lambda_n\}_{n=1}^{\infty}$ of $F(\lambda)$ so that the equality $\lim_{n\to\infty} \lambda_n = \infty$ should hold.

Corollary 1. *Let conditions* (H) *hold, and let the function* $F(\lambda)$ *defined by* (17) *oscillate. Then each solution* $u(x,t) \in C^2(G) \cap C^1(\overline{G})$ *of problem* (1), (2), *oscillates in* G.

In the subsequent theorems we shall investigate the oscillatory properties of the solutions of problem (1), (3). With each solution $u(x,t) \in C^2(G) \cap C^1(\overline{G})$ of problem (1), (3), we associate the function

$$v(t) = \int_\Omega u(x,t)\, dx, \quad t > 0. \tag{18}$$

The following notation is introduced

$$L_1 = \sqrt{K_1^2 + K_2^2}.$$

Lemma 4. *Let conditions* (H) *hold,* $L_1 > 0$, *and let* $u(x,t) \in C^2(G) \cap C^1(\overline{G})$ *be a solution of problem* (1), (3), *satisfying the condition*

$$u(x,t) > 0 \quad \textit{for} \quad (x,t) \in \Omega \times (\lambda - T,\, \lambda + \pi/L), \quad \lambda \geq T. \tag{19}$$

Then the function $v(t)$ *is a positive solution in the interval* $[\lambda,\, \lambda + \pi/L)$ *of the inequality*

$$v''(t) + L_1^2\, v(t) \leq \Phi_1(t), \quad t \in [\lambda,\, \lambda + \pi/L_1), \tag{20}$$

where

$$\Phi_1(t) = \int_\Omega f(x,t)\,dx. \tag{21}$$

Proof. From condition (19) it follows that $u(x,t-\tau) \geq 0$ and $\max_{s\in[t-\sigma,t]} u(x,s) \geq 0$ for $(x,t) \in \Omega \times [\lambda, \lambda + \pi/L_1)$. Integrate both sides of equation (1) with respect to x over the domain Ω. For $t \in [\lambda, \lambda + \pi/L_1)$, we obtain

(22)

$$\frac{d^2}{dt^2}\int_\Omega u(x,t)\,dx = -\Big[\int_\Omega \Delta u(x,t)\,dx + \mu_t \int_\Omega \Delta u(x,t-\tau)\,dx\Big] + \int_\Omega c\big(x,t,u(x), \max_{s\in[t-\sigma,t]} u(x,s)\big)\,dx = \int_\Omega f(x,t)\,dx.$$

From Greens formula and condition H7, it follows that

$$\int_\Omega \Delta u(x,t)\,dx = \int_{\partial\Omega} \frac{\partial u}{\partial n}\,ds = -\int_{\partial\Omega} \gamma(x,t)\,u\,ds \leq 0, \tag{23}$$

(24)

$$\int_\Omega \Delta u(x,t-\tau)\,dx = \int_{\partial\Omega} \frac{\partial u}{\partial n}\,(x,t-\tau)\,ds = -\int_{\partial\Omega} \gamma(x,t-\tau)\,u(x,t-\tau)\,ds \leq 0.$$

From condition H4 and the inequality,

$$\max_{s\in[t,-\sigma,t]} u(x,s) \geq u(x,t),$$

it follows that

$$\int_\Omega c\big(x,t,u(x,t), \max_{s\in[t-\sigma,t]} u(x,s)\big)\,dx \geq (K_1^2 + K_2^2)\int_\Omega u(x,t)\,dx. \tag{25}$$

Using (23)–(25), from (22) we derive

$$v''(t) + (K_1^2 + K_2^2)\,v(t) \leq \int \int_\Omega f(x,t)\,dx$$

which was to be proved. Analogously to the proof of Lemma 4, the following lemma is proved:

Lemma 5. *Let condition* (H) *hold,* $L_1 > 0$*, and let* $u(x,t) \in C^2(G) \cap C^1(\overline{G})$ *be a solution of problem* (1), (3)*, satisfying the condition*

$$u(x,t) < 0 \quad \text{for} \quad (x,t) \in \Omega \times (\lambda - T,\, \lambda + \pi/L_1), \quad \lambda \geq T. \tag{26}$$

Then the function $v_1 t = -\,v(t)$ *is a positive solution of the inequality*

$$v_1''(t) + L_1^2\, v_1(t) \leq -\,\Phi_1(t), \quad t \in [\lambda,\, \lambda + \pi/L_1), \tag{27}$$

where the function $\Phi_1(t)$ *is defined by* (21)*. Analogously to the proof of Theorem* 1*, the following theorem is proved.*

Theorem 2. *Let condition* (H) *hold,* $L_1 > 0$*, and let a number* $\lambda \geq T$ *exist such that the following condition should hold*

$$F_1(\lambda) \equiv \int_\lambda^{\lambda + \pi/L_1} \Phi_1(t) \cdot \sin L_1(t - \lambda)\, dt = 0. \tag{28}$$

Then each solution $u(x,t) \in C^2(G) \cap C^1(\overline{G})$ *of problem* (1), (3)*, has a zero in the domain* $\Omega \times (\lambda - T,\, \lambda + \pi/L_1)$.

Corollary 2. *Let conditions* (H) *hold,* $L_1 > 0$*, and let the function* $F_1(\lambda)$ *defined by* (28) *oscillate. Then each solution* $u(x,t) \in C^2(G) \cap C^1(\overline{G})$ *of problem* (1), (3)*, oscillates in* G.

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